On the absolutely continuous spectrum of Stark Hamiltonians

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Abstract

We study the spectral properties of the Schrődinger operator with a constant electric field perturbed by a bounded potential. It is shown that if the derivative of the potential in the direction of the electric field is smaller at infinity than the electric field then the spectrum of the corresponding Stark operator operator is purely absolutely continuous. In one dimension the absolute continuity of the spectrum is implied by just the boundedness of the derivative of the potential. The sharpness of our criterion for higher dimensions is illustrated by constructing smooth potentials with bounded partial derivatives for which the corresponding Stark operators have a dense point spectrum.

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I Introduction

I.1. Overview

The Hamiltonian of an electron moving in a constant electric field $\overrightarrow{F}$ and subject to an external force given by a potential V is the Stark operator

$$H = -\Delta - x_1 + V(x)$$

acting in $\mathcal{H} = L^2(\mathbb{R}^d)$, where $d \geq 1$ is the dimension, and x_1 is the multiplication operator by the coordinate x_1. It is convenient to regard the potential V as a perturbation of the Stark operator

$$H_0 = -\Delta - x_1,$$

where the electric field is taken as $\overrightarrow{F}= (-1, 0, \cdots, 0)$.

The Stark operator is essentially self-adjoint in $\mathcal{H}$ and its spectrum is purely absolutely continuous:

$$\begin{cases} \sigma_{ac}(H_0) = & \sigma_{ac}(H_0) = \mathbb{R} \\ \sigma_{sc}(H_0) = & \sigma_{pp}(H_0) = \emptyset. \end{cases}$$

Our purpose here is to study the stability of that spectral structure under a perturbation by a (bounded) potential V. The stability here means that the perturbed Hamiltonian H has a purely absolutely continuous spectrum up to a discrete set as a singular spectrum.

Perturbations of interest include the Anderson random potentials

$$V(x) = \sum_{i \in \mathbb{Z}^d} \lambda_i(w) u(x-i) \tag{I.1}$$

where u is the single site potential and $\{\lambda_i(w)\}$ is a collection of i.i.d. random variables. It is well known that when the electric field is zero the operator H has only localized states (almost surely with respect to the disorder) in dimension one. The same is expected in dimension 2. But in dimension $d \geq 3$ there is a region in which the spectrum is pure point associated to the localized states, and a region of absolutely continuous spectrum which corresponds to the extended states, separated by some energies called the mobility edges (see [1, 2, 3, 4] for a precise discussion of such considerations).

But in the presence of a non zero electric field the spectral properties of H change drastically. Indeed (see [5, 6, 7]) in $d = 1$ if V is sufficiently regular (e.g. twice continuously differentiable, bounded with its derivatives) then the spectrum of H is purely absolutely continuous. In particular, this is true for the Anderson potential (see however [8]).

This can be explained by the naive physical intuition: because the electron is uniformly accelerated by the electric field, a bounded potential could not affect its propagation and the spectral properties of its Hamiltonian. However, this is not always true, and therefore the absolute continuity of the spectrum of the Stark operator can be partially or completely destroyed by "small" potential. This has been pointed out by Naboko-Pushnitskii in [9], where a smooth potential V has constructed, that satisfies

$$|V(x)| \leq \frac{C(x)}{(1+|x|)^{1/2}}, \quad C(x) \longrightarrow \infty \ \text{ at infinity arbitrarily slowly}$$

and the corresponding Hamiltonian H has a dense set of eigenvalues. Notice that the derivative of Naboko-Pushnitskii's potential diverges at infinity exactly at the same rate of the function $C(x)$. This illustrates the important role of the derivative in the preservation of the absolutely continuous spectrum in the sense explained above.

We also mention the result of [10] where the one dimensional Krőnig-Penny model with an electric field is studied. The authors establish a transition from purely continuous spectrum to a pure point spectrum as the electric field decreases. There are also several results dealing with strongly singular potentials V for which the corresponding operator H has no absolutely continuous spectrum (see [8, 11]).

Our first purpose in this paper is to describe a class of bounded potentials V for which the corresponding Hamiltonian H has an absolutely continuous spectrum with at most a discrete set as singular spectrum. Basically, our class consists of sufficiently regular potential having a small (at infinity) derivative in the direction of the electric field. In the one dimensional case the derivative does not have to be small but only bounded.

The second goal of our paper is to study the role that the tail of the partial derivative of the potential can play in the occurrence of a (dense) singular spectrum in higher dimension. Indeed, in the case where the dimension $d \geq 2$ we construct a smooth potential V, bounded with its first partial derivatives and leading to a dense point spectrum for H. Again, we notice here that in dimension one this cannot happen, which illustrates the dimension effect in the problem.

I.2. The one dimensional case.

As we already mentioned above, in $d = 1$ if V is sufficiently regular then the absolute continuity of the spectrum of H ([5, 6, 7] is preserved, while it is partially or completely destroyed if the potential is singular (see [9, 10, 8, 11]). What is then the minimal regularity of V preserving the absolute continuity of H? A partial answer is

Theorem 1 *1) Assume that V is of class BC^1 with uniformly continuous first derivative. Then the set $\sigma_p(H)$ of eigenvalues of H is discrete.*
2) Suppose that V is smooth in the Zygmund's sense, i.e.

$$\int_0^1 \sup_{x\in\mathbb{R}} |V(x+\varepsilon) - 2V(x) + V(x-\varepsilon)| \frac{d\varepsilon}{\varepsilon^2} < \infty, \tag{I.2}$$

then H has no singular continuous spectrum.

Remarks.

1. We proved Theorem 1 in [12] under a slightly more restrictive assumption on V. This was be re-obtained partially by Kiselev in [13] by using the Gilbert-Pearson method. The main difference is that the result of [13] does not rule out the possibility to have a singular spectrum also filling the real axis. The slowly decaying potentials are also studied in [13], which combined with [9] provides an example of a Stark operator for which dense point spectrum coexist with the absolutely continuous spectrum.

2. Theorem 1 includes a large class of periodic, quasi-periodic or random potentials, because there is no restriction on the bound of the potential V nor on V'. This is not the case in the multidimensional case (cf. Theorem 3 and 4).

Notice that Theorem 1 only ensures the discreteness of the set of eigenvalues of H but does not exclude the possibility to have embedded eigenvalues. In fact, we have

Theorem 2 *For each real number λ there exist a real-valued potential $V \in C^\infty(\mathbb{R})$, such that V and V' are bounded and λ is an eigenvalue of H.*

Theorem 2 is an essential piece in our construction of a smooth and bounded potential such that H has a dense point spectrum in dimension 2 and more (cf. Theorem 4). To prove it we use the ideas of [9], and the potential V will be sparse.

I.3. The multidimensional case.

In $d > 1$ there is also an important literature on the spectral theory of Stark Hamiltonians ([14, 15, 16] and references therein). The most part of these works assume that V or ∇V tend to zero at infinity. This can be explained partially as follows. When $d = 1$ the derivative of a bounded regular potential is H_0-compact. Which can be lost in $d \geq 2$ if V or $\partial_1 V$ does not tend to zero at infinity (see however [16]).
For a function W we set $W_+(x) = \sup(0, W(x))$. Put $x = (x_1, x')$ with $x' = (x_2, \cdots, x_d)$.

Theorem 3 *Assume that V and $\partial_1 V$ are bounded and that $\|(\partial_1 V)_+\|_\infty < 1$. Then H has no eigenvalues. If moreover*

$$\int_0^1 \sup_{x\in\mathbb{R}^d} |V(x_1+\varepsilon, x') - 2V(x) + V(x_1-\varepsilon, x')| \frac{d\varepsilon}{\varepsilon^2} < \infty \qquad \text{(I.3)}$$

then H has no singular continuous spectrum.

Remarks

1. Actually V does not have to be bounded, we only need that H defines a self-adjoint operator in $\mathcal{H}$. This is not true in Theorem 1.
2. It is not difficult to show that the assumption $\|(\partial_1 V)_+\|_\infty < 1$ can be replaced by only the smallness at infinity, i.e.

   $$\limsup_{|x|\to\infty} (\partial_1 V)_+(x) < 1.$$

   In which case, the possible singular spectrum that H can have is discrete, and consists of finitely degenerate eigenvalues.

Examples.

1. Theorem 3 obviously includes potentials V such that $\partial_1 V$ tends to zero at infinity, but without asking that V itself tends to zero at infinity (see discussion above)
2. Let $q : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a bounded and sufficiently regular function. Then

   $$H = H_0 + q(\alpha x_1, x'), \quad \alpha \in \mathbb{R}$$

   has a purely absolutely continuous spectrum provided that the scaling constant α is sufficiently small. This includes a large class of periodic or quasiperiodic potentials. An analogous statement holds for the Anderson random potential (I.1).

Let us rephrase the last Theorem as follows. If the force induced by the electric field is stronger than the force induced by the potential then the spectrum of H is purely absolutely continuous. This bring us to ask the natural question: what happens when the electric field is weak?. We have

Theorem 4 *Let λ be a fixed real number. Then there exist a real-valued potential $V \in C^\infty(\mathbb{R}^d)$ such that V and $\partial_1 V$ are bounded and $[\lambda, \infty) \subset \sigma_{pp}(H)$.*

According to this theorem, it is not clear whether or not the Anderson model with a weak electric field or a strong disorder in higher dimension has a purely absolutely continuous spectrum. This should be an interesting question to study.

The paper is organized as follows. In section II we shall describe what we need in our proofs. Sections III and IV will be devoted to the proofs of Theorems 1 and 3 respectively. In section V we prove Theorems 2 and 4.

II Basic notions

Our proofs are based on the conjugate operator method. It is an abstract theory which proves that the Hamiltonian H has an absolutely continuous spectrum if it has a conjugate operator A, i.e. a self-adjoint operator such that the commutator $[H, iA]$ is strictly positive in an adequate sense. In this section we give a short description of the main points of this theory. For more details concerning the results of this section we refer the reader [17, 18, 19]. Let H, A be two self-adjoint operators in a Hilbert space $\mathcal{H}$. The C$_0$-group associated to A will be denoted by $e^{-iAt}, t \in \mathbb{R}$. We shall denote by $R(z) = (H-z)^{-1}$ the resolvent of H for a complex number $z \in \mathbb{C} \setminus \sigma(H)$.

Definition 1 *1. We say that H is of class $C^1(A)$ if the map*

$$\mathbb{R} \ni t \longmapsto e^{-iAt}R(z)e^{iAt} \in B(\mathcal{H})$$

is strongly of class C^1, for some (and so for any) complex $z \in \mathbb{C} \setminus \mathbb{R}$
2. We say that H is A-regular if

$$\int_0^1 \|e^{-iA\varepsilon}R(z)e^{iA\varepsilon} - 2R(z) + e^{iA\varepsilon}R(z)e^{-iA\varepsilon}\| \frac{d\varepsilon}{\varepsilon^2} < \infty.$$

Remark that if H is A-regular then H is of class $C^1(A)$. Assume that H is of class $C^1(A)$. Then the intersection $D(A) \cap D(H)$ is dense in $D(H)$ equipped with the graph topology associated to the norm

$$\|f\|_H = \|f\| + \|Hf\|.$$

Moreover the sesquilinear form defined on $D(A) \cap D(H)$ by

$$< f, [H, A]g > = < Af, Hg > - < Hf, Ag >$$

extends continuously to $D(H)$. Then one can define the open set $\tilde{\mu}^A(H)$ of the real points λ for which there exist a constant $a > 0$, a compact operator K in $\mathcal{H}$ such that

$$E(\Delta)[H, iA]E(\Delta) > aE(\Delta) + K, \tag{II.4}$$

for some open interval Δ containing λ. Here E denotes the spectral measure of H.

Proposition 1 *The eigenvalues of H contained in $\tilde{\mu}^A(H)$ are all finitely degenerate and cannot accumulate in $\tilde{\mu}^A(H)$.*

In particular, the spectrum of H is purely continuous in

$$\mu^A(H) := \tilde{\mu}^A(H) \setminus \sigma_p(H),$$

where $\sigma_p(H)$ denotes the set of eigenvalues of H. One can prove easily that $\mu^A(H)$ is in fact the set of all real points λ for which there exist a constant $a > 0$ and an open interval $\Delta \ni \lambda$ such that

$$E(\Delta)[H, iA]E(\Delta) > aE(\Delta), \tag{II.5}$$

We have

Theorem 5 *Assume that e^{iAt} leaves invariant the domain $D(H)$ and that H is A-regular. Then H is purely absolutely continuous in $\mu^A(H)$.*

Remarks.

1. This Theorem is proved in [17], where one can find another version of this theorem in which the invariance of the domain under the action of e^{iAt} is replaced by the fact that H has a gap in its spectrum (which is clearly inadequate for the model considered here). In [19] we have eliminated this condition on the domain without asking the existence of a spectral gap for H, but we ask H to be (locally) slightly more than $A-$regular.

2. Let us mention that if H is of class $C^1(A)$ and $[H, iA]$ is a bounded operator in $\mathcal{H}$ then e^{iAt} leaves invariant the domain of H (see [20]).

III Proof of Theorem 1

Using Mourre's theory described in the last section the proof of Theorem 1 is reduced to prove that H has a conjugate operator. This section is devoted to that.

(i) By straightforward computations we get

$$[H_0, iA] = 1$$

where $A = -id/dx$ is the translation generator. Then H_0 is A-regular (in fact is even of class $C^\infty(A)$ in the sense that the map of Definition 1 is of class C^∞), and

$$\mu^A(H_0) = \mathbb{R}.$$

On the other hand

$$[V, iA] = -V'(x),$$

which is obviously bounded in $\mathcal{H}$ if and only if V' is bounded as function. To conclude the first part of Theorem 1 we have to prove that the Mourre estimate holds locally on $\mathbb{R}$, i.e. $\tilde{\mu}^A(H) = \mathbb{R}$. For this, and according to

$$E(\Delta)[H, iA]E(\Delta) = E(\Delta) - E(\Delta)V'(x)E(\Delta), \tag{III.6}$$

it is sufficient to prove that $E(\Delta)V'(x)E(\Delta)$ is a compact operator in $\mathcal{H}$. But this property is a simple consequence of the following assertion which is proved by Bentosela and all in [7] (see also [21]). If a function G is uniformly continuous and

$$\lim_{r \longrightarrow \infty} \sup_{x \in \mathbb{R}} \left|\frac{1}{r} \int_{x-r}^{x+r} G(y)dy\right| = 0 \tag{III.7}$$

then $E(\Delta)G(x)E(\Delta)$ is a compact operator in $\mathcal{H}$. Clearly if V' is bounded and uniformly continuous, and V is bounded (which are ensured by our assumptions) then $G = V'$ satisfies (3.7). Then $\tilde{\mu}^A(H) = \mathbb{R}$ follows from (3.6) and (3.7), and Proposition 1 finishes the proof of part 1 of Theorem 1.

(ii) Now on let us prove the second part of Theorem 1. According to Theorem 5 and the last part of our proof we have only to show that H is

A-regular and that $D(H)$ is invariant under the action of e^{iAt}. According to the obvious property

$$e^{-iA\varepsilon} V e^{iA\varepsilon} = V(x-\varepsilon).$$

the operator V is A-regular if and only if the function V is smooth in Zygmund's sense, which is exactly our assumption (1.2). Then the operator H is A-regular.

The invariance of the domain $D(H)$ under the action of e^{iAt} follows from the fact that the commutator $[H, iA] = 1 - V'$ is a bounded operator in $\mathcal{H}$. This finishes the proof of Theorem 1.

IV Proof of Theorem 3

We shall prove that the self-adjoint operator

$$A = -i\partial_{x_1}$$

is strictly conjugate to H on $\mathbb{R}$. Indeed, we have

$$[H_0, iA] = 1$$

In particular H_0 is of class $C^\infty(A)$ and A is strictly conjugate to H_0 on $\mathbb{R}$, i.e. $\mu^A(H_0) = \mathbb{R}$. On the other hand, we have

$$e^{-i\varepsilon A} V e^{i\varepsilon A} = V(x_1 - \varepsilon, x').$$

Then it is clear that the regularity assumptions on the function V of Theorem 3 ensure that V is A-regular, and so H is too (since H_0 is). Moreover, the commutator

$$[H, iA] = 1 - \partial_{x_1} V$$

is obviously bounded operator in $\mathcal{H}$. And so the domain of H is invariant under the action of e^{iAt}. It remains to show that $\mu^A(H) = \mathbb{R}$. But this property follows easily as follows:

$$\begin{aligned} [H, iA] &= 1 - \partial_{x_1} V \\ &\geq 1 - (\partial_{x_1} V)_+(x) \\ &\geq 1 - \|(\partial_{x_1} V)_+\|_\infty > 0 \end{aligned}$$

which is a global and strict Mourre estimate. This finishes the proof of the second part of Theorem 3.

Remarks

1. Let us remark that when we only assume that

   $$\limsup_{|x|\longrightarrow\infty}(\partial_{x_1}V)_+) < 1$$

   we only get a local Mourre estimate:

   $$E(\Delta)[H, iA]E(\Delta) \geq aE(\Delta) + K,$$

   where K is a compact operator, $a = 1 - \limsup_{|x|\longrightarrow\infty}(\partial_{x_1}V)_+ > 0$ and Δ is any compact interval. Which means that $\tilde{\mu}^A(H) = \mathbb{R}$. Thus according to Proposition 1 there is only a discrete set of possible eigenvalues of H and all these eigenvalues are finitely degenerate.

2. We also mention the fact that our argument ignore completely if the particle is relativistic or non-relativistic. More precisely, our proof still valid for any operator of the form

   $$H = h(-i\nabla) - x_1 + V(x)$$

   where h is a divergent continuous function and V are as in Theorem 3. An physical interesting situation is when $h(x) = \sqrt{1 + |x|^2}$. In such case H becomes

   $$H = \sqrt{-\Delta + 1} - x_1 + V(x),$$

   which describes the motion of a charged relativistic particle in a constant electric field. However, it is not clear whether or not Theorem 1 still valid. To do this, we have to describe the relative compact operator V with respect to $H_0 = h(P) - x_1$.

V Proof of Theorem 2 and 4

V.1. Proof of Theorem 4.

Let us start by proving that Theorem 4 follows easily from Theorem 2 combined with Simon's result [22]. Assume that $d = 2$ and that

$$V(x_1, x_2) = V(x_1) + V(x_2).$$

Then by separation of variable we decompose H as follows

$$H = H_1 \otimes 1 + 1 \otimes H_2 \tag{V.8}$$

acting in $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, where

$$\begin{aligned} H_1 &= -\frac{d^2}{dx_1^2} - x_1 + V_1(x_1); & \text{(V.9)} \\ H_2 &= -\frac{d^2}{dx_2^2} + V_2(x_2). & \text{(V.10)} \end{aligned}$$

Let λ be a fixed real number. Theorem 2 tell us that there exists a real-valued potential V_1 of class C^∞ such that V_1 and V_1' are bounded and that λ is an eigenvalue of H_1. On the other hand, let $\{\lambda_i\}_i$ be a sequence of positive numbers. Then there exist (cf. Simon [22]) a potential V_2 of class C^∞ such that each λ_i is an eigenvalue of H_2. In particular, if the sequence $\{\lambda_i\}_i$ is dense in $[0, \infty)$ (which we assume from now) then we get that $[0, \infty) \subset \sigma_{pp}(H_1)$. But the decomposition (V.8) of H implies that the numbers $\lambda + \lambda_i$ belong to the set $\sigma_p(H)$ of the eigenvalues of H. It follows then $[\lambda, \infty) \subset \sigma_{pp}(H)$. On the other hand, the potential $V = V_1 + V_2$ is of class C^∞ and $\partial_1 V(x) = V_1'(x_1)$ which is clearly bounded. Finally, an obvious induction allows us to do the same construction for any dimension d.

V.2. Proof of Theorem 2.

Reduction of the problem. Without loss of generality one can assume that $\lambda = 0$. We recall that our goal is to construct a real-valued potential V of class $C^\infty(\mathbb{R})$ such that:

(i) V and V' are bounded and

(ii) there exist $u \in L^2(\mathbb{R})$ solution of

$$-u'' - xu + Vu = 0.$$

Let us apply the Liouville transformation by setting for $x \geq 1$

$$\xi = \frac{2}{3} x^{\frac{3}{2}}, \qquad w(\xi) = x(\xi)^{\frac{1}{4}} u(x(\xi)).$$

One can show that (cf. [9]) the problem (ii) is equivalent to:
(ii)' there exist a function $w \in L^2((1,\infty), \xi^{\frac{-2}{3}} d\xi)$ solution of

$$-w'' + q(\xi)w = w. \tag{V.11}$$

The relation between q and V is

$$q(\xi) = \frac{V(x(\xi))}{x(\xi)} + \frac{5}{36\xi^2}$$

or equivalently

$$V(x) = x(q(\xi(x)) - \frac{5}{36\xi(x)^2}).$$

Moreover, it is not difficult to see that if q of class $C^\infty(1,\infty)$ such that for each integer $m \geq 0$

$$q^{(m)}(\xi) = O(\xi^{-1}), \quad \xi \longrightarrow \infty$$

then V will be of class $C^\infty(\mathbb{R})$, supp $V \subset [1,\infty)$, and is bounded with its derivative (i.e. (i) is satisfied). Let us mention however that $V''(x) = O(\sqrt{x})$ as $x \to \infty$.

Remark

Notice that Wigner-Von Neuwmann example (cf. [23]) provides a potential satisfying all the requirement of the problem (ii)'. We give, however, another construction which can be extended to construct a potential V (bounded with its derivative) for which the one dimensional Stark operator has a discrete sequence of eigenvalues (cf. Theorem 3). In that case, if $d \geq 2$ then V can be chosen bounded with its partial derivatives and leading to a dense point spectrum on all $\mathbb{R}$.

Construction of q. To construct q we apply the Prüfer transformation by setting

$$\begin{cases} w = R\cos\phi \\ w' = R\sin\phi, \end{cases}$$

Straightforward computations give the equations

$$\begin{cases} \frac{R'}{R} = & \frac{1}{2} q \sin 2\phi \\ \phi' = & -1 + q\cos^2\phi, \end{cases} \tag{V.12}$$

or equivalently

$$\begin{cases} R^2 = & C \exp \int_1^\xi q \sin 2\phi dt \\ \phi' = & -1 + q\cos^2\phi, \end{cases} \tag{V.13}$$

The potential q will have the form

$$q(x) = \sum_{k\geq 1} q_k j(\frac{\xi - \xi_k}{\Delta})$$

where ξ_k and q_k are two adequate sequences we have to construct, while $\Delta > 0$ will be choosed sufficiently small and $j \in C^\infty((0,1)), j(x) \geq 0, \int j(x) = 1$. Assume that

$$q \sin 2\phi \leq -\frac{1}{2} q,$$

which is ensured if $\phi(x) \approx 3\pi/4$ on the interval $[\xi_k, \xi_k + \Delta]$. (Remark that $q = 0$ on $[\xi_k + \Delta, \xi_{k+1}]$). Then

$$\begin{aligned} \int_1^\infty R^2 \frac{d\xi}{\xi^{2/3}} &= C \int_1^\infty [\exp \int_1^\xi q \sin 2\phi dt] \frac{d\xi}{\xi^{2/3}} \\ &\leq C_1 + C_2 \sum_{k>0} \int_{\xi_k}^{\xi_{k+1}} \exp(\int_1^\xi q \sin 2\phi dt) \frac{d\xi}{\xi^{2/3}} \\ &\leq C_1 + C_2 \sum_{k>0} \int_{\xi_k}^{\xi_{k+1}} \exp(-\frac{1}{2} \int_1^\xi q dt) \frac{d\xi}{\xi^{2/3}} \\ &\leq C_1 + C_2 \sum_{k>0} [\xi_{k+1} - \xi_k] \exp(-\frac{1}{2} \int_1^{\xi_k} q dt) \\ &\leq C_1 + C_2 \sum_{k>0} [\xi_{k+1} - \xi_k] \exp -[\frac{\Delta}{2} \sum_{j=1}^{j=k} q_j]. \end{aligned}$$

Then if

$$\begin{cases} \xi_{k+1} - \xi_k = O(1) \\ q_k = \frac{C}{\xi_k} \end{cases}, \tag{V.14}$$

where C is sufficiently large constant, then the right member of the last inequality is convergent, which means that w lies in $L^2((1,\infty), \xi^{\frac{-2}{3}} d\xi)$. Moreover, by construction the potential q satisfies

$$q^{(m)} = O(1/\xi), \quad \text{as } x \longrightarrow \infty, \quad \forall m \geq 0.$$

Thus, it is sufficient to construct by induction a sequence ξ_k such that: $\phi(\xi_k) = 3\pi/4(\text{mod } \pi)$ and $q(\xi) = q_k j(\frac{\xi-\xi_k}{\Delta})$ on the interval $[\xi_k, \xi_k + \Delta]$; and $q(\xi) = 0$ on the interval $[\xi_k + \Delta, \xi_{k+1}]$.
Assume that ξ_k is constructed, and let us set $q(\xi) = 0$ for $\xi > \xi_k + \Delta$. Integrating the equation (V.13) between $\xi_k + \Delta < \xi$ we get

$$\phi(\xi) = -\xi + \xi_k + \Delta + \phi(\xi_k + \Delta).$$

Let us choose ξ_{k+1} as the nearest point on the right of ξ_k such that $\phi(\xi_{k+1}) = 3\pi/4(\text{mod } \pi)$. We also have,

$$\xi_{k+1} - \xi_k = \phi(\xi_k + \Delta) - \phi(\xi_{k+1}) - \Delta = O(1).$$

Let us integrate (V.13) between ξ_k and $\xi \leq \xi_k + \Delta$ to get

$$\mid \phi(\xi) - \phi(\xi_k) \mid \leq \Delta + \int_{\xi_k}^{\xi} q dt \leq \Delta + \Delta q_k.$$

For Δ sufficiently small we ensure the fact that $\phi(\xi) \approx 3\pi/4(\text{mod } \pi)$ on $[\xi_k, \xi_k + \Delta]$. The construction is now completed.

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References

[1] Aizenman, M.,*Localization at Weak Disorder: Some Elementary Bounds*, Rev. Math. Phys. 6, 1163 (1994).

[2] Carmona, R.; Lacroix, J. *Spectral theory of random Schrdinger operators. Probability and its Applications.* Birkhuser Boston, Inc., Boston, MA, 1990.

[3] Klein, A., *Spreading of wave packets in the Anderson model on the Bethe lattice.* Comm. Math. Phys. 177 (1996), no. 3, 755–773.

[4] Klein, A. *Extended states in the Anderson model on the Bethe lattice.* Adv. Math. 133 (1998), no. 1, 163–184.

[5] Titchmarsh, E.C., *Eigenfunction expansions, I and II*, Oxford, 1958.

[6] Walter, *Absolute continuity of the essential spectrum of* $-d^2/dx^2 + q(x)$ *without monotony of* q, Math. Z. 129(1) (1972), 83-94.

[7] Bentosela, F., Carmona, R., Duclos, P., Simon, B., Souillard, B., Weder, R., *Schrödinger Operator with an Electric Field and Random or Deterministic Potentials,* Commun. Math. Phys. **88**(1983), 387-397.

[8] Hislop, P.D.; Nakamura, S., *Stark Hamiltonian with unbounded random potentials.* Rev. Math. Phys. 2 (1990), no. 4, 479–494.

[9] Naboko, S. N., Pushnitskii, A. B., *Point Spectrum on a continuous spectrum for Weakly Perturbed Stark Type Operators*, Functional Analysis and its Applications, **29** N°4, (1995), 248-257.

[10] Delyon, F.; Simon, B. and Souillard, B. *From power pure point to continuous spectrum in disordered systems.* Ann. Inst. H. Poincaré Phys. Thór. 42 (1985), no. 3, 283–309.

[11] Exner, P., *The absence of the absolutely continuous spectrum for* δ' *Wannier-Stark ladders,* J. Math. Phys. 36(9)(1995), 4561-4570.

[12] Sahbani, J., *Propagation theorems for some classes of pseudo-differential operators.* J. Math. Anal. Appl. 211 (1997), no. 2, 481–497.

[13] Kiselev, A. *Absolutely continuous spectrum of perturbed Stark operators.* Trans. Amer. Math. Soc. 352 (2000), no. 1, 243–256.

[14] Avron, Y., Herbst, I., *Spectral and scattering theory of Schrdinger operators related to the Stark effect.* Comm. Math. Phys. 52 (1977), no. 3, 239–254.

[15] Jensen, A. *Scattering Theory for Stark Hamiltonians*, Proc. Indian Acad. Sci. (Math. Sci.) Vol 104, n° **4** (1994) 599-651.

[16] Jensen, A. *Commutator methods and asymptotic completeness for a new class of Stark effect Hamiltonians.* Differential equations and mathematical physics (Birmingham, Ala., 1986), 215–221, Lecture Notes in Math., 1285, Springer, Berlin-New York, 1987.

[17] Amrein, W., Boutet de Monvel, A., Georgescu, V., *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhauser, Progress in Math. Ser., nr. 135 1996.

[18] Boutet de Monvel, A.; Georgescu, V.; Sahbani, J., Boundary values of regular resolvent families. Helv. Phys. Acta 71 (1998), no. 5, 518–553.

[19] Sahbani, J., *The Conjugate Operator Method For Locally Regular Hamiltonians*, J. Operator Theory 38 (1997), no. 2, 297–322.

[20] Georgescu, V., Gerard, C. *On the Virial Theorem in Quantum Mechanics*, Comm. Math. Phys. 208 (1999), no. 2, 275–281.

[21] Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B. *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry*, Springer, Berlin 1987.

[22] Simon, B., *Some Schrődinger operators with dense point spectrum.* Proc. Amer. Math. Soc. 125 (1997), no. 1, 203–208.

[23] M. Reed and B. Simon, Methods of Modern Mathematical Physics, IV. Analysis of Operators, Academic Press, New York, 1978.