PERTURBATIONS OF MAGNETIC SCHRÖDINGER OPERATORS

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Abstract. We study Schrödinger operators $H(a, V) := (P - a)^2 + V$ acting in $L^2(\mathbb{R}^3)$. We assume that the magnetic field $B = \text{rot}a$ may be decomposed as $B = B^0 + \tilde{B}$, where B^0 is a very general field having constant direction. The perturbations $\tilde{B}$ and V will be small in a certain sense in the direction of B^0, but in the orthogonal plane they may even grow for certain fields B^0. Commutator methods are used to derive spectral properties of $H(a, V)$.

1. Introduction

The present article is dedicated to the analysis of partial differential operators of the form

$$H(a, V) := (P - a)^2 + V, \tag{1-1}$$

acting in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^3)$. Here $P := -i\nabla$ and $a : \mathbb{R}^3 \to \mathbb{R}^3$, $V : \mathbb{R}^3 \to \mathbb{R}$ are functions which will be called, respectively, *the vector* and *the scalar potential.* The main features of this type of operators come from the fact that the vector potential a generates a magnetic field $B = \text{rot}a$, satisfying the Maxwell equation $divB = 0$. In fact, (1.1) is the quantum Hamiltonian describing in suitable units a non-relativistic particle placed in the magnetic field B and in the electric potential V. Many important properties of $H(a, V)$, for example the spectral ones, cannot be deduced by general principles from the behaviour of the components a_j; the physical quantity is the magnetic field B and this comes into play in a specific way.

Our purpose is to perform the spectral analysis of $H(a, V)$ in a framework which we now expose very briefly. Let us assume that we have a decomposition $B = B^0 + \tilde{B}$ and that the magnetic field B^0 points in a constant direction, which we take with no loss of generality to be the direction of the third axis z. The Maxwell equation $divB^0 = 0$ makes B^0 z-independent: $B^0(x, y, z) = (0, 0, B_z^0(x, y))$ and it is possible

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to choose a vector potential a^0 of the form $a^0(x, y, z) = (a^0_x(x, y), a^0_y(x, y), 0)$ such that $B^0_z = \partial_x a^0_y - \partial_y a^0_x$. Then the operator $H(a^0, 0)$ will be factorized

$$H(a^0, 0) = [(P_x - a^0_x)^2 + (P_y - a^0_y)^2] \otimes 1 + 1 \otimes P_z^2 \tag{1-2}$$

with respect to the tensor decomposition $\mathcal{H} = \mathcal{H}_{xy} \otimes \mathcal{H}_z$, where $\mathcal{H}_{xy} = L^2(\mathbb{R}^2_{xy})$ and $\mathcal{H}_z = L^2(\mathbb{R}_z)$. It follows that its spectrum will be purely absolutely continuous, coinciding with the semiaxis $[\Sigma^0, \infty)$; we set $\Sigma^0 := \inf\sigma^0$, where σ^0 is the spectrum of the operator $H^0_{xy} := (P_x - a^0_x)^2 + (P_y - a^0_y)^2$, acting in $\mathcal{H}_{xy}$.

We are looking for conditions upon $\tilde{B}$ and V such that, outside σ^0, $H(a, V)$ has no singular spectrum and its point spectrum be locally finite. Of course, with suitable hypothesis on B^0_z, one can also get informations about the nature of the spectrum of $H(a, V)$ inside σ^0; but our intention is to leave B^0_z as general as possible. It follows that our analysis is interesting only in situations when there are consistent parts of the spectrum of $H(a, V)$ outside σ^0. Now, a minimal guess would be that $\tilde{B}$ and V must be small at infinity in some suitable sense in order to treat $H(a, V)$ as a perturbation of $H(a^0, 0)$. The main interest of our paper is the fact that we are able to impose smallness conditions only in the direction z of the field B^0. In orthogonal directions $\tilde{B}$ and V may even grow if they are controled by B^0_z in a suitable sense and if some restrictions on the derivatives are imposed. The exact statements are given in the next section, which is devoted to the description of our Hamiltonian and to the exposition of the main result.

We mention that the phenomenon described above was put into evidence in [2] and in [14] for the particular case when B^0_z is constant and $\tilde{B}$ is null. [14] also treats the special situation of an axially symmetric V. Then V may even have a certain exponential growth when $|(x, y)|$ tends to infinity; it seems that this kind of result is not available when B^0_z is general.

Technically, our work relies on the commutator method of E. Mourre ([12], [1]). We shall use a rather rough version, which we describe briefly in Section 3. As a conjugate operator we choose the generator of dilations in the direction z, which was also used in [8] for the study of long-range (electric) perturbations of a Hamiltonian with constant magnetic field.

The Section 4 is devoted to the proofs. The methods of Section 3 are applied by taking care not to loose the anisotropy permitted by the function B^0_z. This is reflected in the introduction of some extra spaces which take B^0_z into account. A crucial point is to prove the compactness of $[H(a, V) + i]^{-1} - [H(a^0, 0) + i]^{-1}$ without asking smallness conditions on $H(a, V) - H(a^0, 0)$.

2. Perturbed Magnetic Hamiltonians

We begin by describing our magnetic Hamiltonians. The technical assertions we make along the road are quite standard and can be found in a much more general form in [2], [3] or [11].

The main data will be a variable magnetic field $B : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies the Maxwell equation $\mathrm{div}B := \partial_x B_x + \partial_y B_y + \partial_z B_z = 0$ and which we consider, for simplicity, of class C^∞. We assume that it is the sum of two C^∞ components $B^0, \tilde{B} : \mathbb{R}^3 \to \mathbb{R}^3$ and that B^0 is parallel to the third axis. It follows that it depends only on the variables x and y, hence it has the form $B^0(x,y,z) = (0,0,B^0_z(x,y))$. Magnetic fields can be deduced from vector potentials $a : \mathbb{R}^3 \to \mathbb{R}^3$ by the formula $B = \mathrm{rot} a$. Since the ambiguity of the choice of a reflects itself in a harmless unitary equivalence of the associated Hamiltonians, we shall work with *the transversal gauge* (see [15] for instance):

$$a(r) = \int_0^1 ds\ B(sr) \wedge sr, \qquad r := (x,y,z) \in \mathbb{R}^3 \tag{2-1}$$

which is also C^∞. Similar expressions are given for a^0 (in terms of B^0) and for $\tilde{a}$ (in terms of $\tilde{B}$). Strictly speaking, the transversal gauge is really useful only for $\tilde{B}$; for B^0 any C^∞ vector potential of the form $a^0(x,y,z) = (a^0_x(x,y), a^0_y(x,y), 0)$ will do.

Our magnetic Schrödinger operator is given formally by

$$H(a,0) = \Pi_x^2 + \Pi_y^2 + \Pi_z^2 = \Pi^2, \tag{2-2}$$

where $\Pi_j := P_j - a_j$; we set $P_j := -i\partial_j$ and denote by the same symbol functions and multiplication operators. More precisely, we introduce first on $\mathcal{D} := \mathcal{C}_0^\infty(\mathbb{R}^3)$ the differential expression $\sum_{j=x,y,z}(P_j - a_j)^2$, which is an essentially self-adjoint operator. Its closure is a positive operator denoted by $H(a,0)$. We also denote by Π_j the closure of $P_j - a_j$ defined on $\mathcal{D}$. Its domain is composed of the vectors $f \in \mathcal{H}$ such that the distribution $P_j f - a_j f$ is also in $\mathcal{H}$. One has the following descriptions of the form domain of $H(a,0)$:

$$\mathcal{G} = \{f \in \mathcal{H} \mid (P_j - a_j)f \in \mathcal{H}, \ \ j = x,y,z\}$$

and of the operator domain of $H(a,0)$:

$$\mathcal{K} = \{f \in \mathcal{G} \mid \sum_{j=x,y,z} (P_j - a_j)^2 f \in \mathcal{H}\}.$$

We endow $\mathcal{G}$ and $\mathcal{K}$ with the usual graph norms and denote by $\mathcal{G}^*$ and $\mathcal{K}^*$ their topological antiduals. All these spaces are Hilbert spaces contained in $\mathcal{D}^*$ (the distributions) and having $\mathcal{D}$ as a dense subspace. If we identify $\mathcal{H}$ with $\mathcal{H}^*$ by the Riesz isomorphims we get the following continuous, dense embeddings:

$$\mathcal{K} \hookrightarrow \mathcal{G} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{G}^* \hookrightarrow \mathcal{K}^*.$$

The operators Π_j are bounded from $\mathcal{G}$ to $\mathcal{H}$ and by extension they may be considered bounded operators between $\mathcal{H}$ and $\mathcal{G}^*$: $\Pi_j \in B(\mathcal{G},\mathcal{H}) \cap B(\mathcal{H},\mathcal{G}^*)$. In a similar way we associate to the vector potential a^0 the objects Π^0_j (closure of $P_j - a^0_j$), $H(a^0,0) := (\Pi^0)^2$, $\mathcal{K}^0$, $\mathcal{G}^0$, $(\mathcal{G}^0)^*$ and $(\mathcal{K}^0)^*$. One has

$$H(a^0,0) = H^0_{xy} \otimes 1 + 1 \otimes P_z^2.$$

The operator $H^0_{xy} := (\Pi^0_x)^2 + (\Pi^0_y)^2$ is assumed here (by a slight abuse of notation) to act in the Hilbert space $\mathcal{H}_{xy} = L^2(\mathbb{R}^2_{xy})$.

Let us put $\sigma^0 := \sigma(H^0_{xy})$ = the spectrum of H^0_{xy} and $\Sigma^0 = \inf\sigma^0$. Then the spectrum of $H(a^0, 0)$ is purely absolutely continuous and coincides with $[\Sigma^0, \infty)$.

The object we study is the perturbed magnetic Hamiltonian

$$H(a;V) := H(a;0) + V, \tag{2-3}$$

with $H(a;0)$ given by (2-2) and V the operator of multiplication by a function $V : \mathbb{R}^3 \to \mathbb{R}$. Our assumptions will imply that $H(a;V)$ is a self-adjoint operator having form domain $\mathcal{G}$, which also coincides with $\mathcal{G}^0$ (see Lemma 4.2).

We shall list now our hypothesis on the functions B^0, $\tilde{B}$ and V, which we assume C^∞. Unimportant constants are sometimes denoted by the same letter even when they may be different. We recall that for $\tilde{B}$ we have chosen the transversal gauge:

$$\tilde{a}(r) = \int_0^1 ds \tilde{B}(sr) \wedge sr.$$

Hypothesis 2.1. $(B^0/1)$ *B^0_z is lower bounded. We shall fix a constant β such that $\mathbb{B} := B^0_z + \beta \geq 1$.*

$(B^0/2)$ $|\partial_j B^0_z| \leq C\mathbb{B}$, *j=x,y.*

$(\tilde{a}/1)$ $\tilde{a}_j \mathbb{B}^{-1/2} = o(1)$, $\quad j = x, y, z$.

$(\tilde{a}/2)$ $|z\partial_z \tilde{a}_j| + |z^2 \partial_z^2 \tilde{a}_j| \leq C\mathbb{B}^{1/2}$.

$(V/1)$ $V\mathbb{B}^{-1} = o(1)$.

$(V/2)$ $|z\partial_z V| + |z^2\partial_z^2 V| \leq C\mathbb{B}$.

We are now in a position to formulate the main result.

Theorem 2.2. *Assume Hypothesis 2.1.*

(a) The point spectrum of $H(a,V)$ in $\mathbb{R} \setminus \sigma^0$ is composed of finitely degenerated eigenvalues with no point of accumulation in $\mathbb{R} \setminus \sigma^0$.

(b) The singular continuous spectrum of $H(a,V)$ in $\mathbb{R} \setminus \sigma^0$ is void.

We note that some other versions are easily available. The excessive smoothness requirements can be removed. With more severe restrictions on $\tilde{B}$ and V, the hypothesis $(B^0/1)$ and $(B^0/2)$ are no longer needed and B^0_z is arbitrary.

Remark 2.3. The hypothesis $(\tilde{a}/1)$ and $(\tilde{a}/2)$ on $\tilde{B}$ are an artifact of our approach based on the the transversal gauge. It seems difficult in this framework to give a completely intrinsic treatement. Notice, however, that they are explicit and not too restrictive. They may be simplified under different assumptions (in order to get rid of the integral, for instance). We prefer to sketch a tipical example. Assume that at infinity $B^0_z(x,y) \sim e^{C(|x|+|y|)}$ (a more radical growth is forbidden by $(B^0/2)$). Then *all* the components $\tilde{B}_x$, $\tilde{B}_y$ and $\tilde{B}_z$ may have (milder) exponential growth in the directions x, y as soon as in the direction z they behave convenably. So that the total magnetic field B can have large (x,y) components far away of the z axis and is quite far of having a fixed direction. One may discuss analogously the polynomial case.

Remark 2.4. Let us turn things around and suppose that we start with a Schrödinger operator $P^2 + V$ with no magnetic field. We choose V smooth, well-behaved in the z direction, but depending very wildly of x and y. For instance, it suffices to ask $(1 + |z|)^\varepsilon |V| + |z\partial_z V| + |z^2\partial_z^2 V| \leq C' e^{C(|x|+|y|)}$, so that V is allowed to grow and to oscillate very strongly in the variables (x, y). It is clear that in general there is no control on the nature of the spectrum. But as soon as some suitable magnetic field pointing in the direction z is turned on (see (V/1) and (V/2)) the spectrum turns gentle: no singular continuous component is possible and eigenvalues can at most accumulate at a discrete set of thresholds. And a suitable extra magnetic field which no longer has direction z leaves this picture unchanged, as discussed in the previous remark.

Examples Since we obtain spectral results only outside σ^0, let us give simple situations when we posses some information about it.

(1) (cf. [7], [4], [11]) Assume that $|B_z^0(x,y)| \to \infty$ when $|(x,y)| \to \infty$. Then H_{xy}^0 has a compact resolvent, hence a purely discrete spectrum. So σ^0 is locally finite, hence there is "plenty of space" outside it. Note that the divergence at infinity of $|B_z^0|$ is by no means necessary. For example if $a_x(x,y) = xy^2$, $a_y(x,y) = 0$, one has $B_z^0(x,y) = -2xy$ but H_{xy}^0 still has a compact resolvent.

(2) (cf. [3], [6], [9]) Assume that B_z^0 depends only on x. Then σ^0 consists of a sequence of bands. In advantageous situations, at least a finite number of gaps will be open. Suppose for example that $B_z^0(x,y) = b(x)$ and that b increases from $b_- := \lim_{x\to-\infty} b(x) > 0$ to $b+ := \lim_{x\to+\infty} b(x)$. Then one has

$$\sigma^0 = \cup_{n=0}^{\infty}[(2n+1)b_-, (2n+1)b_+].$$

If the difference $b_+ - b_-$ is sufficiently small, there are indeed some open gaps in which our Theorem 2.2 can be applied. In the single case $b_- = b_+ := b^0$ all the gaps are open and one has the (infinitely degenerated) Landau levels for the constant magnetic field: $\sigma^0 = (2\mathbb{N}+1)b^0$. If $b_- \neq b_+$ (and also in more general situations) it was shown that in fact H_{xy}^0 is purely absolutely continuous: in [9] a suitable salfadjoint operator A_{xy} was proven to be strictly locally conjugate to H_{xy}^0 outside a discrete set. This might be combined with the technique of the present article to extend the spectral analysis of $H(a, V)$ to the set σ^0.

(3) (cf. [11]) If B_z^0 is periodic and the flux through the unit cell divided by 2π is a rational number, then again the spectrum of H_{xy}^0 has a band structure. [5] put into evidence situations where many gaps are open.

(4) (cf. [3], [10]) If $|B_z^0(x,y)| \to 0$ when $|(x,y)| \to \infty$, then $\sigma^0 = [0, \infty)$. Our treatement gives nothing interesting in this case.

3. THE METHOD OF THE LOCALLY CONJUGATE OPERATOR

We call "the method of the locally conjugate operator" a theory which converts *a Mourre estimate* for the self-adjoint operator H

$$E_H(J)i[H,A]E_H(J) \geq aE_H(J) + K \tag{3-1}$$

($a > 0$, K compact), supplied by some technical assumptions, into qualitative informations on the spectrum of H. The initial result belongs to [12]; see also [1] for a refined version. We shall give here only a brief sketch, insisting only on the facts which will be used in Section 4. Technically, the method consists in deriving a *limiting absorption principle*, i.e. an estimate on the resolvent $(H-z)^{-1}$ which is uniform when the complex number z approaches the real set J, eigenvalues excepted. Operators which are H-smooth locally on J are also available and this implies results on the existence and completeness of some suitable wave operators. For space reasons we do not touch upon these matters here.

Let $\mathcal{G}$ be a Hilbert space continuously and densely embedded in another Hilbert space $\mathcal{H}$. By identifying $\mathcal{H}$ with its topological antidual, we may embed it continuously and densely in $\mathcal{G}^*$, the topological antidual of $\mathcal{G}$, endowed with the usual dual topology (it is also a Hilbert space).

Assume that A is a self-adjoint operator in $\mathcal{H}$ such that $e^{itA}\mathcal{G} \subset \mathcal{G}$ for all $t \in \mathbb{R}$. It is a standard fact that we get C_0-groups in $\mathcal{G}$ and in $\mathcal{G}^*$ for which we use the same notation $\{e^{itA}\}$. For S in $B(\mathcal{G}, \mathcal{G}^*)$ we set $\mathcal{W}_t S := e^{-itA} S e^{itA} \in B(\mathcal{G}, \mathcal{G}^*)$.

Definition 3.1. *We say that S belongs to $C^k(A; \mathcal{G}, \mathcal{G}^*)$ if the function $\mathbb{R} \ni t \mapsto \mathcal{W}_t S \in B(\mathcal{G}, \mathcal{G}^*)$ is strongly of class C^k ($k \in \mathbb{N} \cup \{+\infty\}$).*

Slightly smaller classes C^k_u are obtained by replacing in Definition 3.1 the strong topology on $B(\mathcal{G}, \mathcal{G}^*)$ by the uniform one. The class C^1_u will be needed in Proposition 3.4. But in our applications we shall ask a C^2 condition and this implies C^1_u, so the conclusion of Proposition 3.4. will be available.

We shall constantly assume that $\mathcal{G}$ is the form domain of the self-adjoint operator H acting in $\mathcal{H}$; H extends to an element of $B(\mathcal{G}, \mathcal{G}^*)$. By assuming that H belongs to $C^1(A; \mathcal{G}, \mathcal{G}^*)$ we give a rigorous sense to the commutator $i[H, A] \in B(\mathcal{G}, \mathcal{G}^*)$ as the strong derivative of $\mathbb{R} \ni t \mapsto \mathcal{W}_t H \in B(\mathcal{G}, \mathcal{G}^*)$ taken in $t = 0$. The next definition follows [1] and gives the sharp value of the constant a in the Mourre estimate (3-1) (with or without the compact term K) when J is localized near $\lambda \in \mathbb{R}$ and singles out the case when it is strictly positive. For $\varepsilon > 0$ we set $E(\lambda; \varepsilon) := E_H(\lambda - \varepsilon, \lambda + \varepsilon)$; here E_H is the spectral measure of H.

Definition 3.2. *Assume that $H \in C^1(A; \mathcal{G}, \mathcal{G}^*)$.*

(i) $\rho^A_H(\lambda) := \sup\{a \in \mathbb{R} \mid \exists \varepsilon > 0 \text{ s.t. } E(\lambda; \varepsilon) i[H, A] E(\lambda; \varepsilon) \geq a E(\lambda; \varepsilon)\}$.

(ii) $\tilde{\rho}^A_H(\lambda) := \sup\{a \in \mathbb{R} \mid \exists \varepsilon > 0, \exists K \text{ s.t } E(\lambda; \varepsilon) i[H, A] E(\lambda; \varepsilon) \geq a E(\lambda; \varepsilon) + K\}$.

(iii) We say that A is locally strictly conjugate *to H on the real, open set I if $\rho^A_H(\lambda) > 0$ for all $\lambda \in I$.*

(iv) We say that A is locally conjugate *to H on the real, open set I if $\tilde{\rho}^A_H(\lambda) > 0$ for all $\lambda \in I$.*

Remark 3.3. *It is easy to see that the spectrum of H is exactly the set on which ρ^A_H takes finite values. The function $\tilde{\rho}^A_H$ takes finite values exactly on the essential spectrum of H. In the complement of the point spectrum the two functions coincide.*

The great advantage the function $\tilde{\rho}$ has upon the function ρ is its stability under generalized compact perturbations:

Proposition 3.4. *If $H, H_0 \in C^1_u(A; \mathcal{G}, \mathcal{G}^*)$ and $(H+i)^{-1} - (H_0+i)^{-1}$ is a compact operator, then $\tilde{\rho}^A_H = \tilde{\rho}^A_{H_0}$. In particular, A is locally conjugate on I to H and to H_0 simultaneously.*

The next result gives the effect of the Mourre estimate on the qualitative spectral theory of H.

Proposition 3.5. *Assume that $H \in C^1(A; \mathcal{G}, \mathcal{G}^*)$ and that A is locally conjugate to H on the open set I. Then the point spectrum of H in I is composed of* *eigenvalues of finite multiplicity which does not accumulate inside* I. *If, in addition,* $H \in C^2(A; \mathcal{G}, \mathcal{G}^*)$, *then* H *has no singular continuous spectrum in* I.

4. Proofs

We begin by a short description of this Section. First, a new Hilbert space $\mathcal{B}$ is introduced as the domain in $\mathcal{H}$ of the operator $\mathbb{B}^{1/2}$. Lemma 4.1 compares it with the form domain $\mathcal{G}^0$ of $H(a^0, 0)$, which is shown in Lemma 4.2 to coincide with $\mathcal{G}$, the form domain of the self-adjoint operator $H(a, V)$. In Lemma 4.3 we prove the compactness of the difference of the resolvents. Proposition 4.4 shows that the generator of the dilations in the direction z, denoted by A, is strictly conjugate to the unperturbed operator $H(a^0, 0)$ on $\mathbb{R} \setminus \sigma^0$. Finally, in Proposition 4.5 we show that A is also conjugated to the perturbed Hamiltonian $H(a, V)$ on the same open set. This has Theorem 2.2 as an immediate consequence. For simplicity, from now on, we consider tacitly the whole Hypothesis 2.1 verified. It will be easy to see at every spot which are the assumptions actually used. The notations are those introduced in Section 2.

We shall need an extra space, defined in terms of the magnetic field. It will be nontrivial and useful as soon as the function $\mathbb{B}$ diverges at infinity at least in some direction. We denote it by $\mathcal{B}$ and it is the closure of $\mathcal{D} = C_0^\infty(\mathbb{R}^3)$ in the norm $\| f \|_{\mathcal{B}} := < f, \mathbb{B} f >^{1/2}$. The spaces $\mathcal{B}$ and $\mathcal{B}^*$ are born to fulfil: $\mathbb{B} \in B(\mathcal{B}, \mathcal{B}^*)$ and $\mathbb{B}^{1/2} \in B(\mathcal{B}, \mathcal{H}) \cap B(\mathcal{H}, \mathcal{B}^*)$.

Lemma 4.1. *We have the following continuous, dense embeddings:*

$$\mathcal{G}^0 \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{B}^* \hookrightarrow (\mathcal{G}^0)^*.$$

Proof. One has the following form inequality on $C_0^\infty(\mathbb{R}^2_{xy})$:

$$0 \leq (\Pi_x^0 - i\Pi_y^0)(\Pi_x^0 + i\Pi_y^0) = (\Pi_x^0)^2 + (\Pi_y^0)^2 + i[\Pi_x^0, \Pi_y^0] = H_{xy}^0 - B_z^0.$$

This implies $\mathcal{G}^0 \hookrightarrow \mathcal{B}$.

$\mathcal{B} \hookrightarrow \mathcal{H}$ is clear, since $\mathbb{B} \geq 1$.

The remaining assertions follow by duality.

Lemma 4.2. *The operator* $H(a, V)$ *is essentially self-adjoint on* $\mathcal{D}$. *Its form domain is* $\mathcal{G}$, *which coincides with* $\mathcal{G}^0$.

Proof. The essential self-adjointness of $H(a, V)$ on $\mathcal{D}$ is well-known under much more general assumptions on a and V than ours, cf. [2] for instance.

We recall that $\mathcal{G}^0$ and $\mathcal{G}$ are (respectively) the form domains of $H(a^0, 0)$ and $H(a, 0)$. On $\mathcal{D}$ we may write

$$H(a, 0) = H(a^0, 0) + U,$$

where $U := -\tilde{a} \cdot \Pi^0 - \Pi^0 \cdot \tilde{a} + \tilde{a}^2$ and

$$H(a, V) = H(a^0, 0) + U + V. \tag{4-1}$$

To get our result we only have to show that U and V define on $\mathcal{G}^0$ forms that are infinitesimally small with respect to the form of $H(a^0,0)$. Taking into account conditions $(\tilde{a}/1)$, $(V/1)$ and Lemma 4.1 we see that the operators $\tilde{a}_j$, $|V|^{1/2}$ and Π^0_j are bounded from $\mathcal{G}^0$ to $\mathcal{H}$. Then it will be enough to show for a continuous function $G:\mathbb{R}^3\to\mathbb{R}$ satisfying $G\mathbb{B}^{-1/2}=o(1)$ that for an arbitrary $\varepsilon>0$ there exists a constant C such that

$$\parallel Gf \parallel \leq \ \varepsilon < f, H(a^0,0)f >^{1/2} + C \parallel f \parallel$$

for any $f\in\mathcal{G}^0$. For $R>0$ we denote by χ_R the characteristic function of the ball $\{r\in\mathbb{R}^3 \mid |r|\leq R\}$ and by $\chi_R^\perp$ the function $1-\chi_R$. We have

$$\parallel Gf \parallel \leq \parallel \chi_R^\perp G f \parallel + \parallel \chi_R G f \parallel \leq \parallel \chi_R^\perp G\mathbb{B}^{-1/2} \parallel \cdot \parallel \mathbb{B}^{1/2} f \parallel + \parallel \chi_R G \parallel \cdot \parallel f \parallel \leq$$

$$\leq \ \sup_{|r|>R}|G\mathbb{B}^{-1/2}| \cdot < f, H(a^0,0)f >^{1/2} + \sup_{|r|\leq R}|G(r)| \cdot \parallel f \parallel$$

and this is the estimate we needed if we take R large enough.

Remark. We have shown that $\mathcal{G}^0=\cap_{j=x,y,z}D(\Pi^0_j)$ and $\mathcal{G}=\cap_{j=x,y,z}D(\Pi_j)$ are equal (and have equivalent graph norms). This does not imply that for a fixed j the domains $D(\Pi^0_j)$ and $D(\Pi_j)$ should be equal. If $\mathbb{B}$ is large at infinity, it may happen for instance that $\tilde{a}_z$ is not a small perturbation of $\Pi^0_z=P_z$. Equality is only restaured by taking intersections.

Lemma 4.3. *$[H(a,V)+i]^{-1}-[H(a^0,0)+i]^{-1}$ is a compact operator.*

Proof. (i) We show first that for any continuous function $F:\mathbb{R}^3\to\mathbb{R}$ such that $F(r)\to 0$ when $|r|\to\infty$ and for any $\theta>0$, the operator $F[H(a^0,0)+1]^{-\theta}$ is compact. In fact, the same is true for a^0 replaced by a. Only the case $\theta=1/2$ will be needed in the sequel.

For two operators $S,T\in B(\mathcal{H})$ we write $S\preccurlyeq T$ if for any $f\in\mathcal{H}$ one has $|Sf|\leq T|f|$ almost everywhere. The Kato-Simon inequality ([3]) asserts that $e^{-tH(a^0,0)}\preccurlyeq e^{-tP^2}$ for any $t\geq 0$. By the formula valid for positive operators

$$(A+1)^{-\theta}=1/\Gamma(\theta)\int_0^\infty t^{\theta-1}e^{-t(A+1)}dt$$

one gets $[H(a^0,0)+1]^{-\theta}\preccurlyeq(P^2+1)^{-\theta}$. In [13] it is shown that the algebra of compact operators is hereditary with respect to the relation $\preccurlyeq$, in the sense that $S\preccurlyeq T$ =compact implies S =compact. Also, the relation $\preccurlyeq$ is preserved if we multiply it to the left by the positive function $|F|$. If we combine all these with the well-known fact that $F(P^2+1)^{-\theta}$ is compact for F decaying at infinity, we get the result.

(ii) Since both operators $H(a^0,0)$ and $H(a,V)$ are bounded from $\mathcal{G}$ to $\mathcal{G}^*$, the identity of the resolvent

$$[H(a,V)+i]^{-1}-[H(a^0,0)+i]^{-1}=-[H(a,V)+i]^{-1}(U+V)[H(a^0,0)+i]^{-1}$$

is valid in $B(\mathcal{G}^*,\mathcal{G})$ (see (4-1)).

We need to show that the right hand side is a compact operator in $\mathcal{H}$. By taking into account the fact that $\Pi^0_j \in B(\mathcal{G},\mathcal{H}) \cap B(\mathcal{H},\mathcal{G}^*)$, the explicit form of U and the conditions $(\tilde{B}/1)$ and $(V/2)$, we see that it is enough to prove that $G[H(a^0,0)+i]^{-1}$ and $G[H(a,V)+i]^{-1}$ are compact operators in $\mathcal{H}$ if $G\mathbb{B}^{-1/2} = o(1)$. Both proofs goes the same way.

We set simply $H_0 := H(a^0;0)$, $R_0 := (H_0+1)^{-1}$ and show that GR_0 is compact. We write

$$GR_0 = G\mathbb{B}^{-1/2}\{R_0\mathbb{B}^{1/2} + [\mathbb{B}^{1/2}, R_0]\}.$$

The first term is compact, since $G\mathbb{B}^{-1/2}R_0^{1/2}$ is compact (by (i)) and $R_0^{1/2}\mathbb{B}^{1/2}$ is bounded (by Lemma 4.2). For the second term we write

$$G\mathbb{B}^{-1/2}[\mathbb{B}^{1/2}, R_0] = -G\mathbb{B}^{-1/2}R_0[\mathbb{B}^{1/2}, H_0]R_0$$

and we still have to prove that the operator

$$R_0^{1/2}[\mathbb{B}^{1/2}, H_0]R_0 = \sum_{j=x,y} R_0^{1/2}\{\partial_j\mathbb{B}^{1/2}\Pi^0_j + \Pi^0_j\partial_j\mathbb{B}^{1/2}\}R_0$$

is bounded. But $\partial_j\mathbb{B}^{1/2} = (\partial_j\mathbb{B}\cdot\mathbb{B}^{-1})\mathbb{B}^{1/2}$. By using $(B^0/2)$ and Lemma 4.1 one finishes the proof.

We prove now a Mourre estimate for $H(a^0;0)$. For $t \in \mathbb{R}$, $f \in \mathcal{H}$ and $(x,y,z) \in \mathbb{R}^3$ we set

$$(W_t f)(x,y,z) := e^{t/2} f(x,y,e^t z).$$

$\{W_t\}_{t\in\mathbb{R}}$ forms in $\mathcal{H}$ a strongly continuous unitary group, *the group of dilations in the direction* z. Its generator is given by $A = 1 \otimes A_z$, where $A_z = \frac{1}{2}(P_z Z + Z P_z)$; Z is the operator of multiplication by the variable z. Obviously, W_t leaves the space $\mathcal{G} = \mathcal{G}^0$ invariant, so we are in the framework of Section 3.

Proposition 4.4. *(a)* $H(a^0,0) \in C^\infty(A;\mathcal{G},\mathcal{G}^*)$.

(b) A *is locally strictly conjugate to* $H(a^0,0)$ *on* $\mathbb{R}\setminus\sigma^0$.

Proof. (a) follows immediately from the relation

$$\mathcal{W}_t H(a^0,0) := e^{-itA}H(a^0,0)e^{+itA} = H^0_{xy}\otimes 1 + 1\otimes e^{2t}P_z^2,$$

valid in $B(\mathcal{G},\mathcal{G}^*)$.

(b) In [1], Theorem 8.2.6 it is shown how to calculate the ρ function of an operator of the form $H = H_1\otimes 1 + 1\otimes H_2$ for self-adjoint, bounded from below operators H_1, H_2 with respect to a conjugate operator of the same form $A = A_1\otimes 1 + 1\otimes A_2$. With our notations, it reads

$$\rho^A_{H(a^0,0)}(\lambda) = \inf\{\rho^{0_{xy}}_{H^0_{xy}}(\lambda_1) + \rho^{A_z}_{P_z^2}(\lambda_2) \mid \lambda_1+\lambda_2 = \lambda\}.$$

Short calculations taking into account Remark 3.3 and $i[P_z^2, A_z] = 2P_z^2$ give

$$\rho^{0_{xy}}_{H^0_{xy}}(\lambda_1) = 0 \quad if\ \lambda_1 \in \sigma^0,$$

$$\rho^{0_{xy}}_{H^0_{xy}}(\lambda_1) = \infty \ \ if\ \lambda_1 \notin \sigma^0$$

and

$$\rho^{A_z}_{P_z^2}(\lambda_2) = 2\lambda_2 \ \ if\ \lambda_2 \geq 0,$$

$$\rho^{A_z}_{P_z^2}(\lambda_2) = +\infty \ \ if\ \lambda_2 < 0.$$

By combining the formulae above one gets

$$\rho^{A}_{H(a^0,0)}(\lambda) = 0 \ \ if\ \lambda \in \sigma^0$$

and

$$\rho^{A}_{H(a^0,0)}(\lambda) = 2(\lambda - \sup[\sigma^0 \cap (-\infty, \lambda)]) \ \ if\ \lambda \notin \sigma^0$$

so that $\rho^{A}_{H(a^0,0)}(\lambda)$ is strictly positive if $\lambda \in \mathbb{R} \setminus \sigma^0$.

Proposition 4.5. *(a) $H(a,V)$ belongs to $C^2(A;\mathcal{G},\mathcal{G}^*)$.*
(b) A is locally conjugate to $H(a,V)$ on $\mathbb{R} \setminus \sigma^0$.

Proof. (a) By Proposition 4.4.(a) we know already that $H(a^0,0) \in C^2(A;\mathcal{G},\mathcal{G}^*)$, so it is enough to show that $U, V \in C^2(A;\mathcal{G},\mathcal{G}^*)$.

It is known ([1]) that under the assumption $e^{itA}\mathcal{G} \subset \mathcal{G}$. $t \in \mathbb{R}$, an operator $S \in B(\mathcal{G},\mathcal{G}^*)$ is in $C^1(A;\mathcal{G},\mathcal{G}^*)$ iff the sesquilinear form

$$\varphi_{(S,A)} : D(A;\mathcal{G}) \times D(A;\mathcal{G}) \to \mathbb{C},$$

$$\varphi_{(S,A)}(f,g) := i < Sf, Ag > -i < Af, Sg >$$

can be extended continuously to $\mathcal{G} \times \mathcal{G}$. We set by abuse of notation $< \cdot,\cdot >$ for the natural duality between $\mathcal{G}$ and $\mathcal{G}^*$. $D(A;\mathcal{G})$ is the domain (endowed with the graph-norm) of the infinitesimal generator of the C_0-group $\{e^{itA}\}$ considered in the Hilbert space $\mathcal{G}$. It is given by $D(A;\mathcal{G}) = \{f \in \mathcal{G} \cap D(A) \mid Af \in \mathcal{G}\}$. In the situation described above, the operator $i[S,A] \in B(\mathcal{G},\mathcal{G}^*)$ associated to the extension to $\mathcal{G}\times\mathcal{G}$ of the form $\varphi_{(S,A)}$ will be just the strong derivative $\frac{d}{dt}|_{t=0}e^{-itA}Se^{itA}$. The condition $S \in C^2(A;\mathcal{G},\mathcal{G}^*)$ is treated just by replacing S by $i[S,A]$ in the argument above and iterating.

However, in our concrete framework we may proceed more straightforwardly since $\mathcal{D}$ is dense in all the spaces and invariant under all the operators which come into play. Consequently, what we have to do is to show that the commutators $i[U,A]$, $i[V,A]$, $i[i[U,A],A]$ and $i[i[V,A],A]$ calculated on the invariant domain $\mathcal{D}$ can be extended to elements of $B(\mathcal{G},\mathcal{G}^*)$. By setting $\delta_z := z\partial_z$, we have:

$$i[U,A] + i[V,A] = -P_z\tilde{a}_z - \tilde{a}_zP_z + \Pi \cdot \delta_z\tilde{a} + \delta_z\tilde{a} \cdot \Pi - \delta_zV,$$

$$\begin{aligned} i[i[U,A],A] + i[i[V,A],A] = {} & P_z(2\delta_z\tilde{a}_z - \tilde{a}_z) + (2\delta_z\tilde{a}_z - \tilde{a}_z)P_z - \\ & -\Pi^0 \cdot \delta_z^2\tilde{a} - \delta_z^2\tilde{a} \cdot \Pi^0 - 2\delta_z(\tilde{a} \cdot \delta_z\tilde{a}) + \delta_z^2V \end{aligned}$$

and one has only to take into account $(\tilde{a}/2)$, $(V/2)$ and Lemma 4.1.

(b) We have already shown in Proposition 4.4.(b) that $\rho^{A}_{H(a^0,0)}(\lambda) > 0$ if $\lambda \notin \sigma^0$. This implies, of course, that $\tilde{\rho}^{A}_{H(a^0,0)}(\lambda) > 0$ if $\lambda \notin \sigma^0$. Since $C^2(A;\mathcal{G},\mathcal{G}^*) \subset C^1_u(A;\mathcal{G},\mathcal{G}^*)$, one may apply Proposition 3.4 and Lemma 4.3 to deduce that A is locally conjugate to $H(a,V)$ on $\mathbb{R} \setminus \sigma^0$.

Proof of Theorem 2.2.

Theorem 2.2 follows from Proposition 3.5 and Proposition 4.5.

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