Gibbs measures for Brownian paths under the effect of an external and a small pair potential

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Abstract

We consider Brownian motion in the presence of an external and a weakly coupled pair interaction potential and show that its stationary measure is a Gibbs measure. Uniqueness of the Gibbs measure for two cases is shown. Also the typical path behaviour and some further properties are derived. We use cluster expansion in the small coupling parameter.

KEYWORDS: Brownian motion, Gibbs measures, cluster expansion

1 Introduction

The investigation of certain models in quantum field theory involves the study of Gibbs probability measures on Brownian paths under the effect of some potentials. One example is provided by the case of a charged spinless particle coupled to a Bose field. If $\mathcal{F}$ is the Fock space for the boson field, then in the form of second-quantization the Hamiltonian on $L^2(\mathbb{R}^d) \times \mathcal{F}$ describing this situation is

$$H_{pf} = H_p \otimes 1 + 1 \otimes H_f + \phi(x) \tag{1.1}$$

The first term accounts for the energy contribution of the free particle $H_p = -\frac{1}{2}\Delta + V$ subjected to the external potential $V : \mathbb{R}^d \to \mathbb{R}$. The second term is given by $H_f =$

$\int \omega(k)a^\dagger(k)a(k)dk$ corresponding to the contribution of the free scalar Bose field. The field itself is described by the usual creation and annihilation operators $\{a(k), a^\dagger(k) : k \in \mathbb{R}^d\}$ satisfying the commutation relations $[a(k), a^\dagger(k')] = \delta(k-k')$, $[a(k), a(k')] = 0 = [a^\dagger(k), a^\dagger(k')]$. Here $\omega(k) = \sqrt{m^2 + |k|^2}$ is the dispersion-relation, and $m > 0$ if the bosons are massive, othervise $m = 0$. Finally, the third term comes from the coupling between the particle and the field given by

$$\phi(x) = \frac{\beta}{(2\pi)^{d/2}} \int \left(\frac{\hat{\varrho}(k)}{\sqrt{2\omega(k)}} e^{ikx} \otimes a^\dagger(k) + \frac{\hat{\varrho}(-k)}{\sqrt{2\omega(k)}} e^{-ikx} \otimes a(k) \right) dk \tag{1.2}$$

Here $x \in \mathbb{R}^d$ is the position of the particle, $\hat{\varrho}$ denotes the Fourier transform of its charge distribution and $\beta \in \mathbb{R}$ is the coupling constant. For more details on the definition and use of the model, and for recent results we refer to [2, 8] and the appropriate references therein. By studying its properties one is led to evaluate certain "matrix elements" given by the left hand side of the formula below. This formula is known since Nelson's work [10] and it says that these matrix elements can be obtained by computing an expectation over a Brownian motion under an external potential V and pair interaction potential W:

$$(f \otimes \Omega, e^{-tH} g \otimes \Omega) = \int f(X_0)g(X_t)e^{-\int_0^t V(X_s)ds + \beta^2 \int_0^t \int_0^t W(X_s - X_{s'}, s-s')dsds'} d\mathcal{W}_\xi(X)d\xi \tag{1.3}$$

with

$$W(X_s - X_{s'}, s-s') = \frac{1}{4} \int \frac{|\hat{\varrho}(k)|^2}{\omega(k)} \cos\left(k \cdot (X_s - X_{s'})\right)\, e^{-|s-s'|\omega(k)} dk \tag{1.4}$$

Ω stands for the vacuum state in Fock-space, and f and g are suitable test functions on $\mathbb{R}^d$. The integration in (1.3) is made with respect to the Wiener measure $d\mathcal{W}_\xi$ on the space of continuous paths $X_t \in C(\mathbb{R}_+, \mathbb{R}^d)$, $X_0 = \xi$, and the scalar product at the left hand side is taken on the joint particle-field space $L^2(\mathbb{R}^d) \times \mathcal{F}$. This remarkable formula is obtained by representing the field as a Gaussian Markov-field over $\mathcal{S}'(\mathbb{R}^{d+1})$, and integrating over the field variables which in this case can be done explicitly. W is an effective action for the particle resulting from this procedure. After normalization in (1.3) the right hand side can be interpreted as an expectation with respect to a finite time Gibbs measure on path space for the potentials (V, W) and with Wiener measure as reference measure. One basic question is then if this remains true in the $t \to \infty$ limit. As it turns out [2, 8], having an infinite time Gibbs measure at hand brings about interesting informations on the system as a number of relevant quantities can be obtained as Gibbs averages of suitable functions.

In this context the existence of infinite time Gibbs measures was investigated in [11, 1, 2, 8] before, however the understanding of Gibbs measures on path space for general potentials is nowhere near complete and the available results were obtained under various restrictions. In particular, in [11] certain monotonicity, convexity and smoothness conditions are drawn on the potentials. [1] looked at the case of $P(\phi)_1$-processes in which only an external potential was present. In [2] the problem is considered for the potentials quoted above.

Here we address the question whether infinite time Gibbs measures exist for a larger class of pair potentials than in the above example. In order to show existence more generally we however restrict here to sufficiently weak couplings and use a cluster expansion. By taking this approach no prior knowledge on the existence of a ground state is needed. We choose the pair potential in two ways: One is allowing it to grow at most quadratically in the space component and decay with a power larger than 2 of time, the other is choosing it uniformly bounded in the spatial variable and falling off in time with a power larger than 1. The external potential will be assumed sufficiently strong-binding so that the associated Schrödinger operator defines an intrinsically ultracontractive semigroup.

The paper is organized as follows. In the next section we define the Gibbs measure as a perturbation with a weakly coupled pair interaction of a given Markovian reference process. For simplicity we use free boundary conditions. In Section 3 we derive the cluster representation of the partition function. Section 4 contains the cluster estimates. We give a detailed proof for the growing pair potential and indicate how it becomes simpler for the bounded pair potential. In the first subsection we establish the estimates on the weigths of clusters, and in the second we derive another fundamental cluster estimate that will lead up to showing the convergence of the cluster expansion. In Section 5 we look at the typical path behaviour and derive a logarithmic estimate on paths. In the concluding Section 6 we restrict attention to bounded pair potentials and show uniqueness and some further properties of the Gibbs measure (this material being closely related in particular with Nelson's scalar field model).

2 Gibbs measure for the external and pair potential

Let $\mathcal{X} = C(\mathbb{R}, \mathbb{R}^d)$ be the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, endowed with the σ-field $\mathcal{A} = \sigma(\pi_t : t \in \mathbb{R})$ generated by the point evaluations $\pi_t : \mathcal{X} \to \mathbb{R}^d$, $X \mapsto \pi_t(X) = X_t$. $(\mathcal{X}, \mathcal{A})$ is the measurable space we will use in what follows. For a subset $[T_1, T_2] \subset \mathbb{R}$ we put $\mathcal{X}_{[T_1,T_2]} = C([T_1, T_2], \mathbb{R}^d)$ and for the corresponding σ-field we write $\mathcal{A}_{[T_1,T_2]} = \sigma(\pi_t : t \in [-T, T]) \subset \mathcal{A}$. For the interval $[-T, T]$ we will use the notations $\mathcal{X}_T$ respectively $\mathcal{A}_T$. The Lebesgue measure on $\mathbb{R}^d$ will be denoted $d\lambda^d$.

Next, consider two functions $V : \mathbb{R}^d \to \mathbb{R}$ and $W : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ called respectively external potential and pair interaction potential, subject to the following conditions:

(A1) External potential

V is bounded from below and continuous, moreover $V(x) = a|x|^{2s} + o(|x|^{2s})$, with some $s > 1$ and $a > 0$.

(A2) Classes of pair interaction potential

$W(x_1, x_2, t - s) = W(x_1, x_2, |t - s|)$, and it is symmetric with respect to interchanging x_1 with x_2. Moreover,

(A2-1) There is $R > 0$ and $\alpha > 2$ such that

$$|W(x_1, x_2, t - s)| \;\leq\; R\frac{|x_1|^2 + |x_2|^2}{|t - s|^\alpha + 1} \tag{2.1}$$

for every $x_1, x_2 \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$.

(A2-2) There is $R>0$ and $\alpha>1$ such that

$$|W(x_1,x_2,t-s)| \leq \frac{R}{|t-s|^\alpha+1} \tag{2.2}$$

for every $s,t\in\mathbb{R}$ and uniformly in $x_1,x_2\in\mathbb{R}^d$.

Under assumption (A1) for the associated Schrödinger operator

$$H=-\frac{1}{2}\Delta+V(x) \tag{2.3}$$

we have that $C_0^\infty(\mathbb{R}^d)$ is a form core on which the operator is essentially self-adjoint and bounded from below. The bottom of the spectrum E_0 of H is a simple eigenvalue, and the corresponding eigenfunction ψ_0 (ground state) is strictly positive. The semigroup $\{\exp(-tH), t\geq 0\}$ exists on $L^2(\mathbb{R}^d, d\lambda^d)$, and it is an integral operator with positive, continuous, uniformly bounded kernel $G_t(x,y)$. Moreover, the semigroup is intrinsically ultracontractive meaning the following (see [3]). Take the probability measure $d\nu=\psi_0^2 d\lambda^d$ on $\mathbb{R}^d$, and define the isometry $j: L^2(\mathbb{R}^d,d\nu)\to L^2(\mathbb{R}^d,d\lambda^d)$, $f\mapsto \psi_0 f$. Take now in $L^2(\mathbb{R}^d,d\nu)$ the operator H_ν with $\operatorname{Dom} H_\nu = j^{-1}(\operatorname{Dom} H)$ and

$$H_\nu f=(j^{-1}(H-E_0)j)f=\frac{1}{\psi_0}(H-E_0)(\psi_0 f)=-\frac{1}{2}\Delta f-\left(\frac{\nabla\psi_0}{\psi_0},\nabla f\right)_{\mathbb{R}^d} \tag{2.4}$$

for every $f\in\operatorname{Dom} H_\nu$. Then the associated semigroup

$$e^{-tH_\nu}f=\frac{1}{\psi_0}e^{-t(H-E_0)}\psi_0 f \tag{2.5}$$

exists for all $f\in L^2(\mathbb{R}^d,d\nu)$ and $t\geq 0$. Intrinsic ultracontractivity of e^{-tH} means that e^{-tH_ν} is ultracontractive, i.e. it maps $L^2(\mathbb{R}^d,d\nu)$ into $L^\infty(\mathbb{R}^d,d\nu)$ continuously, or, equivalently, $||e^{-tH_\nu}||_{2,\infty}<\infty$, $\forall t\geq 0$.

Consider now a Markov process with stationary measure $d\nu=\psi_0^2 d\lambda^d$ and transition probability density

$$g_t(x|y)=\frac{G_t(x,y)}{\psi_0(x)\psi_0(y)e^{-E_0 t}} \tag{2.6}$$

Denote the probability distribution of this process by dP, and by $dP_{[T_1,T_2]}$ its restriction to the field $\mathcal{A}_{[T_1,T_2]}$. We take this as reference process for constructing the finite time Gibbs specification $\mu_{[T_1,T_2]}$ on $\mathcal{X}_{[T_1,T_2]}$ with free boundary conditions

$$d\mu_{[T_1,T_2]}(X)=\frac{1}{Z_{[T_1,T_2]}}e^{-\lambda\int_{T_1}^{T_2}\int_{T_1}^{T_2}W(X_s,X_t,s-t)dsdt}dP(X) \tag{2.7}$$

for every $X\in\mathcal{X}_{[T_1,T_2]}$, $T_1<T_2\in\mathbb{R}$, and with parameter $\lambda\in\mathbb{R}$. Here $Z_{[T_1,T_2]}$ is the partition function

$$Z_{[T_1,T_2]}=\int e^{-\lambda\int_{T_1}^{T_2}\int_{T_1}^{T_2}W(X_s,X_t,s-t)dsdt}dP(X) \tag{2.8}$$

turning $\mu_{[T_1,T_2]}$ into a probability measure.

Our main result is

Theorem 2.1 *Suppose V and W satisfy assumptions (A1) respectively either (A2-1) or (A2-2). Take an arbitrary decreasing sequence $T_1^{(n)}$ and an increasing sequence $T_2^{(n)}$ of real numbers, $T_1^{(n)} < T_2^{(n)}$, such that $T_1^{(n)} \to -\infty$, $T_2^{(n)} \to \infty$, and $|\lambda| \leq \lambda^*$ with λ^* small enough. Then the weak local limit $\lim_{n\to\infty} \mu_{[T_1^{(n)},T_2^{(n)}]} = \mu$ exists and it is a Gibbs probability measure on $(\mathcal{X}, \mathcal{A})$. Moreover, μ does not depend on the sequences $T_1^{(n)}, T_2^{(n)}$.*

3 Cluster expansion for the partition function

For simplicity, from now on we take the symmetric interval $[-T, T]$, and simplify the corresponding subscripts to T. Consider a division of $[-T, T]$ into disjoint intervals $\tau_k = (t_k, t_{k+1})$, $k = 0, ..., N-1$, with $t_0 = -T$ and $t_N = T$, each of length b, i.e. fix $b = 2T/N$; for convenience we choose N to be even so that the origin is endpoint to some intervals. Denote by X_{τ_k} the piece of the path X obtained by restriction to interval τ_k. The energy contribution of the pair interaction can be written in terms of the sum

$$W_T := \int_{-T}^{T} \int_{-T}^{T} W(X_t, X_s, s-t) ds dt = \sum_{0 \leq i < j \leq N-1} W_{\tau_i, \tau_j} \tag{3.1}$$

where with the notation

$$\mathcal{J}_{ij} = \int_{\tau_i} dt \int_{\tau_j} W(X_s, X_t, s-t) ds \tag{3.2}$$

we have

$$W_{\tau_i,\tau_j} = \begin{cases} \mathcal{J}_{ij} + \mathcal{J}_{ji} & \text{if } |i-j| \geq 2 \\ \frac{1}{2}(\mathcal{J}_{ii} + \mathcal{J}_{jj}) + \mathcal{J}_{ij} + \mathcal{J}_{ji} & \text{if } |i-j| = 1, \text{ and } i \neq 0,\ j \neq N-1 \\ \mathcal{J}_{ij} + \mathcal{J}_{ji} + \frac{1}{2}\,\mathcal{J}_{00} & \text{if } i = 0 \text{ and } j = 1 \\ \mathcal{J}_{ij} + \mathcal{J}_{ji} + \frac{1}{2}\,\mathcal{J}_{N-1\ N-1} & \text{if } i = N-1 \text{ and } j = N-2 \end{cases} \tag{3.3}$$

We prove here an estimate on the double integrals to be used below.

Lemma 3.1 *For $j - i \geq 1$ we have*

$$\int_{\tau_i} dt \int_{\tau_j} W(X_t, X_s, s-t) ds \ \leq \begin{cases} C_1 b\, \frac{\int_{\tau_i} X^2_{\tau_i}(t)dt + \int_{\tau_j} X^2_{\tau_i}(s)ds}{((j-i-1)b)^\alpha + 1} & \textit{(A2-1) case} \\ \frac{C_2 b^2}{((j-i-1)b)^\alpha + 1} & \textit{(A2-2) case} \end{cases} \tag{3.4}$$

with some $C_1, C_2 > 0$ in each case respectively.

Proof: Take the (A2-1) case; clearly it suffices to consider one term of the sum. We have

$$\int_{\tau_i} dt \int_{\tau_j} \frac{X^2_{\tau_i}(t)}{|t-s|^\alpha + 1} ds = \int_{\tau_i} \left(\int_{\tau_j} \frac{ds}{|t-s|^\alpha + 1} \right) X^2_{\tau_i}(t) dt \tag{3.5}$$

Furthermore (since $i < j$)

$$\begin{aligned}
\int_{\tau_j} \frac{ds}{|t-s|^\alpha + 1} &\leq \int_{bj}^{b(j+1)} \frac{ds}{|s-(i+1)b|^\alpha + 1} \\
&= \int_{b(j-i)}^{b(j-i-1)} \frac{du}{|u|^\alpha + 1} \\
&\leq \frac{1}{\alpha - 1} \frac{(1+1/k)^{\alpha-1} - 1}{(b(k+1))^{\alpha-1}}
\end{aligned}$$

where $k = j - i$. Since $(1+1/k)^{\alpha-1} - 1 \leq C/k$ with some number $C > 0$, we obtain

$$\int_{\tau_j} \frac{ds}{|t-s|^\alpha + 1} \leq \frac{C_1 b}{(b(k-1))^\alpha + 1} \tag{3.6}$$

with a suitable number $C_1 > 0$. From this the statement follows. For the case (A2-2) the result is obtained in the same way. $\square$

By using (3.1) we have

$$e^{-\lambda W_T} = \prod_{0 \leq i < j \leq N-1} (e^{-\lambda W_{\tau_i, \tau_j}} + 1 - 1) = 1 + \sum_{\mathcal{R} \neq \emptyset} \prod_{(\tau_i, \tau_j) \in \mathcal{R}} (e^{-\lambda W_{\tau_i, \tau_j}} - 1) \tag{3.7}$$

Here the summation is performed over all nonempty sets of different pairs of intervals, i.e. $\mathcal{R} = \{(\tau_i, \tau_j) : (\tau_i, \tau_j) \neq (\tau_{i'}, \tau_{j'}) \text{ whenever } (i,j) \neq (i', j')\}$.

In order to keep this and the forthcoming summations in hand we need a few more notations. Two distinct pairs of intervals (τ_i, τ_j) and $(\tau_{i'}, \tau_{j'})$ are called *directly connected* and denoted $(\tau_i, \tau_j) \sim (\tau_{i'}, \tau_{j'})$ if one interval of the pair (τ_i, τ_j) coincides with one interval of the pair $(\tau_{i'}, \tau_{j'})$. A set of connected pairs of intervals is a collection $\{(\tau_{i_1}, \tau_{j_1}), ..., (\tau_{i_n}, \tau_{j_n})\}$ in which each pair of intervals is connected to another through a sequence of directly connected pairs, i.e., for any $(\tau_i, \tau_j) \neq (\tau_{i'}, \tau_{j'})$ there exists $\{(\tau_{k_1}, \tau_{l_1}), ..., (\tau_{k_m}, \tau_{l_m})\}$ such that $(\tau_i, \tau_j) \sim (\tau_{k_1}, \tau_{l_1}) \sim ... \sim (\tau_{k_m}, \tau_{l_m}) \sim (\tau_{i'}, \tau_{j'})$. A set of connected pairs of intervals is called a *contour* and denoted by γ. We denote by $\bar{\gamma}$ the set of all intervals that are elements of the pairs of intervals belonging to a contour γ, and by γ^* the set of time-points of intervals appearing in $\bar{\gamma}$. Clearly, $\mathcal{R}$ can be decomposed into maximal connected components, i.e. contours: $\mathcal{R} = \{\gamma_1, ..., \gamma_r\}$ with $\bar{\gamma}_i \cap \bar{\gamma}_j = \emptyset$, $i \neq j$; $i, j = 1, ..., r$.

The sum in (3.7) is then further expanded as

$$\sum_{\mathcal{R} \neq \emptyset} \prod_{(\tau_i, \tau_j) \in \mathcal{R}} (e^{-\lambda W_{\tau_i, \tau_j}} - 1) = \sum_{r \geq 1} \sum_{\gamma_1, ..., \gamma_r} \prod_{k=1}^{r} \prod_{(\tau_i, \tau_j) \in \gamma_k} (e^{-\lambda W_{\tau_i, \tau_j}} - 1) \tag{3.8}$$

where now summation goes over collections $\{\gamma_1, ..., \gamma_r\}$ of contours such that $\bar{\gamma}_k \cap \bar{\gamma}_{k'} = \emptyset$ unless $k = k'$.

A collection of consecutive intervals $\{\tau_j, \tau_{j+1} ..., \tau_{j+k}\}$, $j \geq 0$, $j + k \leq N - 1$ is called a *chain*. As in the case of contours, $\bar{\varrho}$ and ϱ^* mean the set of intervals belonging to the chain ϱ and the set of time-points in ϱ, respectively. We call two contours γ_1, γ_2 disjoint if they have no intervals in common, i.e. $\bar{\gamma}_1 \cap \bar{\gamma}_2 = \emptyset$. Two chains ϱ_1, ϱ_2 are called disjoint if they have no common timepoints, i.e. $\varrho_1^* \cap \varrho_2^* = \emptyset$. Take now a non-ordered set of disjoint contours and disjoint chains, $\Gamma = \{\gamma_1, ..., \gamma_r; \varrho_1, ..., \varrho_s\}$, with some $r \geq 1$ and $s \geq 0$. Note that such contours and chains may have common timepoints. We use the notation $\Gamma^* = (\cup_i \gamma_i^*) \cup (\cup_j \varrho_j^*)$ for the set of all timepoints appearing as beginnings or ends of intervals belonging to some contour or chain in Γ. Also, we put $\bar{\Gamma} = (\cup_i \bar{\gamma}_i) \cup (\cup_j \bar{\varrho}_j)$ for the set of intervals appearing in Γ through entering some contours or chains. Denote by $\partial^- \gamma$ resp. $\partial^+ \gamma$ the leftmost resp. rightmost timepoints belonging to γ, and the similar objects for ϱ. Γ is called a *cluster* if $\{\gamma_1^*, ..., \gamma_r^*; \varrho_1^*, ..., \varrho_s^*\}$ is a connected collection of sets and for every $\varrho \in \Gamma$ we have that $\partial^- \varrho_l, \partial^+ \varrho_l \in \cup_{j=1}^r \gamma_j^*$. This means that in a cluster chains have no loose ends.

Next we fix the positions of path X at the time-points of the division, i.e we put $X_{t_k} = q_k$, for all $k = 0, ..., N$, with $-T = t_0 < t_1 < ... < t_N = T$. The distribution of path X in interval $[-T, T]$ conditional on the positions attained at the fixed times becomes

$$dP_T(X_{\tau_0}, ..., X_{\tau_{N-1}} | X_{t_0} = q_0, X_{t_1} = q_1, ..., X_{t_N} = q_N) = \prod_{k=0}^{N-1} dP_{\tau_k}(X_{\tau_k} | q_k, q_{k+1}) \tag{3.9}$$

We use the shorthand at the right hand side for the corresponding conditional probabilities for easing the notation. Let $p_{t_0,...,t_N}(q_0, ..., q_N)$ be the density with respect to the measure $\prod_{k=0}^N d\nu_k(q_k)$ of the joint distribution of positions of path X measured at the time-points $t_0, ..., t_N$. Here we mean by $d\nu_k$ a copy of $d\nu$ for $k = 0, ..., N$. We have then by the Markov property

$$\begin{aligned} p_{t_0,...,t_N}(q_0, ..., q_N) &= \prod_{k=0}^{N-1} g_b(q_{k+1}|q_k) = \prod_{k=0}^{N-1} (g_b(q_{k+1}|q_k) - 1 + 1) \\ &= 1 + \sum_{\mathcal{M}} \prod_{k:\tau_k \in \mathcal{M}} ((g_b(q_{k+1}|q_k) - 1) \end{aligned}$$

The summation is extended over all nonempty sets $\mathcal{M} = \{\tau_k = (t_k, t_{k+1})\}$ of different pairs of consecutive time-points. In a similar way as before, we rewrite the latter formula in the form

$$\sum_{\mathcal{M}} \prod_{k:\tau_k \in \mathcal{M}} (g_b(q_{k+1}|q_k) - 1) = \sum_{s \geq 1} \sum_{\varrho_1, ..., \varrho_s} \prod_{j=1}^s \prod_{k:\tau_k \in \varrho_j} (g_b(q_{k+1}|q_k) - 1) \tag{3.10}$$

Here $\{\varrho_1, ... \varrho_s\}$ is a collection of connected sets of intervals τ_k such that $\varrho_k^* \cap \varrho_{k'}^* = \emptyset$ unless $k = k'$, i.e. disjoint chains, and this formula explains the way we defined them before.

For every cluster $\Gamma = \{\gamma_1, ..., \gamma_r; \varrho_1, ..., \varrho_s\}$ define the function

$$\kappa_\Gamma = \prod_{l=1}^r \prod_{(\tau_i, \tau_j) \in \gamma_l} (e^{-\lambda W_{\tau_i \tau_j}} - 1) \prod_{i=1}^s \prod_{k:\tau_k \in \varrho_i} (g_b(q_{k+1}|q_k) - 1) \tag{3.11}$$

Also, introduce the auxiliary probability measure on $\mathcal{X}_T$

$$d\,\mathcal{P}_T(X) = \prod_{k=0}^{N-1} dP_{\tau_k}(X_{\tau_k}|q_k, q_{k+1}) \prod_{k=0}^{N} d\nu_k(q_k) \tag{3.12}$$

and look at

$$K_\Gamma = \mathbb{E}_{\mathcal{P}_T}[\kappa_\Gamma] \tag{3.13}$$

Note that $\int(g_b(q_{k+1}|q_k) - 1)d\nu(q_{k+1}) = \int(g_b(q_{k+1}|q_k) - 1)d\nu(q_k) = 0$. This is the reason why from a cluster we rule out chains having loose ends; for any such chain $\mathbb{E}_{\mathcal{P}_T}[\kappa_\Gamma] = 0$.

By putting (2.8), (3.8), (3.9), (3.10), (3.11) and (3.13) together we obtain on the partition function the following

Proposition 3.2 *For every $T > 0$ the cluster representation for the partition function is given by the sum*

$$Z_T = 1 + \sum_{n\geq 1} \sum_{\{\Gamma_1,...,\Gamma_n\}} \prod_{l=1}^{n} K_{\Gamma_l} \tag{3.14}$$

Here the summation is performed over non-ordered collections of clusters $\{\Gamma_1, ..., \Gamma_n\} \neq \emptyset$ for which $\Gamma_i^* \cap \Gamma_j^* = \emptyset$ whenever $i \neq j$.

As soon as the the cluster representation for Z_T is established, the existence of the weak limit measure $\mu := \lim_{T\to\infty} \mu_T$ follows by the cluster estimates below and the general arguments of [9], Chapter 3.

4 Cluster estimates

The first crucial estimate for the cluster expansion is given by

Proposition 4.1 *For every cluster Γ we have the bound*

$$|K_\Gamma| \;\leq\; \prod_{\varrho\in\Gamma} (c_1|\lambda|^{1/3})^{|\bar{\varrho}|} \prod_{\gamma\in\Gamma} \prod_{(\tau_i,\tau_j)\in\gamma} \frac{c_2|\lambda|^{1/3}}{(|i-j-1|b)^\delta + 1} := E_\Gamma(\lambda, \delta) \tag{4.1}$$

with some constants $c_1, c_2, \Lambda > 0$ and exponent $\delta > 1$.

The details of proof of Proposition 4.1 depend on the particular assumptions we make on the pair interaction potential. We give below the proof for the more complicated case and indicate how it becomes simpler for the other one.

4.1 Proof of cluster estimate

We start with potentials V and W satisfying to assumptions (A1) and (A2-1), respectively.

The cluster estimate (4.1) is based on the following generalized version of the Hölder inequality.

Lemma 4.2 *Let $(Y_i, \mathcal{Y}_i, \varphi_i)$, $i = 1, ..., n$ be a collection of probability spaces and take the product $(Y, \mathcal{Y}, \varphi) = \times_{i=1}^{n}(Y_i, \mathcal{Y}_i, \varphi_i)$. Let $\{\beta_1, ..., \beta_m\}$, $\beta_k \subset [1, ..., n] := \mathcal{N}_n$, be a collection of subsets of the interval $\mathcal{N}_n$. Suppose $\{f_{\beta_i}, i = 1, ..., m\}$ are functions on $(Y, \mathcal{Y}, \varphi)$ measurable with respect to the sub-σ-field $\mathcal{Y}_{\beta_k} = \times_{i \in \beta_k} \mathcal{Y}_i \subset \mathcal{Y}$. Furthermore, let $\{n_{\beta_1}, ..., n_{\beta_m}\}$ be numbers larger than 1, $i = 1, ..., m$, such that for any $i \in \mathcal{N}_n$*

$$\sum_{\beta_k : i \in \beta_k} \frac{1}{n_{\beta_k}} \leq 1 \tag{4.2}$$

Then

$$|\int_Y \prod_{j=1}^{m} f_{\beta_k} d\varphi| \leq \prod_{j=1}^{m} \left(\int_{Y_{\beta_k}} |f_{\beta_k}|^{n_{\beta_k}} d\varphi_{\beta_k} \right)^{1/n_{\beta_k}} \tag{4.3}$$

with $(Y_{\beta_k}, \varphi_{\beta_k}) = \times_{i \in \beta_k}(Y_i, \varphi_i)$.

For a proof see [4].

First we estimate the integrals in (3.13) for the X_{τ_k} under fixed $q_0, ..., q_N$. We take in the lemma above $Y_i = C(\tau_i, \mathbb{R}^d)$, i.e. the space of Brownian paths defined on interval τ_i with the field $\mathcal{A}_{\tau_i} = \mathcal{Y}_i$, and $d\varphi_i = dP_{\tau_i}(\,\cdot\,|q_i, q_{i+1})$, the conditional distribution on $C(\tau_i, \mathbb{R}^d)$ appearing in (3.12). Then we write $\beta_k = (i, j)$ if $(\tau_i, \tau_j) \in \gamma_l$, $\gamma_l \in \Gamma$, and $f_{\beta_k} = e^{-\lambda W_{\tau_i, \tau_j}} - 1$. Choose $n_{\beta_k} = n_{ij} = A|i - j|^{\Delta}$, with some $\alpha > \Delta > 1$ to be specified below, and $A \in \mathbb{N}$ such that

$$\sum_{\beta_k : i \in \beta_k} \frac{1}{n_{\beta_k}} \;\leq\; \sum_{j:\, j \neq i} \frac{1}{n_{ij}} \;=\; \frac{1}{A} \sum_{j:\, j \neq i} \frac{1}{|i - j|^{\Delta}} \;\leq\; \frac{2}{A} \sum_{k=1}^{\infty} \frac{1}{k^{\Delta}} \;\leq\; \frac{1}{4} \tag{4.4}$$

Then use Lemma 4.2 to get for any $\gamma_l \in \Gamma$, $l = 1, 2, ...$

$$\begin{aligned} &|\int \prod_{(\tau_i, \tau_j) \in \gamma_l} (e^{-\lambda W_{\tau_i, \tau_j}} - 1) \prod_{i=0}^{N} dP_{\tau_i}(X_{\tau_i}|q_i, q_{i+1})| \;\leq \\ &\prod_{(\tau_i, \tau_j) \in \gamma_l} \left(\int |e^{-\lambda W_{\tau_i, \tau_j}} - 1|^{n_{ij}} dP_{\tau_i}(X_{\tau_i}|q_i, q_{i+1}) dP_{\tau_j}(X_{\tau_j}|q_j, q_{j+1}) \right)^{1/n_{ij}} \end{aligned} \tag{4.5}$$

As a shorthand for

$$\int |e^{-\lambda W_{\tau_i, \tau_j}} - 1|^{n_{ij}} dP_{\tau_i}(X_{\tau_i}|q_i, q_{i+1}) | dP_{\tau_j}(X_{\tau_j}|q_j, q_{j+1}) \tag{4.6}$$

we write $F_{n_{i,j}}(q_i, q_{i+1}, q_j, q_{j+1})$ if $j > i + 1$, and $F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2})$ if $j = i + 1$. Thus for $\Gamma = \{\gamma_1, ..., \gamma_r; \varrho_1, ..., \varrho_s\}$,

$$\begin{aligned} |K_\Gamma| \;\leq\; &|\int \prod_{k=0}^{N} d\nu_k(q_k) \prod_{l=1}^{s} \prod_{\tau_k \in \varrho_l} (g_b(q_{k+1}|q_k) - 1) \times \\ &\times \prod_{\gamma_l \in \Gamma} \prod_{(\tau_i, \tau_{i+1}) \in \gamma_l} \left(F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2})\right)^{1/n_{i,i+1}} \prod_{\substack{(\tau_i, \tau_j) \in \gamma_l \\ j > i+1}} \left(F_{n_{i,j}}(q_i, q_{i+1}, q_j, q_{j+1})\right)^{1/n_{ij}} | \end{aligned} \tag{4.7}$$

follows. Now we apply again Lemma 4.2 to this integral. Take $Y_i = \mathbb{R}^d$, $d\varphi_i = d\nu$, and pick β_k according to the following classes. Put $\beta_k^{(2)} = \{i, i+1\}$, $\beta_k^{(3)} = \{i, i+1, i+2\}$ and $\beta_k^{(4)} = \{i, i+1, j, j+1\}$, $j > i+1$. Take, moreover, $f_{\beta_k^{(2)}} = g_b(q_{i+1}|q_i) - 1$, $f_{\beta_k^{(3)}} = F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2})$ and $f_{\beta_k^{(4)}} = F_{n_{i,j}}(q_i, q_{i+1}, q_j, q_{j+1})$. By choosing the corresponding Hölder exponents $n_{\beta_k^{(2)}} = 4$, $n_{\beta_k^{(3)}} = A = n_{j,j+1}$, and $n_{\beta_k^{(4)}} = A|j-i|^{\Delta}$, and applied with the correct multiplicities we have then by (4.4)

$$\frac{2}{4} + \frac{3}{A} + \sum_{j; j>i+1} \frac{4}{A|i-j|^{\Delta}} \leq \frac{1}{2} + \frac{4}{A} \sum_{k=1}^{\infty} \frac{1}{k^{\Delta}} \leq 1 \tag{4.8}$$

so Lemma 4.2 is applicable. We thus obtain

$$\begin{aligned} |K_\Gamma| \leq & \prod_{l=1}^{s} \prod_{(\tau_k \in \varrho_l} \left(\int |g_b(q_{k+1}|q_k) - 1|^4 d\nu_{k+1}(q_{k+1}) d\nu_k(q_k) \right)^{1/4} \times \\ & \prod_{l=1}^{r} \prod_{(\tau_i, \tau_{i+1}) \in \gamma_l} \left(F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2}) d\nu_i(q_i) d\nu_{i+1}(q_{i+1}) d\nu_{i+2}(q_{i+2}) \right)^{1/A} \times \\ & \prod_{\substack{(\tau_i, \tau_j) \in \gamma_l \\ j > i+1}} \left(F_{n_{ij}}(q_i, q_{i+1}, q_j, q_{j+1}) d\nu_i(q_i) d\nu_{i+1}(q_{i+1}) d\nu_j(q_j) d\nu_{j+1}(q_{j+1}) \right)^{1/n_{ij}} \end{aligned} \tag{4.9}$$

We are going to estimate now each factor separately.

Lemma 4.3 *For sufficiently large $b > 0$ there exist constants $C > 0$ and $\Lambda > 0$ independent of b such that*

$$|g_b(q|q') - 1| \leq Ce^{-\Lambda b} \tag{4.10}$$

uniformly in q, q'.

Proof: As discussed in the previous section, for a potential V satisfying assumption (A1) the associated Schrödinger semigroup $\exp(-tH)$ is intrinsically ultracontractive. By the semigroup property $e^{-bH_\nu} = e^{-H_\nu} e^{-(b-2)H_\nu} e^{-H_\nu}$ we have then that

$$\begin{aligned} |g_b(q|q') - 1| &= \left| \int \int g_1(q|x)(g_{b-2}(x|y) - 1) g_1(y|q') d\nu(x) d\nu(y) \right| \\ &\leq N_1^2 \left(\int \int (g_{b-2}(x|y) - 1)^2 d\nu(x) d\nu(y) \right)^{1/2} \\ &= N_1^2 \, e^{-\Lambda(b-2)} \left(1 + \sum_{n=2}^{\infty} e^{-2(E_n - E_1)(b-2)} \right)^{1/2} \\ &\leq Ce^{-\Lambda b} \end{aligned} \tag{4.11}$$

By taking b large enough, $C > 0$ can be chosen independently of b. Here $N_1 = ||e^{-H_\nu}||_{2,\infty}$, $E_0 < E_1 \leq E_2 \leq \ldots$ are the eigenvalues, and $\Lambda = E_1 - E_0$ is the spectral gap of H. We used that the integral $(\int\int(g_{b-2}(x|y)-1)^2 d\nu(x)d\nu(y))^{1/2}$ is the Hilbert-Schmidt norm of $g_{b-2}(x|y)-1$, moreover that the infinite sum $\sum_n e^{-(E_n-E_1)t}$ converges for every $t>0$; for details see [12]. $\square$

Next we deal with the second factor in (4.9). We have by (3.4)

$$|e^{-\lambda W_{\tau_i,\tau_{i+1}}(X_{\tau_i},X_{\tau_{i+1}})}-1|^A \;\le\; \left(|\lambda||W_{\tau_i,\tau_{i+1}}|\right)^A e^{|\lambda||W_{\tau_i,\tau_{i+1}}|A} \;\le \tag{4.12}$$
$$(c_1 b|\lambda|)^A\left(\int_{\tau_i} X_{\tau_i}(t)^2dt+\int_{\tau_{i+1}} X_{\tau_{i+1}}(t)^2dt\right)^A e^{Ac_1b|\lambda|(\int_{\tau_i}X_{\tau_i}(t)^2dt+\int_{\tau_{i+1}}X_{\tau_{i+1}}(t)^2dt)}$$

with some $c_1>0$. By using (4.11) we get for sufficiently large b that $|g_b(q|q')-1|\le Ce^{-\Lambda b}\le C'<1$, and thus $g_b(q|q')\ge C''>0$, with some C',C''. Hence we also obtain

$$p_{t_i,t_{i+1},t_{i+2}}(q_i,q_{i+1},q_{i+2})=g_b(q_{i+1}|q_i)g_b(q_{i+2}|q_{i+1})>\text{ const} \tag{4.13}$$

where the bound is independent of q_i,q_{i+1},q_{i+2} and b, and this leads to the estimate

$$\int\frac{F_{n_{i,i+1}}(q_i,q_{i+1},q_{i+2})}{p_{t_i,t_{i+1},t_{i+2}}(q_i,q_{i+1},q_{i+2})}\,p_{t_i,t_{i+1},t_{i+2}}(q_i,q_{i+1},q_{i+2})d\nu(q_i)d\nu(q_{i+1})d\nu(q_{i+2}) \;\le$$
$$\text{const }(|\lambda|b)^A\mathbb{E}_P[\left(\int_0^{2b}X_t^2dt\right)^A e^{Ac_1b|\lambda|\int_0^{2b}X_t^2dt}] \tag{4.14}$$

Take now the case $j>i+1$ and write for a shorthand $I_j(X)=\int_{\tau_j}X_t^2dt$. By using (3.4) we arrive at

$$|F_{n_{ij}}(q_i,q_{i+1},q_j,q_{j+1})| \;\le \tag{4.15}$$
$$\int\left(\frac{|\lambda|c_1b\,(I_i(X)+I_j(X))}{(|i-j-1|b)^\alpha+1}\right)^{n_{ij}} e^{\frac{c_1b|\lambda|n_{ij}(I_i(X)+I_j(X))}{(|i-j-1|b)^\alpha+1}}dP_b(X_{\tau_i}|q_i,q_{i+1})dP_b(X_{\tau_j}|q_j,q_{j+1})$$
$$\le\;\frac{1}{2}\left(\frac{2|\lambda|C_1b}{(|i-j-1|b)^\alpha+1}\right)^{n_{ij}}\;\times$$
$$\times\,[\int I_i(X)^{n_{ij}}e^{|\lambda|bI_i(X)}dP_b(X_{\tau_i}|q_i,q_{i+1})\int e^{|\lambda|bI_j(X)}dP_b(X_{\tau_j}|q_j,q_{j+1})\;+$$
$$+\int I_j(X)^{n_{ij}}e^{|\lambda|bI_j(X)}dP_b(X_{\tau_j}|q_j,q_{j+1})\int e^{|\lambda|bI_i(X)}dP_b(X_{\tau_i}|q_i,q_{i+1})]$$

Here we used the notation

$$D=\max\{Ac_1,\frac{c_1n_{ij}}{(|i-j-1|b)^\alpha+1}=\frac{c_1A|i-j|^\Delta}{(|i-j-1|b)^\alpha+1}\}\;<\;\infty \tag{4.16}$$

and the inequality for any $x_1,x_2\ge 0$, $n\ge 1$

$$(x_1+x_2)^n\;\le\;2^{n-1}(x_1^n+x_2^n) \tag{4.17}$$

Thus for $j>i+1$ we obtain

$$\int F_{n_{ij}}(q_i,q_{i+1},q_j,q_{j+1})d\nu(q_i)d\nu(q_{i+1})d\nu(q_j)d\nu(q_{j+1})\;\le \tag{4.18}$$
$$\text{const }\left(\frac{2|\lambda|c_1b}{(|i-j-1|b)^\alpha+1}\right)^{n_{ij}}\mathbb{E}_P[(\int_0^b X_t^2dt)^{n_{ij}}e^{|\lambda|Db\int_0^bX_t^2dt}]\,\mathbb{E}_P[e^{|\lambda|Db\int_0^bX_t^2dt}]$$

The constant prefactor comes about by dividing and multiplying with the densities as in the estimate leading up to (4.14).

We now turn to estimating the P-averages appearing above. Consider the complex function h with parameter $m \in \mathbb{R}$

$$h(z;m) = \mathbb{E}_P[e^{z\int_0^m X_t^2 dt}] = \sum_{n=0}^{\infty} \frac{z^n}{n!}\mathbb{E}_P[(\int_0^m X_t^2 dt)^n] \tag{4.19}$$

for $z \in \mathbb{C}$. We have

$$\begin{aligned} \mathbb{E}_P[(\int_0^m X_t^2 dt)^n] &\leq m^{n-1}\mathbb{E}_P[\int_0^m X_t^{2n} dt] \\ &= m^n \int_{\mathbb{R}^d} |x|^{2n} d\nu(x) \\ &\leq m^n c \int_{\mathbb{R}^d} |x|^{2n} e^{-\epsilon|x|^{s+1}/(s+1)} d\lambda^d(x) \\ &= \frac{2m^n C_d}{s+1}\left(\frac{s+1}{\epsilon}\right)^{(2n+d)/(s+1)} \Gamma\left(\frac{2n+d}{s+1}\right) \end{aligned} \tag{4.20}$$

Here $C_d > 0$ is a constant dependent on the dimension coming from integrating over $d-1$ angular coordinates, and in the inequality above we used the bound for the ground state

$$|\psi_0(x)|^2 \leq ce^{-\epsilon|x|^{s+1}/(s+1)} \tag{4.21}$$

with appropriate constants $c, \epsilon > 0$, see [4]. From here and an application of Stirling's formula to the Gamma-function it follows that $h(z;m)$ is an entire function of order $(s+1)/(s-1)$ [7], and

$$|h(z;m)| \leq c_1 e^{c_2(m|z|)^{(s+1)/(s-1)}} \tag{4.22}$$

with appropriate constants $c_1, c_2 > 0$. Thus

$$\mathbb{E}_P[e^{|\lambda|Db\int_0^{kb} X_t^2 dt}] \leq c_1 e^{c_2(kb^2|\lambda|D)^{(s+1)/(s-1)}} \tag{4.23}$$

for both $k = 1$ and $k = 2$. By (4.22) we get also

$$\begin{aligned} \mathbb{E}_P[(\int_0^{kb} X_t^2 dt)^n e^{|\lambda|Db\int_0^{kb} X_t^2 dt}] &= \left(\frac{d^n}{dz^n}h(z;kb)\right)_{z=|\lambda|Db} \\ &= \frac{n!}{2\pi i}\int_{|\varpi-|\lambda|Db|=b\sqrt{|\lambda|}} \frac{h(\varpi;kb)}{(|\lambda|Db-\varpi)^{n+1}} d\varpi \\ &\leq \frac{c_1 n!}{(|\lambda|b^2)^{n/2}} e^{c_2(kb(|\lambda|Db+b\sqrt{|\lambda|}))^{(s+1)/(s-1)}} \\ &\leq \frac{c_1 n!}{(|\lambda|b^2)^{n/2}} e^{\bar{c}_2(b^2\sqrt{|\lambda|})^{(s+1)/(s-1)}} \end{aligned} \tag{4.24}$$

for both $k = 1$ and $k = 2$. In the latter bound we used that $|\lambda|$ is small enough. Combining (4.14) with (4.24) we find (since $b > 1$)

$$\left(\int F_{n_{i,i+1}}(q_i, q_{i+1}, q_{i+2}) d\nu(q_i) d\nu(q_{i+1}) d\nu(q_{i+2})\right)^{1/A} \leq B\sqrt{|\lambda|} e^{\bar{c}_2(b^2\sqrt{|\lambda|})^{(s+1)/(s-1)}} \quad (4.25)$$

where $B > 0$ is a constant. Similarly, by putting together (4.18), (4.23) and (4.24) we find

$$\left(\int F_{n_{ij}}(q_i, q_{i+1}, q_j, q_{j+1}) d\nu(q_i) d\nu(q_{i+1}) d\nu(q_j) d\nu(q_{j+1})\right)^{1/n_{ij}} \leq \quad (4.26)$$

$$\frac{c_1\sqrt{|\lambda|}(n_{ij}!)^{1/n_{ij}}}{(|i-j-1|b)^\alpha + 1} e^{c_2(b^2\sqrt{|\lambda|})^{(s+1)/(s-1)}} \leq \bar{B} b^2 \sqrt{|\lambda|} \frac{|i-j|^\Delta}{(|i-j-1|b)^\alpha + 1} e^{c_2(b^2\sqrt{|\lambda|})^{(s+1)/(s-1)}}$$

with some constant $\bar{B} > 0$. Take $\Delta > 1$ such that $\delta := \alpha - \Delta > 1$. Finally, choose

$$b = \frac{1}{3\Lambda} \log \frac{1}{|\lambda|} \quad (4.27)$$

From Lemma 4.3 we then have

$$\begin{aligned} \prod_{l=1}^{s} \prod_{\tau_k \in \varrho_l} \left(\int |g_b(q_{k+1}|q_k) - 1|^4 d\nu_{k+1}(q_{k+1}) d\nu_k(q_k)\right)^{1/4} &\leq \prod_{l=1}^{s} (Ce^{-\Lambda b})^{|\bar{\varrho}_l|} \quad (4.28) \\ &\leq \prod_{l=1}^{s} (c_1|\lambda|^{1/3})^{|\bar{\varrho}_l|} \end{aligned}$$

with $|\bar{\varrho}_l|$ denoting the number of intervals $\tau_k \in \varrho_l$. Thus moreover $b^2\sqrt{|\lambda|} \leq |\lambda|^{1/3}$ for small $|\lambda|$. This and estimates (4.25), (4.26) and (4.28) combined complete then the proof of Proposition 4.1. □

For pair potentials of type (A2-2) the proof becomes simpler. In particular, Lemma 4.3 holds unchanged, and the estimates following (4.9) become more straightforward as the pair potential is uniformly bounded in the path.

4.2 Convergence of the cluster expansion

The second fundamental estimate leading up to the proof of Theorem 2.1 and based on (4.1) is

Proposition 4.4 *There is a constant $c > 0$ independent of λ, and a number $0 < \eta(\lambda) < 1$ (with $\eta \to 0$ as $\lambda \to 0$) such that*

$$\sum_{\substack{\Gamma:\Gamma^* \ni 0 \\ |\bar{\Gamma}| = n}} |K_\Gamma| \leq c\, \eta^n \quad (4.29)$$

Proof: In this section we put for a shorthand

$$\varepsilon = c_1|\lambda|^{1/3} \ \text{ and } \ \mathcal{D}(\gamma) = \prod_{(\tau_i,\tau_j)\in\gamma} \frac{\varepsilon}{|i-j-1|^\delta+1} \tag{4.30}$$

Consider the complex function

$$H(z;\lambda) = \sum_{\Gamma:\Gamma^*\ni 0} K_\Gamma z^{|\bar\Gamma|} = \sum_{\substack{\Gamma:\Gamma^*\ni 0\\ \Gamma\supset \text{ one contour}}} K_\Gamma z^{|\bar\Gamma|} + \sum_{\substack{\Gamma:\Gamma^*\ni 0\\ \Gamma\supset \text{ more than one contour}}} K_\Gamma z^{|\bar\Gamma|} \tag{4.31}$$

We show that for sufficiently small $|\lambda|$ this is an analytic function of z in a large circle of radius $R(\lambda)$, with $R(\lambda)\to\infty$ as $\lambda\to 0$. Moreover, we show that within this circle $H(z;\lambda)$ is uniformly bounded in λ. This will then imply (4.29) by choosing $\eta(\lambda) = 1/R(\lambda)$.

Let us start by estimating the second sum. Let $\mathcal{V} = \{\gamma_1,...,\gamma_r\}$, $r\geq 2$ be a collection of pairwise disjoint contours. We view $\mathcal{V}$ as a fixed vertex set and consider all possible connected graphs $\mathcal{G}$ associated with it. Then by Proposition 4.1 the estimate becomes

$$|\sum_{\substack{\Gamma:\Gamma^*\ni 0\\ \Gamma\supset \text{ more than one contour}}} K_\Gamma z^{|\bar\Gamma|}| \leq \tag{4.32}$$

$$\sum_{r\geq 2}\sum_{\{\gamma_1,...,\gamma_r\}}\sum_{\mathcal{G}}\sum_{s\geq 0}\sum_{\substack{\{\hat\varrho_1,...,\hat\varrho_s\}\\ \text{compatible with } \mathcal{G} \text{ and } 0\in\cup_i\gamma_i^*\cup\hat\varrho_i^*}} \prod_{i=1}^r ((|z|(1+\varepsilon|z|))^{|\bar\gamma_i|}\mathcal{D}(\gamma_i)\prod_{j=1}^s(\varepsilon|z|)^{|\bar{\hat\varrho}_j|}$$

The sum over $\mathcal{G}$ is meant to be taken over all connected graphs with vertex set $\mathcal{V}$ and set of edges $\{(\gamma_i,\gamma_j) : \gamma_i,\gamma_j\in\mathcal{G}; i,j=1,...,r\}$; connected graphs are those for which either $\gamma_i^*\cap\gamma_j^* = \emptyset$ or there exists $\varrho_l\in\Gamma$ such that $\varrho_l^*\cap\gamma_i^*\neq\emptyset\neq\varrho_l^*\cap\gamma_j^*$. We denoted by $\{\hat\varrho_1,...,\hat\varrho_s\}$ the collection of chains satisfying to the conditions below:

1. for every pair $(\gamma_i,\gamma_j)\in\mathcal{G}$, $\gamma_i^*\cap\gamma_j^*=\emptyset$, there is at least one chain $\hat\varrho_l$ of this collection connecting γ_i and γ_j (i.e. $\bar{\hat\varrho}_l\cap\bar\gamma_i = \emptyset = \bar{\hat\varrho}_l\cap\bar\gamma_i$ and $\hat\varrho_l^*\cap\gamma_i^*\neq\emptyset\neq\hat\varrho_l^*\cap\gamma_j^*$), and for any pair $(\gamma_i,\gamma_j)\notin\mathcal{G}$ such a chain does not occur;

2. the sets $\cup_i\bar\gamma_i$ and $\cup_j\bar{\hat\varrho}_j$ are disjoint, and $\cup_j\{\partial^-\hat\varrho_j,\partial^+\hat\varrho_j\}\subset\cup_i\gamma_i^*$;

3. $0\in(\cup_i\gamma_i^*)\cup(\cup_j\hat\varrho_j^*)$;

4. each chain $\hat\varrho_k$ connects some pair $(\gamma_i,\gamma_j)\in\mathcal{G}$ (we refer to this event by $\gamma_i\leftrightarrow\gamma_j$), or fills a gap within some contour γ_i (referred to as $\gamma_i\leftrightarrow\gamma_i$).

Note that each $\hat\varrho$ can join only one pair of contours. The collection $\{\hat\varrho_1,...,\hat\varrho_s\}$ of chains is then constructed in the following way:

1. first remove all chains $\varrho_l\in\Gamma$ for which $\bar\varrho_l\subset\cup_{i=1}^r\bar\gamma_i$;

2. for all remaining chains $\varrho_k\in\Gamma$, $\bar\varrho_k\not\subset\cup_{i=1}^r\bar\gamma_i$ remove all intervals from the set $\bar\varrho_k\cap(\cup_{i=1}^r\bar\gamma_i)$;

3. of the remaining intervals form all possible collections of nonempty chains denoted by $\{\hat{\varrho}_1, ..., \hat{\varrho}_m\}$.

Note that for fixed $\{\gamma_1, ..., \gamma_r\}$ the collection $\{\hat{\varrho}_1, ..., \hat{\varrho}_m\}$ can be obtained from many possible collections of chains $\{\varrho_1, ..., \varrho_s\}$. This fact accounts for the factor $(1+\varepsilon\,|z|)^{|\bar{\gamma}|}$ appearing at the right hand side of (4.32).

Thus we get that for any graph $\mathcal{G}$ with vertex set $\mathcal{V}$

$$\sum_s \sum_{\substack{\{\hat{\varrho}_1,\dots,\hat{\varrho}_s\} \\ \text{compatible with } \mathcal{G} \text{ and } 0 \in \cup_i \gamma_i^* \cup \varrho_i^*}} \prod_{j=1}^{s} (\varepsilon\,|z|)^{|\bar{\hat{\varrho}}_j|} \;\leq\; \prod_{(\gamma_i,\gamma_j)\in\mathcal{G}} f_2(\gamma_i,\gamma_j) \prod_{\gamma_i\in\mathcal{V}} f_1(\gamma_i)(\varepsilon\,|z|)^{\mathrm{dist}(0,\{\gamma\})/3} \tag{4.33}$$

with $\mathrm{dist}(0,\{\gamma_i\}) = \min_i \mathrm{dist}(0,\gamma_i)$, $\mathrm{dist}(0,\gamma) = \min_{\tau\in\gamma}\mathrm{dist}(0,\tau)$, and $\mathrm{dist}(0,\tau_i) = N/2 - i - 1$ if $N/2 - 1 \geq i$, and $\mathrm{dist}(0,\tau_i) = i - N/2$ if $N/2 \leq i$ (remember, N is even). Moreover

$$f_2(\gamma_i,\gamma_j) = \sum_{\substack{\{\hat{\varrho}_1,\dots,\hat{\varrho}_{s'}\}\gamma_i\leftrightarrow\gamma_j \\ (\cup_k \bar{\hat{\varrho}}_k)\cap(\bar{\gamma}_i\cup\bar{\gamma}_j)=\emptyset}} \prod_{l=1}^{s'} (\varepsilon\,|z|)^{2|\bar{\hat{\varrho}}_l|/3} \tag{4.34}$$

$$f_1(\gamma_i) = \sum_{\substack{\{\hat{\varrho}_1,\dots,\hat{\varrho}_{s''}\}\gamma_i\leftrightarrow\gamma_i \\ (\cup_k \bar{\hat{\varrho}}_k)\cap\bar{\gamma}_i=\emptyset}} \prod_{l=1}^{s''} (\varepsilon\,|z|)^{2|\bar{\hat{\varrho}}_l|/3} \tag{4.35}$$

In case no such collections of chains appear we put

$$f_2(\gamma_i,\gamma_j) = 1 \tag{4.36}$$

In (4.33) we also used that $0 \in (\cup_{l=1}^{s}\hat{\varrho}_l^*) \cup (\cup_i \gamma_i^*)$ and $\sum_{l=1}^{s} |\bar{\hat{\varrho}}_l| \geq \mathrm{dist}(0,\{\gamma_i\})$.

Lemma 4.5 *For every collection of disjoint contours $\{\gamma_1, ..., \gamma_r\}$, $r \geq 2$ we have the estimate*

$$\sum_{\mathcal{G}} \prod_{(\gamma_i,\gamma_j)\in\mathcal{G}} f_2(\gamma_i,\gamma_j) \;\leq\; 2^{\sum_{i=1}^{r}|\bar{\gamma}_i|} \sum_{\mathcal{T}} \prod_{(\gamma_i,\gamma_j)\in\mathcal{T}} f_2(\gamma_i,\gamma_j) \tag{4.37}$$

Here the sum at the right hand side is taken over all trees $\mathcal{T}$ constructed by using the vertex set $\mathcal{V}$.

Proof: Use Lemma 8 in Chapter 2, Section 4 of [9] to get

$$\sum_{\mathcal{G}} \prod_{(\gamma_i,\gamma_j)\in\mathcal{G}} f_2(\gamma_i,\gamma_j) \leq \prod_{(i,j)} (1 + f_2(\gamma_i,\gamma_j)) \sum_{\mathcal{T}} \prod_{(\gamma_i,\gamma_j)\in\mathcal{T}} f_2(\gamma_i,\gamma_j) \tag{4.38}$$

The product $\prod_{(i,j)}$ is taken over all pairs (i,j), $i,j = 1, ..., r$.

For every pair (γ_i,γ_j) denote by $\sigma_1^{(i,j)}, ..., \sigma_m^{(i,j)}$, $m = m(i,j)$ the chains of 'free' intervals adjacent to some interval in $\bar{\gamma}_i$ respectively $\bar{\gamma}_j$. We have then

$$1 + f_2(\gamma_i,\gamma_j) = \sum_{\mathcal{S}\subset[1,\dots,m]} \prod_{l\in\mathcal{S}} (\varepsilon\,|z|)^{2|\sigma_l^{(i,j)}|/3} = \prod_{k=1}^{m(i,j)} (1 + (\varepsilon\,|z|)^{2|\sigma_k^{(i,j)}|/3}) \;\leq\; 2^{m(i,j)} \tag{4.39}$$

whenever $\varepsilon\,|z| \le 1$. From here

$$\prod_{j\neq i_0}(1+f_2(\gamma_{i_0},\gamma_j)) \le 2^{\sum_{j\neq i_0} m(i_0,j)} \le 2^{2|\bar\gamma_{i_0}|} \tag{4.40}$$

since the number of chains adjacent to contour γ_0 does not exceed twice the number of intervals in γ_0. In the case (4.36) $1+f_2(\gamma_i,\gamma_j)=2$, but estimate (4.40) above stays valid. (4.38) and (4.40) then imply the lemma. $\square$

In a similar way we prove also that

$$\prod_{i=1}^{r} f_1(\gamma_i) \ \le\ 2^{\sum_i |\bar\gamma_i|} \tag{4.41}$$

Denote $\mathrm{dist}(\gamma,\gamma') = \min_{\tau\in\gamma,\tau'\in\gamma'} \mathrm{dist}(\tau,\tau')$, $\mathrm{dist}(\tau_i,\tau_j) = |j-i-1|$, $j\ge i$. For every collection $\{\hat\varrho_1,...,\hat\varrho_{s'}\}$ entering the sum in $f_2(\gamma_i,\gamma_j)$ we have

$$\sum_{l=1}^{s'} |\bar\varrho_l| \ge \mathrm{dist}(\gamma_i,\gamma_j) \tag{4.42}$$

From here we have

$$f_2(\gamma_i,\gamma_j) \ \le\ (\varepsilon\,|z|)^{\mathrm{dist}(\gamma_i,\gamma_j)/3}\,\tilde f_2(\gamma_i,\gamma_j) \tag{4.43}$$

where $\tilde f_2$ is defined similarly to f_2 with $(\varepsilon\,|z|)^{2/3}$ replaced by $(\varepsilon\,|z|)^{1/3}$. Repeating the arguments from the proof of Lemma 4.5 and estimates (4.41) and (4.43) we get for any tree $\mathcal{T}$

$$\prod_{(\gamma_i,\gamma_j)\in\mathcal{T}} f_2(\gamma_i,\gamma_j)\prod_{i=1}^{r} f_1(\gamma_i) \ \le\ 2^{4\sum_{i=1}^{r}|\bar\gamma_i|}\left(\prod_{(\gamma_i,\gamma_j)\in\mathcal{T}} (\varepsilon\,|z|)^{\mathrm{dist}(\gamma_i,\gamma_j)/3}\right) \tag{4.44}$$

Thus we arrive at the estimate

$$\sum_{\substack{\Gamma:\Gamma^*\ni 0\\ \Gamma\supset\text{ more than one contour}}} |K_\Gamma||z|^{|\bar\Gamma|} \ \le \tag{4.45}$$

$$\sum_{r=2}^{\infty}\sum_{\{\gamma_1,...,\gamma_r\}}\prod_{i=1}^{r}(16|z|(1+\varepsilon\,|z|))^{|\bar\gamma_i|}\mathcal{D}(\gamma_i)(\varepsilon\,|z|)^{\mathrm{dist}(0,\{\gamma_i\})/3}\sum_{\mathcal{T}}\prod_{(\gamma_i,\gamma_j)\in\mathcal{T}}(\varepsilon\,|z|)^{\mathrm{dist}(\gamma_i,\gamma_j)/3}$$

Next we pass to summation over trees. First, we take the trees constructed from the vertex set $\{1,...,r\}$ obtained through $\gamma_k\mapsto k$, $\forall k=1,...,r$, and denote them by $\tilde{\mathcal{T}}$. Then we resum in (4.45):

$$\text{r.h.s. (4.45)} \ \le \tag{4.46}$$

$$\sum_{r\ge 2}\frac{1}{r!}\sum_{\tilde{\mathcal{T}}}\sum_{(\gamma_1,...,\gamma_r)}\prod_{i=1}^{r}(16|z|(1+\varepsilon\,|z|))^{|\bar\gamma_i|}(\varepsilon\,|z|)^{\mathrm{dist}(0,\{\gamma_i\})/3}\mathcal{D}(\gamma_i)\prod_{(i,j)\in\tilde{\mathcal{T}}}(\varepsilon\,|z|)^{\mathrm{dist}(\gamma_i,\gamma_j)/3}$$

The third sum here is performed over all ordered collections of contours $\{\gamma_i\}$ such that $\bar{\gamma}_i \cap \bar{\gamma}_j = \emptyset$ whenever $i \neq j$. Since

$$(\varepsilon\,|z|)^{\mathrm{dist}(0,\{\gamma_i\})/3} \;\leq\; \sum_{i=1}^{r} (\varepsilon\,|z|)^{\mathrm{dist}(0,\gamma_i)/3} \tag{4.47}$$

we get that the right hand side of (4.46) is less than

$$\sum_{r=2}^{\infty} \frac{1}{r!} \sum_{\tilde{\mathcal{T}}} \sum_{i_0=1}^{r} \sum_{\{\gamma_1,\ldots,\gamma_r\}} \prod_{i=1}^{r} (16|z|(1+\varepsilon\,|z|))^{|\bar{\gamma}_i|} (\varepsilon\,|z|)^{\mathrm{dist}(0,\gamma_{i_0})/3} \mathcal{D}(\gamma_i) \prod_{(i,j)\in\tilde{\mathcal{T}}} (\varepsilon\,|z|)^{\mathrm{dist}(\gamma_i,\gamma_j)/3} \tag{4.48}$$

Fix $\tilde{\mathcal{T}}$ and i_0, and estimate first

$$\sum_{\{\gamma_1,\ldots,\gamma_r\}} \prod_{i=1}^{r} (16|z|(1+\varepsilon\,|z|))^{|\bar{\gamma}_i|} \mathcal{D}(\gamma_i) \prod_{(i,j)\in\tilde{\mathcal{T}}} (\varepsilon\,|z|)^{\mathrm{dist}(\gamma_i,\gamma_j)/3} \tag{4.49}$$

Let $j_0 \neq i_0$ be an extremal vertex of tree $\tilde{\mathcal{T}}$ being joint only with vertex k_0. Then

$$\begin{aligned} &\sum_{\gamma_{j_0}} (16|z|(1+\varepsilon\,|z|))^{|\bar{\gamma}_{j_0}|} \mathcal{D}(\gamma_{j_0}) (\varepsilon\,|z|)^{\mathrm{dist}(\gamma_{k_0},\gamma_{j_0})/3} \;\leq \\ &\qquad \sum_{\tau''\in\bar{\gamma}_{k_0}} \sum_{\gamma_{j_0}} \sum_{\tau'\in\bar{\gamma}_{j_0}} (\varepsilon\,|z|)^{\mathrm{dist}(\tau',\tau'')/3} (16|z|(1+\varepsilon\,|z|))^{|\bar{\gamma}_{j_0}|} \mathcal{D}(\gamma_{j_0}) \;\leq \\ &\qquad \sum_{\tau''\in\bar{\gamma}_{k_0}} \sum_{\tau'} (\varepsilon\,|z|)^{\mathrm{dist}(\tau',\tau'')/3} \sum_{\gamma_{j_0}:\tau'\in\bar{\gamma}_{j_0}} (16|z|(1+\varepsilon\,|z|))^{|\bar{\gamma}_{j_0}|} \mathcal{D}(\gamma_{j_0}) \end{aligned} \tag{4.50}$$

Here we used the estimate $(\varepsilon\,|z|)^{\mathrm{dist}(\gamma,\gamma')/3} \leq \sum_{\tau\in\gamma,\tau'\in\gamma'} (\varepsilon\,|z|)^{\mathrm{dist}(\tau,\tau')/3}$, obtained similarly as in (4.47).

Lemma 4.6 *There exists a constant $\bar{C} > 0$ such that for any interval τ and number $k \geq 2$*

$$\sum_{\substack{\gamma:\bar{\gamma}\ni\tau \\ |\bar{\gamma}|=k}} \mathcal{D}(\gamma) \;\leq\; (\bar{C}\,\varepsilon)^{k-1} \tag{4.51}$$

Proof:

$$\sum_{\substack{\gamma:\tau\in\bar{\gamma} \\ |\bar{\gamma}|=k}} \mathcal{D}(\gamma) = \sum_{\tau_1,\ldots,\tau_k} \sum_{\mathcal{G}} \prod_{(\tau_i,\tau_j)\in\mathcal{G}} \frac{\varepsilon}{|i-j-1|^{\delta}+1} \tag{4.52}$$

Here $\mathcal{G} = \mathcal{G}(\tau_1,...,\tau_k)$ denotes connected graphs with vertices $\tau_1,...,\tau_k$. Note that for fixed i_0

$$\begin{aligned} \sum_{\tau_j:j\neq i_0} \frac{\varepsilon}{|i_0-j-1|^{\delta}+1} &= 2\,\varepsilon + 2\sum_{k=1}^{\infty} \frac{\varepsilon}{k^{\delta}+1} \\ &\leq 2\,\varepsilon + 2\,\varepsilon \sum_{k=1}^{\infty} \frac{1}{k^{\delta}} \\ &\leq \bar{C}\,\varepsilon \end{aligned} \tag{4.53}$$

with some $\bar{C}>0$. Thus by using (4.38) we find

$$\sum_{\mathcal{G}}\prod_{(\tau_i,\tau_j)\in\mathcal{G}}\frac{\varepsilon}{|i-j-1|^{\delta}+1}\leq e^{k\bar{C}\,\varepsilon}\sum_{\mathcal{T}}\prod_{(\tau_i,\tau_j)\in\mathcal{T}}\frac{\varepsilon}{|i-j-1|^{\delta}+1} \tag{4.54}$$

Now order the collection $\{\tau_1,...,\tau_k\}$ to get further

$$\sum_{\{\tau_1,...,\tau_k\}}\sum_{\mathcal{T}}=\frac{1}{k!}\sum_{\tilde{\mathcal{T}}}\sum_{(\tau_1,...,\tau_k)}\prod_{(i,j)\in\tilde{\mathcal{T}}}\frac{\varepsilon}{|i-j-1|^{\delta}+1} \tag{4.55}$$

At the right hand side the summation goes over all trees $\tilde{\mathcal{T}}$ constructed on the vertices $\{1,...,k\}$. By the same argument as before we then obtain

$$\sum_{(\tau_1,...,\tau_k)}\prod_{(\tau_i,\tau_j)\in\mathcal{T}}\frac{\varepsilon}{|i-j-1|^{\delta}+1}\leq(\bar{C}\,\varepsilon)^{k-1} \tag{4.56}$$

Since the number of trees having k vertices is k^{k-2} [9], by using Stirling's formula, (4.54) and (4.55) we get the result. This completes the proof of the lemma. □

From (4.51) we obtain that (4.50) is less than

$$\begin{aligned}\sum_{\tau''\in\bar{\gamma}_{k_0}}\sum_{\tau'}(\varepsilon\,|z|)^{\mathrm{dist}(\tau',\tau'')/3}\sum_{k=2}^{\infty}(16|z|(1+\varepsilon\,|z|))^k(\bar{C}\,\varepsilon)^{k-1}\ \leq \\ |\bar{\gamma}_{k_0}|\frac{16\bar{C}\,\varepsilon\,|z|(1+\varepsilon\,|z|)^2}{(1-(\varepsilon\,|z|)^{1/3})(1-16\bar{C}\,\varepsilon\,|z|(1+\varepsilon\,|z|))}\end{aligned} \tag{4.57}$$

From now on we choose z such that $16\bar{C}\,\varepsilon\,|z|(1+\varepsilon\,|z|)<1$.

Next we go on by taking the next vertex of $\tilde{\mathcal{T}}$, say $j_1\neq i_0$ connecting with k_1 (that is, we repeat the procedure for the new tree obtained by deleting from $\tilde{\mathcal{T}}$ the vertex j_0 and edge (j_0,k_0). If $j_1\neq k_0$, we get again an estimate of the type (4.57). If $j_1=k_0$, then we estimate

$$\begin{aligned}\sum_{\tau''\in\gamma_{k_1}}\sum_{\tau'}(\varepsilon\,|z|)^{\mathrm{dist}(\tau',\tau'')/3}\sum_{\gamma_{j_1}:\tau'\in\gamma}|\bar{\gamma}_{j_1}|(16|z|(1+\varepsilon\,|z|))^{|\bar{\gamma}_{j_1}|}\mathcal{D}(\gamma)\ \leq \\ \frac{2|\bar{\gamma}_{k_1}|}{1-(\varepsilon\,|z|)^{1/3}}\sum_{k=2}^{\infty}(16|z|(1+\varepsilon\,|z|))^k k(\bar{C}\,\varepsilon)^{k-1}\end{aligned}$$

Continuing this procedure inductively we get that after summation over γ_{j_m}, $j_m\neq i_0$, connected with γ_{k_m}, that its net contribution is

$$\begin{aligned}\frac{2|\bar{\gamma}_{k_m}|}{1-(\varepsilon\,|z|)^{1/3}}\sum_{k=1}^{\infty}(16|z|(1+\varepsilon\,|z|))^k k^{l_{j_m}-1}(\bar{C}\,\varepsilon)^{k-1} &\leq \frac{2|\gamma_{k_m}|}{1-\varepsilon\,|z|}\sum_{k=2}^{\infty}(\hat{C}|z|)^k k^{l_{j_m}-1}(\bar{C}\,\varepsilon)^{k-1} \\ &\leq \text{const}\ |\gamma_{k_m}|\hat{C}|z|\sum_{k=2}^{\infty}(\tilde{C}\,\varepsilon\,|z|)^{k-1}k^{l_{j_1}-1}\end{aligned}$$

where l_{j_m} is the degree of vertex j_m, i.e. the number of edges of $\tilde{\mathcal{T}}$ incident to j_m, and $\tilde{C}=\bar{C}\hat{C}$.

Lemma 4.7 *We have the bound*

$$\sum_{k=2}^{\infty}(\tilde{C}\,\varepsilon\,|z|))^{k-1}k^m \leq \frac{2^m\tilde{C}e\,\varepsilon\,|z|m!}{1-\tilde{C}e\,\varepsilon\,|z|} \tag{4.58}$$

Proof: With the notation $x = -\ln(\tilde{C}\,\varepsilon\,|z|)$ to be used below we have

$$\begin{aligned}
\sum_{k=2}^{\infty}(\tilde{C}\,\varepsilon\,|z|)^{k-1}k^m &\leq 2^m\sum_{k=1}^{\infty}(\tilde{C}\,\varepsilon\,|z|)^k k^m \\
&= 2^m(-\frac{d}{dx})^m\sum_{k=1}^{\infty}e^{-kx} \\
&= 2^m\frac{m!}{2\pi i}\int_{|x-\varpi|=1}\frac{e^{-\varpi}}{1-e^{-\varpi}}\,\frac{d\varpi}{(\varpi-x)^{m+1}} \\
&\leq \frac{2^m\tilde{C}\,\varepsilon\,|z|em!}{1-\tilde{C}e\,\varepsilon\,|z|}
\end{aligned} \tag{4.59}$$

In the first inequality above the bound $k+1 \leq 2k$ for $k \geq 1$, and in the second equality Cauchy's integral formula has been used. □

By using Lemma 4.7 above we estimate (4.49) further for fixed γ_{i_0} and $\tilde{\mathcal{T}}$

$$\begin{aligned}
&\sum_{\substack{\gamma_k:\,k\neq i_0\\ k=1,\ldots,r}}\prod_{(i,j)\in\tilde{\mathcal{T}}}(\varepsilon\,|z|)^{\mathrm{dist}(\tau_i,\tau_j)/3}\prod_{i\neq i_0}\mathcal{D}(\gamma_i)(16|z|(1+\varepsilon\,|z|))^{|\bar{\gamma}_i|} \leq \\
&\qquad |\bar{\gamma}_{i_0}|^{l_{i_0}}\prod_{k\neq i_0}2^{l_{j_k}}(l_{j_k}-1)!\;(k_3\,\varepsilon\,|z|^2)^{r-1}
\end{aligned}$$

where we used that $\varepsilon\,|z| \leq 1/(\tilde{C}e)$. Thus we need to estimate (see (4.48))

$$\begin{aligned}
&\sum_{\gamma_{i_0}}|\bar{\gamma}_{i_0}|^{l_{i_0}}\mathcal{D}(\gamma_{i_0})(16|z|(1+\varepsilon\,|z|))^{|\gamma_{i_0}|}(\varepsilon\,|z|)^{\mathrm{dist}(0,\gamma_{i_0})/3} \leq \\
&\qquad \sum_{\tau_k}(\varepsilon\,|z|)^{\mathrm{dist}(0,\tau)}\sum_{\gamma_i:\tau\in\gamma_{i_0}}|\bar{\gamma}_{i_0}|^{l_{i_0}}\mathcal{D}(\gamma_{i_0})(16|z|(1+\varepsilon\,|z|))^{|\gamma_{i_0}|}
\end{aligned}$$

By repeating the arguments above we get

$$\begin{aligned}
\sum_{\gamma_{i_0}:\tau\in\gamma_{i_0}}|\bar{\gamma}_{i_0}|^{l_{i_0}}\mathcal{D}(\gamma_{i_0})(16|z|(1+\varepsilon\,|z|))^{|\gamma_{i_0}|} &\leq \sum_{k=2}^{\infty}k^{l_{i_0}}(\bar{C}\,\varepsilon)^{k-1}(16|z|(1+\varepsilon\,|z|))^k \\
&\leq \bar{k}_3 2^{l_{i_0}}l_{i_0}!\,\varepsilon\,|z|^2
\end{aligned}$$

Summation over τ gives

$$\sum_{\tau}(\varepsilon\,|z|)^{\mathrm{dist}(0,\tau)} \leq \text{ const} \tag{4.60}$$

hence we finally obtain for fixed $\tilde{\mathcal{T}}$ and i_0

$$2^{l_{i_0}} l_{i_0}! \prod_{j_k \neq i_0} 2^{l_k}(l_{j_k}-1)! c(k_3\,\varepsilon\,|z|^2)^{r-1} \quad \leq \quad c(2^2 k_3\,\varepsilon\,|z|^2)^{r-1} \prod_{k=1}^{r} l_{j_k}!$$

where we used the fact that $\sum_{k=1}^{r} l_{j_k} = 2(r-1)$ holding for trees. An upper bound of the number of trees with vertices $\{1,...,r\}$ and incidence numbers $\{l_1,...,l_r\}$ is [9]

$$\frac{2^{r-2}(r-2)!}{\prod_{j=1}^{r} l_j!} \tag{4.61}$$

Moreover we have that the number of collections $\{l_1,...,l_r\}$ such that $l_i > 0$ and $\sum_i l_i = 2(r-1)$ is bounded from above by $2^{2(r-1)}$. By summing over i_0 and putting this estimate together with (4.61), we get

$$\sum_{\substack{\Gamma:\Gamma^*\ni 0 \\ \Gamma\supset \text{ more than one contour}}} K_\Gamma |z|^{|\bar{\Gamma}|} \leq c\,\varepsilon\,|z|^2 \tag{4.62}$$

with some constant $c > 0$.

Note that the first term appearing in (4.31) can be handled in a similar way than the one leading up to Lemma 4.6, and the same result is obtained. We see then that by choosing z such that $\varepsilon\,|z|^2 \leq \text{const}$, the sum $\sum_\Gamma K_\Gamma z^{|\bar{\Gamma}|}$ converges and is bounded. Hence we get that $H(z)$ is an analytic function within a circle of radius $R(\lambda)$ with $R(\lambda) \to \infty$ as $\lambda \to 0$, and is bounded by a constant independent of λ. Thus

$$\sum_{\substack{\Gamma:\Gamma^*\ni 0 \\ |\bar{\Gamma}|=n}} |K_\Gamma| \;\leq\; \text{const } R(\lambda)^{-n} \;:=\; \text{const } \eta(\lambda)^n \tag{4.63}$$

with appropriate constants. This completes the proof of Proposition 4.4 and thus of Theorem 2.1. □

5 Typical path behaviour

An important aspect of the problem is to understand what a typical path configuration looks like under the Gibbs measure. This is answered by the following theorem

Theorem 5.1 *With μ-probability 1 we have*

$$|X_t| \leq (C \log|t| + 1)^{1/(s+1)} + Q(X) \tag{5.1}$$

with a suitable number $C > 0$ and a functional $Q(X)$, independent of t.

The strategy of proving Theorem 5.1 is to transfer the typical behaviour for the reference process to the level of the Gibbs measure.

Lemma 5.2 *Take an arbitrary $a > 0$. Then there exist $C > 0$ and $\theta > 0$ such that*

$$P\left(\{X \in \mathcal{X} : \max_{0 \le t \le 1} |X_t| \ge a\}\right) \le C\, e^{-\theta a} \tag{5.2}$$

Proof: For the reference process we have the Dirichlet operator on $L^2(\mathbb{R}^d, d\nu)$

$$Lf = -\Delta f + 2(\nabla \log \psi_0, \nabla f) \tag{5.3}$$

The corresponding Dirichlet form is

$$\mathcal{E}(f,f) = -\int f \Delta f d\nu + 2\int f(\nabla \log \psi_0, \nabla f) d\nu \tag{5.4}$$

Now, by using Lemma 1.12 in [6] we have for an arbitrary function $f \in L^2(d\nu)$ and every $N > 0$

$$P\left(\{X \in \mathcal{X} : \max_{0 \le t \le 1} |f(X_t)| \ge N\}\right) \le \frac{3}{N}\sqrt{\mathcal{E}(f,f) + (f,f)} \tag{5.5}$$

Choose f in the form of convolution

$$f := f_a = 1_{\{x \in \mathbb{R}^d : |x| \ge a\}} * \varphi \tag{5.6}$$

by picking a mollifier φ (with $||\varphi||_\infty < \infty$) such that the edges of the indicator function become smooth enough so that the above convolution falls in the domain of L. The mollifier can be chosen so that the smoothing actually takes place around some sphere $S(a)$ of radius a centred at the origin. More precisely, we choose it such that with a suitable $\varepsilon > 0$ we have $f_a(x) = 1$ for $x \in \mathbb{R}^d \setminus S(a+\varepsilon)$, $f_a(x) = 0$ for $x \in S(a-\varepsilon)$, and f_a is some smooth enough function $\tilde{f}_a$ otherwise. Denote these three domains by I, II and III, respectively. Moreover, put $N = 1$ in (5.5); this corresponds to the situation of the path having gone beyond the given level set some time within the given time interval. Thus

$$P(\max_{0 \le t \le 1} |X_t| \ge a) \le 3\sqrt{||f_a||^2_{L^2(d\nu)} + (f_a, Lf_a)_{L^2(d\nu)}} \tag{5.7}$$

We have

$$||f_a||^2_{L^2(d\nu)} = \int f_a^2(x)\psi_0^2(x) d\lambda = \int_{\mathrm{I}} \psi_0^2(x) d\lambda + \int_{\mathrm{III}} \tilde{f}_a^2(x)\psi_0^2(x) d\lambda \tag{5.8}$$

By using the bound (4.21) of the ground state, we estimate (and similarly for III)

$$\int_{\mathrm{I}} \psi_0^2(x) d\lambda \le c e^{-\theta a^{1/(s+1)}} \tag{5.9}$$

where $c, \theta > 0$ are independent of a. On the other hand since $\tilde{f}_a$ is smooth enough and $\max_{III}\{|\nabla \tilde{f}_a|, |\Delta \tilde{f}_a|, \Delta \tilde{f}_a^2\} \le m < \infty$, we get

$$(\tilde{f}_a, L\tilde{f}_a) \le c' e^{-\theta a^{1/(s+1)}} \tag{5.10}$$

with suitable $c' > 0$. A similar estimate is obtained also for the other two domains. $\square$

Theorem 5.1 will be proven once we will have shown

Lemma 5.3 *Suppose there exist some numbers $c, \theta > 0$ such that for any $a > 0$*

$$P(\{X \in \mathcal{X} : \max_{0\le t\le 1} |X_t| \ge a\}) \ \le c\, e^{-\theta a^{1/(s+1)}} \tag{5.11}$$

Then there exist $c' > 0$ and $\theta' > 0$ such that for any $a > 0$

$$\mu(\{X \in \mathcal{X} : \max_{0\le t\le 1} |X_t| \ge a\}) \ \le c' e^{-\theta' a^{1/(s+1)}} \tag{5.12}$$

Proof: Note that

$$\mu(\{X \in \mathcal{X} : \max_{0\le t\le 1} |X_t| \ge a\}) = \int \chi_a(X) d\mu(X) := \mathcal{M}(a) \tag{5.13}$$

where

$$\chi_a(X) = \begin{cases} 0 & \text{if } \max_{0\le t\le 1} |X_t| < a \\ 1 & \text{if } \max_{0\le t\le 1} |X_t| \ge a \end{cases} \tag{5.14}$$

On the other hand

$$\mathcal{M}(a) = \lim_{T\to\infty} \frac{\int e^{-\lambda \int_{-T}^{T}\int_{-T}^{T} W(X_t,X_s,t-s)dtds} \chi_a(X) d\mathcal{P}(X)}{\int e^{-\lambda \int_{-T}^{T}\int_{-T}^{T} W(X_t,X_s,t-s)dtds} d\mathcal{P}(X)} := \lim_{T\to\infty} \frac{Z_T(\chi_a)}{Z_T} \tag{5.15}$$

Here $\mathcal{P}$ is the measure introduced by (3.12). By the cluster expansion we have

$$\begin{aligned} Z_T(\chi_a) &= \mathbb{E}_{\mathcal{P}}[\chi_a](1 + \sum_{r\ge 1} \sum_{\substack{\Gamma_1,\dots,\Gamma_r \\ \Gamma_i^* \cap \Gamma_j^* = \emptyset,\ i\ne j \\ \tau_{N/2} \not\in \cup_i \bar\Gamma_i}} \prod_{i=1}^{r} K_{\Gamma_i}) \\ &+ \sum_{\Gamma_0 : \bar\Gamma_0 \ni \tau_{N/2}} \mathbb{E}_{\mathcal{P}}[\kappa_\Gamma^a](1 + \sum_{n\ge 1} \sum_{\substack{\Gamma_1,\dots,\Gamma_r \\ \Gamma_i^* \cap \Gamma_j^* = \emptyset,\ i\ne j \\ \Gamma_0^* \cap (\cup_i \Gamma_i^*) = \emptyset}} \prod_{i=1}^{r} K_{\Gamma_i}) \\ &= \mathbb{E}_{\mathcal{P}}[\chi_a] Z_T^{\tau} + \sum_{\Gamma : \bar\Gamma \ni \tau_{N/2}} \mathbb{E}_{\mathcal{P}}[\kappa_\Gamma^a] Z_T^{\Gamma} \end{aligned} \tag{5.16}$$

where $\tau_{N/2} = [0, b]$ and $\kappa_\Gamma^a = \chi_a \kappa_\Gamma$. Here Z_T^τ and Z_T^Γ mean the partition functions over $[-T, T] \setminus \tau_{N/2}$ and $[-T, T] \setminus (\bar\Gamma \cup \bar\Gamma^\partial)$, respectively. Here we denoted by $\bar\Gamma^\partial$ the union of intervals having a common time-point with intervals of $\bar\Gamma$. From (5.16)

$$\mathbb{E}_\mu[\chi_a] = \mathbb{E}_{\mathcal{P}}[\chi_a]\, \frac{Z_T^\tau}{Z_T} + \sum_{\Gamma_0 : \bar\Gamma_0 \ni \tau_{N/2}} \mathbb{E}_{\mathcal{P}}[\kappa_{\Gamma_0}^a]\, \frac{Z_T^{\Gamma_0}}{Z_T} \tag{5.17}$$

follows. By general results in [9] the limits $\lim_{T\to\infty} Z_T^\tau / Z_T$ respectively $\lim_{T\to\infty} Z_T^{\Gamma_0}/Z_T$ exist, and moreover they are estimated by

$$\frac{Z_T^\tau}{Z_T} \ < \ 2^3, \quad \frac{Z_T^{\Gamma_0}}{Z_T} \ \le \ 2^{|\bar\Gamma_0 \cup \bar\Gamma_0^\partial|} \ \le \ 2^{3|\bar\Gamma_0|} \tag{5.18}$$

The latter estimate comes from the fact that with every interval in $\bar{\Gamma}_0$ at most two intervals from $\bar{\Gamma}_0^\partial$ can be connected. Hence

$$\mathbb{E}_\mu[\chi_a] \;\leq\; 2^3\mathbb{E}_{\mathcal{P}}[\chi_a] + \sum_{\Gamma_0:\bar{\Gamma}_0\ni\tau_{N/2}} 2^{3|\bar{\Gamma}_0|}\mathbb{E}_{\mathcal{P}}[\kappa_{\Gamma_0}^a] \tag{5.19}$$

On the other hand we have

$$\mathbb{E}_{\mathcal{P}}[\chi_a] = \int \chi_a \frac{dP(X_{\tau_0}|q_0,q_1)}{g_b(q_0|q_1)} g_b(q_0|q_1)d\nu(q_0)d\nu(q_1) \;\leq\; \text{const}\; \mathbb{E}_P[\chi_a] \tag{5.20}$$

where we used that $g_b(q_1|q_0) \geq \text{const}$ in the similar way as at Lemma 4.6. Furthermore,

$$\mathbb{E}_{\mathcal{P}}[\kappa_{\Gamma_0}^a] \leq (\mathbb{E}_{\mathcal{P}}[\chi_a])^{1/2}\; \left(\mathbb{E}_{\mathcal{P}}[\kappa_{\Gamma_0}^2]\right)^{1/2} \tag{5.21}$$

By the same arguments as in Section 4.3 before we obtain the bound

$$\sum_{\Gamma_0:\bar{\Gamma}_0\ni\tau_{N/2}} \mathbb{E}_{\mathcal{P}}[\kappa_{\Gamma_0}^2]^{1/2}\; 2^{3|\bar{\Gamma}_0|} \;\leq\; \text{const} \tag{5.22}$$

with some constant. Hence we get that

$$\mathbb{E}_\mu[\chi_a] \;\leq\; \text{const}_1\; \mathbb{E}_P[\chi_a] + \text{const}_2\; \mathbb{E}_P[\chi_a]^{1/2} \tag{5.23}$$

Lemma 5.2 then implies (5.12). □

We have thus for the stationary measure μ that

$$\mu\left(\max_{n\leq t\leq n+1}|X_t| \;\geq\; (k\log n)^{1/(s+1)}\right) \;\leq\; \text{const}\frac{1}{n^{s\theta'}} \tag{5.24}$$

Now choose k so that $s\theta' > 1$. An application of the Borel-Cantelli lemma implies then that μ-almost surely

$$|X(t)| \;\leq\; (k\log t)^{1/(s+1)} \tag{5.25}$$

for $t \geq T^*$, with $T^* = T^*(X)$ sufficiently large. Then put $Q(X) = \max_{|t|\leq T^*}|X(t)|$. This then completes the proof of the theorem. □

Remark: It may be conjectured that a result similar to Theorem 5.1 can be obtained from the following fact proven in [1] on the reference measure. The set of paths satisfying

$$\liminf_{|t|\to\infty} |t|^\delta\psi_0(X_t) > 0, \quad \forall\delta > 1 \tag{5.26}$$

has P-measure 1. Then we expect that a similar property holds also for the Gibbs measure μ.

6 Some additional properties of Gibbs measures for bounded pair potentials

We devote this final section to proving some further properties of the Gibbs measure for (A2-2) type pair potentials. This case in particular covers Nelson's scalar field model which is discussed further in [8, 2]. In the first subsection we discuss certain properties of the Gibbs measure like its single-time distributions, conditional distributions and covariances. In the second subsection we discuss its uniqueness separately for the cases of $\alpha > 2$ with no restriction on the coupling constant, respectively $\alpha > 1$ with sufficiently weak couplings.

6.1 Some further properties of the Gibbs measure

Theorem 6.1 *Consider the Gibbs measure μ on $(\mathcal{X}, \mathcal{A})$ for V satisfying assumption (A1) and W satisfying (A2-2). Then the following hold:*

1. *μ is invariant with respect to time shift and time reflection, i.e.*

$$\begin{aligned} &\mu \circ \tau_t = \mu, \quad \forall t \in \mathbb{R}, \quad \textit{where} \quad (\tau_s X)_t = X_{s+t} \\ &\mu \circ \vartheta = \mu, \quad \textit{where} \quad (\vartheta X)_t = X_{-t} \end{aligned}$$

2. *The distributions φ_T of positions X_0 at time $t = 0$ generated μ_T are absolutely continuous with respect to ν, i.e. there exist $C_1, C_2 \in \mathbb{R}$, independent of T and X such that*

$$C_1 \; \leq \; \frac{d\varphi_T}{d\nu}(q) \; \leq \; C_2 \tag{6.1}$$

for every $q \in \mathbb{R}^d$ and $T > 0$. Moreover $\lim_{T\to\infty}(d\varphi_T/d\nu)(q) = (d\varphi/d\nu)(q)$ exists pointwise.

3. *The conditional distributions $\mu_T(\,\cdot\,|X_0 = q)$ converge locally weakly to $\mu(\,\cdot\,|X_0 = q)$, for all $q \in R^d$.*

4. *For any bounded functions $F^{(1)}, F^{(2)}$ on $\mathbb{R}^d$ we have the estimate on the covariance*

$$\begin{aligned} &\operatorname{cov}_\mu(F_s^{(1)}; F_t^{(2)}) = \mathbb{E}_\mu[F_s^{(1)} F_t^{(2)}] - \mathbb{E}_\mu[F_s^{(1)}]\, \mathbb{E}_\mu[F_t^{(2)}] \\ &|\operatorname{cov}_\mu\, (F_s^{(1)}; F_t^{(2)})| \; \leq \; const\, \frac{\sup|F_s^{(1)}| \sup|F_t^{(2)}|}{|t-s|^\beta + 1} \end{aligned}$$

where $\beta > 0$, $F_s^{(1)} := F^{(1)}(X_s)$, $F_t^{(2)} := F^{(2)}(X_t)$, and the constant prefactor is independent of s, t and $F^{(1)}, F^{(2)}$.

Proof: (1) The invariance properties are obvious; they carry over from the reference process and the invariance properties of the pair interaction:

$$\begin{aligned} W_{[T_1,T_2]}(X) &= W_{[T_1+t,T_2+t]}(\tau_t^{-1} X) \\ W_{[T_1,T_2]}(X) &= W_{[-T_2,-T_1]}(\vartheta X) \end{aligned}$$

with arbitrary $T_1 < T_2, t$. Indeed, look at the measures $\mu_{[T_1+t,T_2+t]}$ and $\mu_{[-T_2,-T_1]}$; since the limits $T_1 \to -\infty$, $T_2 \to \infty$ yield the same measure μ, the statements follow.

(2) Notice that

$$\frac{d\varphi_T}{d\nu}(q) = \frac{Z_T(q)}{Z_T} \tag{6.2}$$

where

$$Z_T(q) = \int e^{-\int_{-T}^{T}\int_{-T}^{T} W(X_t,X_s,s-t)dsdt} d\mathcal{P}_T(X|X_0=q) \tag{6.3}$$

By the cluster expansion we obtain

$$Z_T(q) = 1 + \sum_{n\geq 1} \sum_{\substack{\Gamma_1,\dots,\Gamma_n \\ 0\notin\cup_i\Gamma_i^*}} \prod_{i=1}^{n} K_{\Gamma_i} + \sum_{\Gamma_0:0\ni\Gamma_0^*} K_{\Gamma_0}(q)(1+\sum_{p\geq 1} \sum_{\substack{\Gamma_1,\dots,\Gamma_p \\ \Gamma_0^*\cap(\cup_i\Gamma_i^*)=\emptyset}} \prod_{i=1}^{p} K_{\Gamma_i}) \tag{6.4}$$

The first term gives the contribution of all clusters disjoint from $t=0$. In the second we take Γ_0 containing $t=0$ and sum over all collections of clusters disjoint from Γ_0. The weight for Γ_0 is obtained like in (3.11) by integration with respect to all variables $\{X_{\tau_i}\}$ and $\{q_k\}$ except for $q_0 = q$. Evidently, for all $q \in \mathbb{R}$, estimate (4.1) for $K_{\Gamma_0}(q)$ stays true. Next we use the general representation for the logarithm of the partition function (see [9]):

$$\log Z_T = \sum_{n\geq 1} \sum_{\eta_n:=\{\Gamma_1,\dots,\Gamma_n\}} \mathcal{D}_{\eta_n} \prod_{\Gamma\in\eta_n} K_\Gamma \tag{6.5}$$

The second summation is performed over all connected collections of n clusters. The coefficients $\mathcal{D}_\eta$ are such that (6.5) is absolutely convergent whenever the clusters satisfy estimate (4.1). Moreover, for any fixed $m \in [-N,N]$

$$\sum_{n\geq 1} \sum_{\substack{\eta_n \\ m\in\eta_n^*}} |\mathcal{D}_{\eta_n}| \prod_{\Gamma\in\eta_n} |K_\Gamma| \leq C \tag{6.6}$$

holds, with some number $C > 0$. (We denoted $\eta^* = \cup_{\Gamma\in\eta}\Gamma^*$.) Let η^0 denote the collection of all possible Γ_0 containing the origin; the corresponding weights are K_{Γ_0}. Hence we obtain from(6.5)

$$\log Z_T(q) - \log Z_T = \sum_{n\geq 1} \sum_{\substack{\eta_n:\eta_n\cap\eta^0\neq\emptyset \\ \eta_n^*\subset[-N,N]}} \mathcal{D}_{\eta_n} \prod_{\Gamma\in\eta_n} \hat{K}_\Gamma - \sum_{n\geq 1} \sum_{\substack{\eta_n:\eta_n\cap\eta^0\neq\emptyset \\ \eta_n^*\subset[-N,N]}} \mathcal{D}_{\eta_n} \prod_{\Gamma\in\eta_n} K_\Gamma \tag{6.7}$$

where

$$\hat{K}_\Gamma = \begin{cases} K_\Gamma & \text{if } \Gamma^* \not\ni 0 \ \ (\Gamma \notin \eta^0) \\ K_\Gamma(q) & \text{if } \Gamma^* \ni 0 \ \ (\Gamma \in \eta^0) \end{cases} \tag{6.8}$$

Clearly,

$$\sum_{n\geq 1} \sum_{\substack{\eta_n:\eta_n\cap\eta^0\neq\emptyset \\ \eta_n^*\subset[-N,N]}} \mathcal{D}_{\eta_n} \prod_{\Gamma\in\eta_n} \hat{K}_\Gamma = \sum_{n\geq 1} \sum_{\substack{\eta_n:\eta_n^*\ni 0 \\ \eta_n^*\subset[-N,N]}} \mathcal{D}_{\eta_n} \prod_{\Gamma\in\eta_n} \hat{K}_\Gamma \tag{6.9}$$

and by (6.6) we get

$$|\sum_{n\geq 1}\sum_{\substack{\eta_n:\eta_n\cap\eta^0\neq\emptyset\\ \eta^*\subset[-N,N]}}\mathcal{D}_{\eta_n}|\prod_{\Gamma\in\eta_n}|\hat{K}_\Gamma|\ \leq\ C \tag{6.10}$$

A similar estimate is obtained for the second term appearing in (6.7). Hence (6.1) follows.

The existence of the limit claimed in (2) follows by (6.7) and

$$\sum_{n\geq 1}\sum_{\substack{\eta_n:\eta_n\cap\eta^0\neq\emptyset\\ \eta_n^*\subset[-N,N]}}\mathcal{D}_{\eta_n}\prod_{\Gamma\in\eta_n}\hat{K}_\Gamma\ \to\ \sum_{n\geq 1}\sum_{\eta_n:\eta_n\cap\eta^0\neq\emptyset}\mathcal{D}_{\eta_n}\prod_{\Gamma\in\eta_n}\hat{K}_\Gamma\quad\text{as}\quad N\to\infty \tag{6.11}$$

The convergence above comes about by combining (6.6) with (6.9). The second term in (6.7) can be treated completely similarly.

(3) The convergence of $\mu_T(\,\cdot\,|X_0=q)$ to $\mu(\,\cdot\,|X_0=q)$ is proved in a similar way as the convergence of μ_T to μ was proven above. In this case K_Γ has to be changed for $\hat{K}_\Gamma$, and as established before, $\hat{K}_{\Gamma_0}(q_0)$ satisfies estimate (4.1).

(4) First note that for any bounded function F on $\mathbb{R}^d$

$$\mathbb{E}_\mu[F_0]=\mathbb{E}_\nu[F_0]f_{\{0\}}+\sum_{\Gamma_0:0\ni\Gamma_0^*}K_{\Gamma_0}(F_0)f_{\{\Gamma_0^*\}} \tag{6.12}$$

and

$$K_{\Gamma_0}(F_0)=\mathbb{E}_{\mathcal{P}}[F_0\kappa_{\Gamma_0}] \tag{6.13}$$

where $F_0=F(X_0)$, see (3.11) and (3.13). Furthermore, $f_A=\lim_{T\to\infty}Z_T^A/Z_T$, and is estimated as before like $|f_A|\leq 2^{|A|}$. For $A_1\cap A_2=\emptyset$ we moreover have

$$|f_{A_1\cup A_2}-f_{A_1}f_{A_2}|\ \leq\ \text{const}\ \frac{2^{|A_1|+|A_2|}}{\text{dist}(A_1,A_2)^\zeta} \tag{6.14}$$

with some $\zeta>0$. This estimate can easily be obtained by the general results from [9]. Now we can write

$$\begin{aligned}\mathbb{E}_\mu[F_s^{(1)}F_t^{(2)}]\quad&=\quad\mathbb{E}_\nu[F_s^{(1)}]\,\mathbb{E}_\nu[F_t^{(2)}]f_{\{s,t\}}\\
&+\sum_{\substack{\Gamma_1\\ \Gamma_1^*\ni s,\Gamma_1^*\not\ni t}}\mathbb{E}_{\mathcal{P}}[F_s^{(1)}\kappa_{\Gamma_1}]\,\mathbb{E}_\nu[F_t^{(2)}]f_{\{t\cup\Gamma_1^*\}}\\
&+\sum_{\substack{\Gamma_2\\ \Gamma_2^*\ni t,\Gamma_2^*\not\ni s}}\mathbb{E}_\nu[F_s^{(1)}]\,\mathbb{E}_{\mathcal{P}}[F_t^{(2)}\kappa_{\Gamma_2}]f_{\{s\cup\Gamma_2^*\}}\\
&+\sum_{\substack{\Gamma_1,\Gamma_2:\Gamma_1^*\cap\Gamma_2^*=\emptyset\\ \Gamma_1:\Gamma_1^*\ni s,\Gamma_2:\Gamma_2^*\ni t}}\mathbb{E}_{\mathcal{P}}[F_s^{(1)}\kappa_{\Gamma_1}]\,\mathbb{E}_{\mathcal{P}}[F_t^{(2)}\kappa_{\Gamma_2}]f_{\{\Gamma_1^*\cup\Gamma_2^*\}}\\
&+\sum_{\Gamma:\Gamma^*\ni s,t}\mathbb{E}_{\mathcal{P}}[F_s^{(1)}F_t^{(2)}\kappa_\Gamma]f_{\{\Gamma^*\}}\end{aligned}$$

From this formula and (6.12) we obtain

$$
\begin{aligned}
\operatorname{cov}_\mu(F_s^{(1)};F_t^{(2)}) \quad = \quad & \mathbb{E}_\nu[F_s^{(1)}]\,\mathbb{E}_\nu[F_t^{(2)}](f_{\{s,t\}} - f_{\{s\}}f_{\{t\}}) \\
& + \sum_{\substack{\Gamma_1 \\ \Gamma_1^* \ni s, \Gamma_1^* \not\ni t}} \mathbb{E}_\nu[F_t^{(2)}]\,\mathbb{E}_{\mathcal{P}}[F_s^{(1)}\kappa_{\Gamma_1}](f_{\{t\cup\Gamma_1^*\}} - f_{\{t\}}f_{\{\Gamma_1^*\}}) \\
& + \sum_{\substack{\Gamma_2 \\ \Gamma_2^* \ni t, \Gamma_2^* \not\ni s}} \mathbb{E}_\nu[F_s^{(1)}]\,\mathbb{E}_{\mathcal{P}}[F_t^{(2)}\kappa_{\Gamma_2}](f_{\{s\cup\Gamma_2^*\}} - f_{\{s\}}f_{\{\Gamma_2^*\}}) \\
& + \sum_{\substack{\Gamma_1,\Gamma_2:\Gamma_1^*\cap\Gamma_2^*=\emptyset \\ \Gamma_1^*\ni s, \Gamma_2^*\ni t}} \mathbb{E}_{\mathcal{P}}[F_s^{(1)}\kappa_{\Gamma_1}]\,\mathbb{E}_{\mathcal{P}}[F_t^{(2)}\kappa_{\Gamma_2}](f_{\{\Gamma_1^*\cup\Gamma_2^*\}} - f_{\{\Gamma_1^*\}}f_{\{\Gamma_2^*\}}) \\
& + \sum_{\Gamma:\Gamma^*\ni s,t} \mathbb{E}_{\mathcal{P}}[F_s^{(1)}F_t^{(2)}\kappa_\Gamma]f_{\{\Gamma^*\}} \\
& - \sum_{\Gamma_2:\Gamma_2^*\ni s,t} \mathbb{E}_\nu[F_s^{(1)}]\,\mathbb{E}_{\mathcal{P}}[F_t^{(2)}\kappa_{\Gamma_2}]f_{\{s\}}f_{\{\Gamma_2^*\}} \\
& - \sum_{\Gamma_1:\Gamma_1^*\ni s,t} \mathbb{E}_\nu[F_t^{(2)}]\,\mathbb{E}_{\mathcal{P}}[F_s^{(1)}\kappa_{\Gamma_1}]f_{\{t\}}f_{\{\Gamma_1^*\}} \\
& - \sum_{\substack{\Gamma_1,\Gamma_2:\Gamma_1^*\cap\Gamma_2^*\neq\emptyset \\ \Gamma_1^*\ni s, \Gamma_2^*\ni t}} \mathbb{E}_{\mathcal{P}}[F_s^{(1)}\kappa_{\Gamma_1}]\,\mathbb{E}_{\mathcal{P}}[F^{(2)}\kappa_{\Gamma_2}]f_{\{\Gamma_1^*\}}f_{\{\Gamma_2^*\}}
\end{aligned}
$$

For estimating the first four terms at the r.h.s. of the above sum we use (6.14) together with the bound

$$
|\mathbb{E}_{\mathcal{P}}[F^{(i)}\kappa_{\Gamma_i}]| \;\le\; \frac{\sup|F^{(i)}|}{(\operatorname{diam}\Gamma^*)^{\zeta'}+1}E_{\Gamma_i}(\delta',\lambda') \tag{6.15}
$$

$i=1,2$, where $E_{\Gamma_i}(\delta',\lambda')$ is the function appearing at the r.h.s. of estimate (4.1) with slightly modified entries (δ',λ' instead of δ,λ; $\delta'>1$) so that (4.29) still holds. Here $\zeta'=\delta-\delta'>0$, and we used in addition that

$$
\frac{1}{(\operatorname{diam}\Gamma^*)^{\zeta'}+1}\;\frac{1}{\operatorname{dist}(s,\Gamma^*)^{\zeta}+1} \;\le\; \frac{1}{|s-t|^{\beta}+1} \tag{6.16}
$$

whenever $t\in\Gamma^*$ (and similarly for $s\in\Gamma^*$), and

$$
\frac{1}{(\operatorname{diam}\Gamma_1^*)^{\zeta'}+1}\;\frac{1}{(\operatorname{diam}\Gamma_2^*)^{\zeta'}+1}\;\frac{1}{\operatorname{dist}(\Gamma_1^*,\Gamma_2^*)^{\zeta}} \;\le\; \frac{1}{|s-t|^{\beta}+1} \tag{6.17}
$$

for $s\in\Gamma_1^*, t\in\Gamma_2^*$, and $\beta=\min\{\zeta,\zeta'\}>0$.

Next, in the fifth term above we used that $\operatorname{diam}\Gamma^*\ge|t-s|$, if $s,t\in\Gamma^*$, and that

$$
|\mathbb{E}_{\mathcal{P}}[F_s^{(1)}F_t^{(2)}\kappa_\Gamma]| \;\le\; \sup|F_s^{(1)}|\sup|F_t^{(2)}|\frac{E_\Gamma(\delta',\lambda')}{(\operatorname{diam}\Gamma^*)^{\zeta'}+1} \tag{6.18}
$$

For the remaining three terms in the sum above we apply the same argument. Thus for the full sum the corresponding bounds become

$$\mathrm{const}\ \frac{\sup|F_s^{(1)}|\sup|F_t^{(2)}|}{|s-t|^\beta+1} E_{\Gamma_1}(\delta',\lambda')E_{\Gamma_2}(\delta',\lambda')\ 2^{|\Gamma_1^*|+|\Gamma_2^*|} \tag{6.19}$$

whenever $s \in \Gamma_1^*, t \in \Gamma_2^*$, respectively

$$\mathrm{const}\ \frac{\sup|F_1|\sup|F_2|}{|s-t|^\beta+1} E_\Gamma(\delta',\lambda')\ 2^{|\Gamma^*|} \tag{6.20}$$

$s \in \Gamma_1^*$ or $t \in \Gamma_2^*$ or $s,t \in \Gamma^*$. Then by summing over Γ_1, Γ_2 respectively Γ, we finally obtain the similar results as in Section 4.3. □

6.2 Uniqueness of the Gibbs measure

In this section we prove uniqueness of the Gibbs measure for two cases. One is for pair potentials of (A2-2) type with exponent $\alpha > 2$ in which case we apply a general argument involving no restriction on the coupling constant. The other is for the same class of pair potentials with exponent $\alpha > 1$ in which case we use cluster expansion and need to restrict to sufficiently small values $|\lambda|$.

First we deal with the $\alpha > 2$ case. Consider $\mu_T(\,\cdot\,|Y)$ given by

$$d\mu_T(X|Y) = \frac{1}{Z_T(Y)} e^{-\lambda W_T(X|Y)} d\mathcal{P}_T(X|Y) \tag{6.21}$$

Here

$$W_T(X|Y) = W_T(X) + W_T^Y(X) \tag{6.22}$$

is the interaction energy in configuration $X \in \mathcal{X}_{[-T,T]}$ given the boundary configuration $Y = Y^- \cup Y^+$, with $Y^- \in \mathcal{X}_{(-\infty,-T]}$ resp. $Y^+ \in \mathcal{X}_{[T,\infty)}$. W_T is as given by the integral in (3.1), and

$$W_T^Y(X) = 2\int_{-\infty}^{-T} dt \int_{-T}^{T} ds\ W(Y_t^-, X_s, t-s) + 2\int_T^\infty dt \int_{-T}^{T} ds\ W(Y_t^+, X_s, t-s) \tag{6.23}$$

$\mathcal{P}_T(\,\cdot\,|Y) = \mathcal{P}_T(\,\cdot\,|Y_{-T}^-, Y_T^+)$ is the conditional distribution of the reference measure for the given boundary condition Y which depends only on the positions attained at $\pm T$ since $\mathcal{P}$ is Markovian. $Z_T(Y)$ is the partition function for $\mu_T(\,\cdot\,|Y)$. It is easily checked that $\{\mu_T(\,\cdot\,|Y)\}$, with $Y \in C(\mathbb{R} \setminus [-T,T], \mathbb{R}^d)$, $0 < T < \infty$, is a compatible and regular family of conditional probability measures defining a specification in the sense of DLR theory. The limiting Gibbs measure μ on $\mathcal{X}$ is well defined also in DLR sense and it is consistent with this specification (see [5] for terminology and details).

Theorem 6.2 *Suppose V is of (A1) class and W is of (A2-2) class with $\alpha > 2$. Then whenever the Gibbs measure μ exists, it is unique in DLR sense.*

Proof: We show that there exists a number $L > 0$ such that for any cylinder function f on $C(\mathbb{R}, \mathbb{R}^d)$ and any pair of Gibbs measures μ_1 and μ_2 consistent with (6.21)

$$\mathbb{E}_{\mu_1}[f] \;\leq\; L\, \mathbb{E}_{\mu_2}[f] \tag{6.24}$$

First we show that once having (6.24), uniqueness follows. Indeed, suppose that the set of Gibbs measures consistent with (6.21) contains at least two elements. Then, since the set of Gibbs measures consistent with a given specification is convex [5], by the Krein-Milman theorem it would then follow that at least two extremal Gibbs measures $\mu_1 \neq \mu_2$ existed. Then some event $E \in \mathcal{A}$ would separate them (see [5]), i.e. we would have $\mu_1(E) = 1$ and $\mu_2(E) = 0$. Equivalently, for any $\varepsilon > 0$ a cylinder event E' would exist such that $\mu_1(E') > 1 - \varepsilon$ and $\mu_2(E') < \varepsilon$. Now choose f above to be the indicator function of this event. Then putting $\varepsilon < 1/(L+1)$ would contradict (6.24).

We now turn to proving (6.24). By using estimate (2.2) with $\alpha > 2$ it is easy to obtain that

$$|W_T^Y(X)| \;\leq\; c_1 \tag{6.25}$$

uniformly in T and in configurations X and Y, with some constant $c_1 > 0$. Now take $T' = T - \tau$, with some $\tau > 0$ to be specified later on. We take an arbitrary path $X \in \mathcal{X}$ and break it up into $\bar{X}_\tau = X|_{[-T',T']}$, $X_\tau^- = X|_{(-T,-T')}$ and $X_\tau^+ = X|_{[T',T]}$. We have then

$$W_T(X) = W_{T'}(\bar{X}_\tau) + W_{T'}^{X_\tau^- \cup X_\tau^+}(\bar{X}_\tau) \tag{6.26}$$

where in the second term integration is made over $[-T,T]^2 \setminus [-T',T']^2 \subset \mathbb{R}^2$ using the pieces X_τ^-, X_τ^+ of path X. Again, by estimate (2.2) we get

$$|W_{T'}^{X_\tau^- \cup X_\tau^+}(\bar{X}_\tau)| \;\leq\; c_2 \tau \tag{6.27}$$

uniformly in the pieces of X, with some $c_2 > 0$.

Lemma 6.3 *There is a number $M = M(\tau) > 0$ such that for any Y and $T > 0$*

$$\frac{1}{M} \;\leq\; \frac{Z_T(Y)}{Z_{T'}} \;\leq\; M \tag{6.28}$$

where $Z_{T'}$ is defined by (2.8).

Proof: By (6.25) and (6.27) we obtain

$$\begin{aligned} Z_T(Y) \;&=\; \int e^{-\lambda W_T(X|Y)} d\mathcal{P}_T(X|Y) \\ &\leq\; e^{|\lambda| c_1} \int e^{-\lambda W_T(X)} d\mathcal{P}_T(X|Y) \\ &\leq\; e^{|\lambda| c_1 + |\lambda| c_2 \tau} \int e^{-\lambda W_{T'}(\bar{X}_\tau)} d\mathcal{P}_T(X|Y) \end{aligned} \tag{6.29}$$

Put for a shorthand $q^{\pm} = (\bar{X}_\tau)_{\pm T'}$, i.e. the positions of $\bar{X}_\tau$ at the endpoints of the interval $[-T', T']$. Then by using the Markov property of the reference measure we have furthermore

$$\int e^{-\lambda W_{T'}(\bar{X}_\tau)} d\mathcal{P}_T(X|Y) =$$
$$\int e^{-\lambda W_{T'}(\bar{X}_\tau)} d\mathcal{P}_{T'}(\bar{X}_\tau|q^-, q^+) g_\tau(q^-|Y^-_{-T}) g_\tau(q^+|Y^+_T) d\nu(q^-) d\nu(q^+) \tag{6.30}$$

with g_τ as under (2.6). Using now (4.11) we get that

$$\frac{1}{2} \leq 1 - Ce^{-\Lambda\tau} \leq g_\tau(q^{\pm}|Y^{\pm}_{\pm T}) \leq 1 + Ce^{-\Lambda\tau} \leq 2 \tag{6.31}$$

for large enough τ. Fixing now such a τ we obtain by (6.29), (6.30) and (6.31) that

$$Z_T(Y) \leq 2e^{|\lambda|(c_1+c_2\tau)} Z_{T'} \tag{6.32}$$

Similarly, we get the converse inequality

$$Z_T(Y) \geq \frac{1}{2} e^{|\lambda|(c_1+c_2\tau)} Z_{T'} \tag{6.33}$$

(6.32) and (6.33) taken together imply then (6.28) with $M = 2e^{|\lambda|(c_1+c_2\tau)}$. □

Now we return to showing (6.24). Let $f = f_{T'}$ be the cylinder function $f_{T'} = f_{T'}(X|_{[-T',T']})$. By using the DLR equation we obtain

$$\begin{aligned} \mathbb{E}_{\mu_1}[f_{T'}] &= \int \frac{f_{T'}(\bar{X}_\tau) e^{-\lambda W_T(X|Y)}}{Z_T(Y)} d\mathcal{P}_T(X|Y) d\mu_1(Y) \\ &\leq M \frac{1}{Z_{T'}} \int f_{T'}(\bar{X}_\tau) e^{-\lambda W_T(X|Y)} d\mathcal{P}_T(X|Y) d\mu_1(Y) \end{aligned}$$

By the same arguments as above we obtain furthermore

$$\int f_{T'}(\bar{X}_\tau) e^{-\lambda W_T(X|Y)} d\mathcal{P}_T(X|Y) d\mu_1(Y) \leq M \int f_{T'}(\bar{X}_\tau) e^{-\lambda W_{T'}(\bar{X}_\tau)} d\mathcal{P}_{T'}(\bar{X}_\tau) \tag{6.34}$$

and from here

$$\mathbb{E}_{\mu_1}[f_{T'}] \leq M^2 \mathbb{E}_{\mu_{T'}}[f_{T'}] \tag{6.35}$$

This completes the proof of the theorem. □

Now we turn to the $\alpha > 1$ case.

Theorem 6.4 *Suppose V is of class (A1) and W is of class (A2-2) with $\alpha > 1$. Then for sufficiently small $|\lambda|$ the limiting Gibbs measure μ is unique.*

Proof: We consider the class of bounded local functions F_S on $C(\mathbb{R}, \mathbb{R}^d)$ indexed by finite intervals $S \subset \mathbb{R}$ (that is, F_S is measurable with respect to $\mathcal{A}_S$). For the uniqueness of the Gibbs measure it suffices to prove that for any increasing sequence $\{T_n\}$, $T_n \to \infty$, and any corresponding sequence of boundary conditions $\{Y_n\}$

$$\lim_{n\to\infty} \mathbb{E}_{\mu_{T_n}(\,\cdot\,|Y_n)}[F_S] = \mathbb{E}_\mu[F_S] \tag{6.36}$$

for arbitrary F_S of the above class. In order to show this we express the conditional expectations appearing above in terms of the cluster representation. We suppose without loss of generality that S consists of a finite union of intervals of the partition of $[-T,T]$.

From now on we follow the steps of the cluster expansion we explained in Sections 3 and 4 before. Take the same partition of the interval $[-T,T]$ into disjoint segments as before. The interaction energy can then be written as

$$W_T(X|Y) = \sum_{0\le i<j\le N} W_{\tau_i,\tau_j}(X_{\tau_i}, X_{\tau_j}) + \sum_{0\le i\le N-1} W^Y_{\tau_i,T}(X_{\tau_i}) \tag{6.37}$$

with the same notations as before, and with

$$W^Y_{\tau_i,T}(X_{\tau_i}) = 2\int_{-\infty}^{T} ds \int_{\tau_i} dt\, W(X_t, Y_s^-, s-t) + 2\int_T^\infty ds \int_{\tau_i} dt\, W(X_t, Y_s^+, s-t) \tag{6.38}$$

By (2.2) the estimate

$$|W^Y_{\tau_i,T}| \;\le\; \frac{Cb}{(\mathrm{dist}(\tau_i,[-T,T]^c)+1)^{\alpha-1}} \tag{6.39}$$

easily follows. (Here $[-T,T]^c = \mathbb{R}\setminus[-T,T]$ and $\mathrm{dist}(\tau_k,[-T,T]^c) = \min\{kb,(N-k-1)b\}$.) We fix the positions $X_{t_0} = Y^-_{-T} = q_0$, $X_{t_1} = q_1$, ..., $X_{t_{N-1}} = q_{N-1}$, $X_{t_N} = Y^+_T = q_N$. Similarly to (3.12) we introduce the auxiliary measure

$$d\mathcal{P}^Y_T = \prod_{k=0}^{N-1} \frac{e^{-W^Y_{\tau_k,T}(X_k)}}{\mathcal{Z}^T_{\tau_k}(Y|q_k,q_{k+1})} dP_b(X_{\tau_k}|q_k,q_{k+1}) \prod_{k=1}^{N-1} d\nu_k(q_k) \tag{6.40}$$

where

$$\mathcal{Z}^T_{\tau_k}(Y|q_k,q_{k+1}) = \mathbb{E}_{P_b(\,\cdot\,|q_k,q_{k+1})}\left[e^{-W^Y_{\tau_k,T}(X_{\tau_k})}\right] \tag{6.41}$$

Also, for every cluster Γ we consider the function κ^Y_Γ defined similarly to (3.11) by changing the measure $\mathcal{P}_T$ for $\mathcal{P}^Y_T$. If $\pm T \notin \Gamma^*$ then κ^Y_Γ does not depend on Y. If $-T \in \Gamma^*$ and/or $T \in \Gamma^*$ then κ^Y_Γ depends on $Y^-_{-T} = q_0$ and/or $Y^+_T = q_N$, respectively. Next we define the weights like under (3.13):

$$K^Y_\Gamma = \mathbb{E}_{\mathcal{P}^Y_T}[\kappa^Y_\Gamma] \tag{6.42}$$

The partition function $Z_T(Y)$ can be expressed then similarly to (3.14) with these adapted objects. Since Lemma 4.3 holds unchanged and gives a uniform estimate in paths, κ^Y_Γ is estimated in the same way as κ_Γ. Combining thus (6.39) with (4.27) we obtain

$$\begin{aligned}\exp\left(-\frac{C|\lambda|^{1/3}}{(\mathrm{dist}(\tau_k,[-T,T]^c)+1)^{\alpha-1}}\right) \;&\le\; \frac{\exp\left(-W^Y_{\tau_k,T}(X_{\tau_k})\right)}{\mathcal{Z}^Y_{\tau_k}(Y|q_k,q_{k+1})} \\ &\le\; \exp\left(\frac{C|\lambda|^{1/3}}{(\mathrm{dist}(\tau_k,[-T,T]^c)+1)^{\alpha-1}}\right)\end{aligned} \tag{6.43}$$

This implies then that

$$|K_\Gamma| \prod_{\substack{\tau_k \in \cup\bar{\gamma} \\ \gamma\in\Gamma}} \exp\left(-\frac{C|\lambda|^{1/3}}{(\mathrm{dist}(\tau_k,[-T,T]^c)+1)^{\alpha-1}}\right) \le |K_\Gamma^Y| \tag{6.44}$$

$$\le |K_\Gamma| \prod_{\substack{\tau_k \in \cup\bar{\gamma} \\ \gamma\in\Gamma}} \exp\left(\frac{C|\lambda|^{1/3}}{(\mathrm{dist}(\tau_k,[-T,T]^c)+1)^{\alpha-1}}\right)$$

At this point it is useful to make two remarks:

1. For sufficiently small $|\lambda|$ the bound

$$|K_\Gamma^Y| \le 2^{|\bar{\Gamma}|}|K_\Gamma| \tag{6.45}$$

holds (the factor 2 can actually be replaced by any number larger than 1). Thus the cluster estimate (4.29) obtained in Proposition 4.4 stays essentially valid:

$$\sum_{\substack{\Gamma:\Gamma^*\ni 0 \\ |\bar{\Gamma}|=n}} |K_\Gamma^Y| \le c\eta'(\lambda) \tag{6.46}$$

with $\eta'(\lambda) = 2\eta(\lambda)$, going to zero as $\lambda \to 0$.

2. For any fixed Γ we have

$$\lim_{T\to\infty} \kappa_\Gamma^Y = \kappa_\Gamma \tag{6.47}$$

as noted before, and thus

$$\lim_{T\to\infty} K_\Gamma^Y = K_\Gamma \tag{6.48}$$

both converging uniformly in Y.

Then in (6.36) we have

$$\mathbb{E}_{\mu_T}[F_S|Y] = \mathbb{E}_{\mathcal{P}_T^Y}[F_S]\ \frac{Z_T^S(Y)}{Z_T(Y)} + \sum_{n\ge 1} \sum_{\substack{\{\Gamma_1,\dots,\Gamma_m\}:\Gamma_i^*\cap\Gamma_j^*\neq\emptyset \\ \Gamma_i^*\cap S\neq 0,\Gamma_i\subset[-T,T],i=1,\dots,m}} \mathbb{E}_{\mathcal{P}_T^Y}\left[F_S \prod_{i=1}^m \kappa_T^Y\right] \frac{Z_T^{\cup\bar{\Gamma}}(Y)}{Z_T(Y)} \tag{6.49}$$

with the same notations as in (5.16), and specifically $Z_T^{\cup\bar{\Gamma}} = Z_{[-T,T]\setminus\cup_{i=1}^m\bar{\Gamma}_i}$.

Take now a collection of intervals $\{\tau_i\} = \mathcal{U}$; the partition function $Z_{[-T,T]\setminus\cup_{\tau_i\in\mathcal{U}}\tau_i}(Y) \equiv Z_T^{\mathcal{U}}(Y)$ can then be written like in (3.14) only by changing K_Γ for K_Γ^Y and summing over collections of clusters lying inside $[-T,T]\setminus\cup_{\tau_i\in\mathcal{U}}\tau_i$ (i.e., $\Gamma^*\cap\partial\mathcal{U}=\emptyset$).

Lemma 6.5 *For sufficiently small $|\lambda|$ we have the following properties of $f_T^{\mathcal{U}}(Y) := \frac{Z_T^{\mathcal{U}}(Y)}{Z_T(Y)}$. On the one hand,*

$$|f_T^{\mathcal{U}}(Y)| \le 2^{|\mathcal{U}|} \tag{6.50}$$

$|\mathcal{U}|$ denoting the number of intervals contained by $\mathcal{U}$. On the other hand,

$$\lim_{T\to\infty} f_T^{\mathcal{U}}(Y) = f^{\mathcal{U}} \tag{6.51}$$

uniformly in Y. Moreover, $f^{\mathcal{U}}$ also satisfies (6.50) above.

Proof: The estimate (6.50) follows by the same token as explained around (5.18). To obtain (6.51) we use a similar decomposition as in (6.5) to write

$$f_T^{\mathcal{U}}(Y) = 1 + \sum_{\substack{\eta=\{\Gamma\} \\ \Gamma^* \subset [-T,T]}} \tilde{\mathcal{D}}_{\mathcal{U}}(\eta) \prod_{\Gamma \in \eta} K_\Gamma^Y \tag{6.52}$$

and

$$f^{\mathcal{U}} = 1 + \sum_{\eta=\{\Gamma\}} \tilde{\mathcal{D}}_{\mathcal{U}}(\eta) \prod_{\Gamma \in \eta} K_\Gamma \tag{6.53}$$

where both summations go over nonempty collections of clusters $\eta = \{\Gamma\}$ for which $\cup_{\Gamma \in \eta} \Gamma^* \cap \mathcal{U}^* \neq \emptyset$ ($\mathcal{U}^*$ is the set of all time points occurring in $\mathcal{U}$), and moreover in the first term the clusters are required to lie inside $[-T,T]$. The coefficients appearing in both sum are such that the series (6.53) is absolutely convergent whenever for the K_Γ occurring there (4.29) is satisfied with some small enough bound $\eta(\lambda)$. Hence by (6.45) we see that the series (6.52) is term by term dominated by $\sum_\eta |\tilde{\mathcal{D}}(\eta)| \prod_{\Gamma \in \eta} 2^{|\bar{\Gamma}|} |K_\Gamma|$ (here summation goes over all collections of clusters crossing $\mathcal{U}$). Then by putting

$$\hat{K}_\Gamma^Y = \begin{cases} K_\Gamma^Y & \text{if } \Gamma \text{ lies inside } [-T,T] \\ 0 & \text{otherwise} \end{cases} \tag{6.54}$$

and using Lebesgue's dominated convergence theorem we obtain that $f_{\mathcal{U}}^Y \to f_{\mathcal{U}}$ uniformly as $T \to \infty$. □

We now return to the expression (6.49). By the ergodicity of the reference measure

$$\lim_{T\to\infty} \mathbb{E}_{P_T^Y}[F_S] \to \mathbb{E}_P[F_S] \tag{6.55}$$

and hence the first term of (6.49) converges to $\mathbb{E}_{\mathcal{P}}[F_S] f_S$. By the same argument as above we obtain also

$$\lim_{T\to\infty} \mathbb{E}_{\mathcal{P}_T^Y}\left[F_S \prod_{i=1}^m \kappa_{\Gamma_i}^Y\right] = \mathbb{E}_{\mathcal{P}}\left[F_S \prod_{i=1}^m \kappa_{\Gamma_i}\right] \tag{6.56}$$

uniformly in Y, and we also have

$$|\mathbb{E}_{\mathcal{P}_T^Y}[F_S \prod_{i=1}^m \kappa_{\Gamma_i}^Y]| \;\leq\; \max |F_S| \prod_{i=1}^m 2^{|\Gamma_i|} |K_{\Gamma_i}| \tag{6.57}$$

By (6.57) we then obtain

$$\begin{aligned} \sum_{\substack{\{\Gamma_1,\dots,\Gamma_m\} \\ \Gamma_j^* \cap S^* = \emptyset, j=1,\dots,m}} \prod_{i=1}^m 2^{|\bar{\Gamma}_i|} |K_{\Gamma_i}| 2^{|\cup_i \bar{\Gamma}_i|} 2^{|S^*|} \;&\leq\; 2^{|S^*|} \sum_{m=1}^{|S^*|} \binom{|S^*|}{m} \left(\sum_{\Gamma:\Gamma \ni 0} 4^{|\bar{\Gamma}|} |K_\Gamma|\right)^m \\ &\leq\; 2^{|S^*|} \sum_{m=1}^{|S^*|} \binom{|S^*|}{m} \left(\sum_{n=2}^{\infty} 4^n c \eta(\lambda)^n\right)^m < \infty \end{aligned}$$

Here S^* denotes the time points occurring in S. By using now this estimate together with (6.56), and applying Lebesgue's dominated convergence theorem once again we arrive to

$$\mathbb{E}_{\mathcal{P}}[F_S]f_S + \sum_{m=1}^{\infty} \sum_{\substack{\{\Gamma_1,\dots,\Gamma_m\} \\ \Gamma_i^* \cap S^* \neq \emptyset, i=1,\dots,m}} \mathbb{E}_{\mathcal{P}}\left[F_S \prod_{i=1}^{m} \kappa_{\Gamma_i}\right] f_{S\cup(\cup_i \bar{\Gamma}_i)} = \mathbb{E}_{\mu}[F_S] \tag{6.58}$$

This completes the proof of the theorem. □

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