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Correlation Function of the Spin-1/2 XXX Antiferromagnet

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Abstract

We consider a special correlation function in the isotropic spin-$\frac{1}{2}$ Heisenberg antiferromagnet. It is the probability of finding a ferromagnetic string of (adjacent) spins in the antiferromagnetic ground state. We give two different representations for this correlation function. Both of them are exact at any distance, but one becomes more effective for the description of long distance behaviour, the other for the description of short distance behaviour.

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1 Introduction

We consider the spin-$\frac{1}{2}$ Heisenberg XXX antiferromagnet in a magnetic field. The hamiltonian of the model is given by

$$H = \sum_{j\epsilon Z} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z - h\sigma_j^z \ . \tag{1}$$

Here j labels the sites of the lattice and runs through all integers, σ_j are Pauli matrices, and h is the magnetic field directed along the z direction in spin space. Eigenfunctions of the model were constructed by H. Bethe [1]. The antiferromagnetic ground state $|AFM\rangle$ at zero temperature was determined by Hulthén in [2]. The ground state as well as excitations are described by linear integral equations. The integral equation for the energy $\epsilon(\lambda)$ as a function of the spectral parameter λ of the elementary excitation (a spin-$\frac{1}{2}$ kink)[3] reads

$$\epsilon(\lambda) + \tfrac{1}{2\pi} \int_{-\Lambda}^{\Lambda} K(\lambda,\mu)\epsilon(\mu)d\mu = 2h - \frac{2}{\frac{1}{4}+\lambda^2} \tag{2}$$

Here $K(\lambda,\mu) = \frac{2}{1+(\lambda-\mu)^2}$. The energy should be equal to zero at the Fermi edges $\pm\Lambda$, *i.e.* $\epsilon(\Lambda) = \epsilon(-\Lambda) = 0$. This shows the Λ depends on h. In the limit $h \to 0$ Λ tends to ∞ according to

$$\Lambda = \tfrac{1}{2\pi} \ln \tfrac{(2\pi)^3}{eh^2} \tag{3}$$

On the other hand $\Lambda \to 0$ as $h \to 4$, and $\Lambda = 0$ for $h \geq 4$.

In order to define the correlation function of interest we consider the operators

$$P_j = \frac{1}{2}(\sigma_j^z + 1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \tag{4}$$

which project on the state with spin up at site number j.

The so-called "Emptiness-Formation Probability" correlation function is defined as

$$P(x) = \langle AFM| \prod_{j=1}^{x} P_j \ |AFM\rangle. \tag{5}$$

The physical meaning of $P(x)$ is the probability of finding x (where x is an integer) adjacent spins up in the antiferromagnetic vacuum.

Below we give two different representations for this probability based on two different methods of determining correlation functions. The reason for studying $P(x)$ rather than correlators of local spins is that $P(x)$ is the simplest correlation function from a computational point of view in *both* approaches.

2 The Dual Field Approach

The first approach is based on the papers [4-7]. It is described in detail in the book [8]. There are three main steps:

- Starting from the solution of the model by means of the Algebraic Bethe Ansatz, correlation functions are expressed as determinants of Fredholm integral operators.
- The determinants are described by means of integro-differential equations (see Sect. XIV.7. of [8]).
- The large-distance asymptotics of the correlation functions are obtained from the solution of a Riemann-Hilbert problem [8-12].

Let us now consider step 1. In order to express $P(x)$ as a determinant we need to make use of dual quantum fields $\varphi(\lambda)$, which are linear combinations of canonical Bose fields $a(\lambda)$ and $a^\dagger(\lambda)$ with standard commutation relations

$$\begin{aligned} \Big[a(\lambda), a^\dagger(\mu)\Big] &= \delta(\lambda-\mu) \\ \Big[a(\lambda), a(\mu)\Big] &= 0 = \Big[a^\dagger(\lambda), a^\dagger(\mu)\Big] \ . \end{aligned} \tag{6}$$

They are acting in the standard Fock space with reference state $|0)$ (Fock vacuum) defined *via*

$$a(\lambda)|0) = 0 \quad , \quad 0 = (0|a^\dagger(\mu) \ . \tag{7}$$

The dual fields $\varphi(\lambda)$ are given by the following expressions:

$$\varphi(\lambda) = a(\lambda) - \int_{-\infty}^{\infty} d\nu \ln\Big[1 + (\lambda - \nu)^2\Big] a^{\dagger}(\nu) \ . \tag{8}$$

One should note that by construction the dual fields always commute

$$\Big[\varphi(\lambda), \varphi(\mu)\Big] = 0 \ . \tag{9}$$

In the book [8] it is explained in detail for the case of the Bose gas with delta-function interaction how to derive an expression for $P(x)$ in terms of Fredholm determinants. $P(x)$ is found to be a ratio of two determinants (see [8], formula (1.28) on page 246.). For the XXX model a similar expression is valid

$$P(x) = \frac{(0|\det(1+\hat{V})|0)}{\det(1+\frac{1}{2\pi}\hat{K})} \ . \tag{10}$$

Here $\hat{V}$ and $\hat{K}$ are integral operators acting on the interval $[-\Lambda, \Lambda]$, and $|0)$ is the vacuum of the bosonic Fock space defined in (7). The kernel of $\hat{K}$ is the same as in the integral equation for the dressed energy of the elementary excitation (2)

$$K(\lambda, \mu) = \tfrac{2}{1+(\lambda-\mu)^2} \ . \tag{11}$$

The operator $\hat{V}$ also acts in the Fock space of the dual fields and has a kernel

$$V(\lambda, \mu) = \tfrac{1}{2\pi}\Big\{\frac{e(\lambda)e^{-1}(\mu)}{(\lambda-\mu)(\lambda-\mu+i)} + \frac{e^{-1}(\lambda)e(\mu)}{(\mu-\lambda)(\mu-\lambda+i)}\Big\} \tag{12}$$

$$e(\lambda) = \Big(\frac{2\lambda+i}{2\lambda-i}\Big)^{x/2} \exp(\varphi(\lambda)/2) \ . \tag{13}$$

Note that the expression for $V(\lambda, \mu)$ involves only commuting operators.

Formulas (10)-(13) are our main result and conclude step 1 of the dual field approach to correlation functions.

A.R. Its is considering the Riemann-Hilbert problem generated by the integral operator (12). The solution of the Riemann-Hilbert will give an explicit formula for large distance asymptotics [13].

3 The Vertex Operator Approach

There exists another approach to the problem of determining correlation functions at $h = 0$. It was invented recently by the RIMS group [14]. In that paper correlation functions in the XXZ model were considered. We took the XXX limit and obtained the following representation for $P(x)$ in terms of x multiple integrals (recall that x is integer).

$$P(x) = \int_C \frac{d\lambda_1}{2\pi i\lambda_1} \int_C \frac{d\lambda_2}{2\pi i\lambda_2} \dots \int_C \frac{d\lambda_x}{2\pi i\lambda_x} \Big\{ \prod_{a=1}^{x} \Big(1 + \frac{i}{\lambda_a}\Big)^{x-a} \Big(\frac{\pi\lambda_a}{\sinh(\pi\lambda_a)}\Big)^{x}$$

$$\times \prod_{1\le j<k\le x} \frac{\sinh(\pi(\lambda_k - \lambda_j))}{\pi(\lambda_k - \lambda_j - i)} \Big\} . \qquad (14)$$

Here each integral is performed along a contour C, which goes from $-\infty$ to $+\infty$ below the real axis. Formula (14) (as well as formula (10)) is exact for all distances x, but it becomes most efficient at small x. Formula (14) is valid at zero magnetic field $h = 0$ only. The first few values are $P(1) = \frac{1}{2}$, $P(2) = \frac{1}{3}(1 - \ln(2))$.

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