Edge current channels and Chern numbers in the integer quantum Hall effect

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Abstract

A quantization theorem for the edge currents is proven for discrete magnetic half-plane operators. Hence the edge channel number is a valid concept also in presence of a disordered potential. Under a gap condition on the corresponding planar model, this quantum number is shown to be equal to the quantized Hall conductivity as given by the Kubo-Chern formula. For the proof of this equality, we consider an exact sequence of C*-algebras (the Toeplitz extension) linking the half-plane and the planar problem, and use a duality theorem for the pairings of K-groups with cyclic cohomology.

1 Introduction

In quantum Hall effect (QHE) experiments, one observes the quantization of the Hall conductance of an effectively two-dimensional semiconductor in units of the universal constant e^2/h [35, 45]. As the Hall conductance is a macroscopic quantity, this effect is of completely different nature than any quantization in atomic physics resulting from Bohr-Sommerfeld rules. Although also a pure quantum effect, the quantum numbers of the QHE rather turn out to be global topological invariants of the underlying magnetic Hamiltonian. These invariants become apparent only in the strong localization regime.

For an explanation of the integer QHE, the only situation studied in this work, a one-particle framework is widely excepted to be sufficient. The three main existing theoretical approaches are respectively based on the Laughlin Gedankenexperiment [38], on the edge channel picture introduced by Halperin [31] and Büttiker [17] and on the Kubo-Chern formula for the Hall conductivity first derived by TKN$_2$ [50]. Laughlin's argument was rigorously analyzed by Avron, Seiler, Simon and Yaffe even for multiparticle Hamiltonians and in presence of a disordered potential [6, 8, 7]. Bellissard [10, 11] and Kunz [37] and others [13, 2] generalized the TKN$_2$ work in order to show quantization of the Hall conductivity also in presence of a disordered potential as long as the Fermi level lies in a region of dynamically localized states. The mathematical connection between these two works is well understood [7, 13, 2]. Although several recent works concern boundary conditions for magnetic half-plane operators [4] as well as spectral questions for these operators [39, 24, 30], there was up to now no microscopic mathematical theory of edge channel conduction, except for the case without disorder, of course. Furthermore, a conceptual understanding of the link of bulk and edge theory was lacking even though the beautiful work of Hatsugai [32, 33] was an important step in this direction. The results of this work, announced in [49], fill these gaps.

For a description of our main results in a particular case, let $H_H : \ell^2(\mathbf{Z}^2) \to \ell^2(\mathbf{Z}^2)$ denote the Harper Hamiltonian describing the motion of a tight-binding electron in a plane submitted to a constant (but arbitrary) magnetic flux per unit cell (see Section 2.1 for its definition). Further let V be an Anderson type disorder potential, then we consider the stochastic Hamiltonian $H = H_H + V$. In presence of an external electric field, all extended states below the Fermi level undergo the Lorentz drift. The associated (bulk) Hall conductivity $\sigma^b_\perp(\beta, \mu)$ at inverse temperature β and Fermi energy μ is calculated in linear response theory by the Kubo formula. The main result of [50, 13, 2] is that $\sigma^b_\perp(\mu) = \sigma^b_\perp(\infty, \mu)$ is equal to a constant integer multiple of q^2/h as long as the Fermi level μ varies in a given interval Δ of dynamically localized states (q is the particle charge and h Planck's constant). In particular, if Δ is a gap of the spectrum of H, the conclusion holds. The appearing integer is actually given by the Chern number of the Fermi projection P_μ of H. It is hence a topological invariant of the planar model.

In a macroscopic Hall bar, there is now another conduction mechanism by edge currents. For a classical two-dimensional electron gas (2DEG), the cyclotron orbits are intercepted by the boundary and this leads to a net current along the boundary. In order to calculate the corresponding quantum mechanical edge current, we will study the restriction $\hat{H}$ of H to the half-plane Hilbert space $\ell^2(\mathbf{Z} \times \mathbf{N})$, together with a given boundary condition. All operators on the half-plane will carry a hat throughout this work. Let $\hat{J}_x = q\imath[X, \hat{H}]/\hbar$ be the electrical current operator along the boundary (here X is the position operator of the direction parallel to the boundary) and $\hat{P}_\Delta$ the spectral projection of $\hat{H}$ on an interval $\Delta \subset \mathbf{R}$. The edge current carried by the states in Δ is then given by

$$j^e(\Delta) \;=\; \mathrm{Tr}_y \mathcal{T}_x(\hat{P}_\Delta \hat{J}_x) \;, \qquad (1)$$

where $\mathcal{T}_x$ the trace per unit volume parallel to the boundary and Tr_y the trace in the direction perpendicular to the boundary. Note that the calcultion of edge currents is mere equilibrium quantum mechanics: no electric field, linear approximation or dissipation mechnism is needed. Our main result is then the following:

Theorem *Let Δ be a gap of the (almost sure) spectrum of $H = H_H + V$. Then, for any interval $\Delta' \subset \Delta$, one has almost surely*

$$j^e(\Delta') \;=\; -\sigma^b_\perp(\mu)\, \frac{1}{q}\, |\Delta'| \;, \qquad \forall \;\mu \in \Delta \;,$$

where $|\Delta'|$ denotes the length of Δ'.

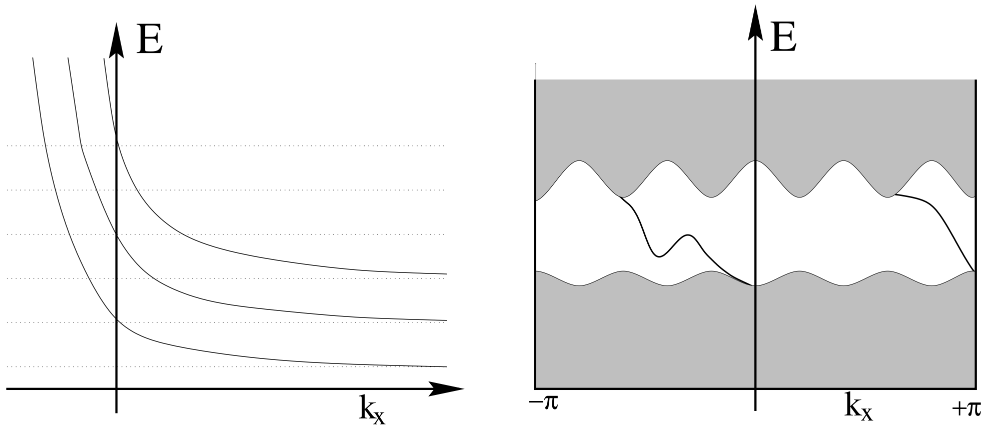

Figure 1: l.h.s.: *schematic representation of the spectrum of the half plane operator* $\hat{H}_L$ (*solid lines*)*, the dashed lines show the Landau bands;* r.h.s.: *spectrum of* $\hat{H}_H$*, the solid lines are the Dirichlet bands and the shaded regions are the Bloch bands*)*for numerical results, see* [32]).

The theorem states that the edge channel number in the sense of [31, 17] of a magnetic Hamiltonian in the half-plane remains a valid concept in presence of a disordered potential, and that this number is moreover equal to the Chern number of the planar Hamiltonian. In the following we will call the limit of the quotient $-qj^e(\Delta)/|\Delta|$ as $\Delta \to \{\mu\}$ also the edge Hall conductivity $\sigma^e_\perp(\mu)$ at μ. It is well known [17] that its quantization can only hold for a magnetic operator on a semi-infinite space because for a strip geometry the backscattering, notably tunneling from upper to lower edge, will destroy quantization.

For the unperturbed Landau Hamiltonian H_L, the continuous analogon of the theorem just states that in the nth gap of H_L, the half-plane operator $\hat{H}_L$ has exactly n bands (compare Fig. 1). In fact, if $\hat{H}_L = \int_{\mathbf{R}}^{\oplus} dk_x \ \hat{H}_L(k_x)$ is the Bloch decomposition in the x-direction and $E_j(k_x)$ are the corresponding edge channels, then for an interval Δ in the nth gap, one has

$$j^e(\Delta) \;=\; \frac{q}{h}\sum_{j\geq 0}\int_{\mathbf{R}} dk_x \;\chi(E_j(k_x)\in\Delta)\;\frac{dE_j(k_x)}{dk_x} \;\;=\; -\frac{q}{h}\;n\;|\Delta|\;,$$

where χ denotes the characteristic function. For the Harper Hamiltonian H_H, the theorem asserts that the sum of the Chern numbers of the lowest n bands is equal to the number of edge channels within the nth gap (also Dirichlet bands in [32]) of the half-plane operator multiplied by their orientation. In the commensurate case of rational flux per unit cell, this result is due to Hatsugai [32, 33]. His proof is completely unrelated to ours though and cannot be generalized to a situation with broken translation invariance due to either an irrational magnetic flux or to a disordered potential.

Our proof of the above theorem splits into two parts. In the first purely analytic step, we consider only half-plane operators and prove that the edge Hall conductivity is equal to the Fredholm index of a certain unitary operator. This can be viewed at as an odd index theorem resulting from a pairing between an odd cyclic cohomology class with the K_1-class of the above unitary operator. Similarly, the quantized Kubo-Hall conductivity is given by a pairing of a certain even cyclic cohomology class with the K_0-class defined by the Fermi projection. These two pairings are over two different C*-algebras which are linked by a short exact sequence called the Toeplitz extension [44]. In the second step, we use the K-theoretic six-term exact sequence of the Toeplitz extension as well as its dual K-homologic counterpart to prove the equality of the pairings. This duality theorem unveals that the equality of edge and bulk Hall conductivities is a consequence of a fundammental topological concept, namely of Bott periodicity. Hence this work fully exploits (and thereby justifies) the use of the non-commutative C*-algebraic framework developed by Bellissard [10].

Let us now comment on variations and generalizations of the above theorem. First of all, its continuous conterpart is under preperation. It uses the Wiener-Hopf rather than the Toeplitz extension [20, 16] and the duality theorem corresponding to Connes' Thom isomorphism [27]. Further we believe the consequences of the theorem to hold under the weaker condition that Δ is an interval containing only dynamically localized states of H. Under precisely this condition it is possible to prove quantization of the bulk Hall conductivity [13]. The gap condition imposed above only allows to deal with the weak disorder regime (unsufficient in order to explain the QHE). In this regime the Hamiltonian $\hat{H}$ has purely absolutely continuous spectrum as show positive commutator estimates for the boundary current operator (the arguments of [24] directly transpose to our discrete case as long as the edge bands of the free magnetic operator have a definite sign). In the regime of intermediate disorder, the gaps of H fill with dynamically localized states. In presence of a boundary, the Tunnel effect to the edge states should turn all these localized states into resonances so that we expect the spectrum to remain absolutely continuous. At the same time, the edge channel number should remain an integer equal to the Chern number of the Fermi pojection of the planar model. In the high disorder regime, the system completely localizes [40, 3]; in between there is a cascade of metal-insulator transitions [13].

The article is organized as follows. In the next Chapter 2, we develop the mathematical framework, then state our main results more precisely in Chapter 3, and explain in detail why they lead to a consistent physical picture of the QHE in a Hall bar. It shows that the macroscopic Hall conductance is quantized no matter what proportion of the current flows by bulk or edge states respectively. We present furthermore a simple argument showing that in a typical QH experiment, at most 10% of the current is carried by edge states. In Chapter 4 we give a complete direct proof of the edge current quantization. Only in Chapter 5 we then use K-theoretic arguments in order to show the equality between edge and bulk Hall conductivities. In order to clearly exhibit the concepts involved, we chose a slightly more general formulation than needed in our context. A short Appendix resuming the most basic results of K-theory is included for the convenience of the reader.

We would like to express our deep gratitude to our teachers Jean Bellissard and Ruedi Seiler. Their works set the stage for the present article which hereby inherits their spirit. We moreover profited from discussions with many colegues, beneeth them Y. Avron, J.-M. Combes, S. DeBievre, G. Elliott, F. Germinet, S. Jitormirskaya, A. Klein and M. Seifert. We acknowledge support of the SFB 288.

2 Observable algebras

In this chapter, after having defined the (geometric) hull of a homogeneous tight-binding Hamiltonian in absense of a magnetic field, we construct the observable algebras as the Toeplitz extension of a twisted crossed product, the twisting being introduced by the magnetic field. We then develop the non-commutative analysis tools of differentiation and integration on these algebras. All these constructions are done in the symmetric gauge, and we show in Section 2.6 how a gauge transformation to the Landau gauge allows to work with an iterated crossed product without twisting, a more convenient formulation for the K-theoretic arguments in Chapter 5. For further details and for physical motivation of the following definitions and subsequent constructions, we refer to [10, 12, 13, 48, 14].

2.1 Homogeneous operators and their hull

As in [14] we define the (geometric) hull of a tight-binding operator in absense of a magnetic field. This seems more adapted than its definition in [13] because the use of magnetic translations leads to a non-trivial hull even for a free magnetic operator as the Harper Hamiltonian (for the Landau Hamiltonian on continuous physical space this problem does not appear due to its invariance w.r.t. the magnetic translations).

Definition 1 *Let $U_0(a)$, $a \in \mathbf{Z}^2$, denote the translations on $\ell^2(\mathbf{Z}^2)$ defined by $U_0(A)\psi(n) = \psi(n-a)$ for $n \in \mathbf{Z}^2$ and $\psi \in \ell^2(\mathbf{Z}^2)$. Consider a self-adjoint bounded operator $H : \ell^2(\mathbf{Z}^2) \to \ell^2(\mathbf{Z}^2)$ of the form*

$$H \;=\; \sum_{n,m\in\mathbf{Z}^2} H_{n,m}\, |n\rangle\langle m| \;, \qquad H_{n,m} \in \mathbf{R} \;,$$

where $\langle m|$ and $|n\rangle$ are the usual bra-ket notations for states localized at $m \in \mathbf{Z}^2$ and $n \in \mathbf{Z}^2$. Then H is called homogeneous if the set $\{U_0(a)HU_0(a)^* \,|\, a \in \mathbf{Z}^2\}$ has a compact closure Ω in the strong operator topology. By continuity, U_0 extends to an action T of $\mathbf{Z}^2$ on Ω. The dynamical system $(\Omega, T, \mathbf{Z}^2)$ is then called the hull of H.

Example: Let H_0 be translation invariant, that is $U_0(a)H_0U_0(a) = H_0$ for all $a \in \mathbf{Z}^2$. A typical example is the discrete Laplacian. Let further $\Omega = [-1,1]^{\otimes \mathbf{Z}^2}$ be furnished with the Tychonov topology as well as the shift action T of $\mathbf{Z}^2$. We fix an invariant and ergodic probability measure $\mathbf{P}$ on Ω with the property $\mathbf{P}(\{\omega \in \Omega \,|\, V_\omega(n) \in I\}) > 0$ for all $n \in \mathbf{Z}^2$ and all intervals $I \subset [-1,1]$ of non-vanishing length. Finally we denote by $V_\omega : \ell^2(\mathbf{Z}^2) \to \ell^2(\mathbf{Z}^2)$ the Anderson-type onsite potential corresponding the disorder configuration $\omega \in \Omega$. Then, for $\mathbf{P}$-almost all $\omega \in \Omega$, the operator $H_\omega = H_0 + \lambda V_\omega$, $\lambda \in \mathbf{R}$, is homogeneous and its hull is equal to $(\Omega, T, \mathbf{Z}^2)$. $\diamond$

From a mathematical point of view, this article is about the analysis of certain topological invariants of the dynamical system $(\Omega, T, \mathbf{Z}^2)$. Associated to it is the C*-dynamical system $(C(\Omega), \alpha, \mathbf{Z}^2)$ where $(\alpha^a f)(\omega) = f(T^{-a}\omega)$ of continuous functions $f \in C(\Omega)$. This gives rise to a natural C*-algebra, the twisted crossed product [42] with a twisting given by the group cocycle $\mathbf{Z}^2 \times \mathbf{Z}^2 \to S^1 : (a,b) \mapsto \theta a \wedge b \bmod(2\pi)$ where $a \wedge b = a_x b_y - a_y b_x$ and $\theta = q\mathcal{B}/\hbar$ is the magnetic flux per unit cell. This algebra has a natural extension, the (twisted) Toeplitz extension. If moreover an ergodic probability measure $\mathbf{P}$ on Ω is given, we have a W*-dynamical system $(L^\infty(\Omega, \mathbf{P}), \alpha, \mathbf{Z}^2)$ with an associated von Neumann crossed product [42]. This also has a Toeplitz extension. These exact sequences describe the link between magnetic operators in the plane and in the half-plane.

2.2 Algebraic twisted crossed product and its Toeplitz extension

The magnetic translations in symmetric gauge $U(a)$, $a = (a_x, a_y) \in \mathbf{Z}^2$, are defined by

$$U(a)\psi(n) \;=\; \exp\left(\frac{\imath}{2}\theta\, a \wedge n\right) \psi(n-a) \;, \qquad n = (n_x, n_y) \in \mathbf{Z}^2 \;, \qquad \psi \in \ell^2(\mathbf{Z}^2) \;,$$

where $a \wedge n = a_x n_y - a_y n_x$ and $\theta = q\mathcal{B}/\hbar$ is the flux per unit cell. U is a projective unitary representation of the group $\mathbf{Z}^2$ on $\ell^2(\mathbf{Z}^2)$, notably $U(a+b) = \exp(\frac{\imath}{2}\theta a \wedge b) U(a) U(b)$. We denote $U_x = U(1,0)$ and $U_y = U(0,1)$, then $U_y U_x = e^{\imath\theta} U_x U_y$. The C*-algebra generated by U_x and U_y is called the rotation algebra $\mathcal{A}_\theta$.

Now we explicitly construct the algebraic twisted crossed product and its Toeplitz extension. The formulas below are chosen such that the magnetic translation $U(a)$ is the representation of the function $(\omega, n) \mapsto \delta_{n,-a}$. Let $\mathcal{A}_0 = C_K(\Omega \times \mathbf{Z}^2)$ be the continuous complex valued functions on $\Omega \times \mathbf{Z}^2$ with compact support. With the following definitions, $\mathcal{A}_0$ becomes a $*$-algebra ($A, B \in \mathcal{A}_0$, $\omega \in \Omega$, $m \in \mathbf{Z}^2$):

$$AB(\omega, m) \;=\; \sum_{l \in \mathbf{Z}^2} A(\omega, l) B(T^{-l}\omega, m-l) \exp\left(\frac{\imath}{2}\theta l \wedge m\right) \;, \qquad A^*(\omega, m) \;=\; \overline{A(T^{-m}\omega, -m)} \;. \tag{2}$$

Further let $\mathcal{E}_0 = C_K(\Omega \times \mathbf{Z} \times \mathbf{N} \times \mathbf{N})$ and $T(\mathcal{A})_0 = \mathcal{E}_0 \oplus \mathcal{A}_0$. As $\mathcal{E}_0$ and $T(\mathcal{A})_0$ will describe half-plane operators, their elements will carry a hat. In order to simultaneously define the $*$-algebra structure on $\mathcal{E}_0$ and $T(\mathcal{A})_0$, we identify elements $(\hat{A}_1, A_2) \in T(\mathcal{A})_0$ with a function $\hat{A} \in C(\Omega \times \mathbf{Z} \times \mathbf{N} \times \mathbf{N})$ by means of the formula ($m_x \in \mathbf{Z}$, $n_y, m_y \in \mathbf{N}$):

$$\hat{A}(\omega, m_x, n_y, m_y) \;=\; \hat{A}_1(\omega, m_x, n_y, m_y) + A_2(\omega, m_x, m_y - n_y) \;. \tag{3}$$

For $\hat{A}, \hat{B} \in T(\mathcal{A})_0$, the product and involution are then defined by

$$\begin{aligned} \hat{A}\hat{B}(\omega, m_x, n_y, m_y) \;&=\; \sum_{(l_x, l_y) \in \mathbf{Z} \times \mathbf{N}} \hat{A}(\omega, l_x, n_y, l_y) \hat{B}(T^{(-l_x, n_y - l_y)}\omega, m_x - l_x, l_y, m_y) \cdot \\ &\qquad \cdot \exp\left(\frac{\imath}{2}\theta(l_x(m_y - n_y) - m_x(l_y - n_y))\right) \;, \end{aligned} \tag{4}$$

$$\hat{A}^*(\omega, m_x, n_y, m_y) \;=\; \overline{\hat{A}(T^{(-m_x, n_y - m_y)}\omega, -m_x, m_y, n_y)} \;. \tag{5}$$

Let us further introduce the inclusion $i : \mathcal{E}_0 \to T(\mathcal{A})_0$ sending $\hat{A} \in \mathcal{E}_0$ to $(\hat{A}, 0) \in T(\mathcal{A})_0$ as well as the projection $\pi : T(\mathcal{A})_0 \to \mathcal{A}_0$ defined by

$$\pi(\hat{A})(\omega, m_x, m_y) \;=\; \lim_{k\to\infty} \hat{A}(\omega, m_x, k, m_y + k) \;, \qquad \hat{A} \in T(\mathcal{A})_0 \;. \tag{6}$$

One directly verifies $\pi(\hat{A}) = A_2$ if $\hat{A} = (\hat{A}_1, A_2) \in T(\mathcal{A})_0$. Hence the operation π eliminates the compact perturbation on the boundary by shifting the system away from it.

The above constructions are made such that the following algebraic lemma holds.

Lemma 1 *We have the following exact sequence of $*$-algebras:*

$$0 \;\to\; \mathcal{E}_0 \;\overset{i}{\to}\; T(\mathcal{A})_0 \;\overset{\pi}{\to}\; \mathcal{A}_0 \;\to\; 0 \;,$$

notably i and π are $*$-morphisms satisfying $\mathrm{Im}(i) = \mathrm{Ker}(\pi)$.

We point out that this exact sequence of $*$-algebras is never split exact even though it is split exact as an exact sequence of vector spaces. More precisely, there does not exist a $*$-morphism $\rho : \mathcal{A}_0 \to T(\mathcal{A})_0$ such that $\rho \circ \pi = \mathbf{1}$.

Proof of Lemma 1. First of all, $i : \mathcal{E}_0 \to T(\mathcal{A})_0$ is clearly a $*$-morphism because the operations (4) and (5) are the same in $\mathcal{E}_0$ and $T(\mathcal{A})_0$ and moreover leave $\mathcal{E}_0$ invariant in $T(\mathcal{A})_0 = \mathcal{E}_0 \oplus \mathcal{A}_0$. Let us check that π is also a $*$-morphism ($\hat{A} = (\hat{A}_1, A_2)$, $\hat{B} = (\hat{B}_1, B_2) \in T(\mathcal{A})_0$):

$$\begin{aligned}\pi(\hat{A}\hat{B})(\omega, m_x, m_y) \;&=\; \lim_{k\to\infty} \sum_{l_x \in \mathbf{Z}} \sum_{l_y \geq 0} \hat{A}(\omega, l_x, k, l_y)\hat{B}(T^{-(l_x, l_y - k)}\omega, m_x - l_x, l_y, m_y + k) \cdot \\ &\qquad\qquad\cdot \exp\left(\frac{\imath}{2}\theta(l_x m_y - m_x(l_y - k))\right) \;.\end{aligned}$$

Now because both $\hat{A}_1, \hat{B}_1 \in \mathcal{E}_0$ have compact support, they will vanish for k sufficiently big. Then we make the change of variables $l_y \leftrightarrow l_y - k$ and right out the identification (3):

$$\begin{aligned}\pi(\hat{A}\hat{B})(\omega, m_x, m_y) \;&=\; \lim_{k\to\infty} \sum_{l_x \in \mathbf{Z}} \sum_{l_y \geq -k} A_2(\omega, l_x, l_y) B_2(T^{-(l_x, l_y)}\omega, m_x - l_x, l_y - m_y) \cdot \\ &\qquad\qquad\cdot \exp\left(\frac{\imath}{2}\theta(l_x m_y - m_x l_y)\right) \;,\end{aligned}$$

Now because A_2 is of compact support, we can replace the summution over $l_y \geq -k$ by that over $l_y \in \mathbf{Z}$ for k sufficiently large. Comparing with (2) we see that $\pi(\hat{A}\hat{B}) = \pi(\hat{A})\pi(\hat{B})$ as $\pi(\hat{A}) = A_2$ and $\pi(\hat{B}) = B_2$. The identity $\pi(\hat{A})^* = \pi(\hat{A}^*)$ can be immediately checked. This moreover shows that $\mathrm{Im}(i) = \mathrm{Ker}(\pi)$. □

These algebras will be called the bulk algebra $\mathcal{A}_0$, its Toeplitz extension $T(\mathcal{A})_0$ and the edge algebra $\mathcal{E}_0$. The algebra $\mathcal{A}_0$ and its various completions will describe operators which are homogeneous in the plane, $\mathcal{E}_0$ models operators which are homogeneous along the boundary (x-direction), but compact in the y-direction perpendicular to the boundary, whereas $T(\mathcal{A})_0$ contains both of these operators, notably homogeneous half-plane operators with compact boundary contributions.

By means of the following formulas, we introduce two families of physical representations π_ω and $\hat{\pi}_\omega$, $\omega \in \Omega$, of the $*$-algebras $\mathcal{A}_0$ and $T(\mathcal{A})_0$ on $\ell^2(\mathbf{Z}^2)$ and $\ell^2(\mathbf{Z} \times \mathbf{N})$ respectively (both $n = (n_x, n_y)$ and $m = (m_x, m_y)$ are in $\mathbf{Z}^2$ or $\mathbf{Z} \times \mathbf{N}$ respectively):

$$\langle n|\pi_\omega(A)|m\rangle \;=\; A(T^{-n}\omega, m-n)\exp\left(\frac{\imath}{2}\theta\; n \wedge m\right) \;, \qquad A \in \mathcal{A}_0 \;, \tag{7}$$

and

$$\langle n|\hat{\pi}_\omega(\hat{A})|m\rangle \;=\; \hat{A}(T^{-n}\omega, m_x - n_x, n_y, m_y)\exp\left(\frac{\imath}{2}\theta\; n \wedge m\right) \;, \qquad \hat{A} \in T(\mathcal{A})_0 \;. \tag{8}$$

It is in fact elementary to verify that π_ω and $\hat{\pi}_\omega$ are $*$-morphisms from $\mathcal{A}_0$ and $T(\mathcal{A})_0$ to the algebras of bounded operators on $\ell^2(\mathbf{Z}^2)$ and $\ell^2(\mathbf{Z} \times \mathbf{N})$ respectively and further that the following covariance relations hold ($a = (a_x, a_y) \in \mathbf{Z}^2$):

$$U(a)\pi_\omega(A)U(a)^* \;=\; \pi_{T^a\omega}(A) \;, \qquad \hat{U}(a_x,0)\hat{\pi}_\omega(\hat{A})\hat{U}(a_x,0)^* \;=\; \pi_{T^{(a_x,0)}\omega}(\hat{A}) \;. \tag{9}$$

where $\hat{U}$ denotes the restriction of U to $\ell^2(\mathbf{Z} \times \mathbf{N})$. Covariance under $\hat{U}(0, a_y)$ only holds for $a_y \geq 0$; for $a_y < 0$ there are corrections by operators in $\mathcal{E}_0$.

2.3 Exact sequence of C*-algebras

Using these representations, we can now introduce norms on $\mathcal{A}_0$ and $T(\mathcal{A})_0$ ($A \in \mathcal{A}_0$ and $\hat{A} \in T(\mathcal{A})_0$):

$$\|A\| \;=\; \sup_{\omega\in\Omega} \|\pi_\omega(A)\| \;, \qquad \|\hat{A}\| \;=\; \sup_{\omega\in\Omega} \|\hat{\pi}_\omega(\hat{A})\| \;. \tag{10}$$

It is elementary to verify the C*-equation for these norms, hence they are actually the unique C*-norms on $\mathcal{A}_0$ and $T(\mathcal{A})_0$. The C*-algebras $\mathcal{A}$, $T(\mathcal{A})$ and $\mathcal{E}$ are defined to be the closure of $\mathcal{A}_0$, $T(\mathcal{A})_0$ and $\mathcal{E}_0$ with respect to these norms. Now, because any $*$-morphism between pre-C*-algebras can be extended by continuity to their C*-closures, the representations π_ω and $\hat{\pi}_\omega$ extend to covariant representations of the C*-algebras and furthermore Lemma 1 leads to an exact sequence of C*-algebras:

$$0 \;\to\; \mathcal{E} \;\overset{i}{\to}\; T(\mathcal{A}) \;\overset{\pi}{\to}\; \mathcal{A} \;\to\; 0 \;. \tag{11}$$

We note that $\mathcal{A}$ is the twisted crossed product $C(\Omega) \times_T \mathbf{Z}^2$ of the dynamical system $(\Omega, T, \mathbf{Z}^2)$ associated to the magnetic twisting introduced above. The structure of this exact sequence will be further analysed in Section 2.6.

Example: Let us consider this exact sequence explicitly for the case of the rotation algebra $\mathcal{A}_\theta$, namely the C*-algebra generated by the unitary operators U_x and U_y. Then the Toeplitz extension is generated by $\hat{U}_x$ and $\hat{U}_y$ which still satisfy the same commutation relation $\hat{U}_y\hat{U}_x = e^{\imath\theta}\hat{U}_x\hat{U}_y$, but while $\hat{U}_x$ remains unitary, $\hat{U}_y$ satisfies $\hat{U}_y^*\hat{U}_y = 1$ and $\hat{U}_y\hat{U}_y^* = 1 - \Pi_0$ where $\Pi_0 = \sum_{l\in\mathbf{Z}} |l,0\rangle\langle l,0|$. The edge C*-algebra is given by the C*-tensor product $C(S^1)$, notbly the C*-algebra generated by $\hat{U}_x$, with the compact operators K in y-direction (this is not quite true, because the multiplication in this product contains the phases as in (4)). The projection is given by $\pi(\hat{U}_{x,y}) = U_{x,y}$. ◇

The following proposition shows that the Hamiltonian which we started out with is in the C*-algebra $\mathcal{A}$. For its proof we refer the reader to [12, 13].

Proposition 1 *Supose that $\tilde{H}$ is a homogeneous bounded self-adjoint operator on $\ell^2(\mathbf{Z}^2)$ the off-diagonal coefficients of which fall off like $|\langle n|\tilde{H}|m\rangle| \leq c|n-m|^{-2}$ for some constant $c > 0$. Let $(\Omega, T, \mathbf{Z}^2)$ be its hull. Then there exists an $H \in \mathcal{A}$ and a $\omega_0 \in \Omega$ such that $\pi_{\omega_0}(H) = \tilde{H}$.*

2.4 von Neumann algebra and its extension

Let $\mathbf{P}$ be a probability measure on Ω which is invariant and ergodic with respect to both T_x and T_y. We suppose that its support is all of Ω. All our results hold for any such measure, but its choice is of physical importance (in particular, the almost sure values of the Hall conductivities may be different for different measures). We now construct the (twisted) von Neumann crossed product $L^\infty(\mathcal{A}, \mathbf{P})$ of the W*-dynamical system $(L^\infty(\Omega, \mathbf{P}), T, \mathbf{Z}^2)$ by the standard procedure [42, 22] using the direct integral representation:

$$\pi_\Omega = \int^\oplus d\mathbf{P}(\omega)\, \pi_\omega \;:\; \mathcal{A} \;\to\; \mathcal{B}(L^2(\Omega \times \mathbf{Z}^2, \mathbf{P} \otimes \lambda)) \;,$$

where λ is the counting measure on $\mathbf{Z}^2$ and $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded operators on the Hilbert space $\mathcal{H} = L^2(\Omega \times \mathbf{Z}^2, \mathbf{P} \otimes \lambda)$. Then $L^\infty(\mathcal{A}, \mathbf{P})$ is defined as the weak closure of $\pi_\Omega(\mathcal{A})$ in $\mathcal{B}(L^2(\Omega \times \mathbf{Z}^2, \mathbf{P} \otimes \lambda))$. Elements of $L^\infty(\mathcal{A}, \mathbf{P})$ are weakly-measurable covariant operator families and can hence be described by measurable functions on $\Omega \times \mathbf{Z}^2$ satisfying $A(\omega, n) \to 0$ as $|n| \to \infty$. We continue to use the notation π_ω for the (fiberwise) representation of these functions on $\ell^2(\mathbf{Z}^2)$. The C*-norm on $L^\infty(\mathcal{A}, \mathbf{P})$ is given by

$$\|A\|_\infty \;=\; \mathbf{P}\text{-}\operatorname*{ess\,sup}_{\omega\in\Omega} \|\pi_\omega(A)\| \;.$$

Similarly, one can construct the W*-crossed product $L^\infty(C(\Omega) \times_{\alpha_x} \mathbf{Z}, \mathbf{P})$ of the W*-dynamical system $(L^\infty(\Omega, \mathbf{P}), \alpha_x, \mathbf{Z})$ where $(\alpha_x f)(\omega) = f(T_x^{-1}\omega)$ for $f \in L^\infty(\Omega, \mathbf{P})$. Elements therein can be represented by functions on $\Omega \times \mathbf{Z}$. Let $L^\infty(\mathcal{E}, \mathbf{P})_0$ denote the finite dimensional matrices with entries in $L^\infty(C(\Omega) \times_{\alpha_x} \mathbf{Z}, \mathbf{P})$. We can represent elements of $L^\infty(\mathcal{E}, \mathbf{P})_0$ as measurable functions on $\Omega \times \mathbf{Z} \times \mathbf{N} \times \mathbf{N}$. Set further $L^\infty(T(\mathcal{A}), \mathbf{P})_0 = L^\infty(\mathcal{E}, \mathbf{P})_0 \oplus L^\infty(\mathcal{A}, \mathbf{P})$. Using the identification (3), the formulas (4) and (5) introduce an algebraic structure on $L^\infty(\mathcal{E}, \mathbf{P})_0$ and $L^\infty(T(\mathcal{A}), \mathbf{P})_0$. Using again the fiberwise representations $\hat{\pi}_\omega$ identifying the functions on $\Omega \times \mathbf{Z} \times \mathbf{N} \times \mathbf{N}$ with operators on $\mathcal{B}(L^2(\Omega \times \mathbf{Z} \times \mathbf{N}, \mathbf{P} \otimes \lambda))$, a C*-norm on both of these algebras is given by

$$\|\hat{A}\|_\infty \;=\; \mathbf{P}\text{-}\operatorname*{ess\,sup}_{\omega\in\Omega} \|\hat{\pi}_\omega(\hat{A})\| \;,$$

and the closure w.r.t. this norm define the C*-algebras $L^\infty(\mathcal{E},\mathbf{P})$ and $L^\infty(T(\mathcal{A}),\mathbf{P})$. Finally the inclusion i and projection π on the pre-C* algebras $L^\infty(\mathcal{E},\mathbf{P})_0$ and $L^\infty(T(\mathcal{A}),\mathbf{P})_0$ can be defined as in (6) and the counterpart of Lemma 1 can be verified, so that the $*$-morphisms i and π can be extended by continuity to the C*-closures giving the the following exact sequence of C*-algebras:

$$0 \;\to\; L^\infty(\mathcal{E},\mathbf{P}) \;\overset{i}{\to}\; L^\infty(T(\mathcal{A}),\mathbf{P}) \overset{\pi}{\to}\; L^\infty(\mathcal{A},\mathbf{P}) \;\to\; 0 \;. \tag{12}$$

2.5 Non-commutative analysis tools

The algebra $L^\infty(\mathcal{A},\mathbf{P})$ admits a differential structure generated by a two-parameter group ρ_k, $k=(k_x,k_y)\in T^2=[-\pi,\pi)^{\times 2}$, of $*$-automorphisms ($A\in L^\infty(\mathcal{A},\mathbf{P})$ and $m=(m_x,m_y)\in\mathbf{Z}^2$):

$$\rho_k(A)(\omega,m) \;=\; \exp(\imath(k_x m_x + k_y m_y))\; A(\omega,m) \;. \tag{13}$$

The associated $*$-derivations $\vec{\nabla}=(\nabla_x,\nabla_y)$ are densly defined and explicitely given by

$$\nabla_x A(\omega,m) \;=\; \imath m_x A(\omega,m) \;, \qquad \nabla_y A(\omega,m) \;=\; \imath m_y A(\omega,m) \;, \tag{14}$$

whenever the right hand sides are in $L^\infty(\mathcal{A},\mathbf{P})$. If $\vec{X}=(X,Y)$ denotes the position operators on $\ell^2(\mathbf{Z}^2)$, then one can verify for such differentiable operators the identities:

$$\pi_\omega(\vec{\nabla}(A)) \;=\; \imath[\pi_\omega(A),\vec{X}] \;. \tag{15}$$

Similarly, there is a two-parameter family of $*$-automorphisms $\hat{\rho}_k$ on $L^\infty(T(\mathcal{A}),\mathbf{P})$ given by

$$\hat{\rho}_k(\hat{A})(\omega,m_x,n_y,m_y) \;=\; \exp(\imath(k_x m_x + k_y(m_y-n_y)))\; A(\omega,m_x,n_y,m_y) \;. \tag{16}$$

The associated $*$-derivations are denoted $\vec{\hat{\nabla}}=(\hat{\nabla}_x,\hat{\nabla}_y)$. They satisfy identities similar to (15). For $n\in\mathbf{N}$, we define $C^n(\mathcal{A})$ to be those elements for which all nth order gradients are in the C*-algebra.

Given an invariant and ergodic probability measure $\mathbf{P}$ with support Ω, a normalized faithful trace $\mathcal{T}$ on $L^\infty(\mathcal{A},\mathbf{P})$ is defined by [42]

$$\mathcal{T}(A) \;=\; \int_\Omega d\mathbf{P}(\omega)\; A(\omega,0) \;, \qquad A\in L^\infty(\mathcal{A},\mathbf{P}) \;. \tag{17}$$

For any increasing sequence $(\Lambda_L)_{L\in\mathbf{N}}$ of squares centered at the origin, Birkhoff's ergodic theorem implies

$$\mathcal{T}(A) \;=\; \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{m\in\Lambda_L} \langle m|\pi_\omega(A)|m\rangle \;, \tag{18}$$

for $\mathbf{P}$-almost all $\omega\in\Omega$ showing that $\mathcal{T}$ is the trace per unit volume.

For $p \in [1,\infty)$, the Banach space $L^p(\mathcal{A},\mathbf{P})$ is defined as the closure of $\mathcal{A}_0$ under the norm $\|A\|_p = (\mathcal{T}(|A|^p))^{1/p}$. If $\pi_{\rm GNS}$ is the GNS representation of $\mathcal{A}$ on $L^2(\mathcal{A},\mathcal{T})$ associated to the state $\mathcal{T}$, then the von Neumann algebra $\pi_{\rm GNS}(\mathcal{A})''$ where $''$ is the bicommutant can easily be seen to be isomorphic to $L^\infty(\mathcal{A},\mathbf{P})$ [22]. A trace $\hat{\mathcal{T}}$ on $L^\infty(T(\mathcal{A}),\mathbf{P})$ is given by $\hat{\mathcal{T}}(\hat{A}) = \mathcal{T}(\pi(\hat{A}))$. However, $\hat{\mathcal{T}}$ is obviously not faithful.

Let now $L^\infty(T(\mathcal{A}),\mathbf{P})_+$ denote the cone of positive operators in $L^\infty(T(\mathcal{A}),\mathbf{P})$ and define $\hat{\mathcal{T}}_{\mathcal{E}} : L^\infty(T(\mathcal{A}),\mathbf{P})_+ \to [0,\infty]$ by

$$\hat{\mathcal{T}}_{\mathcal{E}}(\hat{A}) \;=\; \sum_{m_y\in\mathbf{N}} \int d\mathbf{P}(\omega)\; \hat{A}(\omega,0,m_y,m_y) \;. \tag{19}$$

It follows again from Birkhoff's ergodic theorem that $\hat{\mathcal{T}}_{\mathcal{E}}$ is the trace per unit volume $\mathcal{T}_x$ in the x-direction, followed by the usual trace in the y-direction. Clearly $\hat{\mathcal{T}}_{\mathcal{E}}$ is a weight, namely $\hat{\mathcal{T}}_{\mathcal{E}}(\hat{A}+\lambda\hat{B}) = \hat{\mathcal{T}}_{\mathcal{E}}(\hat{A}) + \lambda\hat{\mathcal{T}}_{\mathcal{E}}(\hat{B})$ for all $\hat{A},\hat{B} \in L^\infty(T(\mathcal{A}),\mathbf{P})_+$ and $\lambda \geq 0$. Moreover one can directly check that $\hat{\mathcal{T}}_{\mathcal{E}}(\hat{A}\hat{A}^*) = \hat{\mathcal{T}}_{\mathcal{E}}(\hat{A}^*\hat{A})$ for all $\hat{A} \in L^\infty(T(\mathcal{A}),\mathbf{P})$. Hence $\hat{\mathcal{T}}_{\mathcal{E}}$ is a trace [25]. In particular, we dispose of its unitary invariance, that is for all $\hat{A} \in L^\infty(T(\mathcal{A}),\mathbf{P})_+$ and unitary $\hat{U} \in L^\infty(T(\mathcal{A}),\mathbf{P})$ one has $\hat{\mathcal{T}}_{\mathcal{E}}(\hat{U}\hat{A}\hat{U}^*) = \hat{\mathcal{T}}_{\mathcal{E}}(\hat{A})$.

As usual (*e.g.* [25, 42]), one now says that an operator $\hat{A} \in L^\infty(T(\mathcal{A}),\mathbf{P})$ is $\hat{\mathcal{T}}_{\mathcal{E}}$-traceclass if $\hat{\mathcal{T}}_{\mathcal{E}}(|\hat{A}|) < \infty$. The set of all $\hat{\mathcal{T}}_{\mathcal{E}}$-traceclass operators form an ideal in $L^\infty(T(\mathcal{A}),\mathbf{P})$ onto which $\hat{\mathcal{T}}_{\mathcal{E}}$ can be extended. Further let us define $L^p(\mathcal{E},\mathbf{P})$ as the closure of $\mathcal{E}_0$ with respect to the norm $\|\hat{A}\|_p = \left(\hat{\mathcal{T}}_{\mathcal{E}}(|\hat{A}|^p)\right)^{1/p}$. The ideal of $\hat{\mathcal{T}}_{\mathcal{E}}$-traceclass operators is then $L^1(\mathcal{E},\mathbf{P}) \cap L^\infty(\mathcal{E},\mathbf{P})$.

It can be directly verified that the traces $\mathcal{T}$ and $\hat{\mathcal{T}}_{\mathcal{E}}$ are invariant under the automorphism groups ρ_k and $\hat{\rho}_k$ respectively. Thus we have for any $A \in C^1(\mathcal{A})$ and any $\hat{A} \in L^1(\mathcal{E},\mathbf{P}) \cap C^1(\mathcal{E})$:

$$\mathcal{T}(\nabla_{x,y}A) \;=\; 0 \;, \qquad \hat{\mathcal{T}}_{\mathcal{E}}(\hat{\nabla}_x\hat{A}) \;=\; 0 \;. \tag{20}$$

2.6 Iterated crossed product and Landau gauge

Let $C(\Omega) \times_{\alpha_x} \mathbf{Z}$ be the C*-crossed product without twisting [42] of the C*-dynamical system $(C(\Omega),\alpha_x,\mathbf{Z})$ where $(\alpha_x f)(\omega) = f(T_x^{-1}\omega)$ for $f \in C(\Omega)$. Elements of $C(\Omega) \times_{\alpha_x} \mathbf{Z}$ can be represented by continuous functions on $\Omega \times \mathbf{Z}$. This C*-algebra inherits a dynamics by the action α_y defined by

$$(\alpha_y f)(\omega,m_x) \;=\; \exp(-\imath\theta m_x)\, f(T_y^{-1}\omega,m_x) \;, \qquad f \in C(\Omega) \times_{\alpha_x} \mathbf{Z} \;.$$

Hence we obtain an iterated crossed product C*-algebra $C(\Omega) \times_{\alpha_x} \mathbf{Z} \times_{\alpha_y} \mathbf{Z}$. Its elements are given by functions on $\Omega \times \mathbf{Z}^2$ and the multiplication is explicitely given by

$$AB(\omega,m) \;=\; \sum_{l\in\mathbf{Z}^2} A(\omega,l)B(T^{-l}\omega,m-l)\exp\left(\imath\theta l_y(l_x-m_x)\right) \;,$$

mimicing the Landau gauge. The iterated crossed product has a Toeplitz extension w.r.t. the action α_y in the precise sense of Pimsner and Voiculescu [44] and this extension turns out to be isomorphic to the exact sequence (11):

$$\begin{tikzcd}
0 \arrow[r] & C(\Omega) \times_{\alpha_x} \mathbf{Z} \otimes K \arrow[r, "\psi"] \arrow[d, "\rho"'] & T \arrow[r, "\pi"] \arrow[d, "\kappa"'] & C(\Omega) \times_{\alpha_x} \mathbf{Z} \times_{\alpha_y} \mathbf{Z} \arrow[r] \arrow[d, "\eta"'] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r, "i"] & T(\mathcal{A}) \arrow[r, "\pi"] & \mathcal{A} \arrow[r] & 0
\end{tikzcd} \tag{21}$$

Here $\otimes K$ denotes the C*-algebraic tensor product with the compact operators on $\ell^2(\mathbf{N})$ and ψ, π and T are the mappings and the Toeplitz extension as defined in [44]. In fact, it is elementary to verify that

$$(\rho\hat{A})(\omega, m_x, n_y, m_y) \;=\; \exp\left(\frac{\imath}{2}\theta m_x(m_y + n_y)\right) \hat{A}(T^{(0,n_y)}\omega, m_x, n_y, m_y) \;, \tag{22}$$

and

$$(\eta A)(\omega, m) \;=\; \exp\left(-\frac{\imath}{2}\theta m_x m_y\right) A(\omega, m) \;, \tag{23}$$

are (gauge) isomorphisms. As it is not used later on, we do not write out a explicit formula for κ here.

3 Main results and their interpretation

3.1 Edge Hall conductivity

Let $\hat{H} = (\hat{K}, H) \in T(\mathcal{A})$ be some lift of the planar (stochastic) self-adjoint Hamiltonian $H \in \mathcal{A}$. For the representation corresponding to the disorder configuration ω, this means:

$$\hat{\pi}_\omega(\hat{H}) \;=\; \Pi\, \pi_\omega(H) + \hat{\pi}_\omega(\hat{K}) \;,$$

where Π denotes the projection from $\ell^2(\mathbf{Z}^2)$ onto $\ell^2(\mathbf{Z} \times \mathbf{N})$. We suppose $\hat{K}$ to have non-zero entries only a finite distance from the boundary (see Section 4.1 for more precise and general hypothesis). Choosing such a lift means that we fix some boundary condition for the operator on the half-plane which is homogenous in the x-direction. For example, if one wishes to impose the α-boundary condition $\cos(\alpha)\psi(m_x, 0) + \sin(\alpha)\psi(m_x, 1) = 0$ for all $m_x \in \mathbf{Z}$, this gives rise to a projectin Π_α on the subspace of all states in $\ell^2(\mathbf{Z} \times \mathbf{N})$ satisfying this condition; then $\hat{\pi}_\omega(\hat{H}) = \Pi_\alpha\, \pi_\omega(H)$, so that $\hat{K} \in \mathcal{E}$ has to satisfy $\hat{\pi}_\omega(\hat{K}) = (\Pi_\alpha - \Pi)\pi_\omega(H)$.

Following the discussions in the introduction, let us now define the edge Hall conductivity as follows:

$$\sigma_\perp^e(\mu) \;=\; -\frac{q^2}{\hbar} \lim_{\Delta \to \{\mu\}} \frac{1}{|\Delta|}\, \hat{\mathcal{T}}_{\mathcal{E}}(\hat{P}_\Delta \hat{\nabla}_x \hat{H}) \;. \tag{24}$$

Recall (see *e.g.* [23]) that the spectrum of $\pi_\omega(H)$ is $\mathbf{P}$-almost surely independent of ω. Our results below show that, under the two hypothesis that (i) $H \in C^3(\mathcal{A})$ and (ii) Δ is within a $\mathbf{P}$-almost sure gap of the spectrum of $\pi_\omega(H)$ (equivalently, Δ is in a gap of the density of states), the operator $\hat{P}_\Delta \in L^\infty(\mathcal{E}, \mathbf{P})$ is in the ideal of $\hat{\mathcal{T}}_\mathcal{E}$-traceclass operators. If Δ is not in a gap, the operator $\hat{P}_\Delta$ is definitely not $\hat{\mathcal{T}}_\mathcal{E}$-traceclass. Nevertheless, the edge current may be well-defined and finite if Δ is a dynamically localized region of the spectrum of H (*cf.* Remark 2 in Section 4.1).

Further we let Π_x denote the projection in $\ell^2(\mathbf{Z} \times \mathbf{N})$ onto $\ell^2(\mathbf{N} \times \mathbf{N})$, namely the subspace spanned by the states $|m_x, m_y\rangle$ with $m_x \geq 0$.

Theorem 1 *Suppose that $\hat{H} \in C^3(T(\mathcal{A}))$. Let the interval $\Delta \subset \mathbf{R}$ be $\mathbf{P}$-almost surely in a gap of the spectrum of $\pi_\omega(H)$. Set*

$$\hat{U}(\Delta) \;=\; \exp\left(-2\pi\imath\, \frac{\hat{H} - E'}{|\Delta|} \hat{P}_\Delta\right) \;, \qquad E' \;=\; \inf(\Delta) \;. \tag{25}$$

Then the operator $\Pi_x \pi_\omega(\hat{U}(\Delta))^* \Pi_x$ is $\mathbf{P}$-almost surely a Fredholm operator on $\ell^2(\mathbf{N} \times \mathbf{N})$ and its index is $\mathbf{P}$-almost surely constant. For any $\mu \in \Delta$, the almost sure value is equal to the edge Hall conductivity:

$$\sigma_\perp^e(\mu) \;=\; \frac{q^2}{h} \,\mathrm{Ind}\, \left(\Pi_x\, \pi_\omega(\hat{U}(\Delta)^*)\, \Pi_x\, |_{\ell^2(\mathbf{N}\times\mathbf{N})}\right) \;.$$

3.2 Link to the Kubo-Chern bulk Hall conductivity

In relaxation time approximation, the Kubo formula for the bulk Hall conductivity $\sigma_\perp^b(\beta, \mu, \tau)$ of a two-dimensional electron gas in the plane at inverse temperature β, chemical potential μ and relaxation time τ reads as follows [13, 48]:

$$\sigma_\perp^b(\beta, \mu, \tau) \;=\; \frac{q^2}{h} \mathcal{T}\left(\nabla_x(f_{\beta,\mu}(H))\, \frac{1}{1/\tau - \mathcal{L}_H} (\nabla_y H)\right) \;,$$

where $f_{\beta,\mu}$ is the Fermi-Dirac function and $\mathcal{L}_H$ the Liouville (super) operator defined by $\mathcal{L}_H(A) = \imath[H, A]/\hbar$ for $A \in \mathcal{A}$. The direct conductivity $\sigma_{//}^b(\beta, \mu, \tau)$ is given by the same formula after replacement of ∇_y by ∇_x. The zero dissipation limit $\tau \to \infty$ of the direct conductivity can only give a finite value (other than 0 or ∞) when the quantum motion is diffusive (weak localization regime) [47, 48]. However, as the Lorentz drift in a magnetic field is a transverse diffusive motion, the Hall conductivity is due to non-dissipative transport. Consequently the zero dissipation limit always gives a finite value. This can actually be proven as shows the following proposition.

Proposition 2 [13, Corollary 2] *Suppose that $H \in C^1(\mathcal{A})$ and that the Fermi level μ is in a gap of the spectrum of H. Then the bulk Hall conductivity $\sigma_\perp^b(\mu)$ at zero temperature ($\beta = \infty$) and zero dissipation ($\tau = \infty$) is*

$$\sigma_\perp^b(\mu) \;=\; \frac{q^2}{h}\, 2\pi\imath\, \mathcal{T}(P_\mu[\nabla_x P_\mu, \nabla_y P_\mu]) \;. \tag{26}$$

The second main result of this work is the following theorem.

Theorem 2 *Suppose that $H \in C^3(\mathcal{A})$ and that the interval Δ is in a gap of the spectrum of H. Then for any $\mu \in \Delta$,*

$$\sigma_\perp^b(\mu) \;=\; \sigma_\perp^e(\mu) \;, \tag{27}$$

and their common value is constant in Δ and equal to an integer multiple of q^2/h.

The Hall conductivity can either be calculated by the index in Theorem 1 or by that of reference [13]. Hence this work gives another alternative proof of the quantization of the bulk Hall conductivity, once one acceptes (26) as its definition.

3.3 Physical interpretation

In this section, we explain how our mathematical results allow to develop a physically consistent, but not mathematically rigorous picture of the QHE. We focus on a so-called Hall bar experiment pictured schematically in Fig. 2. We also use the model with continuous physical space for these explanations, although the mathematical results of this work are restricted to the discret case.

As stressed by Büttiker [17], the size of such a bar has to be much larger than the magnetic length $l \approx 100$Å in order to suppress tunneling between the edge states at the upper and lower edge, otherwise the QH regime cannot be attained. In von Klitzing's experiments [35], the sample size was $400\mu m \times 50\mu m$, but nowadays samples 20 times larger are in use [55]. There are six contacts on the Hall bar. The contacts 1 and 4 are the source and drain contacts for the current I. If $V_{k,l}$ denotes the measured tension between contacts k and l, the associated resistance is denoted by $R_{k,l} = V_{k,l}/I$. The Hall tensions are then $R_{2,5}$, $R_{2,6}$, $R_{3,5}$ and $R_{3,6}$, the longitudinal or direct resistances $R_{2,3}$ and $R_{5,6}$, while $R_{1,4}$ is called the two-terminal (contact) resistance. The QH regime is characterized by quantization of the Hall tension and vanishing of the longitudinal resistance (it is quantized to zero). Typically the experiment is made at about $1K$, but for the explanations below we suppose the temperature to be zero.

An answer to the question what the tensions measured really are was given by Büttiker [17] using the Landauer conductance formalism. The contacts are metallic reservoirs close to thermodynamic equilibrium and $V_{k,l}$ is the difference of the electrochemical potentials of the two reservoirs k and l. These contacts interact with the 2DEG. Let us first suppose that the contacts are ideal in the following sense [17], a hypothesis justified later on:

- the contacts absorb any incoming electron independent of its energy and phase;
- the contacts emit into all those states of the sample that carry a current away from the contact and that have an energy less than the electro-chemical potential of the contact reservoir.

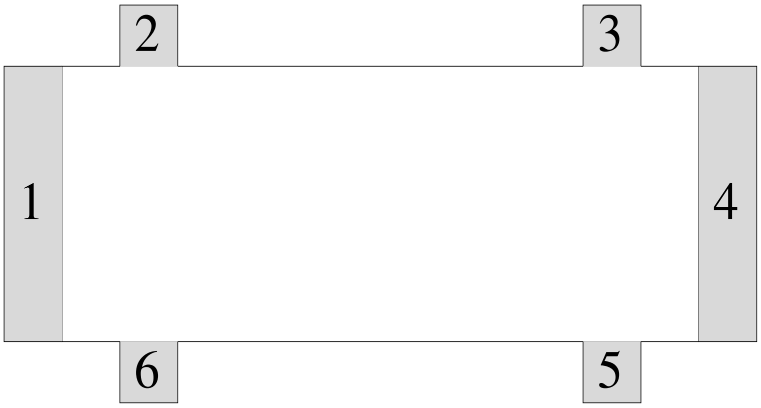

Figure 2: *Schematic representation of the Hall bar sample with 6 contacts.*

These conditions are supposed to hold for all contacts, under the supplementary condition that there is a net current I from contact 1 to 4, but no current leaving any of the other four contacts. Hence the 2DEG is an open conductor with considerable particle exchange with the contacts.

In the QH regime, the 2DEG is in a steady non-equililibrium state interacting with the ideal contacts according to the above two rules and showing quantized Hall and longitudinal resistances. The theoretical picture developed here is based on the following assumptions widely accepted in the physical community, but possibly not that of mathematical physicists:

1. the macroscopic sample can be devided into a (continuously indexed) family of mesoscopic volumes in such a way that the equilibrium physics in each of these mesoscopic volumes can be described by an infinite volume one-particle Hamiltonian;
2. each mesoscopic volume is near thermodynamical equilibrium; it hence has a well-defined local particle density;
3. the properties of the non-equilibrium state can locally be described by the equilibrium Hamiltonian; in particular, we suppose that the response to an external electro-static field is linear and given by Kubo's formula.

Thus we suppose that, for every point $\vec{r} = (x, y)$ in the sample, there are a homogenous one-particle Hamiltonian $H(\vec{r})$ and a local particle density $n(\vec{r})$.

An important physical question concerns the size of the mesoscopic volumes. Several conditions should be satisfied. Their size should be much larger that the magnetic length, but small enough for the particle density to be assumed to be constant. Furthermore they should be smaller than the inelastic scattering length, notably the typical length scale over which the electrons do not collide with phonons or other electrons. Otherwise the one-particle description breaks down (of course, this does not exclude elastic collisions with static disorder due to impurities). We believe these conditions to hold for mesoscopic volumes of about $1\mu m$ diameter. Now we clearly have to distinguish two different infinite volume models for the mesocscopic volumes:

- The bulk model at position $\vec{r}$ is given by a Hamiltonian $H(\vec{r}) = H + qV_{\mathcal{E}}(\vec{r})$ acting on $L^2(\mathbf{R}^2)$. Here H is the sum of the magnetic Landau Hamiltonian H_L and a disordered potential V_{dis}, and $V_{\mathcal{E}}(\vec{r})$ is the external electro-static potential which is supposed to be constant within the mesoscopic volume at $\vec{r}$. Its origin will be discussed below. When submitted to an external electro-static field $\mathcal{E}(\vec{r}) = dV_{\mathcal{E}}(\vec{r})/d\vec{r}$ causing a non-equilibrium situation within the mesoscopic volume, the current-carrying states aquire a drift velocity due to the Lorentz force. This non-dissipative current is linear in the electric field with a proportionality constant given by the Kubo-Chern formula. Note that we describe here the non-equilibrium state due to a small electric field $\mathcal{E}(\vec{r})$ by a local equilibrium Hamiltonian $H(\vec{r})$ containing a constant electro-static potential $V_{\mathcal{E}}(\vec{r})$.

- The edge Hamiltonian at position $\vec{r}$ is $\hat{H}(\vec{r}) = \hat{H} + V_{\mathcal{E}}(\vec{r})$ acting on $L^2(\mathbf{R} \times \mathbf{R}_+)$ where $\hat{H}$ is the restriction of H to the right half-plane. The basic characteristic of this model is the presence of n current-carrying edge channels at the Fermi level whenever the latter is between the nth and $(n+1)$th Landau level. We further note that due to the compressible nature of the 2DEG near the boundary, screening does not allow an electric field to be build up within several magnetic lengths off the edge [19].

To study the total current through the sample, we next have to discuss the form of the electrical potential $V_{\mathcal{E}}(\vec{r})$, the associated electric field $\mathcal{E}(\vec{r}) = dV_{\mathcal{E}}(\vec{r})/d\vec{r}$ and the local chemical potential $\mu(\vec{r})$ defined by the relation

$$n(\vec{r}) \;=\; \mathcal{T}(P_{\mu(\vec{r})}) \;=\; \mathcal{N}((-\infty, \mu(\vec{r}))\ ,$$

where $P_{\mu(\vec{r})}$ is the Fermi projector of H to the energy $\mu(\vec{r})$, $\mathcal{T}$ the trace per unit volume and $\mathcal{N}$ the density of states of H. Note that this definition of $\mu(\vec{r})$ is not with respect to $H(\vec{r})$, but to H. We next suppose that the Hall bar is sufficiently long so that all these quantities only depend on y. Furthermore the projection of the electric field in the x-direction then vanishes. Fig. 3 shows a possible profile of $qV_{\mathcal{E}}(x, y)$ and $qV_{\mathcal{E}}(x, y) + \mu(x, y)$. Before discussing the origin of these curves, let us calculate the associated currents. In each mesoscopic volume of the bulk, the Hall current density $j(\vec{r})$ in y-direction can be calculated by Kubo's formula and is given by

$$j^b(\vec{r}) \;=\; \sigma^b_\perp(\vec{r})\, \mathcal{E}(\vec{r})\ ,$$

where $\sigma^b_\perp(\vec{r})$ is the bulk Hall conductivity at $\vec{r}$ calculated using the Hamiltonian H and chemical potential $\mu(\vec{r})$. In order to calculate the total bulk current I^b, we keep x fixed and have to integrate $j^b(x, y)$ over all y between the left and right edge. Under the hypothesis that $\mu(\vec{r})$ varies in the same region of dynamically localized states of H, $\sigma^b_\perp(\vec{r})$ is constant and equal to $\sigma^b_\perp$ [13]. Thus, if we set $V^b = \int_0^L dy\, \mathcal{E}(x, y)$ as in Fig. 3,

$$I^b \;=\; \int_0^L dy\, \sigma^b_\perp(x, y)\, \mathcal{E}(x, y) \;=\; \sigma^b_\perp\, V^b\ .$$

Next the total edge current I^e is equal to the sum of the left edge and right edge currents. Both can be calculated with the Hamiltonian $\hat{H}$ if one takes account for the different orientation of

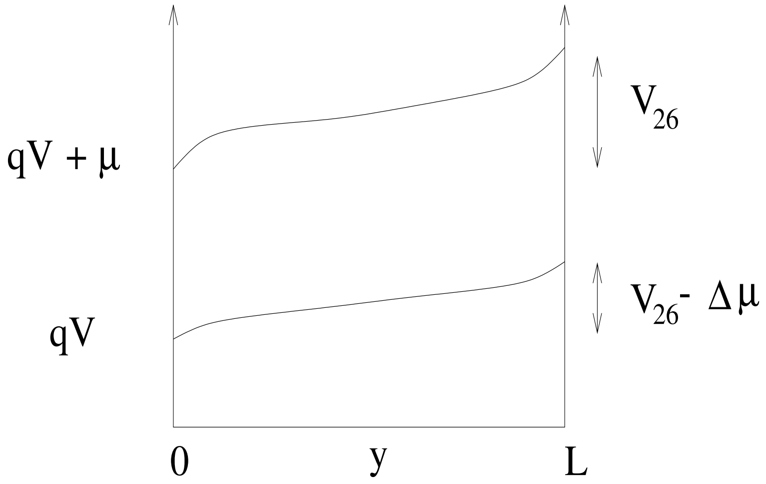

Figure 3: *Plot of the electrostatic potential and its sum with the local chemical potential as a function of the position of the position y in the Hall bar.*

these currents by a minus sign. Supposing again that the interval $\Delta = [\mu(0), \mu(L)]$ is contained in an interval of dynamically localized states of H, edge current quantization then implies

$$I^e \;=\; -q\hat{\mathcal{T}}_{\mathcal{E}}(\hat{P}_{\mu(L)}\nabla_x\hat{H}) + q\hat{\mathcal{T}}_{\mathcal{E}}(\hat{P}_{\mu(0)}\nabla_x\hat{H}) \;=\; -q\hat{\mathcal{T}}_{\mathcal{E}}(\hat{P}_{\Delta}\nabla_x\hat{H}) \;=\; \sigma^e_\perp\,|\Delta|\;.$$

Finally, according to Fig. 3 and the above assumptions of the ideal contacts, the measured Hall tension $V_{2,6}$ (or one of the others) is equal to the sum of V^b and Δ/q. As now $\sigma^b_\perp = \sigma^e_\perp = \sigma_\perp$, because the Fermi level stays in the same region of dynamically localized states of H, so that one obtains the desired quantization:

$$R_{2,6} \;=\; \frac{V_{2,6}}{I} \;=\; \frac{V_{2,6}}{I^b + I^e} \;=\; \frac{V_{2,6}}{\sigma^b_\perp V^b + \sigma^e_\perp|\Delta|} \;=\; \frac{1}{\sigma_\perp}\;.$$

The proportion of current carried by the bulk states is equal to $V^b/(V^b + |\Delta|)$ and the slope of $V_{\mathcal{E}}$ gives the bulk current distribution. Hence the profile of the electro-static potential and the chemical potential in Fig. 3 is of great interest. Considerable effort has been made to experimentally measure and numerically calcuate these curves. In early experiments [45], supplementary leads were introduced into the sample in order to measure the tension profile accross the sample. It turned out to be linear so that one concluded that the current is uniformly distributed through the bulk. However, the contacts modify the electronic structure within the sample too heavily as that this conclusion can be drawn. More recent Pockels effect measurements [29] and inductive coupling measurements [55] (in a sample with back gate) show that about 80% of the Hall tension falls off within a strip of about $400\mu m$ width close to the boundaries. Hence an important portion of the bulk current flows close to the boundaries, but this does *not* mean that an important portion flows by the edge states in the above sense. In fact, edge states are localized within a few magnetic lengths ($l \approx 100$Å) off the boundary and the measured current flows in a strip three orders of magnitude wider. The authors of [55] further conclude that, within their experimental resolution, at most 5% of the total current flows by edge states.

We obtain a similar result from the following simple theoretical reasoning. In the above argument, it was crucial for the quantization to hold that the local chemical potential $\mu(\vec{r})$ stays within the same gap of extended states of H for all $\vec{r}$ in the sample. For the existence of the plateaux, it is furthermore necessary that this condition holds for all filling factors corresponding to the plateaux. This imposes that the energetic interval $\Delta = [\mu(0), \mu(L)]$ has to be smaller than the energetic distance $\hbar\omega_c(1-q)$ (here ω_c is the cyclotron frequency so that $\hbar\omega_c$ is the distance between two Landau levels, and q is the quotient of the energetic width of the plateaux and $\hbar\omega_c$). As the size of Δ determines the proportion of the edge currents, we can estimate this condition using the experimental data for a typical QH regime from [45, Chapter 2] for the $\sigma_\perp = 4$ plateau: $B \approx 6\,T$, $V_{2,6} \approx 170\,mV$ and $q \approx 0.6$. Using the data for the effective electron mass ($m_* \approx 0.07 m_e$) and the electron charge, we obtain $\hbar\omega_c \approx 48\,meV$ and a maximal proportion of edge currents of 10%.

Next we discuss the physical origin of the electrostatic potential $V_{\mathcal{E}}$ and of the variations of local chemical potential $\mu(\vec{r})$. In several numerical and theoretical works [34, 54, 15], the screening within the 2DEG (without impurities) is analyzed in a self-consistent way. Screening in the vicinity of the edge was studied separately in [19]. On the other hand, the experimental results of [55] clearly indicate that an important screening effect can be due to the back gates used in many QHE experiments. Such a back gate is usually a metallic plate on which the 3D semiconductor (doped $GaAs$) is mounted and it can be used to vary the particle density of the 2DEG. These gates are within a mm of the 2DEG, therefore the screening of their electrons can lead to an important electric field in the 2DEG. Another important effect not sufficiently analyzed to our knowledge is the electric field build up by the occupied localized states in the sample. In fact, if the chemical potential varies from left to right border within a region of localized states, the corresponding variation of the occupation rate of the localized states must lead to important electric fields within the 2DEG.

We conclude this section with a brief discussion of the role of the contacts. It is based on the works of Büttiker [17, 18]. If the contacts are ideal as in the above sense, then the chemical potential at the whole left edge is equal to the chemical potential μ_1 of contact 1, hence $\mu_2 = \mu_3 = \mu_1$. Similarly, $\mu_5 = \mu_6 = \mu_4$. Thus not only the Hall and direct resistance are quantized, but also the two-terminal contact resistance $R_{1,4}$ is quantized and equal to the Hall resistance if all the contacts are ideal. This quantization of $R_{1,4}$ was actually observed [28]. However, Büttiker further argued that the Hall and direct resistance are also quantized if either only the contact 1 and 4 or only the contacts 2,3,5 and 6 are ideal. The other contacts in these situations can then be disordered or dirty in the sense that they only absorb a portion of the incoming particles and that the distribution of the outgoing particles can be a non-equilibrium one. For example, if contacts 1 and 4 are dirty in this sense, there would be no equilibrium distribution beneath the edge states between contacts 1 and 2 as well as 4 and 5. Here one could not work with Fermi projections as in all the arguments above. However, the ideal contacts 2 and 5 would then absorb the whole current and reemit it in an equilibrium distribution. Hence the Hall and direct tension are quantized as before. Furthermore, all contacts could be dirty if the inelastic scattering within the sample is effective. This inelastic scattering would then equilibrate all the edge channels, for example, along the path between contacts 1 and 2, no matter what the occupation of the edge states at contact 1 is. This latter picture is, however, physically incorrect. Several experiments [18] have shown that the equilibration length between the edge channels is very long and typically of the size of the sample. Such a long equilibration length is also theoretically consistent [19]. Furthermore, in the situation with dirty contacts, the two terminal resistance $R_{1,4}$ should not be quantized as it experimentally is. Hence the QHE is due to ideal contacts and as all of them are more or less identical, the above picture with six ideal contacts is probably justified.

4 Edge current theory

4.1 Traceclass property near the Fermi surface

The following summability result will be used in the next section. It also implies that the edge current is actually a well-defined mathematical quantity under a gap condition on the plane operator. The precise conditions on the boundary condition of $\hat{H} = (H, \hat{K})$ are the following:

Hypothesis: *$\hat{K} = \hat{K}_1 + \hat{K}_2 \in C^3(\mathcal{E})$ and there is an $L < \infty$ such that for all $\omega \in \Omega$:*

$$\langle n_x, n_y | \hat{\pi}_\omega(\hat{K}_1) | m_x, m_y \rangle \;=\; 0 \quad \textit{for } n_y > L \;, \qquad \langle n_x, n_y | \hat{\pi}_\omega(\hat{K}_2) | m_x, m_y \rangle \;=\; 0 \quad \textit{for } m_y > L \;.$$

Let further the x-sign operator on $\ell^2(\mathbf{Z} \times \mathbf{N})$ be defined by

$$\frac{X}{|X|} |m_x, m_y\rangle \;=\; \mathrm{sgn}(m_x) |m_x, m_y\rangle \;=\; \left\{ \begin{array}{rl} |m_x, m_y\rangle \;, & m_x \geq 0 \;, \\ -|m_x, m_y\rangle \;, & m_x < 0 \;, \end{array} \right. \qquad (m_x, m_y) \in \mathbf{Z} \times \mathbf{N} \;.$$

Proposition 3 *Suppose that $\hat{H} = (\hat{K}, H) \in C^3(T(\mathcal{A}))$ where $\hat{K}$ satisfies the above hypothesis. Let the interval $\Delta \subset \mathbf{R}$ be contained in a gap of the spectrum of $\pi_\omega(H)$ for all $\omega \in \Omega_g$ where $\mathbf{P}(\Omega_g) = 1$. Let f be a real C^4-function supported in Δ. Then $f(\hat{H}) \in C^3(\mathcal{E}) \cap L^1(\mathcal{E}, \hat{\mathcal{T}}_{\mathcal{E}})$ and moreover the Hilbert-Schmidt norm of the operator $[\frac{X}{|X|}, \hat{\pi}_\omega(f(\hat{H}))]$ on $\ell^2(\mathbf{Z} \times \mathbf{N})$ is uniformly bounded for all $\omega \in \Omega_g$.*

Proof. First recall that every operator $A \in C^\gamma(\mathcal{A})$ satisfies

$$|\langle n | \pi_\omega(A) | m \rangle| \;\leq\; \frac{c}{1 + |n - m|^\gamma} \;, \tag{28}$$

where we use the notation $|n - m|^\gamma = |n_x - m_x|^\gamma + |n_y - m_y|^\gamma$. A similar bound holds for operators in $C^\gamma(T(\mathcal{A}))$. As $\hat{H} \in C^3(T(\mathcal{A}))$ and f is a C^4-function, we deduce that $f(\hat{H}) \in C^3(T(\mathcal{A}))$ (this follows from a straight-forward generalization of [46, Theorem 3.3.7]) so that (28) holds for $f(\hat{H})$ with $\gamma = 3$.

Furthermore, given $E \in \Delta$, let $E' \mapsto g_E(E')$ be some C^4-function equal to $(E - E')^{-1}$ outside of Δ. By the same argument as above, the operator $g_E(H)$ is in $C^3(\mathcal{A})$ as long as $E \in \Delta$. Maximising over $E \in \Delta$, we conclude that the bound (28) holds for $A = (E - H)^{-1}$ for a constant c which can be chosen uniform in $E \in \Delta$ and $\omega \in \Omega_g$. As last preliminary, let us note that the operator $(\hat{H} - \Pi H) = \hat{K}_1 + (\hat{K}_2 - \Pi H(1 - \Pi))$ from $\ell^2(\mathbf{Z}^2)$ to $\ell^2(\mathbf{Z} \times \mathbf{N})$ also satisfies (28) for all $\omega \in \Omega$ and that

$$\langle n|\pi_\omega(\hat{K}_2 - \Pi H(1 - \Pi))|m\rangle \;=\; 0 \quad \text{for } m_y > L \;.$$

The main idea is now to use these bounds in Stone's formula for $f(\hat{H})$ in which the resolvant is developed according to the geometric resolvant formula. As all the above bounds hold for all $\omega \in \Omega_g$, we suppress the index $\omega \in \Omega_g$ throughout the rest of the proof.

We replace the geometric resolvant identity

$$\frac{1}{z\Pi - \hat{H}} \;=\; \Pi \frac{1}{z - H} + \frac{1}{z\Pi - \hat{H}} (\hat{H} - \Pi H) \frac{1}{z - H} \;. \tag{29}$$

twice into Stone's formula for $f(\hat{H})$ and develop $(E \pm \imath\epsilon - H)^{-1}$ into a power series in ϵ. Of this series only the constant term, namely $(E - H)^{-1}$, remains after the limit $\epsilon \to 0$:

$$f(\hat{H}) \;=\; \frac{1}{2\pi\imath} \operatorname*{s-lim}_{\epsilon \to 0} \int_\Delta dE\; f(E) \left(\frac{1}{E - \imath\epsilon - \hat{H}} - \frac{1}{E + \imath\epsilon - \hat{H}} \right) (\hat{H} - \Pi H) \frac{1}{E - H} \;, \tag{30}$$

and therefore

$$|\langle n|f(\hat{H})|m\rangle| \;\leq\; c \sum_{l \in \mathbf{Z} \times \mathbf{N}} \sum_{k \in \mathbf{Z}^2} |\langle n|f(\hat{H})|l\rangle|\, |\langle l|\hat{H} - \Pi H|k\rangle| \;\sup_{E \in \Delta}\; |\langle k|\frac{1}{E - H}|m\rangle| \;. \tag{31}$$

As pointed out above, the operator $\hat{H} - \Pi H$ is localized near the boundary. We decompose it into two parts $\hat{K}_1$ and $\hat{K}_2 - \Pi H(1 - \Pi)$ for which we do the following bounds seperately. The contribution C_2 of $\hat{K}_2 - \Pi H(1 - \Pi)$ can be bounded by

$$\begin{aligned}
C_2 \;&\leq\; c \sum_{l \in \mathbf{Z} \times \mathbf{N}} \sum_{k_x \in \mathbf{Z}, k_y \leq w} \frac{1}{1 + |n - l|^3} \, \frac{1}{1 + |l - k|^3} \, \frac{1}{1 + |k - m|^3} \\
&\leq\; c \sum_{l_x, k_x \in \mathbf{Z}} \sum_{l_y \geq 0} \frac{1}{1 + |n - l|^3} \, \frac{1}{1 + |l_x - k_x|^2} \, \frac{1}{1 + |k_x - m_x|^3 + m_y^3} \\
&\leq\; c \sum_{l_x, k_x \in \mathbf{Z}} \frac{1}{1 + |n_x - l_x|^2} \, \frac{1}{1 + |l_y - k_y|^2} \, \frac{1}{1 + |k_x - m_x|^3 + m_y^3} \\
&\leq\; c \left(\frac{1}{1 + |n_x - m_x|^3 + m_y^3} + \frac{1}{1 + |n_x - m_x|^2} \, \frac{1}{1 + m_y^2} \right) \;,
\end{aligned} \tag{32}$$

where the constants c vary, and we have used various elementary estimations such as $(\eta, \gamma > 1)$:

$$\sum_{k\in\mathbf{Z}} \frac{1}{|a|+|k|^\eta}\,\frac{1}{|b|+|k-m|^\gamma} \;\leq\; c\,\left(a^{-1+1/\eta}\frac{1}{|b|+|m|^\gamma}+b^{-1+1/\gamma}\frac{1}{|b|+|m|^\eta}\right)\;.$$

Similarly, one can bound C_1. Hence $\langle n|f(\hat{H})|m\rangle$ can be bounded by the r.h.s. of (32). This in particular implies that for all $\omega\in\Omega_g$:

$$|\langle 0,m_y|f(\hat{H})|0,m_y\rangle|\;\leq\;\frac{c}{1+m_y^2}\;.$$

As this holds $\mathbf{P}$-almost surely with a uniform constant, it directly implies that $f(\hat{H})$ is $\hat{\mathcal{T}}_{\mathcal{E}}$-traceclass. Replacing (32) twice into

$$\mathbf{Tr}_{\ell^2(\mathbf{Z}\times\mathbf{N})}\left(\left|\left[\frac{X}{|X|},f(\hat{H})\right]\right|^2\right)\;=\;\sum_{n,m\in\mathbf{Z}\times\mathbf{N}}(\mathrm{sgn}(m_x)-\mathrm{sgn}(m_x+n_x))^2\,|\langle n|f(\hat{H})|m\rangle|^2\;,$$

the second assertion follows from a short calculation. □

Remark 1 The above proof can be slightly modified in order to prove that $\hat{P}_\Delta\in L^\infty(\mathcal{E},\mathbf{P})$ and is $\hat{\mathcal{T}}_{\mathcal{E}}$-traceclass under the slightly stronger hypothesis that $H\in C^4(\mathcal{A})$. For that purpose one has to replace f in (30) by the indicator function on Δ and then bound $\langle n|\hat{P}_\Delta|l\rangle$ in (31) trivially by 1. However, the proof of Theorem 1 below shows that the $\hat{\mathcal{T}}_{\mathcal{E}}$-traceclass property of $\hat{P}_\Delta$ holds whenever $H\in C^3(\mathcal{A})$. ◇

Remark 2 Replacing the geometric resolvant identity into the Stone formula for $\hat{P}_\Delta$ just as above shows that in general $\hat{P}_\Delta=\Pi P_\Delta+\hat{R}$ where P_Δ is the spectral projection of H onto Δ and $\hat{R}$ is a rest. Under the hypothesis that Δ is a dynamically localized region of the spectrum (or that in Δ the localization bound [2, Lemma 4] on the 4-point Green's function holds), we expect $\hat{R}$ to be $\hat{\mathcal{T}}_{\mathcal{E}}$-traceclass. Furthermore note that the term ΠP_Δ does note contribute to the current because

$$\hat{\mathcal{T}}_{\mathcal{E}}(\Pi P_\Delta\hat{\nabla}_x\hat{H})\;=\;\sum_{m_y\geq 0}\mathcal{T}(P_\Delta\nabla_x H)\;=\;0\;,$$

where in the last step we used [13, Proposition 3] showing that $\mathcal{T}(P_\Delta\nabla_x H)=0$. Note that it is essential in this calculation to take the trace per unit volume in the x-direction before taking the usual trace in the y-direction. If $\hat{R}$ is $\hat{\mathcal{T}}_{\mathcal{E}}$-traceclass, the definition of the edge current is thus meaningful in this sense. ◇

Remark 3 In a more general context one can show that, if $\hat{A}\in\mathcal{E}$ is $\hat{\mathcal{T}}_{\mathcal{E}}$-traceclass and an element of $C^2(\mathcal{E})$, then $[\frac{X}{|X|},\hat{\pi}_\omega(\hat{A})]$ is a Hilbert-Schmidt operator on $\ell^2(\mathbf{Z}\times\mathbf{N})$ for $\mathbf{P}$-almost all $\omega\in\Omega$. This follows from the remarkable identity

$$\frac{1}{4}\int d\mathbf{P}(\omega)\;\mathbf{Tr}_{\ell^2(\mathbf{Z}\times\mathbf{N})}\left(\left|\left[\frac{X}{|X|},\hat{\pi}_\omega(\hat{A})\right]\right|^2\right)\;=\;\frac{1}{\imath}\,\hat{\mathcal{T}}_{\mathcal{E}}((\sigma\hat{A})^*\hat{\nabla}_x\hat{A})\;,\qquad(33)$$

holding for any $\hat{A}\in\mathcal{E}_0$ where

$$(\sigma\hat{A})(\omega, m_x, n_y, m_y) \;=\; \mathrm{sgn}(m_x)\,\hat{A}(\omega, m_x, n_y, m_y)\;.$$

As $\hat{A} \in C^2(\mathcal{E})$ implies that $(\sigma\hat{A}) \in \mathcal{E}$, the formula (33) can be extended to $C^2(\mathcal{E}) \cap L^1(\mathcal{E}, \hat{\mathcal{T}}_{\mathcal{E}})$, implying hence the above result. Note that this does not give us a uniform bound in ω though and is thus a weaker result (not sufficient for our purposes below). A proof of (33) can be given along the lines of that of Proposition 4. $\diamond$

4.2 1-cocycles over the edge algebra

We will show in the next section that the edge Hall conductivity is given by $\hat{\mathcal{T}}_{\mathcal{E}}((\hat{U}^*-1)\hat{\nabla}_x\hat{U})$ for some unitary operator $\hat{U} \in \tilde{\mathcal{E}}$. Hence it can be interpreted as a non-commutative winding number. In order to explain and further develop this analogy, let us recall that the winding number of a continuous unitary function $f : S^1 \to \mathbf{C}$ is given by the similar expression $\int_{S^1} f^* df/(2\pi\imath)$; it is furthermore equal to the Fredholm index of the associated Toeplitz operator on Hardy space. There is a corresponding index theorem for the edge Hall conductivity, once we have linked it to a 1-cocycle of a standard Hilbert space framework. This is done in Proposition 4 below. From then on, one only has to follow the various steps of Connes' index calculation [22].

The strategy in all these calculations is to first restrict oneself to a dense sub-algebra where all algebraic manipulations can be easily justified, and then to extend the index theorem to suitable unitary elements of the C*-algebra.

The following expression is well-defined and finite for $\hat{A}, \hat{B} \in \mathcal{E}_0$:

$$\xi_1(\hat{A}, \hat{B}) \;=\; \imath\, \hat{\mathcal{T}}_{\mathcal{E}}(\hat{A}\hat{\nabla}_x\hat{B})\;. \tag{34}$$

Actually ξ_1 can be defined on a much wider class of operators. For the purpose of this work, it is sufficient to consider elements in $\mathcal{D} = C^2(\mathcal{E}) \cap L^1(\mathcal{E}, \hat{\mathcal{T}}_{\mathcal{E}})$. Note that the product rule for $\hat{\nabla}_x$ and the ideal property of the $\hat{\mathcal{T}}_{\mathcal{E}}$-traceclass operators imply that $\mathcal{D}$ is an algebra. It becomes a normed algebra when furnished with the norm:

$$\|\hat{A}\|_{\mathcal{D}} \;=\; \|\hat{A}\| + \|\hat{\nabla}_x^2\hat{A}\| + \hat{\mathcal{T}}_{\mathcal{E}}(|\hat{A}|)\;.$$

Lemma 2 *ξ_1 is a cyclic 1-cocycle on $\mathcal{D}$, notably it is cyclic and closed under the Hochschild boundary operator b:*
(i) $\xi_1(\hat{A}, \hat{B}) = -\xi_1(\hat{B}, \hat{A})$ *for all* $\hat{A}, \hat{B} \in \mathcal{D}$.
(ii) $0 = b\xi_1(\hat{A}, \hat{B}, \hat{C}) = \xi_1(\hat{A}\hat{B}, \hat{C}) - \xi_1(\hat{A}, \hat{B}\hat{C}) + \xi_1(\hat{C}\hat{A}, \hat{B})$ *for all* $\hat{A}, \hat{B}, \hat{C} \in \mathcal{D}$.

Proof. Both algebraic identities can be verified using the product rule for the derivation $\hat{\nabla}_x$ and the invariance of the trace $\hat{\mathcal{T}}_{\mathcal{E}}$ under $\hat{\nabla}_x$. $\Box$

Next we introduce another 1-cocycle on the unitization $\tilde{\mathcal{D}}$ of $\mathcal{D}$ by setting

$$\zeta_1(\hat{A}, \hat{B}) \;=\; \int d\mathbf{P}(\omega)\; \zeta_1^\omega(\hat{A}, \hat{B})\;, \qquad \hat{A}, \hat{B} \in \tilde{\mathcal{D}}\;,$$

where

$$\zeta_1^\omega(\hat{A},\hat{B}) \;=\; \frac{1}{4}\,\mathbf{Tr}_{\ell^2(\mathbf{Z}\times\mathbf{N})}\left(\frac{X}{|X|}\left[\frac{X}{|X|},\hat{\pi}_\omega(\hat{A})\right]\left[\frac{X}{|X|},\hat{\pi}_\omega(\hat{B})\right]\right)\;. \tag{35}$$

According to Remark 3 of the last section, this cocycle is actually well-defined and finite on $\mathcal{D}$.

Proposition 4 *On $\mathcal{D}$, we have $\zeta_1 = \xi_1$.*

Proof. It is sufficient to show the equality for the dense subalgebra $\mathcal{E}_0 \subset \mathcal{D}$ because both ζ_1 and ξ_1 are continuous with respect to $\|\,.\,\|_{\mathcal{D}}$. A direct calculation shows that for $\hat{A},\hat{B} \in \mathcal{E}_0$:

$$\zeta_1(\hat{A},\hat{B}) \;=\; -\frac{1}{4}\int d\mathbf{P}(\omega)\sum_{m\in\mathbf{Z}\times\mathbf{N}}\;\sum_{l\in\mathbf{Z}\times\mathbf{N}}\mathrm{sgn}(m_x)(\mathrm{sgn}(m_x)-\mathrm{sgn}(l_x))^2\langle m|\hat{\pi}_\omega(\hat{A})|l\rangle\,\langle l|\hat{\pi}_\omega(\hat{B})|n\rangle\;.$$

Because $\hat{A} \in \mathcal{E}_0$, the sum over $m_x \in \mathbf{Z}$ actually only contains a finite number of elements, and can thus be exchanged with the integral over $\mathbf{P}$. Then we make the change of variables $n_x = l_x - m_x$ and use the covariance relation (9) in order to obtain:

$$\begin{aligned}\zeta_1(\hat{A},\hat{B}) \;=\; & -\frac{1}{4}\sum_{m_x\in\mathbf{N}}\int d\mathbf{P}(\omega)\sum_{m_y,l_y\in\mathbf{N}}\;\sum_{n_x\in\mathbf{Z}}\mathrm{sgn}(m_x)\;(\mathrm{sgn}(m_x)-\mathrm{sgn}(m_x+n_x))^2\;\cdot\\ & \cdot\;\langle 0,m_y|\hat{\pi}_{T^{(-m_x,0)}\omega}(\hat{A})|n_x,l_y\rangle\;\langle n_x,l_y|\hat{\pi}_{T^{(-m_x,0)}\omega}(\hat{B})|0,m_y\rangle\;.\end{aligned}$$

Next, by invariance of the measure $\mathbf{P}$, we can replace $T^{(-m_x,0)}\omega$ by ω. Then we change the sum over m_x and the integral over $\mathbf{P}$ again and use the identity

$$\sum_{m_x\in\mathbf{N}}\mathrm{sgn}(m_x)\;(\mathrm{sgn}(m_x)-\mathrm{sgn}(m_x+n_x))^2 \;=\; -\,4\,n_x\;.$$

By definition of $\hat{\nabla}_x$, one therefore has

$$\zeta_1(\hat{A},\hat{B}) \;=\; \imath\int d\mathbf{P}(\omega)\sum_{m_y\in\mathbf{N}}\langle 0,m_y|\hat{\pi}_\omega(\hat{A})\hat{\pi}_\omega(\hat{\nabla}_x\hat{B})|0,m_y\rangle\;,$$

which is precisely $\xi_1(\hat{A},\hat{B})$. □

Finally let us introduce a further 1-cocyle on $\tilde{\mathcal{D}}$ using the projection operator $\Pi_x = (1 + X/|X|)/2$ on $\ell^2(\mathbf{Z}\times\mathbf{N})$:

$$\eta_1(\hat{A},\hat{B}) \;=\; \int d\mathbf{P}(\omega)\;\eta_1^\omega(\hat{A},\hat{B})\;,\qquad \hat{A},\hat{B}\in\tilde{\mathcal{D}}\;,$$

where

$$\eta_1^\omega(\hat{A},\hat{B}) \;=\; \mathbf{Tr}_{\ell^2(\mathbf{Z}\times\mathbf{N})}(\Pi_x[\hat{\pi}_\omega(\hat{B}),\hat{\pi}_\omega(\hat{A})]\Pi_x - [\Pi_x\hat{\pi}_\omega(\hat{B})\Pi_x,\Pi_x\hat{\pi}_\omega(\hat{A})\Pi_x])\;. \tag{36}$$

Proposition 5 *On $\tilde{\mathcal{D}}$, we have $\eta_1^\omega = \zeta_1^\omega$ for all $\omega \in \Omega$.*

Proof. Some algebra shows

$$\Pi_x \hat{\pi}_\omega(\hat{A})\hat{\pi}_\omega(\hat{B})\Pi_x - \Pi_x\hat{\pi}_\omega(\hat{A})\Pi_x\hat{\pi}_\omega(\hat{B})\Pi_x \;=\; -\frac{1}{4}\Pi_x \left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{A})\right]\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{B})\right] . \qquad (37)$$

Note that if $\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{A})\right]\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{B})\right]$ is traceclass, so is the left-hand side (this is the case for $\hat{A}, \hat{B} \in \tilde{\mathcal{D}}$ by Remark 3). Hence using the same identity with $\hat{A}$ and $\hat{B}$ exchanged, we obtain:

$$\begin{aligned}\eta_1^\omega(\hat{A},\hat{B}) \;&=\; \frac{1}{4}\,\mathbf{Tr}_{\ell^2(\mathbf{Z}\times\mathbf{N})}\left(\Pi_x\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{A})\right]\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{B})\right] - \Pi_x\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{B})\right]\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{A})\right]\right)\\ &=\; \frac{1}{2}\,\mathbf{Tr}_{\ell^2(\mathbf{Z}\times\mathbf{N})}\left(\frac{X}{|X|}\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{A})\right]\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{B})\right] - \frac{X}{|X|}\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{B})\right]\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{A})\right]\right) .\end{aligned}$$

Note here that the second equality holds because both $\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{A})\right]$ and $\left[\frac{X}{|X|}, \hat{\pi}_\omega(\hat{B})\right]$ are Hilbert-Schmidt [42]. From the above we deduce that

$$\eta_1^\omega(\hat{A},\hat{B}) \;=\; \frac{1}{2}\left(\zeta_1^\omega(\hat{A},\hat{B}) - \zeta_1^\omega(\hat{B},\hat{A})\right) ,$$

and the cyclicity property of ζ_1^ω allows to conclude. □

Corollary 1 *On $\mathcal{D}$, we have $\xi_1 = \zeta_1 = \eta_1$.*

It is now immediate to deduce the two following odd index theorems for stochastic operators.

Proposition 6 *Suppose (only for the purpose of this proposition) that $\mathbf{P}$ is ergodic w.r.t. the $\mathbf{Z}$-action T_x. Let $\hat{A} \in \tilde{\mathcal{D}}$ be invertible. Then $\Pi_x\hat{\pi}_\omega(\hat{A})\Pi_x$ is $\mathbf{P}$-almost surely a Fredholm operator on $\ell^2(\mathbf{N}\times\mathbf{N})$ the index of which is $\mathbf{P}$-almost surely independent of $\omega \in \Omega$. The almost sure value is equal to $\xi_1(\hat{A}-\alpha, \hat{A}^{-1}-\alpha^{-1})$ whenever $\hat{A}-\alpha \in \mathcal{D}$, $\alpha \in \mathbf{C}$, $\alpha \neq 0$.*

Proof. Because $\hat{A} \in \tilde{\mathcal{D}}$, Proposition 3 implies that $\eta_1^\omega(\hat{A},\hat{A}^{-1}) < \infty$ for $\mathbf{P}$-almost all $\omega \in \Omega$. Thus (36) and the Fedosov formula (*e.g.* [13, Prop. 8]) imply that $\Pi_x\hat{\pi}_\omega(\hat{A})\Pi_x$ is a Fredholm operator on $\Pi_x\ell^2(\mathbf{Z}\times\mathbf{N}) = \ell^2(\mathbf{N}\times\mathbf{N})$ and that its index is equal to $\eta_1^\omega(\hat{A},\hat{A}^{-1})$. Because

$$\Pi_x\hat{\pi}_{T^{(a,0)}\omega}(\hat{A})\Pi_x \mid_{\ell^2(\mathbf{N}\times\mathbf{N})} \;=\; \Pi_x\hat{\pi}_\omega(\hat{A})\Pi_x + K \mid_{\hat{U}(a,0)\ell^2(\mathbf{N}\times\mathbf{N})\cong\ell^2(\mathbf{N}\times\mathbf{N})} ,$$

where K is a compact operator on $\ell^2(\mathbf{N}\times\mathbf{N})$ and the Fredholm index is invariant under compact perturbations, we see that the index is T_x-translation invariant in $\omega \in \Omega$. Hence it is $\mathbf{P}$-almost surely constant by the ergodicity of $\mathbf{P}$ with respect to T_x. As $\eta_1^\omega(\hat{A},\hat{A}^{-1}) = \eta_1^\omega(\hat{A}-\alpha,\hat{A}^{-1}-\alpha^{-1})$, Corollary 1 implies that the almost sure index is equal to $\xi_1(\hat{A}-\alpha,\hat{A}^{-1}-\alpha^{-1})$. □

In our context, the measure $\mathbf{P}$ is only ergodic w.r.t. the $\mathbf{Z}^2$-action T. However, this is sufficient to given an almost sure index for certain elements in $\tilde{\mathcal{D}}$, notably those in the image of the exponential map.

Proposition 7 *Let $\hat{H}$ satisfy the hypothesis of* Proposition 3 *and let g be a real C^4 function with values in $[0,1]$, equal to 0 or 1 outside of Δ. Set $\hat{U} = \exp(-2\pi\imath\, g(\hat{H}))$. Then $\Pi_x \hat{\pi}_\omega(\hat{U})\Pi_x$ is $\mathbf{P}$-almost surely a Fredholm operator on $\ell^2(\mathbf{N}\times\mathbf{N})$ the index of which is $\mathbf{P}$-almost surely independent of $\omega \in \Omega$. The almost sure value is equal to $\xi_1(\hat{U}-1, \hat{U}^{-1}-1)$.*

Proof. By Proposition 3, $\hat{U} \in \tilde{\mathcal{D}}$. Thus from the proof of Proposition 6 follows that $\Pi_x\hat{\pi}_\omega(\hat{U})\Pi_x$ is $\mathbf{P}$-almost surely a Fredholm operator and that its index is T_x-invariant. To conclude, we have to show its T_y-invariance.

Herefore, let $Q : \ell^2(\mathbf{Z}\times\mathbf{N}) \to \ell^2(\mathbf{Z}\times\mathbf{N})$ denote the projection on the subspace spanned by $\{\ |m_x, 0\rangle \mid m_x \in \mathbf{Z}\}$. Now, if $\hat{H} = (\hat{K}, H) \in T(\mathcal{A})$, we can use the equation $\hat{\pi}_\omega(\hat{H}) = \Pi\pi_\omega(H) + \hat{\pi}_\omega(\hat{K})$ and the covariance relation $U_y\pi_{T_y^{-1}\omega}(H)U_y^* = \pi_\omega(H)$ in order to obtain

$$\hat{\pi}_\omega(\hat{H}) \;=\; \hat{U}_y\hat{\pi}_{T_y^{-1}\omega}(\hat{H})\hat{U}_y^* + \hat{R}_\omega \;,$$

where $\hat{R}_\omega$ is an operator family, covariant in the x-direction and compact in the y-direction and given by

$$\hat{R}_\omega \;=\; \hat{\pi}_\omega(\hat{H})Q + Q\hat{\pi}_\omega(\hat{H})(1-Q) + (1-Q)\hat{\pi}_\omega(\hat{K})(1-Q) + \hat{U}_y\hat{\pi}_{T_y^{-1}\omega}(\hat{K})\hat{U}_y^* \;.$$

Now $\hat{R}$ is actually a boundary operator satisfying the hypothesis of Section 4.1. Hence the operator $\hat{U}_y\hat{\pi}_{T_y^{-1}\omega}(\hat{H})\hat{U}_y^* + \lambda\hat{R}_\omega$ is a lift of $\pi_\omega(H)$ and by Proposition 3

$$\hat{U}_\omega(\lambda) \;=\; \exp(-2\pi\imath\, f(\hat{U}_y\hat{\pi}_{T_y^{-1}\omega}(\hat{H})\hat{U}_y^* + \lambda\hat{R}_\omega)) \;.$$

is $\mathbf{P}$-almost surely a norm-continuous (in λ) family of Fredholm operators. Clearly $\hat{U}_\omega(1) = \hat{\pi}_\omega(\hat{U})$ and moreover, because $\hat{U}_y^*\hat{U}_y = 1$, we have $\hat{U}_\omega(0) = \hat{U}_y\hat{\pi}_{T_y^{-1}\omega}(\hat{U})\hat{U}_y^*$. Therefore the homotopy invariance of the Fredholm index implies:

$$\mathrm{Ind}(\Pi_x\hat{\pi}_\omega(\hat{U})\Pi_x) \;=\; \mathrm{Ind}(\Pi_x\hat{U}_y\hat{\pi}_{T_y^{-1}\omega}(\hat{U})\hat{U}_y^*\Pi_x) \;=\; \mathrm{Ind}(\Pi_x\hat{\pi}_{T_y^{-1}\omega}(\hat{U})\Pi_x) \;,$$

which concludes the proof. □

Remark. (Not needed for the understanding of the rest of this work.) There are underlying Freholm modules in the above calculations. Recall [22] that a Fredholm module over an algebra $\mathcal{A}$ is a family $(\mathcal{A}, \tilde{\mathcal{H}}, \tilde{\pi}, F)$ where $\tilde{\pi}$ is a trivially graded representation of $\mathcal{A}$ on a graded seperable Hilbert space $\tilde{\mathcal{H}}$ and where F is a self-adjoint operator on $\tilde{\mathcal{H}}$ such that

$$\text{(i)}\;\; FG = -GF \;, \qquad \text{(ii)}\;\; F^2 = 1 \;, \qquad \text{(iii)}\;\; [\tilde{\pi}(A), F] \in K \quad \forall\, A \in \mathcal{A} \;.$$

Here G is the graduation operator, and K denotes the compact operators on $\tilde{\mathcal{H}}$. One then defines the differential (with help of the graded commutator $[\ ,\ ]_s$) and the supertrace of a bounded operator T on $\tilde{\mathcal{H}}$ as usual:

$$dT \;=\; [F, T]_s \;, \qquad \mathrm{Tr}_s(T) \;=\; \frac{1}{2}\,\mathrm{Tr}(GFdT) \;,$$

Let us consider the Fredholm module $(\mathcal{D}, \ell^2(\mathbf{Z}\times\mathbf{N})\otimes\ell^2(\mathbf{Z}\times\mathbf{N}), \tilde{\pi}_\omega, F)$ where

$$\tilde{\pi}_\omega(\hat{A}) \;=\; \begin{pmatrix} \hat{\pi}_\omega(\hat{A}) & 0 \\ 0 & \hat{\pi}_\omega(\hat{A}) \end{pmatrix} \,, \qquad F \;=\; \imath \begin{pmatrix} 0 & \frac{X}{|X|} \\ -\frac{X}{|X|} & 0 \end{pmatrix} \,.$$

It is now a matter of calculation to verify that the 1-cocycle associated to this Fredholm module coincides with the one-cocycle ζ_1^ω defined in (35):

$$\zeta_1^\omega(\hat{A},\hat{B}) \;=\; -\frac{1}{4}\,\mathrm{Tr}_s(\tilde{\pi}_\omega(\hat{A})d\tilde{\pi}_\omega(\hat{B}))\;.$$

4.3 Calculation of the edge current

This section contains the proof of the edge current quantization and some comments.

Proof of Theorem 1. Let $G\in C^2(\mathbf{R})$ be a positive function supported on an interval $\Delta'\subset\Delta$. We set

$$g(E) \;=\; 1-\frac{1}{\int_{\mathbf{R}} dE'\,G(E')}\int_{-\infty}^{E} dE'\,G(E')\;,$$

and the unitary operator $\hat{W}=\exp(-2\pi\imath\, g(\hat{H}))$. By Proposition 7, $\pi_\omega(\hat{W})$ and $\pi_\omega(\hat{W}^*)$ are $\mathbf{P}$-almost surely Fredholm operators with a $\mathbf{P}$-almost sure index. Changing Δ' within Δ (or changing G) does not change their index by the homotopy invariance of the Fredholm index. Hence their indices are actually associated to the gap Δ and can moreover be calculated as the index of $\hat{U}(\Delta)$ and $\hat{U}(\Delta)^*$ respectively. We denote that of $\hat{U}(\Delta)^*$ by Ind so that

$$\mathrm{Ind} \;=\; \imath\,\hat{\mathcal{T}}_{\mathcal{E}}((\hat{W}^*-1)\hat{\nabla}_x\hat{W})\;. \tag{38}$$

Now recall Duhamel's formula for $\hat{\nabla}_x\hat{W}$:

$$\hat{\nabla}_x\hat{W} \;=\; -2\pi\imath\int_0^1 ds\;\hat{W}^{1-s}\,\hat{\nabla}_x g(\hat{H})\,\hat{W}^s\;,$$

where the integral is defined as a norm convergent Riemann sum in $\mathcal{E}$. We replace this into (38), use the continuity of $\hat{\mathcal{T}}_{\mathcal{E}}$ in order to exchange the trace with the integral over s and finally use the unitary invariance of $\hat{\mathcal{T}}_{\mathcal{E}}$ to deduce

$$\mathrm{Ind} \;=\; 2\pi\,\hat{\mathcal{T}}_{\mathcal{E}}((1-\hat{W})\hat{\nabla}_x g(\hat{H}))\;.$$

To calculate this, we develop $g(E)=\sum_k c_k T_k(E)$ in a series of Tchebychev polynomials over Δ (slightly enlarged if necessary). Because $g\in C^4(\mathbf{R})$, this series converges absolutely and uniformly on $\Delta'\subset\Delta$, so the sum over k can be exchanged with $\hat{\mathcal{T}}_{\mathcal{E}}$. On each polynomial in $\hat{H}$, we now apply the gradient, and then use the cyclicity of the trace and $[\hat{W},\hat{H}]=0$ in order to regroup all polynomials in $\hat{H}$ to the left of $\hat{\nabla}_x\hat{H}$. Because $G(E)=-\sum_k c_k T_k'(E)\cdot(\int_{\mathbf{R}} dE'\,G(E'))$, we conclude that

$$\mathrm{Ind} \;=\; \frac{2\pi}{\int_{\mathbf{R}} dE'\,G(E')}\,\hat{\mathcal{T}}_{\mathcal{E}}((\hat{W}-1)\,G(\hat{H})\,\hat{\nabla}_x\hat{H})\;.$$

Now let $G_j \in C^3(\mathbf{R})$ be a sequence of functions converging from below to the indicator function onto Δ'. If $\hat{W}_j$ denotes the associated unitary, then $(\hat{W}_j - 1)G_j(\hat{H})$ converges in norm to $\hat{U}(\Delta') - 1$. Therefore, for any $\Delta' \subset \Delta$,

$$\text{Ind} \;=\; \frac{2\pi}{|\Delta'|}\,\hat{\mathcal{T}}_{\mathcal{E}}((\hat{U}(\Delta') - 1)\,\hat{\nabla}_x \hat{H}) \;.$$

For a given $k \in \mathbf{N}$, $k \geq 1$, we devide Δ into k intervals $\Delta_j = [E_{j-1}, E_j]$, $j = 1 \ldots k$, of equal length $|\Delta|/k$. Note that $E_0 = E' = \inf(\Delta)$. Let us set

$$\hat{U}_j - 1 \;=\; \exp\left(-2\pi\imath\,\frac{\hat{H} - E_j}{|\Delta_j|}\,\hat{P}_{\Delta_j}\right) - 1 \;=\; (\hat{U}(\Delta)^k - 1)\,\hat{P}_{\Delta_j} \;.$$

Now the Fredholm index of each $\hat{U}_j^*$ is equal to that of $\hat{U}(\Delta)^*$, namely Ind. Hence it follows that

$$\text{Ind} \;=\; \frac{1}{k}\sum_{j=1}^{k} \frac{2\pi}{|\Delta_j|}\,\hat{\mathcal{T}}_{\mathcal{E}}((\hat{U}_j - 1)\,\hat{\nabla}_x \hat{H}) \;=\; \frac{2\pi}{|\Delta|}\,\hat{\mathcal{T}}_{\mathcal{E}}((\hat{U}(\Delta)^k - 1)\,\hat{\nabla}_x \hat{H}) \;. \tag{39}$$

Introducing a sign in the above definition of the $\hat{U}_j$'s, it can be verified that (39) also holds for negative integers, so it holds for all $k \in \mathbf{Z}$ except $k = 0$.

Finally we make a Fourier analysis on the interval Δ. Let $\Delta' \subset \Delta$ be an interval such that $\inf(\Delta') > E'$ and let $\Delta'' = \Delta \backslash \Delta'$. The indicator function $\chi_{\Delta''}$ on Δ'' (taking the value 1/2 on the two boundary points of Δ'') can now be written as:

$$\chi_{\Delta''}(E) \;=\; \sum_{k \in \mathbf{Z}} a_k \,\exp\left(-\frac{2\pi\imath}{|\Delta|}\,k(E - E')\right) \;, \qquad a_k \;=\; \int_\Delta \frac{dE}{|\Delta|}\,\chi_{\Delta''}(E)\,\exp\left(-\frac{2\pi\imath}{|\Delta|}\,k(E - E')\right) \;,$$

where the convergence of the Fourier series is pointwise. We have:

$$\sum_{k \in \mathbf{Z}} a_k \;=\; 1 \;, \qquad a_0 \;=\; \frac{|\Delta''|}{|\Delta|} \;, \qquad \sum_{k \neq 0} a_k \;=\; \frac{|\Delta'|}{|\Delta|} \;,$$

so that (except for the unimportant boundary points):

$$\hat{P}_\Delta \sum_{k \neq 0} a_k\,\hat{U}(\Delta)^k \;=\; \frac{|\Delta'|}{|\Delta|}\,\hat{P}_\Delta \;-\; \hat{P}_{\Delta'} \;.$$

Finally, summing (39) we thus obtain:

$$\text{Ind} \;=\; \frac{|\Delta|}{|\Delta'|}\sum_{k \neq 0} a_k\,\frac{2\pi}{|\Delta|}\,\hat{\mathcal{T}}_{\mathcal{E}}(\hat{P}_\Delta\,\hat{U}(\Delta)^k\,\hat{\nabla}_x \hat{H}) - \frac{2\pi}{|\Delta|}\,\hat{\mathcal{T}}_{\mathcal{E}}(\hat{P}_\Delta\,\hat{\nabla}_x \hat{H}) \;=\; -\frac{2\pi}{|\Delta'|}\,\hat{\mathcal{T}}_{\mathcal{E}}(\hat{P}_{\Delta'}\,\hat{\nabla}_x \hat{H}) \;.$$

Δ' being an arbitrary interval in Δ, the result is thus proven. □

We conclude this section with a brief comment on why there actually is an edge current in presence of a magnetic field. Although physically clear, it also follows from the following formal algebraic calculation with the Harper Hamiltonian $H = U_x + U_x^* + U_y + U_y^*$. It is an element of the rotation algebra $\mathcal{A}_\theta$ where θ is the flux per unit cell. First note that $\hat{\nabla}_x \hat{H} = \imath(\hat{U}_x - \hat{U}_x^*)$. Thus:

$$\hat{\mathcal{T}}_{\mathcal{E}}(\hat{H}\hat{\nabla}_x\hat{H}) \;=\; \sum_{m_y \geq 0} \langle 0, m_y | (\hat{U}_y\hat{U}_x + \hat{U}_y^*\hat{U}_x - \hat{U}_y\hat{U}_x^* - \hat{U}_y^*\hat{U}_x^*) | 0, m_y\rangle \;=\; 0 \;.$$

Similarly, one can check with a little more effort and using the algebraic relations of $\hat{U}_x$ and $\hat{U}_y$ that $\hat{\mathcal{T}}_{\mathcal{E}}(\hat{H}^2\hat{\nabla}_x\hat{H}) = 0$, but $\hat{\mathcal{T}}_{\mathcal{E}}(\hat{H}^3\hat{\nabla}_x\hat{H}) = \sin(\theta) \neq 0$ for non-vanishing magnetic field. Other higher powers of $\hat{H}$ and hence functions of $\hat{H}$ also give non-vanishing results. Actually, it is not surprising that the third power gives the first contribution because it corresponds to the walk from one edge site via the interior to the next edge site. The probability (or quantum expectation) for this (either to the right or the left depending on the value of θ) is positive.

5 Duality of pairings of K-theory with cyclic cohomology

Given a Banach algebra $\mathcal{B}$, one can define abelian groups which are dual to the K-groups of $\mathcal{B}$ in the sense that there is a natural pairing between the two which takes values in the integers. The dual of $K_1(\mathcal{B})$ is e.g. given by the group $\mathrm{Ext}(\mathcal{B})$ of extensions of $\mathcal{B}$ and the pairing is quite simple [16]. Using suspension, one can then also obtain a six-term exact sequence of the Ext-groups from a short exact sequence of algebras [16]. But when it comes to pair $K_0(\mathcal{B})$ with $\mathrm{Ext}(S\mathcal{B})$, one realizes that it would be better to work with a different picture of the dual of $K_0(\mathcal{B})$ which is not based on suspension. Such a theory is fully developped in the topological context and goes under the name KK-theory. On the other hand, there is a cohomology theory for algebras, cyclic cohomology, which is defined on the subalgebra of so-called smooth elements of $\mathcal{B}$ and which also pairs with $K_0(\mathcal{B})$, but not always integrally. It is related to the dual of $K_0(\mathcal{B})$ in the sense of KK-theory and produces the same result when both apply. But despite the two apparent disadvantages (only defined on smooth elements, pairing not always integer valued), its advantages overwhelm: first, we find it technically and computationally much simpler to handle in our case, and second, it is not tight to the topological category and the pairing extends to certain projections which are no longer in $\mathcal{B}$. This allowed [13] to prove quantization in the intermediate disorder regime and we hope that it will eventually lead to a proof of Theorem 2 under a similar hypothesis. Our presentation here is adopted from the works of Pimsner [43] and Nest [41], with simplifications adequate to our context.

5.1 Definition of cyclic cohomology

Given a complex algebra $\mathcal{B}$, let $C^n_\lambda(\mathcal{B})$ be the set of $n+1$-linear functionals on $\mathcal{B}$ which are cyclic in the sense that

$$\varphi(A_1, \cdots, A_n, A_0) \;=\; (-1)^n \varphi(A_0, \cdots, A_n) \;.$$

Definition 2 *The cyclic cohomology $HC(\mathcal{B})$ of $\mathcal{B}$ is the cohomology of the complex*

$$0 \to C^0_\lambda(\mathcal{B}) \to \cdots \to C^n_\lambda(\mathcal{B}) \xrightarrow{b} C^{n+1}_\lambda(\mathcal{B}) \to \cdots$$

with

$$\begin{aligned} b\varphi(A_0,\cdots,A_{n+1}) &= \sum_{j=0}^{n}(-1)^j\varphi(A_0,\cdots,A_jA_{j+1},\cdots,A_{n+1}) \\ &\quad +(-1)^{n+1}\varphi(A_{n+1}A_0,\cdots,A_n). \end{aligned} \tag{40}$$

An element $\varphi \in C^n_\lambda(\mathcal{B})$ satisfying $b\varphi = 0$ is called a cyclic n-cocycle.

Example 1 $HC^0(\mathcal{B})$ is the set of traces on $\mathcal{B}$. In fact, let $\mathcal{T} : \mathcal{B} \to \mathbf{C}$ be a linear functional, then

$$b\mathcal{T}(A_0, A_1) \;=\; \mathcal{T}(A_0A_1) - \mathcal{T}(A_1A_0)\ .$$

Example 2 Let ∇_j, $j = 1\cdots,n$ be n commuting derivations on $\mathcal{B}$ (not necessarily fully defined) and $\mathcal{T}$ an invariant trace ($\mathcal{T}\circ\nabla_j = 0$), not necessarily finite. Then $\xi_n : A^{\otimes n+1} \to \mathbf{C}$ defined by

$$\xi_n(A_0,\cdots,A_n) \;=\; \sum_{\sigma\in S_n}(-1)^{\mathrm{sgn}(\sigma)}\mathcal{T}(A_0\nabla_{\sigma(1)}A_1\cdots\nabla_{\sigma(n)}A_n) \;=\; \mathcal{T}(A_0\nabla_{[1}A_1\cdots\nabla_{n]}A_n)\ ,$$

(the second expression is an abbreviation) is an n-cocycle. This will be proved below. The domain of definition of ξ_n depends on the domains of definition of the derivations and that of $\mathcal{T}$.

5.2 Pairing with K-groups

Now let $\mathcal{B}$ be a Banach algebra. The elements of cyclic cohomology of degree j pair with $K_j(\mathcal{B})$ (j mod 2 in the latter case) in the following way. Let ξ be a $2k$-cocycle, $[\xi]$ its class. Then

$$\langle [P]_0 - [P_m]_0, [\xi]\rangle \;=\; c_{2k}\ \mathrm{Tr}\otimes\xi(P,\ldots,P)\ , \qquad P \in M_{m+l}(\mathcal{B})\ ,$$

is a well-defined pairing between $K_0(\mathcal{B})$ and $HC^{ev}(\mathcal{B})$ [22]. If ξ is a $2k+1$-cocycle, then

$$\langle [V]_1, [\xi]\rangle \;=\; c_{2k+1}\ \mathrm{Tr}\otimes\xi(V^{-1}-1, V-1,\ldots,V-1)\ , \qquad V \in M_m(\mathcal{B})\ ,$$

(the entries alternate) is a well-defined pairing between $K_1(\mathcal{B})$ and $HC^{odd}(\mathcal{B})$. Here the normalization constants are (chosen as in [21] and [43], but not as in [22]):

$$c_{2k} = \frac{1}{(2\pi\imath)^k}\frac{1}{k!}\ , \qquad c_{2k+1} = \frac{1}{(2\pi\imath)^{k+1}}\frac{1}{2^{2k+1}}\frac{1}{(k+\frac{1}{2})(k-\frac{1}{2})\cdots\frac{1}{2}}\ .$$

This lets cyclic cohomology appear as a dual theory to K-theory, but there are fundamental differences between the two. The category of algebras for which these invariants are defined is different. K-theory is topological, whereas cyclic cohomology is rather a differential theory. In particular, one often considers functionals (cyclic cocycles) which are merely densely defined on a (local) Banach algebra (or not even densely). Nevertheless, HC has functorial properties which are analogous and compatible with those of K.

5.3 Cycles

There is a convenient reformulation of the description of cyclic cocycles in terms of graded differential algebras (Ω, d) with graded closed traces (we use the widely spread notation Ω here, because misinterpretation as the hull of Section 2.1 seems excluded). A graded algebra is a graded vectorspace $\Omega = \bigoplus_{n\in\mathbf{Z}}\Omega^n$ (we denote by ∂A the degree of a homogeneous element A) such that $\partial(AB) = \partial(A) + \partial(B)$ for homogeneous elements. It is called a graded differential algebra if there exists a graded differential d (a derivation whose square vanishes) of degree 1. A graded trace on Ω^n is a linear functional $\int$ which satisfies $\int w_1 w_2 = (-1)^{\partial w_1 + \partial w_2}\int w_2 w_1$. It is closed if $\int$ vanishes on $d(\Omega^{n-1})$.

Definition 3 *A n-dimensional cycle over $\mathcal{B}$ is a graded differential algebra (Ω, d) of highest non-trivial degree n together with a closed graded trace $\int$ on Ω^n and an algebra homomorphism $\mathcal{B} \to \Omega^0$.*

We will assume here that the homomorphism $\mathcal{B} \to \Omega^0$ is injective and hence identify $\mathcal{B}$ with a subalgebra of Ω^0. See [22] for the proof of the following result.

Proposition 8 *Any cycle of dimension n over $\mathcal{B}$ defines a cyclic n-cocycle through what is called its character:*

$$\xi(A_0, \cdots, A_n) = \int A_0 dA_1 \cdots dA_n\ .$$

Example 3 Consider an action of $\mathbf{R}^n$ on the algebra $\mathcal{B}$ by commuting derivations ∇_j (thus view $\mathbf{R}^n$ as Lie algebra) and suppose that $\mathcal{T}$ is an invariant trace, i.e. $\mathcal{T}(\exp(t\nabla_j)A) = \mathcal{T}(A)$ for all t. Let Ω be the tensor product $\mathcal{B} \otimes \Lambda\mathbf{C}^n$ of $\mathcal{B}$ with the Grassmann algebra $\Lambda\mathbf{C}^n$ with generators e_j, $j = 1, \cdots, n$ and define

$$d(A \otimes v) = \sum_{j=1}^{n} \nabla_j A \otimes e_j v\ .$$

It is straightforwardly checked that (Ω, d) is a differential algebra. Note that $\Lambda^n\mathbf{C}^n \cong \mathbf{C}$ (as vector spaces) and any isomorphism is a graded trace. Taking $\imath(e_1\cdots e_n) = 1$ as such an isomorphism, we define $\int = \tau \otimes \imath$, that is

$$\int A_0 dA_1 \cdots dA_n \;=\; \mathcal{T}(A_0 \nabla_{[1} A_1 \cdots \nabla_{n]} A_n)\ ,$$

a cycle of dimension n associated to our second example. Let us prove that this is actually closed and graded. The graded cyclicity follows directly from the graded cyclicity of $\imath$ and the cyclicity of $\mathcal{T}$. To prove closedness write $\mathcal{T} \otimes \imath = \imath \circ (\mathcal{T} \otimes \mathrm{id})$ on the highest degree. Since d acts trivially on $\mathbf{C} \otimes \Lambda\mathbf{C}^n$, the invariance of $\mathcal{T}$ implies $(\mathcal{T} \otimes \mathrm{id}) \circ d = 0$. This proves the claim. Note that the situation of this example arrises always if we have a dynamical system $(\mathcal{B}, G)$ with an abelian Lie group G.

One might wonder whether one cannot always find a cycle for a cyclic cocycle. The construction of the universal differentiable algebra $\Omega(\mathcal{B})$ over $\mathcal{B}$ gives a positive answer to that question [22].

5.4 Constructing new cycles from old ones

We next extend Example 3. Suppose we have the same situation as in that example except that in place of an invariant trace there is an invariant k-cycle $(\Omega, \delta, \int)$. By this we mean that $\mathbf{R^n}$ acts on Ω by derivations such that δ commutes with this action and the graded trace $\int$ is invariant. The aim is to construct a $k+n$-cycle from these data. Before we do so let us recall that the graded tensor product $\Omega\hat{\otimes}\Lambda$ of two graded algebras is the graded tensor product of the vector spaces with product $(w_1\hat{\otimes}v_1)(w_2\hat{\otimes}v_2) = (-1)^{\partial w_2 \partial v_1} w_1 w_2 \hat{\otimes} v_1 v_2$. In particular, it reduces to the usual tensor product if one of the algebras is trivially graded.

Proposition 9 *Let $(\Omega, \delta, \int)$ be a k-cycle over the algebra $\mathcal{B}$ with an action of $\mathbf{R^n}$ by n derivations ∇_i which commute with δ and leave $\int$ invariant. Taking $\Omega' = \Omega\hat{\otimes}\Lambda\mathbf{C}^n$, the graded tensor product, $d' = \delta\hat{\otimes}1 + d$ with $d(w\hat{\otimes}v) = (-1)^{\partial w}\sum_j \nabla_j w \hat{\otimes} e_j v$, and $\int' = \int\hat{\otimes}\imath$ one obtains a $k+n$-cycle $(\Omega', d', \int')$ over $\mathcal{B}$.*

Proof: Ω' is naturally bigraded and one defines the total degree to be the sum of the components of the bidegree. A straightforward calculation shows that that δ and d are anti-commuting graded differentials which are of total degree 1. Therefore $d' = d + \delta$ is a differential of total degree one and (Ω', d') a graded differential algebra with respect to the total degree. It remains thus to check that $\int'$ is a closed graded trace. First note, that $\int'$ is non-zero precisely at the highest non-trivial (total) degree. Further

$$\begin{aligned}
\int' (w_1\hat{\otimes}v_1)(w_2\hat{\otimes}v_2) &= (-1)^{\partial w_2 \partial v_1} \left(\int w_1 w_2\right) \imath(v_1 v_2) \\
&= (-1)^{\partial w_2 \partial v_1 + \partial w_1 \partial w_2 + \partial v_1 \partial v_2} \left(\int w_2 w_1\right) \imath(v_2 v_1) \\
&= (-1)^{(\partial w_1 + \partial v_1)(\partial w_2 + \partial v_2)} \int' (w_2\hat{\otimes}v_2)(w_1\hat{\otimes}v_1)
\end{aligned}$$

shows that $\int'$ is a graded trace. We have

$$\int' d'(w\hat{\otimes}v) = \left(\int \delta w\right) \imath(v) + \sum_j (-1)^{\partial w} \left(\int \nabla_j w\right) \imath(e_j v).$$

The first term vanishes due to closedness of $\int$ and the second due to its invariance. Thus $\int'$ is closed. $\square$

There are two applications of the above which are important for us.

5.4.1 Suspension of cyclic cocycles

In the first, we consider a Banach algebra $\mathcal{B}$ with an n-cycle $(\Omega, d, \int)$ and look at $C(S^1, \mathcal{B})$ the algebra of continuous functions over $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ with values in $\mathcal{B}$. S^1 is an abelian Lie group thus acting on itself and we take this action and extend it trivially to $C(S^1, \mathcal{B})$. Then the Lie algebra $\mathbf{R}$ of S^1 acts by differentiation along the parameter of S^1 which we denote t. We define for $C(S^1, \mathcal{B})$ an n-cycle $(\Omega', d', \int')$ as follows. $\Omega' = C(S^1, \Omega)$, d is the trivial extension of d and for $\int'$ we combine the old n-cycle for $\mathcal{B}$ with integration (Lebesgue measure) over S^1, namely $\int' = \int_{S^1} dt \int$. Now Proposition 9 provides us with an $n+1$-cycle $(\Omega'', d'', \int'')$ for $C(S^1, \mathcal{B})$. Its restriction to the suspensions $S\mathcal{B}$, sub-algebra of $C(S^1, \mathcal{B})$, plays an important role for us. Concretely, for $A_i : S \to \mathcal{B}$, we get

$$\int'' A_0 d''A_1 \cdots d''A_{n+1} \ = \ \sum_{j=1}^{n+1} (-1)^{n+1-j} \int_{S^1} dt \int A_0 dA_1 \cdots dA_{j-1} \dot{A}_j dA_{j+1} \cdots dA_{n+1}\ .$$

If ξ denotes the character of $(\Omega, d, \int)$ in this example then we denote by ξ^s the character on the suspension $S\mathcal{B}$ obtained from $(\Omega'', d'', \int'')$ in the above way. Note that this suspension construction differs from Connes' (double) suspension [21, 22]. The following difficult theorem is due Pimsner [43].

Theorem 3 *The map $HC^n(\mathcal{B}) \to HC^{n+1}(S\mathcal{B})$: $[\xi] \mapsto [\xi^s]$ is dual to the Bott map β for even cycles and dual to the suspension map Θ for odd cocyles, i.e.*

$$\langle [P]_0 - [P_m]_0, [\xi] \rangle \ = \ \langle \beta([P]_0 - [P_m]), [\xi^s] \rangle$$

for even cocycles ξ over $\mathcal{B}$ and

$$\langle [V]_1, [\xi] \rangle \ = \ -\langle \theta([V]_1), [\xi^s] \rangle$$

for odd ones.

5.4.2 Cyclic cocycles for crossed products

In the second application, we consider a n-cycle $(\Omega, d, \int)$ over an algebra $\mathcal{B}$ on which is given a $\mathbf{Z}$-action (such that d commutes with this action and $\int$ is invariant under it). The aim is to construct a $n+1$-cycle over $\mathcal{B} \times_\alpha \mathbf{Z}$, the algebraic crossed product. To do so we first construct a n-cycle over $\mathcal{B} \times_\alpha \mathbf{Z}$ and then apply Proposition 9. The n-cycle over $\mathcal{B}$ is given by $(\Omega', d', \int')$ where $\Omega' = \Omega \times_\alpha \mathbf{Z}$, the algebraic crossed product, d' the trivial extension of d to functions $A : \mathbf{Z} \to \Omega$, i.e. $(d'A)(m) = dA(m)$, and $\int' A = \int A(0)$. To show that d' is a derivation one needs indeed that it commutes with the $\mathbf{Z}$-action α and the latter is also essential to show that $\int'$ is a graded trace. Now on $\mathcal{B}$ and on Ω we have an action of the dual group of $\mathbf{Z}$ which is S^1. Its Lie algebra acts by the derivation

$$\nabla A(m) \;=\; \imath m A(m) \;.$$

This action commutes with d' and leaves $\int'$ invariant, the latter simply because $\nabla A(0) = 0$. We thus can apply Proposition 9 to obtain an $n+1$-cycle $(\Omega'', d'', \int'')$. More explicitly, for $A : \mathbf{Z} \to \mathcal{B}$ we get

$$\int'' A_0 d'' A_1 \cdots d'' A_{n+1} \;=\; \sum_{j=1}^{n+1} (-1)^{n+1-j} \int \left(A_0 dA_1 \cdots dA_{j-1} \nabla A_j dA_{j+1} \cdots dA_{n+1}\right)(0) \;.$$

Now, if ξ is the character of $(\Omega, d, \int)$, we denote by $\#_\alpha \xi$ the character corresponding to the above $n+1$-cycle $(\Omega'', d'', \int'')$. The proof of the following lemma is a straightforward algebraic calculation.

Lemma 3 *Let $\mathcal{B}$ be an algebra furnished with a $\mathbf{Z}$-action α, further let $(\Omega, d, \int)$ be a α-invariant n-cycle over $\mathcal{B}$ and ξ its character. Then $\#_\alpha(\xi^s) = (\#_\alpha \xi)^s$.*

5.5 Duality theorem for crossed products

The following theorem appears in Nest's article [41, Prop. 12.6], but with a partially erronous proof. We use the homotopy invariance of cyclic cohomology in an analogous way to that of [27] in order to correct it.

Theorem 4 *Let $(\mathcal{B}, \mathbf{Z}, \alpha)$ be a C^*-dynamical system, $(\Omega, d, \int)$ be a α-invariant n-cycle over $\mathcal{B}$ and ξ its character. The map $HC^n(\mathcal{B}) \to HC^{n+1}(\mathcal{B} \times_\alpha \mathbf{Z})$: $[\xi] \mapsto [\#_\alpha \xi]$ is dual to the connecting map of the Pimsner-Voiculescu exact sequence, i.e.*

$$\langle \exp([P]_0 - [P_m]_0), [\xi] \rangle \;=\; 2\pi \, \langle [P]_0 - [P_m]_0, [\#_\alpha \xi] \rangle \;,$$

for odd n, and

$$\langle \mathrm{ind}([V]_1), [\xi] \rangle \;=\; -2\pi \, \langle [V]_1, [\#_\alpha \xi] \rangle \;,$$

for even n.

Proof: We denote the unitary generating the action α in $\mathcal{B} \times_\alpha \mathbf{Z}$ by U, that is $\alpha(A) = UAU^*$ for all $A \in \mathcal{B}$.

Due to the Lemma 3 and Theorem 3, it suffices to consider only the case of even n because the crossed product commutes with the suspension, notably $S\mathcal{B} \times_\alpha \mathbf{Z} = S(\mathcal{B} \times_\alpha \mathbf{Z})$.

Now let $[P]_0 \in K_0(\mathcal{B})$ be in the image of the index map. From the Pimsner-Voiculescu exact sequence we then deduce that $[\alpha(P)]_0 = [P]_0$. Thus there exists a unitary $W \in \mathcal{B}$ which is homotopic to the identity and such that $\alpha(P) = WPW^*$ (if necessary, one passes to matrix algebras over $\mathcal{B}$). According to Proposition 10 of the Appendix, the preimage of $[P]_0$ under the index map is then $[UP + W(1-P)]_1 \in K_1(\mathcal{B} \times_\alpha \mathbf{Z})$. Therefore the second statement of the theorem is equivalent to

$$\langle [P]_0 - [P_m]_0, [\xi] \rangle \;=\; -2\pi \, \langle [UP + W(1-P)]_1, [\#_\alpha \xi] \rangle \;,$$

for an $n = 2k$-cocycle ξ of $\mathcal{B}$. Suppose first that $W = 1$. Then

$$\langle [UP + 1 - P]_1, [\#_\alpha \xi] \rangle \;=\; c_{2k+1} \int'' P(U^* - 1) d''((U-1)P) \cdots d''((U-1)P)$$

with the cycle for the character as above. Since the action α commutes with d' and $[U, P] = 0$, we can collect all U's to the left. Furthermore, $P(d'P)^m P$ vanishes if m is odd and is equal to $P(d'P)^m$ if m is even. Using this the r.h.s. becomes

$$\begin{aligned}
&\imath c_{2k+1}(k+1) \int' U(U-1)^k (U^{-1}-1)^{k+1} P(d'P)^{2k} \\
&\quad = \; -\imath c_{2k+1}(k+1) \sum_{l=0}^{k} \binom{k}{l} \binom{k+1}{l+1} \int P(dP)^{2k} \\
&\quad = \; -\imath c_{2k+1}(k+1) \binom{2k+1}{k} \int P(dP)^{2k} \\
&\quad = \; -\frac{1}{2\pi} \langle [P]_0, [\xi] \rangle \;.
\end{aligned}$$

To treat now the case $W \neq 1$, we use the new action $\alpha'(A) = W^* \alpha(A) W$ and the associated crossed product $\mathcal{B} \times_{\alpha'} \mathbf{Z}$. Further let us introduce an automorphism γ of $M_2(\mathcal{B})$ by

$$\gamma \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;=\; \begin{pmatrix} \alpha(A) & \alpha(B)W \\ W^*\alpha(C) & W^*\alpha(D)W \end{pmatrix} \;.$$

Then there are two embeddings

$$\rho_1 \;:\; \mathcal{B} \times_\alpha \mathbf{Z} \to M_2(\mathcal{B}) \times_\gamma \mathbf{Z} \;, \qquad \rho_2 \;:\; \mathcal{B} \times_{\alpha'} \mathbf{Z} \to M_2(\mathcal{B}) \times_\gamma \mathbf{Z} \;,$$

onto the upper left and lower right corner of the 2×2-matrix respectively. Finally we introduce the C^∞-path $\mathrm{Ad}_{V_t} : M_2(\mathcal{B}) \times_\gamma \mathbf{Z} \to M_2(\mathcal{B}) \times_\gamma \mathbf{Z}$, $t \in [0, 1]$, where

$$V_t \;=\; \begin{pmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix} \;.$$

As Ad_{V_1} interchanges the upper left and lower right corners, we can use the following commutative diagram to define an isomorphism $i : \mathcal{B} \times_\alpha \mathbf{Z} \to \mathcal{B} \times_{\alpha'} \mathbf{Z}$:

$$\begin{tikzcd}
\mathcal{B} \times_\alpha \mathbf{Z} \arrow[r, "\rho_1"] \arrow[d, "i"'] & M_2(\mathcal{B}) \times_\gamma \mathbf{Z} \arrow[d, "\mathrm{Ad}_{V_1}"] \\
\mathcal{B} \times_{\alpha'} \mathbf{Z} \arrow[r, "\rho_2"] & M_2(\mathcal{B}) \times_\gamma \mathbf{Z}
\end{tikzcd}$$

One can now check that $i(A) = A$ for all $A \in \mathcal{B}$ and $i(U) = WU'$ where U' is the generator of the α'-action in $\mathcal{B} \times_{\alpha'} \mathbf{Z}$. Hence $i(UP + W(1-P)) = W(U'P + (1-P))$. Further one checks that

$$\#_\alpha \xi \;=\; \rho_1^*(\#_\gamma(\mathrm{Tr}_2 \otimes \xi)) \;, \qquad \#_{\alpha'} \xi \;=\; \rho_2^*(\#_\gamma(\mathrm{Tr}_2 \otimes \xi)) \;.$$

To verify that the pairings with $[\#_\alpha \xi]_1$ and $[\#_{\alpha'} \xi]_1$ coincide, we now use Connes' homotopy invariance of cyclic cohomology [22] extended to non-unital algebras in [27]. As $V_0 = 1$, it tells us that $[\#_\gamma(\mathrm{Tr}_2 \otimes \xi)]_1 = [\mathrm{Ad}_{V_1}^* \#_\gamma(\mathrm{Tr}_2 \otimes \xi)]_1$ in $HC^{odd}(M_2(\mathcal{B}) \times_\gamma \mathbf{Z})$. Therefore we obtain:

$$\begin{aligned}
\langle [UP + W(1-P)]_1, [\#_\alpha \xi]_1 \rangle \;&=\; \langle [\rho_1(UP + W(1-P))]_1, [\#_\gamma(\mathrm{Tr}_2 \otimes \xi)] \rangle \\
&=\; \langle [\rho_1(UP + W(1-P))]_1, [\mathrm{Ad}_{V_1}^*(\#_\gamma(\mathrm{Tr}_2 \otimes \xi))] \rangle \\
&=\; \langle [i(UP + W(1-P))]_1, [\rho_2^*(\#_\gamma(\mathrm{Tr}_2 \otimes \xi))] \rangle \\
&=\; \langle [U'P + (1-P)]_1, [\#_{\alpha'} \xi] \rangle \;,
\end{aligned}$$

where in the last step we use $[W(U'P + (1-P))]_1 = [U'P + (1-P)]_1$ because W is homotopic to the identity in $\mathcal{B}$. Now this last pairing is in $\mathcal{B} \times_{\alpha'} \mathbf{Z}$ where the above calcuation can be applied in order to conclude. $\Box$

5.6 Duality theory for quantum Hall systems

According to the results of Chapter 4 and of Proposition 2, the definitions in Section 5.2 show that edge and bulk Hall conductivity result from the following pairings of cyclic cohomology with K-theory over $\mathcal{E}$ and $\mathcal{A}$ respectively:

$$\sigma_\perp^e(\Delta) \;=\; -\frac{q^2}{h}\, 2\pi\imath \,\langle [\hat{U}(\Delta)]_1, [\xi_1] \rangle \;, \qquad \sigma_\perp^b(\mu) \;=\; \frac{q^2}{h}\, 4\pi^2\imath \,\langle [P_\mu]_0, [\xi_2] \rangle \;.$$

Here $\hat{U}(\Delta)$ and ξ_1 are defined in (25) and (34), P_μ is the Fermi projection of the plane operator on a Fermi level $\mu \in \Delta$ and ξ_2 is the 2-cocylce over $\mathcal{A}$ defined by

$$\xi_2(A, B, C) \;=\; \imath\,\mathcal{T}(A \nabla_x B \nabla_y C - A \nabla_y B \nabla_x C) \;.$$

We now use the isomorphisms ρ and η defined in (22) and (23) in order to transform these two pairings to pairings over the algebras $C(\Omega) \times_{\alpha_x} \mathbf{Z} \otimes K$ and $C(\Omega) \times_{\alpha_x} \mathbf{Z} \times_{\alpha_y} \mathbf{Z}$ of the Pimsner-Voiculescu exact sequence (21):

$$\sigma^e_\perp(\Delta) \;=\; -\frac{q^2}{h}\, 2\pi\imath\; \langle [\rho^{-1}(\hat{U}(\Delta))]_1, [\rho^*\xi_1]\rangle \;, \qquad \sigma^b_\perp(\mu) \;=\; \frac{q^2}{h}\, 4\pi^2\imath\; \langle [\eta^{-1}(P_\mu)]_0, [\eta^*\xi_2]\rangle \;.$$

Now we may apply the duality Theorem 4 for Pimsner-Voiculesu exact sequences by setting $\mathcal{B} = C(\Omega) \times_{\alpha_x} \mathbf{Z}$ on which the action is given by $\alpha = \alpha_y$. Hence Theorem 2 is proven once we have verified that $\exp([\eta^{-1}(P_\mu)]_0) = [\rho^{-1}(\hat{U}(\Delta))]_1$ and $\eta^*\xi_2 = \#_{\alpha_y}\rho^*\xi_1$. Concerning the latter equality one can immediately verify that, if one defines ξ_1 and ξ_2 by the same formulas on the corresponding algebras of the Pimsner-Voiculescu exact sequence, then $\eta^*\xi_2 = \xi_2$ and $\rho^*\xi_1 = \xi_1$ (use herefore that η and ρ are isomorphisms leaving the traces invariant); then going back to the definition of $\#_{\alpha_y}$ in Subsection 5.4.2, the identity $\xi_2 = \#_{\alpha_y}\xi_1$ can be directly checked.

For the second equality, it is clearly sufficient to show that $\exp([P_\mu]_0) = [\hat{U}(\Delta)]_1$. We first note that P_μ is equal to the continuous function of the Hamiltonian $g(H) = P_{E'} - (H - E')P_\Delta/|\Delta|$ where $E' = \inf(\Delta)$. A selfadjoint lift of P_μ is thus given by $g(\hat{H})$. Now using $[\hat{P}_{E'}, \hat{P}_\Delta] = 0$ and the fact that $\exp(2\pi\imath\hat{P}) = 1$ for any projection $\hat{P}$, we thus obtain from the definition of the exponential map recalled in the Appendix:

$$\exp([P_\mu]_0) \;=\; [\exp(-2\pi\imath\, g(\hat{H}))]_1 \;=\; [\hat{U}(\Delta)]_1 \;.$$

This concludes the proof of Theorem 2.

Appendix: Some help for cooking with K-theory

This is only a very brief summary of the main K-theoretic notions used in this work. We refer to [16, 53] for all the details and even for the definition of the groups $K_0(\mathcal{B})$ and $K_1(\mathcal{B})$ of a given involutive Banach algebra $\mathcal{B}$. Roughly, K_0 is the Grothendieck group generated by homotopy classes of idempotents in matrix algebras over $\mathcal{B}$, and $K_1(\mathcal{B})$ is formed by homotopy classes of invertibles (or equivalently unitaries) therein. However, great care needs to be taken when $\mathcal{B}$ does not have a unit.

The suspension $S\mathcal{B}$ of $\mathcal{B}$ is the algebra of continuous functions $f : S^1 \to \mathcal{B}$ which vanish at a distinguished point that we choose to be 1. This construction provides a link between the K-groups. First, the following map is a group isomorphism:

$$\Theta \;:\; K_1(\mathcal{B}) \to K_0(S\mathcal{B}) \;, \qquad [V]_1 \mapsto \left[W_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_t^*\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0 \;.$$

Here $V \in M_n(\mathcal{B})$ is an unitary representative and $W_t \in M_{2n}(\mathcal{B})$ a homotopy from $\begin{pmatrix} V & 0 \\ 0 & V^* \end{pmatrix}$ to 1. The blocksize in the matrix is $n \times n$. One of the key properties of K-theory is that

$$\beta \;:\; K_0(\mathcal{B}) \to K_1(S\mathcal{B}) \;, \qquad [P]_0 \mapsto [e^{2\pi\imath tP}]_1 \;=\; [(e^{2\pi\imath t} - 1)P + (1 - P)]_1 \;,$$

is also an isomorphism called the Bott map. It shows that $K_0(\mathcal{B}) \cong K_0(SS\mathcal{B})$, making K-theory a periodic theory.

Another crucial property of K-theory is that any short exact sequence of Banach algebras,

$$0 \to \mathcal{J} \overset{i}{\to} \mathcal{B} \overset{\pi}{\to} \mathcal{B}/\mathcal{J} \to 0 \,,$$

gives rise to a six-term exact sequence of K-groups:

$$\begin{tikzcd}
K_0(\mathcal{J}) \arrow[r, "i_*"] & K_0(\mathcal{B}) \arrow[r, "\pi_*"] & K_0(\mathcal{B}/\mathcal{J}) \arrow[d, "\exp"] \\
K_1(\mathcal{B}/\mathcal{J}) \arrow[u, "\mathrm{ind}"] & K_1(\mathcal{B}) \arrow[l, "\pi_*"'] & K_1(\mathcal{J}) \arrow[l, "i_*"']
\end{tikzcd}$$

The definition of the push-forward maps i_* and π_* is immediate. Let us recall those of the index and exponential map. Given a unitary $V \in M_n(\mathcal{B}/\mathcal{J})$ defining a class in $K_1(\mathcal{B}/\mathcal{J})$, let $W \in M_{2n}(\mathcal{B})$ be a unitary lift of $\begin{pmatrix} V & 0 \\ 0 & V^* \end{pmatrix}$. Then

$$\mathrm{ind}([V]_1) \;=\; \left[W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^*\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0 \,.$$

The exponential map is defined by $\exp := \Theta^{-1} \circ \mathrm{ind} \circ \beta$. Explicitly, if $P \in M_n(\mathcal{B}/\mathcal{J})$ defines a class in $K_0(\mathcal{B}/\mathcal{J})$ and $L(P) \in \mathcal{B}$ a self-adjoint lift for it, then

$$\exp([P]_0) \;=\; \left[\exp(-2\pi\imath\, L(P))\right]_1 \,.$$

We furthermore recall some facts about the Toeplitz extension (see [44]) of the C*-crossed product $\mathcal{B} \times_\alpha \mathbf{Z}$ associated to the C*-dynamical system $(\mathcal{B}, \mathbf{Z}, \alpha)$ (for unital $\mathcal{B}$):

$$0 \to \mathcal{B} \otimes K \overset{\psi}{\to} T(\mathcal{B}) \overset{\pi}{\to} \mathcal{B} \times_\alpha \mathbf{Z} \to 0 \,. \tag{41}$$

Here K are the compact operators on $\ell^2(\mathbf{N})$ generated by the finite rank operators $e_{n,m}$, $n, m \in \mathbf{N}$, the matrix units, and ψ is the homorphism defined in [44]. We have met such an extension in Section 2.6. The imbedding $j : \mathcal{B} \to \mathcal{B} \otimes K$, $j(b) = b \otimes e_{11}$ induces an isomorphism $K(\mathcal{B}) \overset{j_*}{\cong} K(\mathcal{B} \otimes K)$ and in [44] Pimsner and Voiculescu have shown that $K(T(\mathcal{B})) \cong K(\mathcal{B})$ as well. Keeping carefully track of these isomorphisms, they obtained from the six-term exact sequence associated to (41) the nowadays called Pimsner-Voiculescu six-term exact sequence:

$$\begin{tikzcd}
K_0(\mathcal{B}) \arrow[r, "i_* - \alpha_*^{-1}"] & K_0(\mathcal{B}) \arrow[r, "i_*"] & K_0(\mathcal{B} \times_\alpha \mathbf{Z}) \arrow[d, "\exp"] \\
K_1(\mathcal{B} \times_\alpha \mathbf{Z}) \arrow[u, "\mathrm{ind}"] & K_1(\mathcal{B}) \arrow[l, "j_*"'] & K_1(\mathcal{B}) \arrow[l, "i_* - \alpha_*^{-1}"']
\end{tikzcd}$$

Strictly speaking the maps ind and exp in this diagram are not the same as the ones in the original sequence, but have to be composed with j_*^{-1}.

For the convenience of the reader, we now produce a direct proof of the following result by Elliott and Natsume [26] used in Section 5.5, which is a discrete (one-way) analogon of Connes' Thom isomorphism [20]. We denote the unitary generator of the action in $\mathcal{B}\times_\alpha \mathbf{Z}$ by U, notably $\alpha(A) = UAU^*$ for all $A \in \mathcal{B}$.

Proposition 10 *Let $P \in M_n(\mathcal{B})$ be a projection defining a class in $K_0(\mathcal{B})$ which is in the image of the index map, and let $W \in M_n(\mathcal{B})$ be a unitary such that $\alpha(P) = WPW^*$. Then a preimage of $[P]_0$ is the class defined by the unitary $X = UP + W(1-P)$.*

Proof: First note that the unitary W exists (possibly after enlarging n) because $[P]_0 = [\alpha(P)]_0$ by the Pimsner-Voiculescu exact sequence. In order to compute $\mathrm{ind}([X]_1)$, we have to find a lift $W \in M_{2n}(T(\mathcal{B}))$ of $\begin{pmatrix} X & 0 \\ 0 & X^* \end{pmatrix}$ and to compute $j_*^{-1}\left(\left[W\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}W^*\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0\right)$. There exists a lift $\hat{U} \in T(\mathcal{B})$ of U with the following properties: $\hat{U}\hat{U}^* = 1$, $\hat{U}^*\hat{U} = 1 - P_0$, where $P_0 = i(1 \otimes e_{11})$ and $\hat{U}A\hat{U}^* = \alpha(A)$ for all $A \in \mathcal{B}$. In particular, P commutes with P_0 and it is not difficult to verify that

$$W = \begin{pmatrix} \hat{U}P + W(1-P) & 0 \\ PP_0 & P\hat{U}^* + (1-P)W^* \end{pmatrix}$$

is a lift of $\begin{pmatrix} X & 0 \\ 0 & X^* \end{pmatrix}$. Furthermore,

$$\left[W\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}W^*\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0 = \left[\begin{pmatrix} 0 & 0 \\ 0 & PP_0 \end{pmatrix}\right]_0 = j_*([P]_0)\ .$$

□

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