SCATTERING THEORY FOR THE DIRAC OPERATOR WITH A LONG-RANGE ELECTROMAGNETIC POTENTIAL

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Abstract

We consider the Dirac operator with a long-range potential $V(x)$. Scalar, pseudo-scalar and vector components of $V(x)$ may have arbitrary power-like decay at infinity. We introduce wave operators with time-independent modifiers. These modifiers are pseudo-differential operators whose symbols are, roughly speaking, constructed in terms of approximate eigenfunctions of the stationary problem. We derive and solve eikonal and transport equations for the corresponding phase and amplitude functions. From analytical point of view, our proof of the existence and completeness of the wave operators relies on the limiting absorption principle and radiation estimates established in the paper. This allows us to fit the long-range scattering theory for the Dirac operator into the framework of smooth perturbations. Finally, we find the asymptotics for large times t of solutions $u(x,t)$ of the time-dependent Dirac equation.

1. INTRODUCTION

1.1. For short-range interactions V (decaying faster than the Coulomb potential), scattering theory for the Dirac operator H is completely analogous to the Schrödinger case. The situation is more complicated for long-range interactions (decaying as the Coulomb potential or slower) when the free dynamics e^{-iH_0t} cannot be used as an approximation to e^{-iHt}. An appropriate modification of e^{-iH_0t} relies always on solutions of an eikonal and a transport equation for the phase and the amplitude functions determining the modified dynamics. In general, in the case of systems these equations are not efficient and are difficult to handle. Nevertheless for the Dirac operator we derive an eikonal equation for the phase function which is quite similar to the Schrödinger case. Moreover, we find an explicit solution of the corresponding transport equation (which can be neglected in the Schrödinger case) for the amplitude function.

The existing papers on scattering theory for the Dirac operator with a long-range perturbation rely mainly on the Enss method which is intrinsically time-dependent. The results on the existence and completeness of modified wave operators are somewhat patchy. In the Coulomb case modified wave operators were constructed in [5] basically in the same way as in [4]. Following the ideas of [3] for the Schrödinger operator, the construction (without proof) for a general class of long-range electrostatic perturbations was given in [22]. The more difficult problem of completeness was first studied in [10] where a purely electrostatic potential with an arbitrary decay was considered. In [15] detailed results on the asymptotic behaviour of observables such as position, velocity or projections on the positive and negative energy states were obtained. These results were used in [11] to study a general electromagnetic potential decaying as $|x|^{-1}$ for $|x| \to \infty$.

Our goal here is to develop a consistent stationary approach to scattering theory for the Dirac operator with a general long-range electromagnetic potential. In particular, we prove the existence and completeness of wave operators for an arbitrary (power-like) decay of perturbation at infinity. We modify the free dynamics in the coordinate x and momentum ξ variables which amounts to introduction of a modifier (called also an "identification") $J_\pm$ intertwining approximately the operators H_0 and H and depending on the sign of the time t. The operators $J_\pm$ emerge naturally as pseudo-differential operators with symbols $j_\pm(x,\xi)$ constructed (up to different cut-off functions) in terms of approximate eigenfunctions of the operator H. In the Schrödinger case, the scheme adopted here was developed in [19] and the construction of the operators $J_\pm$ goes back to [7].

1.2. Let

$$H_0 = \sum_{k=1}^{3} \alpha_k D_k + m\alpha_0, \quad m > 0, \quad D_k = -i\partial/\partial x_k, \tag{1.1}$$

be the self-adjoint operator in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathcal{D}(H_0) = \mathbf{H}^1(\mathbb{R}^3, \mathbb{C}^4)$ (the Sobolev class). Recall that the (4×4)-Dirac matrices $\alpha_k = \alpha_k^*$ satisfy the anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I, \quad j,k = 0,1,2,3. \tag{1.2}$$

They may be chosen as

$$\alpha_0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} \quad \text{and} \quad \alpha_k = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix} \quad \text{if} \quad k = 1,2,3,$$

where

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are the Pauli matrices.

The free Dirac operator H_0 can be easily diagonalized by the Fourier transform $\mathcal{F}$,

$$(\mathcal{F}f)(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\langle \xi, x\rangle} f(x)dx =: \hat{f}(\xi).$$

Actually, $\mathcal{F}H_0\mathcal{F}^*$ acts as multiplication by the matrix

$$h_0(\xi) = \sum_{k=1}^{3} \alpha_k \xi_k + m\alpha_0 \tag{1.3}$$

called the symbol of H_0. A straightforward calculation shows that

$$h_0(\xi) = \nu(\xi) p_0^+(\xi) - \nu(\xi) p_0^-(\xi), \quad \nu(\xi) = \sqrt{|\xi|^2 + m^2}, \tag{1.4}$$

where

$$p_0^\pm(\xi) = 2^{-1} \left(I \pm \nu^{-1}(\xi) (\sum_{k=1}^{3} \alpha_k \xi_k + m\alpha_0) \right) \tag{1.5}$$

are orthogonal projections on the eigenspaces of $h_0(\xi)$ corresponding to its eigenvalues $\pm\nu(\xi)$. In particular, the spectrum of H_0 is absolutely continuous and coincides with $(-\infty, -m] \cup [m, \infty)$. Let $\mathcal{I}^+ = (m, \infty)$, $\mathcal{I}^- = (-\infty, -m)$ and denote by E_0 the spectral family of the operator H_0. Equation (1.4) implies that

$$(\mathcal{F}E_0(\mathcal{I}^\pm) f)(\xi) = p_0^\pm(\xi) (\mathcal{F}f)(\xi), \quad f \in \mathcal{H}.$$

Obviously, functions $u_0^\pm(x, \xi) = p_0^\pm(\xi) e^{i\langle x, \xi \rangle}$ (actually, instead of vectors, we consider matrices whose columns are eigenfunctions of H_0) satisfy the equation $H_0 u_0^\pm(x, \xi) = \pm\nu(\xi) u_0^\pm(x, \xi)$. Below we consider positive energies and omit the upper index "$+$". For negative energies, considerations are quite similar and we indicate necessary changes.

Let us now consider the Dirac operator $H = H_0 + V$ with a long-range matrix interaction

$$V(x) = v(x) + \sum_{k=0}^{3} \alpha_k A_k(x). \tag{1.6}$$

Of course, H is self-adjoint in the space $\mathcal{H}$ on the domain $\mathcal{D}(H) = \mathcal{D}(H_0)$ if functions $v, A_k, k = 0, 1, 2, 3$ are real and, say, bounded. We recall that v is an electrostatic potential, and (A_1, A_2, A_3) is a magnetic potential. Our first goal is to find approximate eigenfunctions of the continuous spectrum of the operator H.

Recall that in the Schrödinger case one constructs approximate eigenfunctions in the form

$$u(x, \xi) = p(x, \xi) e^{i\varphi(x, \xi)}, \tag{1.7}$$

where $\varphi(x, \xi) - \langle x, \xi \rangle = o(|x|)$ and $p(x, \xi) - 1 = o(1)$ as $|x| \to \infty$. Substitution in the equation $-\Delta u + Vu = |\xi|^2 u$ leads to the eikonal equation $|\nabla \varphi|^2 + V(x) = |\xi|^2$ for φ and the transport equation for p. Moreover, one can set $p(x, \xi) = 1$ and thus avoid a consideration of the transport equation. Indeed, the function $u(x, \xi) = e^{i\varphi(x, \xi)}$ satisfies already the Schrödinger equation up to a term $-i\Delta\varphi$ which is short-range. Note however that Ansatz (1.7) seems to fail for the Schrödinger operator with a matrix potential.

In the Dirac case, it is natural to seek approximate eigenfunctions corresponding to the "eigenvalues" $\nu(\xi)$ in the form (1.7) where again $\varphi(x, \xi) - \langle x, \xi \rangle = o(|x|)$ and

$p(x,\xi) - p_0(\xi) = o(1)$ as $|x| \to \infty$. Substituting (1.7) in the Dirac equation $Hu = \nu(\xi)u$ and setting

$$B(x,\xi) = \sum_{j=1}^{3} (\partial\varphi(x,\xi)/\partial x_j + A_j(x))\alpha_j + m(\alpha_0 + A_0(x)) + v(x) - \nu(\xi) \tag{1.8}$$

we obtain the equation $Bp = 0$ (for fixed x, ξ). This implies that $B^2 p = 0$. Then using the algebra of Dirac matrices we deduce from the equations $Bp = 0$ and $B^2 p = 0$ the eikonal equation

$$|\nabla\varphi + A|^2 + (m + A_0)^2 - (\nu - v)^2 = 0, \quad \nabla = \nabla_x, \quad A = (A_1, A_2, A_3), \tag{1.9}$$

for the phase-function φ.

The next step is to choose the matrix-function $p(x,\xi)$. We note that in contrast to the Schrödinger case the choice $p(x,\xi) = p_0(\xi)$ is not sufficient, and the relation

$$(H - \nu(\xi))u(x,\xi) = O(|x|^{-1-\epsilon}), \quad \epsilon > 0, \quad |x| \to \infty \tag{1.10}$$

is satisfied only if $p(x,\xi)$ is a solution of the corresponding transport equation. Fortunately, this equation turns out to be degenerate and admits an explicit solution (see Proposition 3.2). Of course, to satisfy (1.10) it suffices to solve eikonal equation (1.9) also up to a short-range term only.

Equation (1.9) seems to be new. It is convenient to rewrite it in terms of the function

$$\Phi(x,\xi) = \varphi(x,\xi) - \langle x, \xi\rangle. \tag{1.11}$$

Let us introduce the effective (momentum dependent) scalar potential

$$\mathbf{v}(x,\xi) = \langle \xi, A(x)\rangle + \nu(\xi)v(x) + \mathbf{b}(x), \tag{1.12}$$

where

$$\mathbf{b}(x) = mA_0(x) + 2^{-1}(|A(x)|^2 + A_0^2(x) - v^2(x)). \tag{1.13}$$

Then (1.9) is equivalent to the equation

$$\langle \xi, \nabla\Phi(x,\xi)\rangle + \langle A(x), \nabla\Phi(x,\xi)\rangle + 2^{-1}|\nabla\Phi(x,\xi)|^2 + \mathbf{v}(x,\xi) = 0, \tag{1.14}$$

which we use below.

As is well known, equation (1.14) does not have global solutions and we have to remove either a neighbourhood of the incident $-\xi$ or of the outgoing direction ξ. This leads to two solutions $\Phi_\pm$, two approximate eigenfunctions $u_\pm(x,\xi)$ and two different identifications $J_\pm$ defined as PDO

$$(J_\pm f)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle + i\Phi_\pm(x,\xi)} p_\pm(x,\xi)\zeta_\pm(x,\xi)\psi(|\xi|^2)\hat{f}(\xi)d\xi. \tag{1.15}$$

Here $\psi \in C_0^\infty(0,\infty)$ and the cut-off function $\zeta_\pm$ is supported in a cone

$$\mathcal{C}_\pm(\delta) = \{(x,\xi) \in \mathbb{R}^6 : \pm\langle x,\xi\rangle \geq \delta|x||\xi|\}, \quad \delta \in (-1,1), \tag{1.16}$$

so ζ_+ (ζ_-) removes a neighbourhood of the incident $-\xi$ (outgoing ξ) direction. The function $\zeta_\pm$ localizes $u_\pm$ on the part of the phase space where (1.10) is satisfied. Note however that due to $\zeta_\pm$, the "effective" perturbation

$$T_\pm = HJ_\pm - J_\pm H_0 \tag{1.17}$$

is a pseudo-differential operator with symbol vanishing as $|x|^{-1}$ only at infinity.

To use freely the pseudo-differential calculus, we require that $v, A_k, k = 0, 1, 2, 3$, be C^∞-functions such that for all $\alpha \in \mathbb{N}^3$

$$|D^\alpha v(x)| + \sum_{k=0}^{3} |D^\alpha A_k(x)| \le C_\alpha \langle x\rangle^{-\rho-|\alpha|}, \quad \rho > 0, \quad \langle x\rangle = (1 + |x|^2)^{1/2}. \tag{1.18}$$

1.3. Similarly to the Schrödinger case [19], our approach to scattering theory for the Dirac operator relies on the limiting absorption principle and the radiation estimates. The first of these results means that the operator of multiplication by $\langle x\rangle^{-r}$, $r > 1/2$, is locally H-smooth in the sense of T. Kato (see Definition 2.1). The second is related to the critical case $r = 1/2$. Set

$$(\nabla_j^\perp u)(x) = (\partial_j u)(x) - |x|^{-2} x_j \sum_{k=1}^{3} x_k (\partial_k u)(x), \quad j = 1, 2, 3, \quad \partial_k = \partial/\partial x_k, \tag{1.19}$$

(so $\langle x, (\nabla^\perp u)(x)\rangle = 0$). Then the operators $\langle x\rangle^{-1/2}\nabla_j^\perp$ are locally H-smooth. We emphasize that radiation estimates are necessary to treat the part of the perturbation (1.17) decaying as $|x|^{-1}$ at infinity.

These results allow us to factorize $T_\pm$ into a sum of products of locally H- and H_0-smooth operators. Thus the theory of smooth perturbations can be applied to prove the existence of wave operators

$$W_\pm(H, H_0; J_\tau) = s - \lim_{t\to\pm\infty} e^{iHt} J_\tau e^{-iH_0 t} \tag{1.20}$$

for $\tau =$ " $+$ " and $\tau =$ " $-$ " as well as "inverse" wave operators

$$W_\pm(H_0, H; J_\tau^*) = s - \lim_{t\to\pm\infty} e^{iH_0 t} J_\tau^* e^{-iHt} P. \tag{1.21}$$

Here the limits are strong, and P is the projection on the absolutely continuous subspace of H. Under assumption of their existence, operators (1.20) and (1.21) are adjoint to each other. One can replace $J_\pm$ by a simpler operator

$$(\mathbf{J}_\pm f)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle + i\Phi_\pm(x,\xi)} p_0(\xi)\zeta_\pm(x,\xi)\psi(|\xi|^2)\hat{f}(\xi)d\xi, \tag{1.22}$$

constructed in terms of solutions $\Phi_\pm$ of the eikonal equation (1.14) only. Indeed, the difference $J_\pm - \mathbf{J}_\pm$ is compact. Therefore $W_\pm(H, H_0; J_\tau) = W_\pm(H, H_0; \mathbf{J}_\tau)$ and the existence of one of these operators implies the existence of another. However, the preliminary consideration of the more complicated identification $J_\pm$ is necessary because the operator $H\mathbf{J}_\pm - \mathbf{J}_\pm H_0$ does not factorize into a product of locally H- and H_0-smooth operators.

Note that a classical particle propagates as $t \to \pm\infty$ in the cone $\mathcal{C}_\pm(\delta)$ of the phase space. The corresponding quantum arguments show that the wave operator $W_\pm(H, H_0; \mathbf{J}_\pm)$ are isometric on the subspace $E_0(\Lambda)\mathcal{H}$ if $\psi(|\xi|^2) = 1$ for $\nu(\xi) \in \Lambda \subset (m, \infty)$. Similarly, $W_\pm(H, H_0; \mathbf{J}_\mp) = 0$. These wave operators play an auxiliary but important role in our construction of scattering theory. Actually, using the results formulated above and choosing the cut-off functions $\zeta_\pm$ in such a way that the operator

$$(\mathbf{J}_+\mathbf{J}_+^* + \mathbf{J}_-\mathbf{J}_-^* - I)E(\Lambda)$$

is compact, we prove that the operators $W_\pm(H, H_0; \mathbf{J}_\pm)$ are complete, that is the range

$$\mathrm{Ran}(W_\pm(H, H_0; \mathbf{J}_\pm)E_0(\Lambda)) = E(\Lambda)P\mathcal{H}. \tag{1.23}$$

The stationary approach developed here allows one to write down formulas for scattering matrix and to study the diagonal singularity of its kernel. Given the results of this paper, this is essentially similar to the Schrödinger case [19] and therefore we do not dwell upon it here.

2. LIMITING ABSORPTION PRINCIPLE AND RADIATION ESTIMATES

2.1. Let us recall some basic notions of the theory of smooth perturbations (see e.g. [13], [18] for details). Below we denote by c and C different positive constants whose precise values are of no importance for us. The class of bounded operators is denoted $\mathcal{B}$.

Definition 2.1 *Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$, and let K be an H-bounded operator. It is called H-smooth if one of the following three (equivalent) relations is satisfied:*

(i)

$$\gamma_1^2 = (2\pi)^{-1} \sup_{||f||=1,\, f\in\mathcal{D}(H)} \int_{-\infty}^{\infty} ||Ke^{-iHt}f||^2 dt < \infty, \tag{2.1}$$

(ii)

$$\gamma_2^2 = \pi^{-1} \sup_{\varepsilon>0,\, \lambda\in\mathbb{R}} \varepsilon||KR(\lambda \pm i\varepsilon)||^2 < \infty, \tag{2.2}$$

(iii)

$$\gamma_3^2 = \sup_{X\subset\mathbb{R}} |X|^{-1}||KE(X)||^2 < \infty. \tag{2.3}$$

In this case, $\gamma_1 = \gamma_2 = \gamma_3$.

The operator K is said to be H-smooth on a Borel set Λ if $KE(\Lambda)$ is H-smooth.

The following assertion is almost obvious.

Proposition 2.2 *Let K be an H-bounded operator and let $\Lambda \subset \mathbb{R}^+$ be a compact interval. Then K is H-smooth on Λ and $-\Lambda$ if and only if it is H^2-smooth on Λ^2.*

Proof – It suffices to take into account the obvious identity

$$||E_{H^2}(X^2)Kf||^2 = ||E_H(X)Kf||^2 + ||E_H(-X)Kf||^2, \quad X \subset \Lambda,$$

and use definition (2.3). $\Box$

The next well known assertion allows to construct new H-smooth operators given an H-smooth operator K (see, e.g., [20] for its proof), but it is also of interest in the case $K = 0$.

Proposition 2.3 *Let* $\mathbf{M}$ *be an* H*-bounded operator and let* K *be* H*-smooth on a bounded interval* Λ*. Suppose that for any* $u = E_H(\Lambda)u$

$$||Gu||^2 \le (i[H, \mathbf{M}]u, u) + ||Ku||^2.$$

Then G *is also* H*-smooth on* Λ*.*

The theory of smooth perturbations gives a convenient sufficient condition for the existence of wave operators (it can be easily deduced from definition (2.1)).

Proposition 2.4 *Let* H_1, H_2 *be a couple of self-adjoint operators in* $\mathcal{H}$*, and let* J *be a bounded operator. Suppose that the operator* $T = H_2 J - J H_1$ *admits a factorization*

$$T = \sum_{i=1}^{N} K_{2,i}^* K_{1,i},$$

where $K_{j,i}, i = 1, \dots, N$*, are* H_j*-smooth,* $j = 1, 2$*, on a compact interval* $\Lambda \subset \mathbb{R}$*. Then the wave operators*

$$W_\pm(H_2, H_1; J, \Lambda) = s - \lim_{t \to \pm\infty} e^{iH_2 t} J e^{-iH_1 t} E_1(\Lambda) P_1$$

(P_1 *is the orthogonal projection on the absolutely continuous subspace of the operator* H_1*) and* $W_\pm(H_1, H_2; J^*, \Lambda)$ *exist.*

2.2. Let us return to the Dirac operator $H = H_0 + V$ defined by equations (1.1), (1.6). With the help of the Mourre commutator method, the limiting absorption principle for the operator H (cf. [2], [16]) can be obtained practically in the same way as for the Schrödinger operator (see [9], [12]). Actually, it can be reduced to the Schrödinger case but we prefer a direct way. Let us set

$$\mathbf{A} = 2^{-1} \sum_{k=1}^{3} (x_k D_k + D_k x_k).$$

Since

$$i[H_0, \mathbf{A}] = H_0 - m\alpha_0, \tag{2.4}$$

the operator $\mathbf{A}$ (in contrast to [2] where the Mourre method was also used) can be taken for the conjugate operator to H.

The Mourre estimate is introduced in the following

Theorem 2.5 *Suppose that $V(x) = V_L(x) + V_S(x)$ where*

$$|V_L(x)| + \langle x\rangle\, |\partial V_L(x)/\partial |x|| \leq C\langle x\rangle^{-\rho}, \ \ \rho > 0, \tag{2.5}$$

$$|V_S(x)| \leq C\langle x\rangle^{-1-\rho}. \tag{2.6}$$

Then eigenvalues of H are of finite multiplicity (except, possibly, the eigenvalues $\pm m$) and may accumulate at the points $\pm m$ only. For any $\pm\lambda > m$, $\lambda \notin \sigma_p(H)$, there exists a sufficiently small $\eta > 0$ such that the (Mourre) estimate

$$\pm E(X_\eta) i[H, \mathbf{A}] E(X_\eta) \geq c E(X_\eta), \quad c > 0, \quad X_\eta = (\lambda - \eta, \lambda + \eta) \tag{2.7}$$

holds.

Proof – Consider, for example, the positive spectrum of H. Let $\Lambda = (\lambda_0, \lambda_1)$ where $m < \lambda_0 < \lambda_1 < \infty$. It follows from (2.4) that

$$i[H, \mathbf{A}] = (H - m\alpha_0) - V + i[V, \mathbf{A}]. \tag{2.8}$$

Since $\alpha_0 \leq I$, we have that

$$E(\Lambda)(H - m\alpha_0)E(\Lambda) \geq (\lambda_0 - m)E(\Lambda). \tag{2.9}$$

The operator $V(H_0 + i)^{-1}$ is compact because $V(x) \to 0$ as $|x| \to \infty$. Under assumption (2.5) the operator $[V_L, \mathbf{A}]$ is multiplication by the matrix-function $i|x|\partial V_L/\partial |x|$ which tends to zero. Therefore the operator $[V_L, \mathbf{A}](H_0 + i)^{-1}$ is also compact. The commutator

$$[V_S, \mathbf{A}] = V_S \mathbf{A} - \mathbf{A} V_S = \sum_{j=1}^{3} (V_S x_j D_j - D_j x_j V_S) - 3iV_S, \tag{2.10}$$

so under assumption (2.6) the operator $(H_0 + i)^{-1}[V_S, \mathbf{A}](H_0 + i)^{-1}$ is compact. Thus it follows from (2.8), (2.9) that

$$E(\Lambda) i[H, \mathbf{A}] E(\Lambda) \geq (\lambda_0 - m)E(\Lambda) + K \tag{2.11}$$

with a compact operator K.

Similarly to the Schrödinger case (see, e.g., [12]), it is easy to check that the virial theorem holds for eigenfunctions of H. This means that $([H, \mathbf{A}]\psi, \psi) = 0$ if $\psi \in \mathcal{D}(H)$ and $H\psi = \lambda\psi$. Combined with estimate (2.11), this standardly implies that eigenvalues $\lambda > m$ are of finite multiplicity and they cannot accumulate at interior points of (m, ∞).

Let us fix the point $\lambda \in \Lambda$, $\lambda \notin \sigma_p(H)$, and multiply (2.11) by the operator $E(X_\eta)$ from the left and from the right. Since $E(X_\eta)$ converges strongly to zero as $\eta \to 0$, we have that $||KE(X_\eta)|| \to 0$ as $\eta \to 0$. Thus (2.11) implies (2.7) if η is small enough. $\Box$

Remark 2.6 We have shown that under the assumptions of Theorem 2.5 the operator $(H_0 + i)^{-1}[H, \mathbf{A}](H_0 + i)^{-1}$ is bounded.

To deduce the limiting absorption principle from estimate (2.7) one needs some assumptions on the double commutator $[[V,\mathbf{A}],\mathbf{A}]$. As shown in [9], [12] it suffices to require that this operator be bounded. It is easy to see that this condition is satisfied if V is sufficiently regular, that is

$$|\partial^\alpha V(x)| \leq C(1+|x|)^{-\rho-|\alpha|}, \quad \rho > 0, \quad |\alpha| \leq 2. \tag{2.12}$$

Thus we obtain

Theorem 2.7 *Let assumptions* (2.12) *hold. Then the operator-function* $\langle x\rangle^{-r}R(z)\langle x\rangle^{-r}$, $r > 1/2$, *is continuous in norm with respect to the parameter* z *in the closed complex plane cut along* $(-\infty,-m]\cup[m,\infty)$ *with exception of eigenvalues of* H *and the points* $\pm m$. *In particular, the operator* $\langle x\rangle^{-r}$ *for* $r > 1/2$ *is* H*-smooth on any compact interval* $\Lambda \subset (-\infty,-m)\cup(m,\infty)$ *such that* $\Lambda\cap\sigma_p(H) = \emptyset$. *The operator* H *has no singular continuous spectrum.*

2.3. Theorem 2.7 remains true under assumptions of Theorem 2.5 although in this case the double commutator $[[V,\mathbf{A}],\mathbf{A}]$ is no longer bounded. However, as shown in [17], it suffices to check that

$$\langle x\rangle^{-(1-\rho)/2}(H_0+i)^{-1}[f(H)[V,\mathbf{A}]f(H),\mathbf{A}](H_0-i)^{-1}\langle x\rangle^{-(1-\rho)/2} \in \mathcal{B} \tag{2.13}$$

for any $f \in C_0^\infty(\mathbb{R})$. To be more precise, in [17] the Schrödinger operator was considered and the above result was hidden in the proof. Therefore we discuss it in Appendix A.

Let us prove (2.13). Different technical estimates needed for this are collected in the two following lemmas.

Lemma 2.8 *Suppose that functions* v *and* A_k, $k = 0,1,2,3$, *are bounded. Let* $f \in C_0^\infty(\mathbb{R})$ *and* $r \in [0,1]$. *Then*

$$(H_0+i)[(H+i)^{-1},\langle x\rangle^r](H_0+i) \in \mathcal{B}, \quad (H_0+i)[f(H),\langle x\rangle^r](H_0+i) \in \mathcal{B}. \tag{2.14}$$

Proof – Indeed,

$$(H+i)^{-1}\langle x\rangle^r - \langle x\rangle^r(H+i)^{-1} = -(H+i)^{-1}[H,\langle x\rangle^r](H+i)^{-1},$$

where the operator

$$[H,\langle x\rangle^r] = [H_0,\langle x\rangle^r] = -ir\langle x\rangle^{r-2}\sum_{j=1}^{3}\alpha_j x_j \tag{2.15}$$

is bounded. Since $\mathcal{D}(H) = \mathcal{D}(H_0)$, this implies the first inclusion (2.14). The second follows from the first in a standard way (see Lemma 7.4 of [12] for a similar assertion). $\square$

Lemma 2.8 allows us to commute the operators $(H\pm i)^{-1}$ or $f(H)$ with $\langle x\rangle^r$, which leads to

Lemma 2.9 *Under the assumptions of Lemma 2.8, the operators*

$$\langle x\rangle^{-r}(H\pm i)^{-1}\langle x\rangle^{r}, \quad \langle x\rangle^{-r}D_j(H\pm i)^{-1}\langle x\rangle^{r}, \quad j=1,2,3, \tag{2.16}$$

and

$$\langle x\rangle^{-r}(H_0\pm i)^{n}f(H)\langle x\rangle^{r}, \ n=-1,0,1, \tag{2.17}$$

are bounded.

Proposition 2.10 *Let assumption* (2.5), (2.6) *be satisfied and* $f\in C_0^\infty(\mathbb{R})$. *Then inclusion* (2.13) *holds.*

Proof – Let us consider first the long-range part V_L. The commutator

$$\mathcal{U}=[V_L,\mathbf{A}]=i(|x|\partial V_L/\partial|x|)=\langle x\rangle^{(1-\rho)/2}B\langle x\rangle^{-(1+\rho)/2},$$

where $B\in\mathcal{B}$. The double commutator

$$[f(H)[V_L,\mathbf{A}]f(H),\mathbf{A}]=f(H)\mathcal{U}f(H)\mathbf{A}-\mathbf{A}f(H)\mathcal{U}f(H)$$

consists of two terms which are adjoint to one another. For example, for the first we have that

$$\begin{aligned}&\langle x\rangle^{-(1-\rho)/2}(H_0+i)^{-1}f(H)\mathcal{U}f(H)\mathbf{A}(H_0-i)^{-1}\langle x\rangle^{-(1-\rho)/2}=\\ &\quad(\langle x\rangle^{-(1-\rho)/2}(H_0+i)^{-1}f(H)\langle x\rangle^{(1-\rho)/2})B(\langle x\rangle^{-(1+\rho)/2}f(H)\langle x\rangle^{(1+\rho)/2})\\ &\qquad\times(\langle x\rangle^{-(1+\rho)/2}\mathbf{A}(H_0-i)^{-1}\langle x\rangle^{-(1-\rho)/2}).\end{aligned}$$

Every factor here is a bounded operator according to Lemma 2.9.

To consider the short-range part V_S we are obliged to open the double commutator

$$\begin{aligned}[f(H)[V_S,\mathbf{A}]f(H),\mathbf{A}] &= f(H)V_S\mathbf{A}f(H)\mathbf{A}-f(H)\mathbf{A}V_Sf(H)\mathbf{A}\\ &- \mathbf{A}f(H)V_S\mathbf{A}f(H)+\mathbf{A}f(H)\mathbf{A}V_Sf(H).\end{aligned}$$

Here the first and the fourth terms as well as the second and the third are adjoint to one another. Thus we have to prove that

$$\langle x\rangle^{-(1-\rho)/2}(H_0+i)^{-1}f(H)Tf(H)\mathbf{A}(H_0-i)^{-1}\langle x\rangle^{-(1-\rho)/2}\in\mathcal{B}, \tag{2.18}$$

where T is one of the operators $V_S\mathbf{A}$ or $\mathbf{A}V_S$. If $T=V_S\mathbf{A}$, we factorize V_S as $V_S=\langle x\rangle^{(1-\rho)/2}B\langle x\rangle^{-1-(1+\rho)/2}$ with a bounded operator B. Therefore operator (2.18) is a product of operators

$$\langle x\rangle^{-(1-\rho)/2}(H_0+i)^{-1}f(H)\langle x\rangle^{(1-\rho)/2}, \quad \langle x\rangle^{-1-(1+\rho)/2}\mathbf{A}(H_0+i)^{-1}\langle x\rangle^{(1+\rho)/2},$$
$$\langle x\rangle^{-(1+\rho)/2}(H_0+i)f(H)\langle x\rangle^{(1+\rho)/2}, \quad \langle x\rangle^{-(1+\rho)/2}\mathbf{A}(H_0-i)^{-1}\langle x\rangle^{-(1-\rho)/2}.$$

All of them are bounded by Lemma 2.9. Similarly, if $T=\mathbf{A}V_S$ we use that $V_S=\langle x\rangle^{-(1+\rho)/2}B\langle x\rangle^{-(1+\rho)/2}$ with $B\in\mathcal{B}$. Therefore operator (2.18) factorizes in a product of operators

$$\langle x\rangle^{-(1-\rho)/2}(H_0+i)^{-1}\langle x\rangle^{(1-\rho)/2}, \quad \langle x\rangle^{-(1-\rho)/2}f(H)\mathbf{A}\langle x\rangle^{-(1+\rho)/2},$$
$$\langle x\rangle^{-(1+\rho)/2}f(H)\langle x\rangle^{(1+\rho)/2}, \quad \langle x\rangle^{-(1+\rho)/2}\mathbf{A}(H_0-i)^{-1}\langle x\rangle^{-(1-\rho)/2}.$$

All of them are again bounded by Lemma 2.9. □

As was already mentioned (see Appendix A for details), Proposition 2.10 implies

Theorem 2.11 *Under the assumptions of Theorem* 2.5*, all conclusions of Theorem* 2.7 *remain true.*

Corollary 2.12 *Let* $\Lambda \subset (-\infty, -m) \cup (m, \infty)$ *be a compact interval disjoint from* $\sigma_p(H)$*. Under the assumptions of Theorem* 2.5*, for any* $r > 1/2$ *the operators* $\langle x\rangle^{-r} D_j E(\Lambda)$*,* $j = 1, 2, 3$*, are* H*-smooth.*

Proof – Using the Hilbert identity, we get

$$||\langle x\rangle^{-r} D_j R(z)|| \leq ||\langle x\rangle^{-r} D_j R(i)|| + |z - i|\, ||\langle x\rangle^{-r} D_j R(i)\langle x\rangle^{r}||\, ||\langle x\rangle^{-r} R(z)||.$$

Since, by (2.16), the operator $\langle x\rangle^{-r} D_j R(i)\langle x\rangle^{r}$ is bounded, the proof follows from definition (2.2) of H-smoothness. $\square$

Corollary 2.13 *Under the assumptions of Theorem* 2.5*, for any* $r > 1/2$ *the operators* $\langle x\rangle^{-r}$ *and* $\langle x\rangle^{-r} D_j$*,* $j = 1, 2, 3$*, are* H^2*-smooth on any compact interval of* (m^2, ∞) *disjoint from* $\sigma_p(H^2)$*.*

Proof – It suffices to put together Proposition 2.2, Theorem 2.11 and Corollary 2.12. $\square$

Results on the absence of eigenvalues embedded in the continuous spectrum of H can be found in [1], but we do not need them.

2.4. Our proof of the radiation estimates for the Dirac operator relies on a reduction to the Schrödinger case. Then the approach of [20] can be directly applied. Let $\mathbf{m}(x) = |x|$ for $|x| \geq 1$, $\mathbf{m}(x)$ being C^∞ everywhere. Let us also introduce the differential operator

$$\mathbf{M} = \sum_{j=1}^{3} (\mathbf{m}_j D_j + D_j \mathbf{m}_j), \qquad \mathbf{m}_j = \partial \mathbf{m}/\partial x_j.$$

Note that for $|x| \geq 1$, $\mathbf{m}_j(x) = x_j/|x|$ and

$$\mathbf{m}_{jk}(x) = \partial^2 \mathbf{m}(x)/\partial x_j \partial x_k = |x|^{-1}\delta_{jk} - |x|^{-3} x_j x_k, \tag{2.19}$$

where δ_{jk} is the Kronecker symbol.

The case of regular perturbations is quite easy.

Theorem 2.14 *Let assumption* (2.12) *be satisfied, and let the operators* $\nabla_j^\perp$*,* $j = 1, 2, 3$*, be defined by* (1.19)*. Then the operators* $G_j = \langle x\rangle^{-1/2}\nabla_j^\perp$ *are* H*-smooth on any compact interval* $\Lambda \subset (-\infty, -m) \cup (m, \infty)$ *such that* $\Lambda \cap \sigma_p(H) = \emptyset$*.*

Proof – Let us consider

$$H^2 = (H_0 + V(x))^2 = -\Delta + m^2 + \mathbf{V},$$

where

$$\mathbf{V} = VH_0 + H_0V + V^2 =: N_0(x) + \sum_{j=1}^{3} N_j(x)\partial_j \tag{2.20}$$

and the matrices $N_k(x)$ satisfy the conditions

$$|\partial^\alpha N_k(x)| \leq C\langle x\rangle^{-\rho-|\alpha|}, \quad |\alpha| \leq 1, \quad k = 0, 1, 2, 3. \tag{2.21}$$

The operator H^2 is selfadjoint on the domain $\mathbf{H}^2(\mathbb{R}^3, \mathbb{C}^4)$. A direct calculation shows that

$$i[-\Delta, \mathbf{M}] = 4\sum_{j,k=1}^{3} D_j \mathbf{m}_{jk} D_k - (\Delta^2 \mathbf{m}). \tag{2.22}$$

By (2.19), for $|x| \geq 1$

$$\sum_{j,k=1}^{3} \mathbf{m}_{jk} D_k u \overline{D_j u} = |x|^{-1} \sum_{j=1}^{3} |\nabla_j^\perp u|^2, \tag{2.23}$$

and $(\Delta^2 \mathbf{m})(x) = O(|x|^{-3})$. By (2.20), (2.21),

$$[\mathbf{V}, \mathbf{M}] = \mathbf{N}_0(x) + \sum_{j=1}^{3} \mathbf{N}_j(x)\partial_j, \tag{2.24}$$

where $\mathbf{N}_k(x) = O(|x|^{-1-\rho})$, $k = 0, 1, 2, 3$. Combining (2.22) – (2.24), we get that

$$4||G_k u||^2 \leq i([H^2, \mathbf{M}]u, u) + C(||\langle x\rangle^{-r} u||^2 + \sum_{j=1}^{3} ||\langle x\rangle^{-r} D_j u||^2), \tag{2.25}$$

where $r = (1+\rho)/2, k = 1, 2, 3$. By Proposition 2.3, this implies that the operators G_k are H^2-smooth on Λ^2. Proposition 2.2 concludes the proof. □

The radiation estimates are still true in the general case.

Theorem 2.15 *Under the assumptions of Theorem* 2.5*, the conclusions of Theorem* 2.14 *remain true.*

The proof of Theorem 2.15 is technically rather complicated, and therefore it is postponed up to the Appendix B. Although convenient, Theorem 2.15 is not absolutely necessary for construction of scattering theory (see subsection 4.4). Probably it is also of interest in its own sake.

3. THE EIKONAL AND TRANSPORT EQUATIONS

3.1. Our aim here is to construct "approximate" eigenfunctions (1.7) of the operator H. This will lead to the eikonal equation (1.14) for the phase (1.11). We emphasize that all our estimates below are uniform in ξ for $0 < c \leq |\xi| \leq C < \infty$.

Let us introduce the "remainder"

$$\mathbf{q}(x, \xi) = e^{-i\varphi(x,\xi)}((Hu)(x, \xi) - \nu(\xi)u(x, \xi)) \tag{3.1}$$

in the Dirac equation $Hu = \nu u$. Plugging (1.7) into (3.1), we see that

$$\mathbf{q}(x,\xi) = q(x,\xi) - i\sum_{k=1}^{3} \alpha_k \partial_k p(x,\xi), \tag{3.2}$$

where

$$q(x,\xi) = \left(\sum_{k=1}^{3} \alpha_k (\partial_k \varphi)(x,\xi) + m\alpha_0 + V(x) - \nu(\xi) \right) p(x,\xi). \tag{3.3}$$

We will construct functions $\varphi(x,\xi)$ and $p(x,\xi)$ in such a way that, for all multi-indices α, β,

$$|\partial_x^\alpha \partial_\xi^\beta \mathbf{q}(x,\xi)| \le C_{\alpha,\beta} \langle x \rangle^{-1-\epsilon-|\alpha|}, \quad \epsilon > 0, \tag{3.4}$$

in one of the cones $\mathcal{C}(\delta) = \mathcal{C}_\pm(\delta)$ defined by (1.16). In particular, this implies (1.10) in the same cone $\mathcal{C}(\delta)$. Of course, (3.4) is satisfied if both functions $q(x,\xi)$ and $\nabla p(x,\xi)$ satisfy these estimates. The condition on $q(x,\xi)$ "couples" $p(x,\xi)$ and $\Phi(x,\xi)$. So our first goal is to obtain an equation for the function $\Phi(x,\xi)$ alone.

Actually, the function $\Phi(x,\xi)$ will be defined as an approximate solution of the eikonal equation (1.14). Let us set

$$\mathbf{r}(x,\xi) = 2\langle \xi, \nabla\Phi(x,\xi)\rangle + 2\langle A(x), \nabla\Phi(x,\xi)\rangle + |\nabla\Phi(x,\xi)|^2 + 2\mathbf{v}(x,\xi), \tag{3.5}$$

where $\mathbf{v}(x,\xi)$ is effective potential (1.12). We will construct $\Phi(x,\xi)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \Phi(x,\xi)| \le C_{\alpha,\beta} \langle x \rangle^{1-\rho-|\alpha|}, \quad x \in \mathcal{C}(\delta), \tag{3.6}$$

and

$$|\partial_x^\alpha \partial_\xi^\beta \mathbf{r}(x,\xi)| \le C_{\alpha,\beta} \langle x \rangle^{-1-\epsilon-|\alpha|}, \quad x \in \mathcal{C}(\delta), \quad \epsilon > 0, \tag{3.7}$$

for all multi-indices α, β.

Then the matrix-function $p(x,\xi)$ is defined by the following explicit formula. Put

$$K(x,\xi) = K_\Phi(x,\xi) = (2\nu(\xi))^{-1} \left(v(x) + A_0(x)\alpha_0 + \sum_{k=1}^{3} (A_k(x) + \partial_k \Phi(x,\xi))\alpha_k \right). \tag{3.8}$$

Clearly,

$$K(x,\xi) \to 0 \quad \text{as} \quad |x| \to \infty, \quad x \in \mathcal{C}(\delta),$$

provided $V(x) \to 0$ and $\nabla\Phi(x,\xi) \to 0$ as $|x| \to \infty$. Thus the inverse matrix $(I - K(x,\xi))^{-1}$ is well-defined for $x \in \mathcal{C}(\delta)$ with $|x|$ large enough, and we can set

$$p(x,\xi) = (I - K(x,\xi))^{-1} p_0(\xi), \tag{3.9}$$

where $p_0(\xi)$ is matrix (1.5). The values of $p(x,\xi)$ for x from a compact domain remain arbitrary.

Let us find an explicit relation between the remainders $q(x,\xi)$ and $\mathbf{r}(x,\xi)$. We introduce auxiliary functions

$$s_0(x) = A_0(x) + m, \quad s_j(x,\xi) = A_j(x) + \partial_j \varphi(x,\xi), \quad j = 1,2,3,$$

$$s(x,\xi) = v(x) - \nu(\xi). \tag{3.10}$$

Then function (3.5) equals

$$\mathbf{r}(x,\xi) = \sum_{k=0}^{3} s_k^2(x,\xi) - s^2(x,\xi). \tag{3.11}$$

We introduce also the auxiliary matrix-function

$$B(x,\xi) = B_\Phi(x,\xi) = \sum_{k=0}^{3} s_k(x,\xi)\alpha_k + s(x,\xi) \tag{3.12}$$

(this matrix is of course the same as (1.8)) which allows us to rewrite (3.3) as

$$q(x,\xi) = B(x,\xi)p(x,\xi). \tag{3.13}$$

Note the identity

$$\begin{aligned} B^2(x,\xi) &= \sum_{k=0}^{3} s_k^2(x,\xi) + s^2(x,\xi) + 2s(x,\xi)\sum_{k=0}^{3} s_k(x,\xi)\alpha_k \\ &= \mathbf{r}(x,\xi) + 2s(x,\xi)B(x,\xi), \end{aligned} \tag{3.14}$$

following directly from (1.2).

Now we are able to obtain a convenient expression for q via Φ and $\mathbf{r}$. The inverse matrix $(\nu(\xi)K(x,\xi) - s(x,\xi))^{-1}$ appearing below is well-defined for $x \in \mathcal{C}(\delta)$ with $|x|$ large enough.

Proposition 3.1 *Let the functions $q, \mathbf{r}, s$ and the matrix-function K be defined by equalities* (3.3), (3.5), (3.10) *and* (3.8), *respectively. Then*

$$q = 2^{-1}\mathbf{r}(\nu K - s)^{-1}(I-K)^{-1}p_0. \tag{3.15}$$

Proof – Clearly, matrix (3.12) equals

$$B(x,\xi) = B_0(\xi) + 2\nu(\xi)K(x,\xi), \tag{3.16}$$

where

$$B_0(\xi) = \sum_{k=1}^{3} \alpha_k\xi_k + m\alpha_0 - \nu(\xi).$$

It follows from (3.16) that

$$B^2 = B_0^2 + 2\nu B_0 K + 2\nu K B.$$

Using identities $B_0^2 = -2\nu B_0$ and (3.14), we deduce from it that

$$2(\nu K - s)B = 2\nu B_0(I-K) + \mathbf{r}.$$

Therefore

$$B(I-K)^{-1} = \nu(\nu K - s)^{-1}B_0 + 2^{-1}\mathbf{r}(\nu K - s)^{-1}(I-K)^{-1}.$$

Taking into account equality $B_0(\xi)p_0(\xi) = 0$, we find that

$$B(I-K)^{-1}p_0 = 2^{-1}\mathbf{r}(\nu K - s)^{-1}(I-K)^{-1}p_0.$$

By virtue of (3.9), (3.13), this implies (3.15). □

In particular, if eikonal equation (1.14) is satisfied (for example, in some cone), then $\mathbf{r} = 0$ and, by (3.15), $q = 0$. In this case function (3.3) equals zero, and the remainder $\mathbf{q}$ in the Dirac equation reduces to the second term in the right-hand side of (3.2).

The function Φ will be constructed in the next subsection. Let us formulate here a conditional result following directly from Proposition 3.1.

Proposition 3.2 *Let assumption* (1.18) *with some* $\rho \in (0,1)$ *be fulfilled. Suppose that a function* Φ *satisfies estimates* (3.6) *and corresponding function* (3.5) *satisfies estimates* (3.7)*. Define the matrix* p *by formulas* (3.8), (3.9)*, set*

$$\tilde{p}(x,\xi) = p(x,\xi) - p_0(\xi) \tag{3.17}$$

and let u *be function* (1.7)*. Then for all multi-indices* α, β

$$|\partial_x^\alpha \partial_\xi^\beta \tilde{p}(x,\xi)| \le C_{\alpha,\beta}\langle x\rangle^{-\rho-|\alpha|}, \quad x \in \mathcal{C}(\delta), \tag{3.18}$$

and remainder (3.1) *in the Dirac equation satisfies estimates* (3.4).

3.2. To construct a function $\Phi = \Phi_\pm$ satisfying conditions of Proposition 3.2 in a cone $\mathcal{C}_\pm(\delta)$, we consider first the auxiliary linear equation

$$\langle \xi, \nabla\phi_\pm(x,\xi)\rangle + F(x,\xi) = 0. \tag{3.19}$$

Lemma 3.3 *Suppose that for all multi-indices* α, β

$$|\partial_x^\alpha \partial_\xi^\beta F(x,\xi)| \le C_{\alpha,\beta}\langle x\rangle^{-\rho-|\alpha|}, \quad \rho \in (0,1), \quad x \in \mathcal{C}_\pm(\delta).$$

Then the function

$$\phi_\pm(x,\xi) = \pm\int_0^\infty (F(x \pm t\xi, \xi) - F(\pm t\xi, \xi))dt =: (Q_\pm(\xi)F)(x) \tag{3.20}$$

satisfies equation (3.19) *and*

$$|\partial_x^\alpha \partial_\xi^\beta \phi_\pm(x,\xi)| \le C_{\alpha,\beta}\langle x\rangle^{1-\rho-|\alpha|}, \quad x \in \mathcal{C}_\pm(\delta), \ \forall\ \alpha, \beta.$$

The proof can be obtained by a direct differentiation (see [19], for details).

Lemma 3.3 allows us to construct $\Phi_\pm$ by the method of successive approximations. We define inductively functions $\Phi_\pm^{(n)}$ by the equalities

$$\Phi_\pm^{(n+1)} = Q_\pm(\xi)F_\pm^{(n)}, \quad n \ge 0, \tag{3.21}$$

$$F_\pm^{(0)} = \mathbf{v}, \quad F_\pm^{(1)} = \langle A, \nabla\Phi_\pm^{(1)}\rangle + 2^{-1}|\nabla\Phi_\pm^{(1)}|^2, \tag{3.22}$$

and for $n \geq 2$,

$$F_\pm^{(n)} = \langle A, \nabla\Phi_\pm^{(n)}\rangle + \sum_{m=1}^{n-1}\langle\nabla\Phi_\pm^{(m)}, \nabla\Phi_\pm^{(n)}\rangle + 2^{-1}|\nabla\Phi_\pm^{(n)}|^2. \tag{3.23}$$

Then

$$\langle\xi, \nabla\Phi_\pm^{(n+1)}\rangle + F_\pm^{(n)} = 0, \quad n = 0, 1 \dots . \tag{3.24}$$

Using Lemma 3.3, we obtain by induction that

$$|\partial_x^\alpha \partial_\xi \Phi_\pm^{(n)}(x,\xi)| \leq C_{\alpha,\beta}\langle x\rangle^{1-n\rho-|\alpha|}, \quad x \in \mathcal{C}_\pm(\delta). \tag{3.25}$$

For $(N+1)\rho > 1$, we set

$$\Phi_\pm(x,\xi) = \sum_{n=1}^{N}\Phi_\pm^{(n)}(x,\xi). \tag{3.26}$$

Then it follows from (3.22) – (3.24) that function (3.5) equals

$$\mathbf{r}_\pm = \mathbf{r}_\pm^{(N)} = 2\langle A, \nabla\Phi_\pm^{(N)}\rangle + 2\sum_{m=1}^{N-1}\langle\nabla\Phi_\pm^{(m)}, \nabla\Phi_\pm^{(N)}\rangle + |\nabla\Phi_\pm^{(N)}|^2$$

(if $N = 1$ the sum in the right-hand side is absent). Therefore the following assertion is a direct consequence of estimates (3.25).

Proposition 3.4 *Let assumption* (1.18) *with some* $\rho \in (0,1)$ *be fulfilled. Define the function* $\Phi_\pm(x,\xi)$ *by equalities* (3.20), (3.21) – (3.23), (3.26) *and* $\mathbf{r}_\pm(x,\xi)$ – *by equality* (3.5). *Then for all multi-indices* α, β *estimates* (3.6) *and* (3.7) *are satisfied with* $\epsilon = (N+1)\rho - 1$ *in* $\mathcal{C}_\pm(\delta)$.

In particular, we can set $N = 1$ if $\rho > 1/2$ and hence

$$\Phi_\pm(x,\xi) = \pm\int_0^\infty (\mathbf{v}(x \pm t\xi, \xi) - \mathbf{v}(\pm t\xi, \xi))\, dt \tag{3.27}$$

where $\mathbf{v}$ is defined by (1.12), (1.13). Moreover, we can neglect quadratic terms in (1.13) which gives the expression

$$\begin{aligned}\Phi_\pm(x,\xi) \;&=\; \pm\int_0^\infty (\nu(\xi)v(x \pm t\xi) - \nu(\xi)v(\pm t\xi) \\ &\qquad + \langle\xi, A(x \pm t\xi) - A(\pm t\xi)\rangle + mA_0(x \pm t\xi) - mA_0(\pm t\xi))\, dt.\end{aligned}$$

Combining Propositions 3.2 and 3.4, we arrive at the main result of this section.

Theorem 3.5 *Let assumption* (1.18) *with some* $\rho \in (0,1)$ *be fulfilled. Define the phase function* $\Phi_\pm(x,\xi)$ *by equalities* (3.20), (3.21) – (3.23), (3.26), *the matrix* $p_\pm(x,\xi)$ – *by formulas* (3.8), (3.9), *and let* $u_\pm(x,\xi)$ *be function* (1.7). *Then* $\Phi_\pm(x,\xi)$ *satisfies* (3.6), *matrix-function* (3.17) *satisfies* (3.18) *and for the function* $\mathbf{q}_\pm(x,\xi)$ *defined by* (3.1) *estimate* (3.4) *holds.*

The construction above works for positive energies. To consider negative energies, we should replace $\nu(\xi)$ by $-\nu(\xi)$ in all formulas.

4. WAVE OPERATORS

Here we prove the existence and completeness of wave operators (1.20) with identification (1.15).

4.1. In this subsection we collect necessary facts about pseudo-differential operators (PDO) defined by

$$(Af)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i\langle \xi, x\rangle} a(x,\xi) \hat{f}(\xi) d\xi, \tag{4.1}$$

where $f \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$. With respect to (4×4)-matrix symbols $a(x,\xi)$ we everywhere assume that $a \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and $a(x,\xi) = 0$ for $|\xi|$ large enough. Recall that the class $S^m_{\rho,\delta}$, $\rho > 0, \delta < 1$, of symbols is distinguished by the condition

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle x\rangle^{m-\rho|\alpha|+\delta|\beta|}. \tag{4.2}$$

We need also a more special class of oscillating symbols $a(x,\xi)$ satisfying the representation

$$a(x,\xi) = e^{i\Phi(x,\xi)} b(x,\xi) \text{ for } \Phi \in S^r_{1,0}, \text{ where } r \in [0,1) \text{ and } b \in S^m_{1,0}. \tag{4.3}$$

This class will be denoted $C^m(\Phi)$. Clearly, $C^m(\Phi) \subset S^m_{1-r,r}$. We consider PDO (4.1) as operators in the space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$. The adjoint to A operator A^* is given by

$$(\widehat{A^* f})(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\langle \xi, x\rangle} \overline{a(x,\xi)} f(x) dx \tag{4.4}$$

and thus fits into the usual framework of the PDO theory (note however that compared to this theory the roles of the variables x and ξ in (4.4) are interchanged). This allows us to recover all the results of the theory for the operator A itself.

The following assertions are taken from the paper [21].

Proposition 4.1 *Suppose that A is a PDO with symbol $a \in C^m(\Phi)$. Then the operator $A\langle x\rangle^{-m}$ is bounded and $A\langle x\rangle^{-m_1}$, $m_1 > m$, is compact in the space $\mathcal{H}$.*

Proposition 4.2 *Let A_j, $j = 1,2$, be PDO with symbols $a_j \in C^0(\Phi)$, and let A be the PDO with symbol $\overline{a_1}(x,\xi) a_2(x,\xi)$. Then the operator $A_1^* A_2 - A$ is compact.*

Proposition 4.3 *Let A_j, $j = 1,2$, be PDO with symbols $a_j \in C^0(\Phi)$, and let A be the PDO with symbol $a_1(x,\xi)\overline{a_2}(x,\xi)$. Then the operator $A_1 A_2^* - A$ is compact.*

If $\Phi \in S^r_{1,0}$ for $r < 1/2$, then $C^m(\Phi) \subset S^m_{\rho,\delta}$ where $\rho = 1 - r > 1/2$ and $\delta = r < 1/2$. In this case Propositions 4.1 – 4.3 reduce to standard assertions of PDO calculus (see, e.g., the book [14]).

We note also the following elementary

Proposition 4.4 *Suppose that B is a PDO with symbol $b \in C^m(\Phi)$ where $\Phi \in S^r_{1,0}$. Suppose that coefficients of differential operators*

$$G_i = \sum_{|s| \leq s_0} g_s^{(i)}(x) D^s, \quad i = 1,2,$$

satisfy the condition $g_s^{(i)} \in C^\infty(\mathbb{R}^3)$ *and* $\partial^\alpha g_s^{(i)}(x) = O(|x|^{n-\alpha})$ *as* $|x| \to \infty$ *for some* n *and all* α. *Set* $g^{(i)}(x,\xi) = \sum_{|s|\leq s_0} g_s^{(i)}(x)\xi^s$, *and let* T *be the PDO with symbol*

$$t(x,\xi) = \overline{g^{(1)}(x,\xi)}g^{(2)}(x,\xi)b(x,\xi).$$

Then the operator $\langle x\rangle^p(T - G_1^* B G_2)\langle x\rangle^p$ *is bounded if* $2p = -m - 2n + 1 - r$.

4.2. We assume below that $V = V_S + V_L$ where the short-range part V_S satisfies estimate (2.6) and the long-range part V_L satisfies condition (1.18). Let $\Phi_\pm = \Phi_\pm(V_L)$ and $p_\pm = p_\pm(V_L)$ be the functions constructed in Theorem 3.5 with respect to V_L only. The operator $J_\pm$ is defined by formula (1.15) where the cut-off function

$$\zeta_\pm(x,\xi) = \eta(x)\sigma_\pm(\langle\hat{x},\hat{\xi}\rangle), \quad \hat{x} = x/|x|, \quad \hat{\xi} = \xi/|\xi|,$$

$\sigma_\pm \in C^\infty(-1,1)$, $\sigma_\pm(\tau) = 1$ near ± 1, $\sigma_\pm(\tau) = 0$ near ∓ 1, $\eta \in C^\infty(\mathbb{R}^3)$, $\eta(x) = 0$ near 0, $\eta(x) = 1$ for large x and $\psi \in C_0^\infty(\mathbb{R}_+)$. Note that $\zeta_\pm$ is supported in a cone $\mathcal{C}_\pm(\delta)$. The function η is introduced only to avoid the singularity of $\hat{x}$ at $x = 0$. The function ψ allows us to localize our considerations on a compact subinterval of (m,∞). It follows from estimates (3.18) that $b_\pm = p_\pm\zeta_\pm\psi \in S_{1,0}^0$. Let functions $\tilde{\sigma}$, $\tilde{\eta}, \tilde{\psi}$ satisfy the assumptions above on $\sigma_\pm, \eta, \psi$, respectively, and $\tilde{\sigma}_\pm\tilde{\eta}\tilde{\psi} = 1$ on support of $\sigma_\pm\eta\psi$. Set $\tilde{\Phi}_\pm = \Phi_\pm\tilde{\sigma}_\pm\tilde{\eta}\tilde{\psi}$. By (3.6), $\tilde{\Phi}_\pm \in S_{1,0}^{1-\rho}$ and $e^{i\Phi_\pm}b_\pm = e^{i\tilde{\Phi}_\pm}b_\pm$. Therefore the symbol $a_\pm = e^{i\Phi_\pm}b_\pm$ of the PDO $J_\pm$ belongs to the class $C^0(\tilde{\Phi}_\pm)$, and hence, by Proposition 4.1, $J_\pm$ is a bounded operator in the space $\mathcal{H}$.

Our first goal is to show that both triples H_0, H, J_+ and H_0, H, J_- satisfy the assumptions of Proposition 2.4. To that end, we need the following

Lemma 4.5 *Let the operators* $\nabla_j^\perp$, $j = 1,2,3$, *be defined by* (1.19). *Let* T *be a PDO with symbol*

$$t(x,\xi) = g(x,\xi)w(\langle\hat{x},\hat{\xi}\rangle)\eta(x)\psi(|\xi|^2),$$

where $g \in C^{-1}(\Phi)$, $\Phi \in S_{1,0}^{1-\rho}$, $w \in C^\infty(-1-\epsilon, 1+\epsilon)$ *for some* $\epsilon > 0$ *et* $w(\pm 1) = 0$. *Then* T *admits the representation*

$$T = \sum_{j=1}^{3} G_j^* B^{(s)} G_j + \langle x\rangle^{-p} B^{(r)} \langle x\rangle^{-p}, \tag{4.5}$$

where $G_j = \langle x\rangle^{-1/2}\nabla_j^\perp$, $p = (1+\rho)/2$ *and the operators* $B^{(s)}, B^{(r)}$ *are bounded.*

Proof – Let $B^{(s)}$ and T_j, $j = 1,2,3$, be PDO with symbols

$$b^{(s)}(x,\xi) = \langle x\rangle|\xi|^{-2}(1 - \langle\hat{\xi},\hat{x}\rangle^2)^{-1}t(x,\xi)$$

and

$$t_j(x,\xi) = \langle x\rangle^{-1}(\xi_j - |x|^{-2}x_j\langle\xi,x\rangle)^2 b^{(s)}(x,\xi).$$

The function $(1-\tau^2)^{-1}w(\tau)$ is C^∞, so that $b^{(s)} \in C^0(\Phi)$ and the operator $B^{(s)}$ is bounded by Proposition 4.1. Since the symbol of the operator G_j equals $\langle x\rangle^{-1/2}(\xi_j - |x|^{-2}x_j\langle x,\xi\rangle)$, it follows from Proposition 4.4 that the operators $\langle x\rangle^p(G_j^* B^{(s)} G_j - T_j)\langle x\rangle^p$ are bounded.

Now the equality $t(x,\xi)=\sum_{j=1}^3 t_j(x,\xi)$ implies the boundedness of the operator $\langle x\rangle^p(T-\sum_{j=1}^3 G_j^* B^{(s)} G_j)\langle x\rangle^p$. This is equivalent to (4.5). $\Box$

Set $H_L = H_0 + V_L$. Let us calculate the operator $T_\pm^{(L)} = H_L J_\pm - J_\pm H_0$.

Lemma 4.6 *The operator $T_\pm^{(L)}$ is PDO with symbol*

$$t_\pm(x,\xi) = t_\pm^{(s)}(x,\xi) + t_\pm^{(r)}(x,\xi), \tag{4.6}$$

where

$$t_\pm^{(s)}(x,\xi) = e^{i\Phi_\pm(x,\xi)} b_\pm^{(s)}(x,\xi)\psi(|\xi|^2), \quad t_\pm^{(r)}(x,\xi) = e^{i\Phi_\pm(x,\xi)} b_\pm^{(r)}(x,\xi)\psi(|\xi|^2)$$

$$b_\pm^{(s)}(x,\xi) = -i\eta(x)|x|^{-1}\sum_{k=1}^3 (|\xi|^{-1}\xi_k - |x|^{-1}x_k\langle\hat{x},\hat{\xi}\rangle)\sigma'_\pm(\langle\hat{x},\hat{\xi}\rangle)\alpha_k p_0(\xi)$$

and the function $b_\pm^{(r)}(x,\xi)$ satisfies for all α,β the estimates

$$|\partial_x^\alpha \partial_\xi^\beta b_\pm^{(r)}(x,\xi)| \le C_{\alpha,\beta}\langle x\rangle^{-1-\epsilon-|\alpha|}, \quad 0 < c \le |\xi| \le C < \infty, \quad \epsilon > 0. \tag{4.7}$$

Proof – Using notation (3.1) we see that $H_L J_\pm$ is the PDO with symbol

$$e^{i\Phi_\pm(x,\xi)}\psi(|\xi|^2)\left((\nu(\xi)p_\pm(x,\xi) + \mathbf{q}_\pm(x,\xi))\zeta_\pm(x,\xi) - i\sum_{k=1}^3 (\partial_{x_k}\zeta_\pm)(x,\xi)\alpha_k p_\pm(x,\xi)\right). \tag{4.8}$$

The operator $J_\pm H_0$ is also PDO with symbol

$$e^{i\Phi_\pm(x,\xi)}\psi(|\xi|^2)p_\pm(x,\xi)h_0(\xi), \tag{4.9}$$

where the matrix-function $h_0(\xi)$ is defined by (1.3). By (3.9),

$$p_\pm(x,\xi)h_0(\xi) = \nu(\xi)p_\pm(x,\xi),$$

so that (4.9) cancels in $T_\pm^{(L)}$ with the first term in the right-hand side of (4.8). According to (3.4) the second term $\mathbf{q}_\pm(x,\xi)\zeta_\pm(x,\xi)$ in brackets in (4.8) satisfies estimate (4.7). Finally, $b_\pm^{(s)}(x,\xi)$ is the part of order -1 of the last term in these brackets. According to (3.18) the remainder satisfies again estimate (4.7). $\Box$

Proposition 4.7 *The operator $T_\pm = HJ_\pm - J_\pm H_0$ admits the representation* (4.5) *with $p > 1/2$.*

Proof – By (4.6),

$$T_\pm = T_\pm^{(s)} + T_\pm^{(r)} + V_S J_\pm,$$

where $T_\pm^{(s)}$ and $T_\pm^{(r)}$ are PDO with symbols $t_\pm^{(s)}$ and $t_\pm^{(r)}$, respectively. Since $t_\pm^{(s)} \in C^{-1}(\tilde{\Phi}_\pm)$ and $\sigma'_\pm(s) = 0$ in neighbourhoods of the points $s=-1$ and $s=1$, Lemma 4.5 can be directly applied to $T_\pm^{(s)}$. It follows from (4.7) and Proposition 4.1 that the operator

$\langle x\rangle^p T_\pm^{(r)}\langle x\rangle^p$ is bounded if $p=(1+\epsilon)/2$. Similarly, $\langle x\rangle^p V_S\langle x\rangle^p$ is bounded if $p=(1+\rho)/2$. $\Box$

By virtue of Theorems 2.11 and 2.15, the operators $\langle x\rangle^{-p}, p>1/2$, and G_j, $j=1,2,3$, are H_0-smooth on any compact interval $\Lambda\subset(-\infty,-m)\cup(m,\infty)$. Moreover, they are also H-smooth if $\Lambda\cap\sigma_p(H)=\emptyset$. Therefore Proposition 4.7 implies that both triples H_0,H,J_+ and H_0,H,J_- satisfy on such Λ the assumptions of Proposition 2.4. Taking into account that linear combinations (for all admissible Λ) of elements $f_0=E_0(\Lambda)f_0$ and $f=E(\Lambda)f$ are dense in $\mathcal{H}$ and the absolutely continuous subspace $P\mathcal{H}$ of H, respectively, we obtain the following result.

Theorem 4.8 *Suppose that $V=V_S+V_L$ where V_S satisfies* (2.6) *and V_L satisfies* (1.18). *Let $J_\pm=J_\pm(V_L)$ be PDO* (1.15) *where the functions $\Phi_\pm=\Phi_\pm(V_L)$ and $p_\pm=p_\pm(V_L)$ are defined by equalities* (3.20), (3.21) – (3.23), (3.26) *and* (3.8), (3.9) *with V replaced by V_L. Then the wave operators*

$$W_\pm(H,H_0;J_\pm)\qquad W_\pm(H_0,H;J_\pm^*), \tag{4.10}$$

and

$$W_\pm(H,H_0;J_\mp)\qquad W_\pm(H_0,H;J_\mp^*), \tag{4.11}$$

exist. The operators (4.10) *as well as* (4.11) *are adjoint to each other.*

Corollary 4.9 *Let $\mathbf{J}_\pm$ be PDO* (1.22). *Then all wave operators*

$$W_\pm(H,H_0;\mathbf{J}_\pm),\quad W_\pm(H_0,H;\mathbf{J}_\pm^*),\quad W_\pm(H,H_0,\mathbf{J}_\mp),\quad W_\pm(H_0,H,\mathbf{J}_\mp^*)$$

exist and coincide with corresponding operators (4.10), (4.11).

Proof – The operator $J_\pm-\mathbf{J}_\pm$ is PDO with symbol

$$e^{i\Phi_\pm(x,\xi)}\tilde p(x,\xi)\zeta_\pm(x,\xi)\psi(|\xi|^2),\quad \tilde p(x,\xi)=p(x,\xi)-p_0(\xi).$$

By (3.18), Proposition 4.1 implies that the operator $J_\pm-\mathbf{J}_\pm$ is compact. $\Box$

4.3. Our proof of the isometricity and completeness of the wave operators $W_\pm(H,H_0;\mathbf{J}_\pm)$ relies on

Lemma 4.10 *The following relations hold*

$$s-\lim_{t\to\pm\infty}(\mathbf{J}_\pm^*\mathbf{J}_\pm-\psi^2(H_0^2-m^2)P_0)e^{-iH_0t}=0, \tag{4.12}$$

$$s-\lim_{t\to\pm\infty}\mathbf{J}_\mp^*\mathbf{J}_\mp e^{-iH_0t}=0. \tag{4.13}$$

Proof – Proposition 4.2 shows that, up to a compact term, $\mathbf{J}_\mp^*\mathbf{J}_\mp$ is the PDO with symbol $p_0(\xi)\zeta_\mp^2(x,\xi)\psi^2(|\xi|^2)$. It follows that

$$(\mathbf{J}_\mp^*\mathbf{J}_\mp e^{-iH_0t}f)(x)=(2\pi)^{-3/2}\int_{\mathbb{R}^3}e^{i\langle\xi,x\rangle-i\nu(\xi)t}p_0(\xi)\zeta_\mp^2(x,\xi)\psi^2(|\xi|^2)\hat f(\xi)d\xi+\epsilon_\mp(x,t), \tag{4.14}$$

where $||\epsilon_\mp(\cdot,t)|| = o(1)$ as $|t|\to\infty$. The phase $\langle\xi,x\rangle - \nu(\xi)t$ does not have stationary points if $|x|>t$, and the stationary point equals $\operatorname{sgn} t\, mx(t^2-|x|^2)^{-1/2}$ if $|x|<|t|$. This point does not belong to the support of $\zeta_\mp(x,\xi)\psi(|\xi|^2)$ if $t\to\pm\infty$. Therefore, integrating by parts, we estimate, for arbitrary N, integral (4.14) as $t\to\pm\infty$ by $C_N(1+|x|+|t|)^{-N}$. This implies (4.13). Similarly, we find that, up to a compact operator, $\mathbf{J}_\pm^*\mathbf{J}_\pm - \psi(H_0^2 - m^2)P_0$ is the PDO with symbol $p_0(\xi)(\zeta_\pm^2(x,\xi)-1)\psi^2(|\xi|^2)$. Thus the stationary phase arguments can be again applied. □

Remark 4.11 The same arguments show that the wave operators $W_\pm(H,H_0;\mathbf{J}_\pm)$ do not depend on the choice of the functions $\sigma_\pm(\langle\hat{x},\hat{\xi}\rangle)$ (and $\eta(x)$) as long as $\sigma_\pm(s)=1$ in a neighbourhood of the point $s=\pm1$ and $\sigma_\pm(s)=0$ in a neighbourhood of the point $s=\mp1$.

Proposition 4.12 *Under the asumptions of Theorem* 4.8

$$W_\pm(H,H_0,\mathbf{J}_\mp)=0,\qquad W_\pm(H_0,H,\mathbf{J}_\mp^*)=0. \tag{4.15}$$

If $\Lambda\subset(m,\infty)$ is a compact interval and $\psi(\lambda^2-m^2)=1$ for $\lambda\in\Lambda$, then the operators $W_\pm(H,H_0,\mathbf{J}_\pm)$ are isometric on the subspace $E_0(\Lambda)\mathcal{H}$.

Proof – Clearly

$$\begin{aligned}&\| W_\pm(H,H_0,\mathbf{J}_\pm)f\|^2 = \lim_{t\to\pm\infty}\|\mathbf{J}_\pm e^{-iH_0t}f\|^2\\ &= \lim_{t\to\pm\infty}((\mathbf{J}_\pm^*\mathbf{J}_\pm - \psi^2(H_0^2-m^2))e^{-iH_0t}f, e^{-iH_0t}f) + \|\psi(H_0^2-m^2)e^{-iH_0t}f\|^2 .\end{aligned}$$

If $f\in E_0(\Lambda)\mathcal{H}$, then, by (4.12), the right-hand side here equals $||f||^2$. Similarly, (4.13) implies the first equality (4.15). The second one is a consequence of the first because $W_\pm(H_0,H;\mathbf{J}_\mp^*) = W_\pm(H,H_0,\mathbf{J}_\mp)^*$. □

We are now able to check that the wave operators $W_\pm(H,H_0;\mathbf{J}_\pm)$ are complete.

Theorem 4.13 *Under the assumptions of Theorem* 4.8 *and Proposition* 4.12 *the wave operators $W_\pm(H,H_0;\mathbf{J}_\pm)$ exist, are isometric on the subspace $E_0(\Lambda)\mathcal{H}$ and are complete, i.e., equality* (1.23) *holds.*

Proof – By Theorem 4.8, Corollary 4.9 and Proposition 4.12, we have only to prove (1.23). Since the operator $W_\pm(H_0,H;\mathbf{J}_\pm^*)$ exists, for any $f\in E(\Lambda)P\mathcal{H}$,

$$\lim_{t\to\pm\infty}||\mathbf{J}_\pm^* e^{-iHt}f - e^{-iH_0t}f_0|| = 0,\quad f_0 = W_\pm(H_0,H;\mathbf{J}_\pm^*)f\in E_0(\Lambda)\mathcal{H}.$$

This implies the equality

$$\lim_{t\to\pm\infty}||\mathbf{J}_\pm\mathbf{J}_\pm^* e^{-iHt}f - \mathbf{J}_\pm e^{-iH_0t}f_0|| = 0. \tag{4.16}$$

On the other hand, according to the second relation (4.13), $||\mathbf{J}_\mp^* e^{-iHt}f||\to 0$ as $t\to\pm\infty$ and hence

$$\lim_{t\to\pm\infty}||\mathbf{J}_\mp\mathbf{J}_\mp^* e^{-iHt}f|| = 0. \tag{4.17}$$

It follows from Proposition 4.3 that, up to a compact term, $\mathbf{J}_\pm \mathbf{J}_\pm^* + \mathbf{J}_\mp \mathbf{J}_\mp^*$ is the PDO with symbol $p_0(\xi)(\zeta_\pm^2(x,\xi)+\zeta_\mp^2(x,\xi))\psi^2(|\xi|^2)$. Choosing (see Remark 4.11) $\sigma_\pm$ such that $\sigma_\pm^2 + \sigma_\mp^2 = 1$ and taking into account (4.16), (4.17), we obtain that

$$\lim_{t\to\pm\infty} ||\psi^2(H_0^2 - m^2)E_0(\mathcal{I})e^{-iHt}f - \mathbf{J}_\pm e^{-iH_0 t} f_0|| = 0, \quad \mathcal{I} = (m,\infty).$$

Since the operator

$$\psi^2(H^2 - m^2)E(\mathcal{I}) - \psi^2(H_0^2 - m^2)E_0(\mathcal{I})$$

is compact and $\psi^2(H^2 - m^2)E(\mathcal{I})f = f$, this leads to the relation

$$\lim_{t\to\pm\infty} ||e^{-iHt}f - \mathbf{J}_\pm e^{-iH_0 t} f_0|| = 0. \tag{4.18}$$

Thus $f = W_\pm(H, H_0; \mathbf{J}_\pm) f_0$. $\square$

4.4. Wave operators corresponding to the negative part of the spectrum can be considered in the same way. In this case the functions $\Phi_\pm^-$ are again defined by formulas (3.20), (3.21) – (3.23), (3.26), if $\nu(\xi)$ in definition (1.12) is replaced by $-\nu(\xi)$. Similarly, the matrix-functions $p_\pm^-$ are defined by formulas (3.8), (3.9) if $\nu(\xi)$ in (3.8) is replaced by $-\nu(\xi)$. Then the operators $J_\pm^-$ are given by the relations

$$(J_\pm^- f)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle + i\Phi_\mp^-(x,\xi)} p_\mp^-(x,\xi)\eta(x)\sigma_\mp(\langle \hat{x}, \hat{\xi}\rangle)\psi(|\xi|^2)\hat{f}(\xi)d\xi.$$

The operators $\mathbf{J}_\pm^-$ are obtained if $p_\mp^-(x,\xi)$ is replaced here by $p_0^-(\xi)$. Theorem 4.13 remains true for the wave operators $W_\pm(H, H_0; \mathbf{J}_\pm^-)$ if Λ is a compact subinterval of $(-\infty, -m)$.

Finally we note that, in the proof of the existence of wave operators (Theorem 4.8), Theorem 2.15 can be substituted by the technically simpler Theorem 2.14. Indeed, neglecting first the short-range part V_S and setting $H_L = H_0 + V_L$, we obtain that wave operators (4.10) (4.11) exist for the triples H_0, H_L, J_τ where $\tau =$ " + " and $\tau =$ " − ". On the other hand, using only the limiting absorption principle for the operators H_L and $H = H_L + V_S$, we see that the wave operators $W_\pm(H, H_L)$ and $W_\pm(H_L, H_0)$ also exist. Therefore, by the multiplication theorem (see e.g. [20]), the wave operators (4.10), (4.11) exist and

$$W_\pm(H, H_0; J_\tau) = W_\pm(H, H_L)W_\pm(H_L, H_0; J_\tau),$$
$$W_\pm(H_0, H; J_\tau^*) = W_\pm(H_0, H_L)W_\pm(H_L, H; J_\tau^*).$$

Thus the wave operators $W_\pm(H, H_0; J_\pm)$ are isometric and complete.

5. THE ASYMPTOTICS OF SOLUTIONS FOR LARGE TIMES

Here we find the asymptotics as $t \to \pm\infty$ of a solution $u(x,t) = (\exp(-iHt)f)(x)$ of the time-dependent Dirac equation for an arbitrary $f \in \mathcal{H} \ominus \mathcal{H}_H^{(p)}$. As usual we suppose that $f \in E(\mathcal{I}^+)\mathcal{H}$ and make comments on the case $f \in E(\mathcal{I}^-)\mathcal{H}$ at the end of the section.

5.1. Let us proceed from Theorem 4.13. According to (4.18) the problem reduces to the study as $t \to \pm\infty$ of the integral

$$(\mathbf{J}_\pm e^{-iH_0t} f_0)(x) = (2\pi)^{-3/2}\eta(x) \int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle + i\Phi_\pm(x,\xi) - i\nu(\xi)t}\sigma_\pm(\langle \hat{x}, \hat{\xi}\rangle) g_0(\xi) d\xi \qquad (5.1)$$

where

$$g_0(\xi) = \psi(|\xi|^2)p_0(\xi)\hat{f}_0(\xi) \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}).$$

The asymptotics of this integral can be found by the stationary phase method. Set

$$\Theta(x,\xi,t) = \langle x, \xi\rangle + \Phi_\pm(x,\xi) - \nu(\xi)t, \quad \pm t > 0. \qquad (5.2)$$

The stationary point $\xi^{(st)} = \xi^{(st)}(x,t)$ of this function is determined by the equation

$$x + \nabla_\xi \Phi_\pm(x, \xi^{(st)}) - \nu(\xi^{(st)})^{-1}\xi^{(st)}t = 0, \qquad (5.3)$$

which can be rewritten as

$$\xi^{(st)}(x,t) = \pm m(x + \nabla_\xi\Phi_\pm(x,\xi^{(st)}))(t^2 - |x + \nabla_\xi\Phi_\pm(x,\xi^{(st)})|^2)^{-1/2}. \qquad (5.4)$$

Using estimates (3.6) it is easy to check that for sufficiently large $|x|$, $|t|$ and $|x| < |t|$, equation (5.4) has a unique solution which can be obtained by iterations. In particular, starting from $\pm mx(t^2 - |x|^2)^{-1/2}$ (the exact solution for the free case $H = H_0$ when $\Phi_\pm = 0$) we see that

$$\xi^{(st)}(x,t) = \pm mx(t^2 - |x|^2)^{-1/2} + O(|t|^{-\rho}). \qquad (5.5)$$

In the ball $|x| \le c_1|t|$ and outside of the ball $|x| \ge c_2|t|$ where $c_1 = c_1(f)$ and $c_2 = c_2(f)$, we have, for an arbitrary N, the estimate

$$|(\mathbf{J}_\pm e^{-iH_0t} f_0)(x)| \le C_N(1 + |x| + |t|)^{-N}.$$

This estimate can be obtained by a direct integration by parts. Thus for a given function $\hat{f}_0 \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ we may suppose that

$$0 < c_1|t| < |x| < c_2|t|, \quad \text{where } c_2 < 1. \qquad (5.6)$$

Let us introduce the Hessian matrix of function (5.2)

$$\mathfrak{h}(\xi, x, t) = \operatorname{Hess}\Theta(x,\xi,t) = \left\{\frac{\partial^2}{\partial\xi_j\partial\xi_k}\Theta(x,\xi,t)\right\}, \quad j,k = 1,2,3, \qquad (5.7)$$

and set

$$\Omega(x,t) = \Theta(x, \xi^{(st)}(x,t), t). \qquad (5.8)$$

Applying the stationary phase method to integral (5.1), we find that in the region (5.6)

$$\begin{aligned}(\mathbf{J}_\pm e^{-iH_0t} f_0)(x) &= |\det \mathfrak{h}(\xi^{(st)}, x, t)|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sgn}\mathfrak{h}(\xi^{(st)},x,t)} \\ &\times\ e^{i\Omega(x,t)}\zeta_\pm(x, \xi^{(st)}(x,t)) g_0(\xi^{(st)}(x,t))(1 + o(1)). \end{aligned} \qquad (5.9)$$

Clearly,

$$\operatorname{Hess}\nu(\xi) = \nu(\xi)^{-3}\begin{pmatrix} m^2+\xi_2^2+\xi_3^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_1\xi_2 & m^2+\xi_1^2+\xi_3^2 & -\xi_2\xi_3 \\ -\xi_1\xi_3 & -\xi_2\xi_3 & m^2+\xi_1^2+\xi_2^2 \end{pmatrix},$$

and hence

$$\det\operatorname{Hess}\nu(\xi) = m^2(m^2+|\xi|^2)^{-5/2}, \quad \operatorname{sgn}\operatorname{Hess}\nu(\xi) = 3.$$

Relation (5.2) implies that

$$|\det\mathfrak{h}(\xi,x,t)| = m^2(m^2+|\xi|^2)^{-5/2}|t|^3(1+O(|t|^{-\rho})), \quad \operatorname{sgn}\mathfrak{h}(\xi,x,t) = \mp 3. \tag{5.10}$$

In particular, it follows from (5.5) that

$$|\det\mathfrak{h}(\xi^{(st)}(x,t),x,t)| = m^{-3}t^{-2}(t^2-|x|^2)^{5/2}(1+O(|t|^{-\rho})).$$

Furthermore, according to (5.5), $\zeta_\pm(x,\xi^{(st)}(x,t)) = 1$ for sufficiently large t, $\pm t > 0$, and

$$\lim_{t\to\pm\infty}\int_{|x|\le|t|}|g_0(\xi^{(st)}(x,t)) - g_0(\pm mx(t^2-|x|^2)^{-1/2})|^2dx = 0.$$

Let us now introduce a family of unitary operators $U_0(t)$ by the formula

$$(U_0(t)f_0)(x) = m^{3/2}|t|(t^2-|x|^2)^{-5/4}e^{\mp 3i\pi/4}e^{i\Omega(x,t)}\hat{f}_0(\pm mx(t^2-|x|^2)^{-1/2}), \quad \pm t>0, \tag{5.11}$$

for $|x| < |t|$ and $(U_0(t)f_0)(x) = 0$ for $|x| \ge |t|$. We recall that the phases $\Omega(x,t)$ are defined by formulas (5.2), (5.3) and (5.8).

The results obtained allow us to formulate the following result.

Lemma 5.1 *For any $f_0 \in E_0(\mathcal{I})\mathcal{H}$,*

$$\lim_{t\to\pm\infty}||\mathbf{J}_\pm e^{-iH_0t}f_0 - U_0(t)\psi(H_0^2-m^2)f_0|| = 0.$$

Now we can reformulate Theorem 4.13 in the following form.

Theorem 5.2 *Suppose that $V = V_S + V_L$ where V_S satisfies* (2.6) *and V_L satisfies* (1.18)*. Let the phase $\Omega(x,t)$ be defined by formulas* (5.2), (5.8) *where $\xi^{(st)}(x,t)$ is the solution of equation* (5.4)*. Let $U_0(t)$ be the operator* (5.11)*. Then the wave operators*

$$\mathbf{W}_\pm = s-\lim_{t\to\pm\infty} e^{iHt}U_0(t)E_0(\mathcal{I})$$

exist, are isometric and $\operatorname{Ran}\mathbf{W}_\pm = E(\mathcal{I})P\mathcal{H}$*. Furthermore, $\mathbf{W}_\pm$ are related to the wave operators of Theorem* 4.13 *by the formula $\mathbf{W}_\pm\psi(H_0^2-m^2) = W_\pm(H,H_0;\mathbf{J}_\pm)$.*

It follows from Theorem 5.2 that for any $E(\mathcal{I})P\mathcal{H}$

$$\lim_{t\to\pm\infty}||e^{-iHt}f - U_0(t)f_0|| = 0, \quad f_0 = \mathbf{W}_\pm^* f.$$

Thus similarly to the Schrödinger case [20], a long-range perturbation changes for large times only the phase (see (5.11)) of the solution of the Dirac equation. Of course, the leading term of asymptotics of the phase $\Omega(x,t)$ coincides with the phase

$$\Omega_0(x,t) = \mp m\sqrt{t^2-|x|^2} \tag{5.12}$$

for the free problem $H = H_0$.

The case $f \in E(\mathcal{I}^-)\mathcal{H}$ can be considered quite similarly. We have to replace in all formulas $\Phi_\pm = \Phi_\pm^+$ and $\nu(\xi)$ by $\Phi_\mp^-$ and $-\nu(\xi)$, respectively, and change " $\pm$ " in (5.11) to "$\mp$". This gives phase function $\Omega^-(x,t)$ and the operator $U_0^-(t)$. The wave operators

$$\mathbf{W}_\pm^- = s - \lim_{t\to\pm\infty} e^{iHt}U_0^-(t)E_0(\mathcal{I}^-)$$

exist, are isometric, Ran $\mathbf{W}_\pm^- = E(\mathcal{I}^-)P\mathcal{H}$ and $\mathbf{W}_\pm^-\psi(H_0^2-m^2) = W_\pm(H,H_0;\mathbf{J}_\pm^-)$.

5.2 The expressions (5.3), (5.8) for Ω may be simplified. Of course, we need the function $\Omega(x,t)$ only in the cone $|x| < |t|$. Here we obtain the eikonal equation for $\Omega(x,t)$ which then can be solved by iterations. To that end, we first find relations between derivatives of the phase functions $\Phi_\pm$ and Ω. Let us differentiate expression (5.8) in the variables x and t:

$$\Omega_{x_i} = \xi_i^{(st)} + \sum_{j=1}^3 x_j \frac{\partial\xi_j^{(st)}}{\partial x_j} - t\nu(\xi^{(st)})^{-1}\sum_{j=1}^3 \xi_j^{(st)}\frac{\partial\xi_j^{(st)}}{\partial x_i} + \Phi_{x_i} + \sum_{j=1}^3 \frac{\partial\Phi}{\partial\xi_j}\frac{\partial\xi_j^{(st)}}{\partial x_i} \tag{5.13}$$

and

$$\Omega_t = \langle x, \xi_t^{(st)}\rangle - \nu(\xi^{(st)}) - t\nu(\xi^{(st)})^{-1}\langle\xi^{(st)},\xi_t^{(st)}\rangle + \langle\nabla_\xi\Phi_\pm,\xi_t^{(st)}\rangle. \tag{5.14}$$

Replacing $\nabla_\xi\Phi_\pm$ in (5.13), (5.14) by its expression from equation (5.3), we finally get

$$\nabla_x\Omega = \xi^{(st)} + \nabla_x\Phi_\pm, \quad \Omega_t = -\nu. \tag{5.15}$$

Our goal is to rewrite equation (1.14) where $\xi = \xi^{(st)}(x,t)$ in terms of the function Ω. Using expressions (5.15), we find that

$$\mathbf{v}(x,\xi^{(st)}) = \langle\xi^{(st)},A\rangle - v\partial_t\Omega + \mathbf{b}$$

and

$$\begin{aligned}&\langle\xi^{(st)},\nabla\Phi_\pm\rangle + \mathbf{v} + \langle A,\nabla\Phi_\pm\rangle + 2^{-1}|\nabla\Phi_\pm|^2\\ &\qquad = 2^{-1}|\nabla_x\Omega|^2 - 2^{-1}|\partial_t\Omega|^2 + \langle A,\nabla_x\Omega\rangle - v\partial_t\Omega + \mathbf{b} + 2^{-1}m^2,\end{aligned}$$

where $\mathbf{b}$ is given by (1.13). Now it follows from (1.14) that Ω satisfies the condition (cf. [6])

$$|\nabla_x\Omega|^2 - |\partial_t\Omega|^2 - 2v\partial_t\Omega + 2\mathbf{b} + 2\langle A,\nabla_x\Omega\rangle + m^2 = O(|t|^{-1-\epsilon}).$$

This equation can be solved by iterations

$$\Omega = \Omega_0 + \Omega_1 + \ldots + \Omega_n,$$

starting from the phase (5.12) for the free problem $H = H_0$. Indeed, setting $\tilde{\Omega} = \Omega - \Omega_0$, we obtain for $\tilde{\Omega}$ the relation

$$\frac{\mp 2m}{\sqrt{t^2 - |x|^2}} \left(\langle x, \nabla_x \tilde{\Omega} \rangle + t\partial_t \tilde{\Omega} \right) + |\nabla_x \tilde{\Omega}|^2 - |\partial_t \tilde{\Omega}|^2 \mp 2mvt(t^2 - |x|^2)^{-1/2} + 2\mathbf{b}$$
$$\mp 2m(t^2 - |x|^2)^{-1/2} \langle A, x \rangle - 2v\partial_t \tilde{\Omega} + 2\langle A, \nabla_x \tilde{\Omega} \rangle = O(|t|^{-1-\epsilon}). \tag{5.16}$$

Now we use the following obvious

Lemma 5.3 *A solution of the equation*

$$\langle x, \nabla_x \Omega(x,t) \rangle + t\partial_t \Omega(x,t) = f(x,t), \quad |x| \leq |t|,$$

where f is C^1 and $f(0,0) = 0$, can be constructed by the formula

$$\Omega(x,t) = \int_0^1 f(ux, ut) u^{-1} du.$$

This lemma allows us to solve (5.16) by iterations. For example, for the first iteration Ω_1, equation (5.16) gives

$$\langle x, \nabla_x \Omega_1 \rangle + t\partial_t \Omega_1 = vt \mp m^{-1} \mathbf{b}(x)(t^2 - |x|^2)^{1/2} - \langle A(x), x \rangle.$$

Applying Lemma 5.3 we obtain

$$\Omega_1(x,t) = t\int_0^1 v(ux)du \mp m^{-1}(t^2 - |x|^2)^{1/2} \int_0^1 \mathbf{b}(ux)du - \int_0^1 \langle A(ux), x \rangle du. \tag{5.17}$$

This expression is sufficient for $\rho > 1/2$. Moreover, in this case the quadratic terms in (1.13) can be neglected which gives

$$\Omega_1(x,t) = t\int_0^1 v(ux)du \mp (t^2 - |x|^2)^{1/2} \int_0^1 A_0(ux)du - \int_0^1 \langle A(ux), x \rangle du. \tag{5.18}$$

Of course, the phase function Ω^- equals $\Omega^- = \Omega_0^- + \tilde{\Omega}^-$ where Ω_0^- is defined by (5.12) with " $\mp$ " replaced by " $\pm$ " and $\tilde{\Omega}^-$ satisfies (5.16) with " $\mp$ " replaced by " $\pm$ ". In particular, in the case $\rho > 1/2$ " $\mp$ " in (5.17), (5.18) should be replaced by " $\pm$ ".

An immediate consequence of (5.18) and Theorem 5.2 is the well known result of [8] about long-range magnetic scattering.

Proposition 5.4 *Let assumption* (1.18) *be fulfilled for $\rho > 1/2$ and suppose that $v = A_0 = 0$. Suppose additionally that A satisfies the transversal gauge condition, i.e., for any $x \in \mathbb{R}^3$, $\langle A(x), x \rangle = 0$. Then the usual wave operators*

$$W_\pm = s - \lim_{t \to \pm\infty} e^{iHt} e^{-iH_0 t}$$

exist and are complete.

APPENDIX A. PROOF OF THEOREM 2.11

This appendix is devoted to the proof of the limiting absorption principle for the operator $H = H_0 + V$ where $V = V_L + V_S$ satisfies (2.5), (2.6). As shown in Section 2, under these assumptions the Mourre estimate (2.7) and inclusion (2.13) are satisfied. Our arguments here follow essentially those of [9], [12], [17] for the Schrödinger operator. A difference between the Schrödinger and Dirac operators is that their domains are $\mathbf{H}^2$ and $\mathbf{H}^1$, respectively. Since the conjugate operator $\mathbf{A}$ is the same, *a priori* this could have created new difficulties. As we shall see, there are not essential.

Let $\lambda > m$, $\lambda \notin \sigma_p(H)$. Then for sufficiently small $\eta > 0$ estimate (2.7) holds. Let us choose a function $f \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} f \subset X_\eta$ and $f(\mu) = 1$ if $\mu \in [\lambda - \eta_0, \lambda + \eta_0]$, $\eta_0 < \eta$. Let $Y = [\lambda - \eta_0/2, \lambda + \eta_0/2]$. Following [12], we set

$$K_\varepsilon = (1 + |x|)^{-1/2-\sigma}(1 + |\varepsilon x|)^{-1/2+\sigma}, \quad \sigma = r - 1/2 \in (0, 1/2),$$

$$B = [H, i\mathbf{A}], \quad M = f(H)Bf(H), \quad G_\varepsilon = (H - i\varepsilon M - z)^{-1}, \ \varepsilon \geq 0, \quad \mathrm{Im} z > 0,$$

and $F_\varepsilon = K_\varepsilon G_\varepsilon K_\varepsilon$. The proof of Theorem 2.11 hinges on the differential inequality

$$\left|\left|\frac{dF_\varepsilon}{d\varepsilon}\right|\right| \leq C\varepsilon^{-a}(1 + ||F_\varepsilon||), \quad a < 1. \tag{A.1}$$

We emphasize that, here and below, different constants do not depend neither on ε nor on z. The proof of (A.1) requires, in its turn, several technical assertions. The first of them is borrowed from [12] (see Lemmas 7.3 and 7.4). Its proof relies only on estimate (2.7) and the boundedness of M (see Remark 2.6).

Proposition A.1 *For $\varepsilon > 0$ and $\operatorname{Im} z > 0$, the operator $H - i\varepsilon M - z$ is invertible and the operator-function $G_\varepsilon(z)$ is continuously differentiable in $\varepsilon \in (0, \infty)$ and continuous in $\varepsilon \in [0, \infty)$. Moreover, for $\operatorname{Re} z \in Y$, some $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$ the following estimates are valid*

$$||f(H)G_\varepsilon(z)\varphi|| \leq C\varepsilon^{-1/2}|\langle\varphi, G_\varepsilon(z)\varphi\rangle|^{1/2},$$

$$||(H_0 + i)G_\varepsilon(z)|| \leq C\varepsilon^{-1}, \tag{A.2}$$

$$||(H_0 + i)(I - f(H))G_\varepsilon(z)|| \leq C, \tag{A.3}$$

and

$$||K_\varepsilon G_\varepsilon(z)(H_0 + i)|| \leq C\varepsilon^{-1/2}(1 + ||F_\varepsilon||^{1/2}). \tag{A.4}$$

Below we always suppose that $\operatorname{Re} z \in Y$, $\operatorname{Im} z > 0$, $\varepsilon \geq 0$ but all our estimates remain true for $\operatorname{Im} z < 0$, $\varepsilon \leq 0$. We need a generalization of (A.4).

Lemma A.2 *For any $s \in [0, 1]$,*

$$||K_\varepsilon G_\varepsilon(z)(H_0 + i)\langle x\rangle^s|| \leq C\varepsilon^{-1/2-s}(1 + ||F_\varepsilon||^{1/2}). \tag{A.5}$$

Proof – If $s=0$, then (A.5) reduces to (A.4). So, by interpolation arguments, it suffices to check (A.5) for $s=1$. Since the commutator $[H_0,\langle x\rangle]$ is a bounded operator, to that end we need to verify that

$$||K_\varepsilon G_\varepsilon\langle x\rangle(H_0+i)||\leq C\varepsilon^{-3/2}(1+||F_\varepsilon||^{1/2}). \tag{A.6}$$

Commuting the operator G_ε and $\langle x\rangle$, we find that

$$G_\varepsilon\langle x\rangle=\langle x\rangle G_\varepsilon+G_\varepsilon T_\varepsilon G_\varepsilon, \tag{A.7}$$

where

$$T_\varepsilon=[\langle x\rangle,H_0]-i\varepsilon[\langle x\rangle,M]. \tag{A.8}$$

The operator $[\langle x\rangle,H_0]$ is of course bounded. Let us write the second operator as

$$[\langle x\rangle,M]=[\langle x\rangle,f(H)]Bf(H)+f(H)[\langle x\rangle,B]f(H)+f(H)B[\langle x\rangle,f(H)]. \tag{A.9}$$

The first and the third terms here are bounded operators because $[\langle x\rangle,f(H)](H_0+i)\in\mathcal{B}$ by Lemma 2.9 and $(H_0+i)^{-1}Bf(H)\in\mathcal{B}$ by Remark 2.6. Let us check that $[\langle x\rangle,B]\in\mathcal{B}$. Remark that $[V_L,\mathbf{A}]$ is an operator of multiplication and hence $[\langle x\rangle,[V_L,\mathbf{A}]]=0$. The equality $[\langle x\rangle,[V_S,\mathbf{A}]]=0$ follows from representation (2.10). Finally, according to (2.4), $[\langle x\rangle,[H_0,i\mathbf{A}]]=[\langle x\rangle,H_0]$ is a bounded operator. Thus operator (A.9) is bounded and it follows from (A.6) – (A.8) that

$$||K_\varepsilon G_\varepsilon\langle x\rangle(H_0+i)||\leq(||K_\varepsilon\langle x\rangle||+C||K_\varepsilon G_\varepsilon||)||G_\varepsilon(H_0+i)||.$$

By definition of K_ε,

$$||K_\varepsilon\langle x\rangle||\leq C\varepsilon^{\sigma-1/2}. \tag{A.10}$$

Taking into account (A.2) and (A.4), we arrive at (A.6). □

The condition on the double commutator of V with $\mathbf{A}$ is used in the following

Lemma A.3 *Let inclusion* (2.13) *be fulfilled. Then*

$$||K_\varepsilon G_\varepsilon[f(H)[H,\mathbf{A}]f(H),\mathbf{A}]G_\varepsilon K_\varepsilon||\leq C\varepsilon^{-2+\rho}(1+||F_\varepsilon||). \tag{A.11}$$

Proof – By Remark 2.6, the operator $[f(H),\mathbf{A}]$ is bounded (cf. Lemma 2.9). Set $\tilde{H}_0=H_0-m\alpha_0$. Therefore the operator

$$\begin{aligned} i[f(H)[H_0,\mathbf{A}]f(H),\mathbf{A}] &= [f(H)\tilde{H}_0f(H),\mathbf{A}] \\ &= f(H)\tilde{H}_0[f(H),\mathbf{A}]-if(H)\tilde{H}_0f(H)+[f(H),\mathbf{A}]\tilde{H}_0f(H)\end{aligned}$$

is also bounded. Taking into account (2.13), we now obtain that

$$\begin{aligned} ||K_\varepsilon G_\varepsilon[f(H)[V,\mathbf{A}]f(H),\mathbf{A}]G_\varepsilon K_\varepsilon|| &\leq C||K_\varepsilon G_\varepsilon(H_0+i)\langle x\rangle^{(1-\rho)/2}|| \\ &\times\ ||\langle x\rangle^{(1-\rho)/2}(H_0-i)G_\varepsilon K_\varepsilon||.\end{aligned}$$

By Lemma A.2, the right-hand side here is estimated by $C\varepsilon^{-2+\rho}(1+||F_\varepsilon||)$. □

We can now prove (A.1) following step by step [12], [17]. Note that

$$\left|\left|\frac{dF_\varepsilon}{d\varepsilon}\right|\right| \le \left|\left|\frac{dK_\varepsilon}{d\varepsilon}\right|\right| (||G_\varepsilon K_\varepsilon|| + ||K_\varepsilon G_\varepsilon||) + \left|\left|K_\varepsilon \frac{dG_\varepsilon}{d\varepsilon} K_\varepsilon\right|\right|. \tag{A.12}$$

Since $||dK_\varepsilon/d\varepsilon|| \le C\varepsilon^{\sigma-1/2}$, estimate (A.4) shows that the first term in the right-hand side of (A.12) does not exceed $C\varepsilon^{\sigma-1}(1+||F_\varepsilon||)$. To consider the second term, we remark that

$$\begin{aligned}\frac{dG_\varepsilon}{d\varepsilon} &= iG_\varepsilon M G_\varepsilon = iG_\varepsilon(f(H)-I)Bf(H)G_\varepsilon + iG_\varepsilon B(f(H)-I)G_\varepsilon \\ &\quad -G_\varepsilon[H - i\varepsilon M - z, \mathbf{A}]G_\varepsilon - i\varepsilon G_\varepsilon[M, \mathbf{A}]G_\varepsilon. \end{aligned} \tag{A.13}$$

Let us multiply (A.13) by K_ε from the left and from the right and estimate each term separately.

For the first term, we have that according to (A.3), (A.4)

$$\begin{aligned}||K_\varepsilon G_\varepsilon(f(H)-I)Bf(H)G_\varepsilon K_\varepsilon|| &\le ||K_\varepsilon||\,||G_\varepsilon(f(H)-I)(H_0+i)|| \\ &\times\ ||(H_0+i)^{-1}Bf(H)||||G_\varepsilon K_\varepsilon|| \\ &\le C\varepsilon^{-1/2}(1+||F_\varepsilon||^{1/2}) \le C_1\varepsilon^{-1/2}(1+||F_\varepsilon||).\end{aligned}$$

The second term is quite similar. Opening the commutator in the third term we find that

$$\begin{aligned}&||K_\varepsilon G_\varepsilon[H - i\varepsilon M - z, \mathbf{A}]G_\varepsilon K_\varepsilon|| \\ &\qquad \le ||K_\varepsilon\langle x\rangle||\,||\langle x\rangle^{-1}\mathbf{A}(H_0+i)^{-1}||\,(||(H_0+i)G_\varepsilon K_\varepsilon|| + ||K_\varepsilon G_\varepsilon(H_0+i)||) \\ &\qquad \le C\varepsilon^{\sigma-1}(1+||F_\varepsilon||),\end{aligned}$$

where we have used (A.4) and (A.10). Finally, the last term is estimated by (A.11). Putting the results obtained together, we see that (A.12) entails inequality (A.1) with $a = \min\{\sigma, \rho\}$.

Starting from the bound $||F_\varepsilon|| \le C\varepsilon^{-1}$ (which follows from (A.2)) and using (A.1), we obtain by iterative arguments (see [12] for details) that F_ε is actually uniformly bounded, that is

$$\sup_{\operatorname{Re} z\in Y, \operatorname{Im} z>0, 0\le\varepsilon<\varepsilon_0} ||F_\varepsilon(z)|| < \infty.$$

In particular, for $\varepsilon = 0$ this ensures that the operator-function $\langle x\rangle^{-r}R(z)\langle x\rangle^{-r}$, $r > 1/2$, is uniformly bounded for $\operatorname{Re} z \in \Lambda$, $\operatorname{Im} z > 0$, where Λ is a compact interval such that $\Lambda \cap \sigma_p(H) = \emptyset$. Moreover, the same inequality (A.1) ensures continuity of this function with respect to z.

APPENDIX B. PROOF OF THEOREM 2.15

Let $H = H_0 + V$ where $V = V_L + V_S$ satisfies (2.5), (2.6). Our goal here is to prove H-smoothness of the operators $G_j = \langle x\rangle^{-1/2}\nabla_j^\perp$ on any compact interval $\Lambda \subset (-\infty, -m)\cup$

(m, ∞) such that $\Lambda \cap \sigma_p(H) = \emptyset$. By Proposition 2.2 it suffices to verify that the operators G_j are H^2-smooth on Λ^2.

Our proof is similar to that of subsection 2.3 and relies on a milder modification of inequality (2.25). Actually, we shall check that, for all $k = 1, 2, 3$, $r = (1+\rho)/2$ and any $u = E(\Lambda)u$,

$$\begin{aligned} 2||G_k u||^2 &\leq -\mathrm{Im}\,(\mathbf{M}u, H^2 u) \\ &+ C(||\langle x\rangle^{-r}u||^2 + ||\langle x\rangle^{-r}Hu||^2 + \sum_{j=1}^{3}(||\langle x\rangle^{-r}D_j u||^2 + ||\langle x\rangle^{-r}D_j Hu||^2)). \end{aligned} \quad \text{(B.1)}$$

Since, by Corollary 2.12, the operators $\langle x\rangle^{-r}$, $\langle x\rangle^{-r}D_j$, $\langle x\rangle^{-r}H$ and $\langle x\rangle^{-r}D_j H$ are H^2-smooth on Λ^2, Proposition 2.3 implies that the operators G_k are also H^2-smooth on Λ^2.

Below we shall prove the inequality (B.1). To that end, we have to calculate $\mathrm{Im}\,(\mathbf{M}u, H^2 u)$ for $u = E(\Lambda)u$. Note that the domain $\mathcal{D}(H^2)$ of the operator H^2 defined via the spectral theorem remains indetermined. This impedes our calculation of the commutator of H^2 and $\mathbf{M}$. In particular, the terms like $(\mathbf{M}u, H_0^2 u)$ or $(\mathbf{M}u, H_0 V_S u)$ make no sense. To avoid domain problems, we first commute H^2 with a bounded operator

$$\mathbf{M}_n = \sum_{j=1}^{3} \left(\mathbf{m}_j D_j \zeta_n(D) + \zeta_n(D) D_j \mathbf{m}_j\right),$$

where $\zeta_n(\xi) = \zeta(\xi/n)$ and $\zeta \in C_0^\infty(\mathbb{R}^3)$, such that $\zeta(\xi) = 1$ for $|\xi| \leq 1$. Then we pass to the limit $n \to \infty$. Note that $\mathbf{M}_n u \in \mathbf{H}^1(\mathbb{R}^3, \mathbb{C}^4)$ and

$$s - \lim_{n\to\infty} \mathbf{M}_n u = \mathbf{M}u$$

for any $u \in \mathbf{H}^1(\mathbb{R}^3, \mathbb{C}^4)$. Therefore, for any $u = E(\Lambda)u \in \mathbf{H}^1(\mathbb{R}^3, \mathbb{C}^4)$,

$$\begin{aligned} (\mathbf{M}u, H^2 u) &= \lim_{n\to\infty} (\mathbf{M}_n u, H^2 u) = \lim_{n\to\infty} (H\mathbf{M}_n u, Hu) \\ &= \lim_{n\to\infty} [(H_0 \mathbf{M}_n u, H_0 u) + (H_0 \mathbf{M}_n u, V_S u)] + (V_S \mathbf{M}u, H_0 u) \\ &+ (\mathbf{M}u, (H_0 V_L + V_L H_0 + V_L^2)u) + (\mathbf{M}u, (V_S V_L + V_L V_S + V_S^2)u). \end{aligned} \quad \text{(B.2)}$$

In the right-hand side we have already passed to the limit $n \to \infty$ in the terms where this passage is obvious.

Let us consider every term in the right-hand side separately. First, we note obvious estimates

$$\begin{aligned} |(V_S \mathbf{M}u, H_0 u)| &\leq ||\langle x\rangle^{r} V_S \mathbf{M}u||\, ||\langle x\rangle^{-r} H_0 u|| \\ &\leq C\left(||\langle x\rangle^{-r}u||^2 + ||\langle x\rangle^{-r}\nabla u||^2\right), \end{aligned}$$

$$\begin{aligned} |(\mathbf{M}u, (V_S V_L + V_L V_S + V_S^2)u)| &\leq ||\langle x\rangle^{-r}\mathbf{M}u||\, ||\langle x\rangle^{r}(V_S V_L + V_L V_S + V_S^2)u|| \\ &\leq C\left(||\langle x\rangle^{-r}u||^2 + ||\langle x\rangle^{-r}\nabla u||^2\right), \end{aligned}$$

following from the assumption (2.6) (and boundedness of V_L). To treat other terms we have to take the imaginary part.

Lemma B.1 *For any $u \in \mathbf{H}^1(\mathbb{R}^3, \mathbb{C}^4)$,*

$$|\mathrm{Im}\,(\mathbf{M}u, (H_0V_L + V_LH_0 + V_L^2)u)| \leq C\left(||\langle x\rangle^{-r}u||^2 + ||\langle x\rangle^{-r}\nabla u||^2\right).$$

Proof – It suffices to prove this inequality for $u \in \mathbf{H}^2(\mathbb{R}^3, \mathbb{C}^4)$. In this case

$$2|\mathrm{Im}\,(\mathbf{M}u, (H_0V_L + V_LH_0 + V_L^2)u)| = |([H_0V_L + V_LH_0 + V_L^2, \mathbf{M}]u, u)|.$$

Note that, by assumption (2.5), the commutators of $\mathbf{M}$ with V_L and V_L^2 are multiplications by short-range functions. Similarly, the commutators of $\mathbf{M}$ and H_0 is a first order differential operators whose coefficients are $O(|x|^{-1})$. Multiplied by $V_L(x)$ this gives us again a short-range function. □

Next we calculate the limit in the right-hand side of (B.2).

Lemma B.2 *For any $u \in \mathbf{H}^1(\mathbb{R}^3, \mathbb{C}^4)$, there exists*

$$\lim_{n\to\infty} \mathrm{Im}\,(H_0\mathbf{M}_nu, H_0u) = -2\sum_{j,k}(\mathbf{m}_{jk}u_j, u_k) + 2^{-1}((\Delta^2\mathbf{m})u, u), \quad u_j = D_ju. \qquad \text{(B.3)}$$

Proof – Remark that

$$\mathrm{Im}\,(H_0\mathbf{M}_nu, H_0u) = \mathrm{Im}\,(\nabla\mathbf{M}_nu, \nabla u) = \mathrm{Im}\,([\nabla, \mathbf{M}_nu, \nabla u). \qquad \text{(B.4)}$$

Clearly,

$$([\partial_j, \mathbf{m}_k\partial_k\zeta_n(D) + \partial_k\zeta_n(D)\mathbf{m}_k]u, \partial_ju) = ((\mathbf{m}_{kj}\partial_k\zeta_n(D) + \partial_k\zeta_n(D)\mathbf{m}_{kj})u, \partial_ju).$$

We can pass here to the limit $n \to \infty$ which gives

$$((2\mathbf{m}_{kj}\partial_k + \mathbf{m}_{kkj})u, \partial_ju).$$

Summing these expressions over k and j and substituting them in (B.4), we arrive at (B.3). □

Below, we also use that, for any $u \in \mathbf{H}^1(\mathbb{R}^3, \mathbb{C}^4)$,

$$\lim_{n\to\infty}[H_0, \mathbf{M}_n]u = -\sum_{j,k}\alpha_j(\mathbf{m}_{jk}\partial_k + \partial_k\mathbf{m}_{jk})u =: \mathcal{W}u. \qquad \text{(B.5)}$$

Lemma B.3 *For any $u = E(\Lambda)u$, there exists*

$$\lim_{n\to\infty} \mathrm{Im}\,(H_0\mathbf{M}_nu, V_Su) = \mathrm{Im}\,(\mathcal{W}u, V_Su) + \mathrm{Im}\,(\mathbf{M}Hu, V_Su) - \mathrm{Im}\,(\mathbf{M}V_Lu, V_Su). \qquad \text{(B.6)}$$

Moreover, the right-hand side of (B.6) is bounded by

$$C\left(||\langle x\rangle^{-r}u||^2 + ||\langle x\rangle^{-r}Hu||^2 + \sum_{j=1}^{3}(||\langle x\rangle^{-r}D_ju||^2 + ||\langle x\rangle^{-r}D_jHu||^2)\right). \qquad \text{(B.7)}$$

Proof – Clearly,

$$\begin{aligned}(H_0\mathbf{M}_n u, V_S u) &= ([H_0, \mathbf{M}_n]u, V_S u) + (\mathbf{M}_n H u, V_S u) - (\mathbf{M}_n V_S u, V_S u) \\ &- (\mathbf{M}_n V_L u, V_S u). \end{aligned} \tag{B.8}$$

It is important that the "bad" term $(\mathbf{M}_n V_S u, V_S u)$ disappears here if the imaginary part is taken. In all other terms in the right-hand side of (B.8) we can pass to the limit $n \to \infty$. In the first term we use (B.5), in the next – the equality $\mathbf{M}_n H u = (\mathbf{M}_n E(\Lambda))(Hu)$, and in the last – the inclusion $V_L u \in \mathbf{H}^1(\mathbb{R}^3, \mathbb{C}^4)$ for $u \in \mathbf{H}^1(\mathbb{R}^3, \mathbb{C}^4)$. This gives us equality (B.6). It follows from (B.5) and assumptions (2.5), (2.6) that $(\mathcal{W}u, V_S u)$ and

$$(\mathbf{M}V_L u, V_S u) = ([\mathbf{M}, V_L]u, V_S u) + (V_L \mathbf{M}u, V_S u)$$

are estimated by $C(||\langle x\rangle^{-r}u|| + ||\langle x\rangle^{-r}\nabla u||)||\langle x\rangle^{-r}u||$. The term $(\mathbf{M}Hu, V_S u)$ is estimated by $C(||\langle x\rangle^{-r}Hu|| + ||\langle x\rangle^{-r}\nabla Hu||)||\langle x\rangle^{-r}u||$. $\square$

Combining the results obtained, we see that equality (B.2) implies estimate (B.1). As was already explained, this estimate, in its turn, ensures H^2-smoothness of the operator G_k on Λ^2.

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