# A Pair of Optimal Inequalities Related to the Error Function

Mary Beth Ruskai *
Department of Mathematics
University of Massachusetts Lowell
Lowell, MA 01854 USA
bruskai@cs.uml.edu

Elisabeth Werner
Department of Mathematics
Case Western Reserve University
Cleveland, OH 44106 USA
emw2@po.cwru.edu

November 4, 1997

The function

$$V(x) \equiv \sqrt{\pi} e^{x^2}[1 - \mathrm{erf}(x)] \tag{1}$$
$$= \int_0^\infty \frac{e^{-u}}{\sqrt{x^2+u}} du = 2e^{x^2} \int_x^\infty e^{-t^2} dt$$

arises in many contexts, from probability to mathematical physics, and satisifies the differential equation

$$V'(x) = 2xV(x) - 2 \quad \text{with} \quad V(0) = \sqrt{\pi}. \tag{2}$$

In this note, we restrict attention to $x \geq 0$, in which case $0 < V(x) < 1/x$ and $V(x)$ is decreasing. For $x \geq 0$ we show that

$$g_\pi(x) \leq V(x) < g_4(x) \tag{3}$$

*supported by National Science Foundation Grant DMS-97-06981

where

$$g_k(x) = \frac{k}{(k-1)x + \sqrt{x^2+k}}. \tag{4}$$

We also show that these inequalities are optimal for functions of the form (4) with equality only at $g_\pi(0) = V(0) = \sqrt{\pi}$. The bounds in (3) are considerably sharper than the classical inequalities of Komatsu [1]. The upper bound, which implies the convexity of $1/V(x)$, was established independently by Wirth [4] for $x \geq 0$ and by Szarek and Werner [3] for $x > -\frac{1}{\sqrt{2}}$. The weaker lower bound $g_3(x) < V(x)$ was used in [2] to show that the function $\left[1/V(x) - x\right]^2 / V(x)$ is decreasing for $x \geq 0$. It is easy to see that the family of functions $g_k(x)$ is increasing in $k$ and that $0 < g_k(x) < 1/x$.

In order to prove that the upper bound is optimal, we first observe that

$$\begin{aligned} & g_k'(x) > 2[x g_k(x) - 1] \\ \iff & (k-2)x^2 + k(k-3) < (k-2)x\sqrt{x^2+k} \\ \iff & x^2(k-2)(k-4) + k(k-3)^2 < 0 \end{aligned} \tag{5}$$

when $k > 3$. We now restrict attention to $3 \leq k \leq 4$ and let $h_k(x) = g_k(x) - V(x)$. For $k = 4$, the expression (5) implies $g_4'(x) < 2[x g_k(x) - 1]$ so that $h_4'(x) < 2x h_4(x)$ for all $x \geq 0$; whereas for $k < 4$ this holds only for $x < a_k = \sqrt{\frac{k(k-3)^2}{(k-2)(4-k)}}$. Since both $V(x)$ and $g_k(x)$ are positive and bounded above by $1/x$, their difference also satisfies $|h_k(x)| < 1/x \to 0$.

For $k = 4$, if $h_4(x) \leq 0$ for some $x > 0$, then $h_4'(x) < 2x h_4(x)$ is negative and thus $h_4$ is negative and strictly decreasing from a certain $x$ on, which contradicts $\lim_{x\to\infty} h_4(x) = 0$. Thus $h_4(x) > 0$ so that $g_4(x) > V(x)$, for all $x$. Now suppose that for some $k < 4$, $g_k$ is an upper bound, i.e. $h_k(x) \geq 0$ for all $x \geq 0$. In particular, $h_k(x) \geq 0$ for all $x > a_k$. For $k < 4$, we find however that $h_k'(x) > 2x h_k(x)$ holds for $x > a_k$. Thus we get $h_k(x) \geq 0$ and strictly increasing for all $x > a_k$ which contradicts $\lim_{x\to\infty} h_k(x) = 0$. Thus the upper bound can *not* hold when $x > a_k$ and $k < 4$. The lower bound also fails for $k > \pi$ since then $h_k(0) = g_k(0) - V(0) = \sqrt{k} - \sqrt{\pi} > 0$.

To establish the improved lower bound $g_\pi \leq V(x)$ we note that the argument above implies that $h_k(x)$ is negative for $x > a_k$ and $3 < k \leq \pi$. However for $k < \pi$ we have $h_k(0) < 0$ so that $h_k(x)$ is also negative for very small $x$. If $h_k(x)$ is ever non-negative, we can let $b$ denote the first place $h_k(x)$ touches or crosses the x-axis, i.e., $h_k(b) = 0$ and $h_k(x) < 0$ for $x < b$. Then $h_k$ must

be increasing on some interval of the form $(x_0, b)$. However, by the remarks above, $h_k(b) = 0$ implies $b \leq a_k$ so that $h_k'(x) < 2xh_k(x) < 0$ on $(x_0, b)$. Since this contradicts $h_k$ increasing on $(x_0, b)$, we must have $h_k(x) < 0$ for all $x \geq 0$ if $k < \pi$.

Thus we have proved the lower bound $g_k(x) < V(x)$ on $[0, \infty)$ for $k < \pi$. Since $g_k$ is continuous and increasing in $k$, it follows that $g_\pi(x) \leq V(x)$. To show that this inequality is strict except at $x = 0$, note that the right derivative of $h_k$ at 0 satisfies $h_k'(0) = 2 - k$ so that $h_\pi'(0) < 0$ and $h_\pi(x)$ is negative at least on some small interval $(0, x_1)$. Then we can repeat the argument above to show that $h_\pi(x) < 0$ if $x > 0$.

**Acknowledgment** We are grateful for the hospitality of the School of Mathematics at Georgia Tech where this work was done.

# References

[1] K. Ito and H.P. McKean *Diffusion Processes and Their Sample Paths* (Springer-Verlag, 1965).

[2] R. Brummelhuis and M. B. Ruskai, "A One-Dimensional Model for Many-Electron Atoms in Extremely Strong Magnetic Fields: Maximum Negative Ionization" preprint.

[3] S. J. Szarek and E. Werner, "Confidence Regions for Means of Multivariate Normal Distributions and a Non-symmetric Correlation Inequality for Gaussian Measure" MSRI preprint No. 1997-009, to appear in *J. Multivariate Analysis.*

[4] M. Wirth, "On considère la fonction de $\mathbb{R}$ dans $\mathbb{R}$ définie par $f(x) = e^{-x^2/2}$; démontrer que la fonction $g$ de $\mathbb{R}$ dans $\mathbb{R}$ définie par $g(x) = f(x)/\int_x^\infty f(t)dt$ est convexe" *Revue de Mathématiques Spéciales* **104**, 187-88 (1993).