THE LÉVY FLIGHT FORAGING HYPOTHESIS IN BOUNDED REGIONS: SUBORDINATE BROWNIAN MOTIONS AND HIGH-RISK/HIGH-GAIN STRATEGIES

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Abstract. We investigate the problem of the Lévy flight foraging hypothesis in an ecological niche described by a bounded region of space, with either absorbing or reflecting boundary conditions.

To this end, we consider a forager diffusing according to a fractional heat equation in a bounded domain and we define several efficiency functionals whose optimality is discussed in relation to the fractional exponent $s \in (0,1)$ of the diffusive equation.

Such equation is taken to be the spectral fractional heat equation (with Dirichlet or Neumann boundary conditions).

We analyze the biological scenarios in which a target is close to the forager or far from it. In particular, for all the efficiency functionals considered here, we show that if the target is close enough to the forager, then the most rewarding search strategy will be in a small neighborhood of $s = 0$.

Interestingly, we show that $s = 0$ is a global pessimizer for some of the efficiency functionals. From this, together with the aforementioned optimality results, we deduce that the most rewarding strategy can be unsafe or unreliable in practice, given its proximity with the pessimizing exponent, thus the forager may opt for a less performant, but safer, hunting method.

However, the biological literature has already collected several pieces of evidence of foragers diffusing with very low Lévy exponents, often in relation with a high energetic content of the prey. It is thereby suggestive to relate these patterns, which are induced by distributions with a very fat tail, with a high-risk/high-gain strategy, in which the forager adopts a potentially very profitable, but also potentially completely unrewarding, strategy due to the high value of the possible outcome.

Preamble

On the one hand, many popular adages share the idea that *to achieve a prominent goal one has to take risks* (e.g., "no gain without pain", "nothing ventured, nothing gained", "no guts, no glory", just to name a few proverbs). On many occasions, the ambition to a high reward may lead individuals to face potential dangers, and in some situations there is a full master plan centered around a high-risk/high-gain plan: for instance, the blueprint of the European Research Council is to fund high-risk/high-reward research, in which severe conceptual challenges (which, by definition, are prone to scientific failure) are accepted downsides for a research project to be truly successful and impactful.

On the other hand, there is nowadays a great interest in the investigation of *optimal searching strategies*, e.g. in the study of animal behavior, and the research on this topic has necessarily to be somewhat controversial, given the complexity of the phenomenon into consideration.

Our view on this point is that the difficulty of addressing the topic of optimal searching is not only due to the *enormous amount of parameters* which should be accounted for (such as predators and prey distributions, previous knowledge of the territory, interactions with the environment, social factors, different reactions to adverse circumstances, competition phenomena, cooperative behaviors,

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etc.), and not only due to the difficulty of measuring many of these parameters via *objective empirical observations*.

In fact, in our opinion, a core difficulty in this topic stems from the difficulty of assessing unambiguously and indisputably a suitable notion of "gain" which should be maximized by a searching algorithm. This gain cannot be limited to the actual effectiveness of the procedure (i.e., whether or not the predator captures the prey), but it has to take into account the cost of the procedure itself (e.g., the time needed for the task, or the energy spent for it), and, at least on some occasions, the possible value of the outcome of the search.

One of the findings of our research is indeed that the high-risk/high-reward situation may appear naturally even in very simple situations, therefore the notion of "best strategy" requires a *very careful mathematical setting*, in which an *efficiency functional* is chosen and *maximized*, and *the location of the maxima is confronted with that of the minima*.

In doing so, one discovers immediately some interesting features. First of all, different efficiency functionals can produce different results. This already highlights a structural complication towards a full understanding of the notion of optimal searching strategy: for instance, in a biological study, different species, or different individuals of the same species, may, implicitly or explicitly, address a different type of efficiency functional.

In addition, in several concrete situations, the maximizers of some efficiency functional may end up to be dangerously close to the minimizers: this is a clear case of high-risk/high-reward pattern and, in this "unstable" situation, one should expect that the practical outcome of the optimal searching pattern be influenced by intermediate strategies aiming at a balance between top performances and conservative options (e.g., a risk assessment which compromises between the most rewarding and the safest result). Quite likely, in these conditions, different biological species, or different members of the same group, may end up adopting different search strategies.

Interestingly, in our setting, the situation in which the most rewarding strategy is arbitrarily close to a complete failure of the searching pattern is related to Lévy distributions with a very low exponent and a very fat tail. This pattern is known to be related to foraging modes of "ambush" type (see [DGNBD17, DGV22b]). The literature has also collected experimental evidence of some species, such as anglers and blonde skates, which do follow diffusive paths with very low Lévy exponent: remarkably, a correlation has been found between this type of diffusion and the high content of energy of the targets (see [DGNBD17]).

In our setting, this correlation is possibly motivated precisely by the fact that the most rewarding Lévy exponent happens to be very close to the pessimizer. In a sense, it can be significant to imagine that such a high-risk/high-gain strategy becomes particularly suitable when the possible outcome is of exceptional value (in the case of a biological predator, a prey of exceptionally high energetic content).

That is, in an implicit risk assessment, the value of the target may mitigate the prospect of an unsuccessful search, thus favoring the emergence, in these specific situations, of high-risk/high-reward diffusive patterns.

In this work, this general vision will be embodied into a precise mathematical study of the Lévy flight foraging hypothesis, considering the possibility that processes with long jumps (instead of standard Gaussian random movements) can optimize search efficiency by diminishing the repetitions of visits to previously inspected sites. Different efficiency functionals will be taken into account, with a thorough analysis of their optimizers and pessimizers. This phenomenon in which optimizers and pessimizers cluster together will be also explicitly detected and discussed.

The Lévy flights will be modeled via a heat equation of fractional type in bounded domains. We consider the case of a hostile environment (such as a "fence", modeled by homogeneous Dirichlet conditions which "annihilate" a biological species outside a confinement domain) as well as the case of reflecting boundaries (modeled by homogeneous Neumann conditions which maintain a biological species within a niche without altering the number of individuals present in the region).

To implement these boundary conditions in the setting of the fractional heat equation, we will make use of the spectral version of the fractional Laplacian.

In some of the efficiency functionals that we consider, predators and targets are modeled as points in the space. In other cases instead we will model predators and targets as regions of space (assuming e.g. that the biological individuals are uniformly distributed within these regions): this situation can also be considered as a technical and conceptual simplification of the notion of "direct vision" which was previously adopted in the literature, see e.g. [VBH+99]. That is, here we do not introduce an additional parameter to truncate the Lévy distribution in the proximity of its singularity (which entails in itself some delicate issues, see [PV21]) and we do not alter the diffusive equation to account for foragers directly aiming at the prey when they lie at short mutual distance. Instead, the diffusive equation is supposed to hold at every spatial scale and the role of a different region of influence (e.g., induced by uncertainties in the data or by a different hunting pattern at a small scale) is encoded only in the efficiency functional.

Here, we do not restrict our analysis to the one-dimensional case; in fact, we deal with an arbitrary large number n of dimensions. We note that the case of higher dimension is, in many instances, not only a situation of utmost biological interest, but also a source of technical difficulties and scientific controversies, see e.g. [LTBV20, BRB+21, LTBV21].

1. Introduction to the mathematical setting and main results

In the last decades, anomalous diffusion has been investigated as an appropriate substitute for normal diffusion in several branches of science, such as biology and in particular the foraging theory (see for instance [SK86, VAB+96, EPW+07, VBH+99, Rey18]). In this context a special case of anomalous diffusion occurs when a forager in search of food, rather than diffusing according to the classical Brownian motion, performs long-jump patterns characterized by a space and time steps scale invariance, see e.g. [KS05] and the references therein.

This type of searches fits the model of the Lévy flight, according to the probabilistic description given in Section 4.3 of [AV19]. In contrast to what happens with the classical random walk, the forager performing these flights has less chances to revisit intensively the immediate surrounding areas and then being confined in a narrow region. Therefore, in the biological framework, Lévy flights seem to be a better search strategy when the source of food is scarce and sparsely distributed and there is a large area to be covered in order to succeed in the hunt.

These kinds of foraging search strategies have been empirically observed in many ecological systems, see e.g. [VAB+96, ARMA02, RFMM+04, EPW+07, SSH+08, HQD+10, HWQ+12, HWS13]. Moreover, several studies have been made in order to validate the Lévy flight foraging hypothesis from a mathematical and statistical point of view [VBH+99, BCF+02, VAB+00, VBB+02].

In these models a number of assumptions are usually made on the environment, on targets and foragers. For instance, a low prey density is often assumed and the targets are randomly distributed in a wide area; the forager does not keep memory of previous encounters; the forager has scarce information on the area to search and on the prey location. On the one hand, on some occasions, these structural assumptions are introduced in order to simplify the problem, which otherwise would be extremely challenging to be analyzed from a theoretical perspective; on the other hand, some of these conditions can actually be structurally necessary for the convenience of the Lévy flight strategy over more standard type of diffusive processes. In any case, the complexity of the raw problem is a consequence of its dependence on a great number of environmental, evolutionary and biological variables. Even though an oversimplification may lead to a less accurate model in some circumstances, we can evince from a simplified model some remarkable properties, advancing the knowledge on such a complex topic.

In this paper we will investigate the Lévy flight foraging hypothesis relying on a fractional elliptic operator. This is motivated by the fact that in the limit of the time step going to zero, the distribution of a seeker performing Lévy flights converges to the solution of a fractional heat equation, see e.g. [Val09, BV16, AV19].

In order to test the Lévy flights foraging hypothesis, we consider some efficiency functionals, accounting for the random encounter rate between the forager and the target. We maximize these efficiency functionals with respect to the fractional exponent, with the aim of understanding which flight was more advantageous for the forager. From a biological perspective, this optimization with respect to the fractional exponent corresponds to the possibility of a forager to modify its searching strategy by tuning e.g. the average length of a hunting path and the waiting times between different paths.

We will assume that the forager is confined in some bounded region $\Omega \subset \mathbb{R}^n$, which plays the role of an ecological niche. Both Dirichlet and Neumann boundary conditions will be taken into account to describe absorbing and reflecting boundaries.

For us, the choice of a spectral fractional heat equation as a diffusion equation for the forager was motivated by its stochastic interpretation as a subordinate Brownian motion in Ω, see [DGV22a]. See also [MPV13, CDV17, SV17, DV21, DPLV] and the references therein for several applications of fractional elliptic equations to biological problems.

In this paper, we will test the Lévy flight foraging hypothesis by taking into account different biological configurations, such as:

- the case in which the forager starting position and the target location coincide,
- the case in which the forager starting position is located in proximity of the target,
- the case in which the forager and the target, instead of being modeled as material points, are uniformly distributed in some regions of space.

The situation in which the biological population is not confined into a bounded region of space and can travel through the whole of $\mathbb{R}^n$ is technically different and has been treated in the papers [DGV, DGV22b].

The paper is structured as follows. In Section 1.1, we define the *efficiency functionals* for the spectral search in the bounded region $\Omega \subset \mathbb{R}^n$. They will be taken to be proportional to the encounter rate between the forager and the target. Moreover, different "penalizations quantities" will be considered, such as the average distance and the mean square displacement, in order to build physically reliable efficiency functionals.

Sections 1.2 and 1.3 are devoted to the study of the *maximizer* for the aforementioned functionals. These maximizers thus correspond to the most rewarding searching mode. In particular, in Section 1.2 we will assume that the forager starting position and the prey location coincide. This scenario, though physically less relevant, will let us detect some *monotonicity properties* of one of the functionals, when the domain satisfies suitable geometric properties, see Theorem 1.4 below. This result shows how *the search for a maximizer is related to the geometric structure of the play field.*

In Section 1.3, as well as in Section 1.4 for the case of distributed foragers and targets, we analyze the case in which the target is in some small neighborhood of the forager starting position. Here we establish that *if the target position converges to the initial location of the seeker, then the maximizer of the efficiency functionals is located in a neighborhood of* $s = 0$. This is the content of Theorems 1.7, 1.8, 1.15 and 1.16.

Furthermore, in Theorems 1.6 and 1.14 it will be proved that for some of these efficiency functionals the strategy $s = 0$ is the *unique global minimizer*, thus corresponding to the unique pessimizer of the searching mode. This minimality result, together with the convergence of the best strategy, will entail that, roughly speaking, the most rewarding strategy may end up being not reliable, presenting arbitrarily close pessimizers, thus opening the dilemma of whether in practice one should follow the most performant option, or the safest one, or, say, a balanced combination of the two (see Remark 1.9 below).

In Section 2 we collect the main analytical tools that will be employed in order to prove our main results.

Finally in Section 3 we prove the results stated in Sections 1.2 and 1.3.

1.1. **Efficiency Functionals.** To measure the effectiveness of a foraging strategy, one can consider different functionals which account for the rate of hunting "success" for the predator versus the "effort" needed.

The possibility of accounting for different efficiency functionals plays, in our opinion, a crucial role in biology and ethology, since, while the notion of "foraging success" may be somewhat objective (as measured for instance by the amount of food eaten, or by the calories carried by such a food), the notion of "cost spent to achieve the success" is intrinsically more ambiguous and different biological theories may end up measuring this concept in different ways. As an example, we recall the debate about the way honey bees assess how far they have flown (whether based on the energy expended in flying or on the fatigue required by the action, as conjectured in former experiments, or on the image motion of the surrounding landscape through visual perception, as pointed out in the "optic flow hypothesis" and addressed in recent tests, see [SZATJ00]). Related to this, we also recall that in some situations the measure of the distance traveled can be performed according to a number of possible strategies (e.g., in the case of ants, which can use optic flow, pheromone and chemical trails, as well as the "counting" of the number of steps, see [WWW06]). See also [Gad21] and the references therein for further reading on how animals measure distances.

The mathematical setting that we consider here goes as follows. We model a forager moving in some bounded region $\Omega \subset \mathbb{R}^n$ through a spectral fractional diffusion with either Dirichlet or Neumann homogeneous boundary conditions. The domain Ω where the diffusion occurs can be seen as an ecological niche where the forager is confined (the Dirichlet condition corresponding to the case in which the forager is killed at the boundary of the niche, and the Neumann datum corresponding e.g. to fences that prevent the forager to exit the niche).

Specifically, the probability density $u = u(t, x)$ of the forager satisfies the diffusive equation

$$\partial_t u(t,x) = -(-\Delta)^s u(t,x) \qquad \text{for all } (t,x) \in (0,+\infty)\times\Omega, \tag{1.1}$$

with either Dirichlet or Neumann homogeneous boundary conditions.

Here above s is a fractional parameter in $(0, 1)$ and the operator $(-\Delta)^s$ represents the spectral fractional Laplacian, see e.g. Sections 2.3 and 4.3 in [AV19] for the basics of this operator. See also [DGNBD17] for different approaches to the problem of Lévy flights in (one-dimensional) bounded domains.

We also assume that the targets are scattered in Ω according to a distribution $p(t, x)$, where $(t, x) \in [0, +\infty) \times \Omega$.

We consider, as an initial measure of the success of the hunting strategy of the predator, a *foraging success functional* which accounts for the random encounters between the forager following the dispersive equation in (1.1) and the targets.

Specifically, in the situation considered here, given $T \in (0, +\infty)$ and $y \in \Omega$, the foraging success functional takes the form

$$\int_0^T \int_\Omega r^s(t,x,y)\, p(t,x)\, dx\, dt, \tag{1.2}$$

where $r^s(t,x,y)$ represents either the Dirichlet or the Neumann spectral fractional heat kernel, for some fractional parameter $s \in (0,1)$, see for instance [DGV22a] and the beginning of the forthcoming Section 2 for definitions and basic properties of these kernels.

We notice indeed that the quantity in (1.2) is associated with the probability that a forager starting at the position $y \in \Omega$ and following the diffusion process modeled by the fractional heat equation with either Dirichlet or Neumann boundary condition hits a target distributed according to $p(t,x)$ in the time interval $(0,T)$.

To obtain an efficiency functional, we compare this quantity with some other quantities of physical and biological significance that instead provide a penalization for the seeker. Here, we will consider as penalization quantities the *time* T, the *average distance traveled by the forager* $l^y(s,T)$ after a time T and the *mean square displacement* $\mathcal{A}^y(s,T)$ after a time T.

More explicitly, the average distance traveled by the forager at time $T \in (0,+\infty)$ is given by

$$l^y(s,T) := \int_0^T \int_\Omega |\zeta - y|\, r^s(t,\zeta,y)\, d\zeta\, dt. \tag{1.3}$$

The probabilistic interpretation underpinning this definition consists in taking into account the random process Y_t starting at y corresponding to a subordinate Brownian motion which is either killed or reflected at the boundary (the generator of such a process corresponding to the spectral fractional Laplacian with either Dirichlet or Neumann datum).

In this framework, the quantity $|Y_t|$ represents the distance at time t for a single representation of the process, whence it is natural to consider its expected value

$$\mathbb{E}^s_y[|Y_t|] = \int_\Omega |x-y|\, r^s(t,x,y)\, dx$$

as the mean distance traveled at time t. The setting in (1.3) is thus the average over time $t \in (0,T)$ of this quantity.

Similarly, the mean square displacement is given by

$$\mathcal{A}^y(s,T) := \int_0^T \int_\Omega |\zeta - y|^2\, r^s(t,\zeta,y)\, d\zeta\, dt \tag{1.4}$$

and represents the average over time $t \in (0,T)$ of the expected value of the squared distance

$$\mathbb{E}^s_y[|Y_t|^2] = \int_\Omega |x-y|^2\, r^s(t,x,y)\, dx.$$

Interestingly, subordinators related to waiting times may have an intimate connection to biology, since spontaneous patterns of waiting times are known to occur in nature, and they can be species-specific, depend on body size, foraging modes, prey preference, etc., see [WMH+14].

While the notations in (1.3) and (1.4) are the same for the Dirichlet and the Neumann cases (the difference being only in the fractional heat kernel, which is sensitive to the boundary conditions), it is convenient to distinguish explicitly between the two types of boundary data and for this we add the subscript D or N to the notation, namely we write $l^y_D(s,T)$, $l^y_N(s,T)$, $\mathcal{A}^y_D(s,T)$ and $\mathcal{A}^y_N(s,T)$ to emphasize the dependence of the average distance traveled and of the mean square displacement with respect to the Dirichlet or the Neumann boundary condition.

As a special case of target distribution $p(t,\xi)$, we consider the situation in which there is only one target located at $x \in \Omega$. In this case, the distribution $p(t,\xi)$ reduces to the Dirac's delta $\delta_x(\xi)$ and the foraging success functional in (1.2) will be denoted (depending on the boundary condition) by

$$\begin{aligned}
\Phi_D^{x,y}(s,T) &= \int_0^T \int_\Omega r_D^s(t,\zeta,y)\delta_x(\zeta)\,d\zeta\,dt = \int_0^T r_D^s(t,x,y)\,dt \\
\text{or}\qquad \Phi_N^{x,y}(s,T) &= \int_0^T \int_\Omega r_N^s(t,\zeta,y)\delta_x(\zeta)\,d\zeta\,dt = \int_0^T r_N^s(t,x,y)\,dt.
\end{aligned} \tag{1.5}$$

In this paper we focus on the optimal foraging strategy according to the following efficiency functionals:

$$\begin{aligned}
&\mathcal{E}_{1,D}^{x,y}(s,T) := \frac{\Phi_D^{x,y}(s,T)}{T}, && \mathcal{E}_{1,N}^{x,y}(s,T) := \frac{\Phi_N^{x,y}(s,T)}{T}, \\
&\mathcal{E}_{2,D}^{x,y}(s,T) := \frac{\Phi_D^{x,y}(s,T)}{l_D^y(s,T)}, && \mathcal{E}_{2,N}^{x,y}(s,T) := \frac{\Phi_N^{x,y}(s,T)}{l_N^y(s,T)}, \\
&\mathcal{E}_{3,D}^{x,y}(s,T) := \frac{\Phi_D^{x,y}(s,T)}{\mathcal{A}_D^y(s,T)} \quad \text{and} && \mathcal{E}_{3,N}^{x,y}(s,T) := \frac{\Phi_N^{x,y}(s,T)}{\mathcal{A}_N^y(s,T)}.
\end{aligned} \tag{1.6}$$

In addition to the functionals in (1.6), we consider the following set-dependent functionals. Here, the exact initial positions of target and forager are replaced by uniform densities in two subregions of Ω. Namely, we assume that the targets are distributed in Ω according to

$$p(t,x) := \frac{\chi_{\Omega_1}(x)}{|\Omega_1|},$$

for some measurable set $\Omega_1 \subset \Omega$, where χ_{Ω_1} is the characteristic function of Ω_1 and $|\Omega_1|$ denotes the Lebesgue measure of Ω_1.

The forager diffusing via the spectral fractional heat equation is initially uniformly distributed in some measurable set $\Omega_2 \subset \Omega$ and therefore, dropping for the moment the subscript D and N, its density in $(t,x) \in (0,+\infty) \times \Omega$ is given by

$$f^s(t,x) := \frac{1}{|\Omega_2|} \int_{\Omega_2} r^s(t,x,y)\,dy,$$

see e.g. Lemmas 2.14 and 3.11 in [DGV22a].

With this notation, the *set-dependent forager success functional* takes the form

$$\begin{aligned}
\widetilde{\Phi}^{\Omega_1,\Omega_2}(s,T) &:= \int_0^T \int_\Omega f^s(t,x)p(t,x)\,dx\,dt \\
&= \frac{1}{|\Omega_1||\Omega_2|} \int_0^T \int_{\Omega_1\times\Omega_2} r^s(t,x,y)\,dx\,dy\,dt.
\end{aligned} \tag{1.7}$$

Furthermore, in this framework, the average distance traveled by the forager and the mean square displacement are given by

$$\begin{aligned}
\widetilde{l}^{\Omega_2}(s,T) &:= \int_0^T \int_\Omega |\xi - y| f^s(t,\xi)\,d\xi\,dt \\
&= \frac{1}{|\Omega_2|} \int_0^T \int_{\Omega\times\Omega_2} |\xi - y| r^s(t,\xi,y)\,d\xi\,dy\,dt \\
\text{and}\qquad \widetilde{\mathcal{A}}^{\Omega_2}(s,T) &:= \int_0^T \int_\Omega |\xi - y|^2 f^s(t,\xi)\,d\xi\,dt \\
&= \frac{1}{|\Omega_2|} \int_0^T \int_{\Omega\times\Omega_2} |\xi - y|^2 r^s(t,\xi,y)\,d\xi\,dy\,dt.
\end{aligned} \tag{1.8}$$

Therefore, with these set-dependent foraging success functional and penalization quantities, we define the *set-dependent efficiency functionals* as

$$\begin{aligned}
&\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s,T) := \frac{\widetilde{\Phi}^{\Omega_1,\Omega_2}_{D}(s,T)}{T}, &&\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,N}(s,T) := \frac{\widetilde{\Phi}^{\Omega_1,\Omega_2}_{N}(s,T)}{T},\\
&\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{2,D}(s,T) := \frac{\widetilde{\Phi}^{\Omega_1,\Omega_2}_{D}(s,T)}{\widetilde{l}^{\Omega_2}_{D}(s,T)}, &&\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{2,N}(s,T) := \frac{\widetilde{\Phi}^{\Omega_1,\Omega_2}_{N}(s,T)}{\widetilde{l}^{\Omega_2}_{N}(s,T)},\\
&\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{3,D}(s,T) := \frac{\widetilde{\Phi}^{\Omega_1,\Omega_2}_{D}(s,T)}{\widetilde{\mathcal{A}}^{\Omega_2}_{D}(s,T)} \quad \text{and} &&\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{3,N}(s,T) := \frac{\widetilde{\Phi}^{\Omega_1,\Omega_2}_{N}(s,T)}{\widetilde{\mathcal{A}}^{\Omega_2}_{N}(s,T)}.
\end{aligned} \tag{1.9}$$

1.2. **Prey at forager starting position and change of monotonicity.** In this section we will assume that the forager starts its search from the prey location. In this case, all the efficiency functionals in (1.6) diverge if $n \geqslant 2$ or $n = 1$ and $s \in \left(0, \frac{1}{2}\right]$, as better specified in the following proposition. For this reason, in this scenario where the forager starting position coincides with the target location, we will only work in one dimension.

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected, $x \in \Omega$ and $\mathcal{E}$ be any of the efficiency functionals in* (1.6) *with $x = y$.*

Then, for each $T \in (0, +\infty)$, if either $n \geqslant 2$ or $n = 1$ and $s \in \left(0, \frac{1}{2}\right]$ it holds that $\mathcal{E}(s, T) = +\infty$.

In the one-dimensional framework, the connectedness hypothesis on Ω forces the domain to be an interval. Thus, up to a translation, we can suppose that $\Omega = (0, a)$ for some $a \in (0, +\infty)$. In this case, several results can be obtained at the same time for all the efficiency functionals in (1.6).

In the following proposition we establish that the range of the fractional exponent in which these functionals achieve a finite value coincides with $\left(\frac{1}{2}, 1\right]$, and that in this interval they are continuous in s.

Proposition 1.2. *Let $a \in (0, +\infty)$, $\Omega = (0, a)$, $x \in \Omega$, $T \in (0, +\infty)$ and $\mathcal{E}$ be any of the efficiency functionals in* (1.6) *with $x = y$.*

Then, $\mathcal{E}(s, T) \in (0, +\infty)$ for all $s \in \left(\frac{1}{2}, 1\right]$ and $\mathcal{E}(\cdot, T) \in C\left(\left(\frac{1}{2}, 1\right]\right)$.

In terms of detecting the most rewarding foraging strategy with respect to the Lévy exponent s, we show that *if the initial position of the forager coincides with the location of the target then $s = 1/2$ is the optimizer* for all the efficiency functionals in (1.6):

Theorem 1.3. *Let $a \in (0, +\infty)$, $\Omega = (0, a)$, $x \in \Omega$ and $\mathcal{E}$ be any of the efficiency functionals in* (1.6) *with $x = y$.*

Then, for all $T \in (0, +\infty)$, the supremum over $s \in \left(\frac{1}{2}, 1\right]$ of $\mathcal{E}$ is attained at $s = \frac{1}{2}$, with

$$\lim_{s \searrow \frac{1}{2}} \mathcal{E}(s, T) = +\infty. \tag{1.10}$$

Even though the environmental scenario of a forager starting its search precisely from the target location is physically less relevant than the other cases, it can serve as an example of the complexity of the optimization problem and its dependence on external factors, such as the geometrical properties of the domain.

In what follows, we provide an example of *change of monotonicity* for the functionals in equation (1.5). Specifically, we show that *if the interval in which we consider the motion is small enough, then the functionals are strictly decreasing in s.* On the other hand, we prove that *if the interval is large enough, then there is a region of this interval such that if the search starts there, then the monotonicity property is violated* in a neighborhood of the Brownian strategy $s = 1$, see Figure 1.

Theorem 1.4. *Let $a \in (0, +\infty)$, $\Omega = (0, a)$, $T \in (0, +\infty)$ and $x \in \Omega$. Let Φ be any of the foraging success functional in* (1.5) *with $x = y$.*

Then, if $a \in (0,\pi]$, for every $s_0 \in \left(\frac{1}{2},1\right]$ and $s_1 \in (s_0,1]$, we have that

$$\Phi(s_0,T) > \Phi(s_1,T). \tag{1.11}$$

Also, for every $\nu \in \left(0,\frac{1}{2}\right)$ there exists $a_\nu \in (\pi,+\infty)$ such that if $a \in (a_\nu,+\infty)$ then, for every $T \in [\nu a^{2s},+\infty)$, $x \in (\nu a,(1-\nu)a)$, $s_0 \in \left(\frac{1+\nu}{2},1\right)$ and $s_1 \in (s_0,1]$, it holds that

$$\Phi_D^{x,x}(s_1,T) > \Phi_D^{x,x}(s_0,T). \tag{1.12}$$

Furthermore, for every $\nu \in \left(0,\frac{1}{2}\right)$ there exists $a_\nu \in (\pi,+\infty)$ such that if $a \in (a_\nu,+\infty)$ then, for every $T \in [\nu a^{2s},+\infty)$, $x \in \left(0,\frac{(1-\nu)a}{2}\right) \cup \left(\frac{(1+\nu)a}{2},a\right)$, $s_0 \in \left(\frac{1+\nu}{2},1\right)$ and $s_1 \in (s_0,1]$, it holds that

$$\Phi_N^{x,x}(s_1,T) > \Phi_N^{x,x}(s_0,T). \tag{1.13}$$

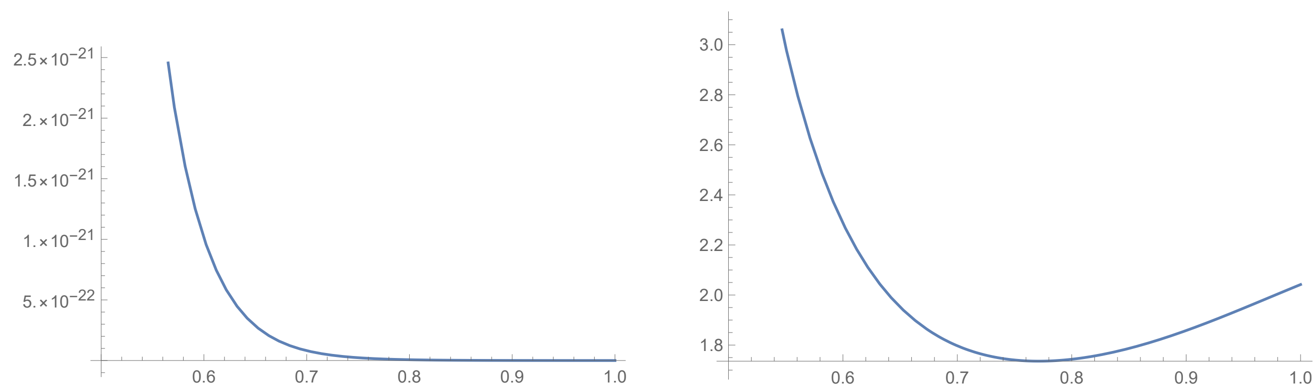

FIGURE 1. *Plot of $\left(\frac{1}{2},1\right) \ni s \mapsto \Phi_D^{x,x}(s,T)$ for $\Omega=(0,a)$ with $x=2.5$, $T=100$ and $a\in\{3,10\}$. We have approximated Φ_D, as explicitly given in (3.5), by summing to the 5×10^5th term.*

In [DGV22a] we studied the monotonicity properties of the fractional heat kernel $r^s(t,x,x)$ with respect to the fractional parameter s and we showed that these properties depend on the geometry of the domain. This dependence is expressed via the eigenvalues of either the Dirichlet or the Neumann Laplacian, which are well-known to depend on geometric features of the domain, like its measure or the Hausdorff measure of its boundary. For further details on this relation see the comments after Theorems 1.11 and 1.23 in [DGV22a] and the references therein.

More precisely, in Theorem 1.10 of [DGV22a] we established that if the first eigenvalue of the Dirichlet Laplacian is greater than 1, then the fractional heat kernel $r_D^s(t,x,x)$ is strictly decreasing in s. Analogously, in Theorem 1.22 in [DGV22a] we proved that if the first nonvanishing eigenvalue $\mu_{k(x)}$ of the Neumann Laplacian associated to a nonvanishing eigenfunction in x is greater than 1, then $r_N^s(t,x,x)$ is strictly decreasing in s. The monotonicity property given in (1.11) is thus a consequence of Theorems 1.10 and 1.22 of [DGV22a] and the definitions in (1.5).

On the other hand, in Theorems 1.11 and 1.23 of [DGV22a] we proved that under some circumstances there is a change of monotonicity for $r^s(t,x,x)$. Indeed, we showed that if the first eigenvalue of the Dirichlet Laplacian, or $\mu_{k(x)}$ as described above for the Neumann case, is smaller than 1, then for every $s_0,s_1\in(0,1)$ such that $s_0<s_1$ there exists some $T\in(0,+\infty)$ such that $r^{s_0}(t,x,x) < r^{s_1}(t,x,x)$ for all $t\in(T,+\infty)$. This latter change of monotonicity in relation to the size of the eigenvalues inspired the search for a change of monotonicity also for the efficiency functionals $\Phi_D^{x,x}$ and $\Phi_N^{x,x}$, which is proved to be true, as expressed by equations (1.12) and (1.13) above.

1.3. **Prey in proximity of the forager.** We now turn our attention to the efficiency functionals in (1.6) when the initial position of the forager $y\in\Omega$ is different from the target location $x\in\Omega$. We begin by stating the following *continuity result* with respect to the fractional exponent s.

Proposition 1.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. For every $(x, y, T) \in \Omega \times \Omega \times (0, +\infty)$ such that $x \neq y$, let us denote by $\mathcal{E}^{x,y}$ any of the efficiency functionals in* (1.6).

Then, $\mathcal{E}^{x,y}(s, T) \in (0, +\infty)$ for all $s \in (0, 1]$ and $\mathcal{E}^{x,y}(\cdot, T) \in C((0, 1])$.

In the following result we establish that for each $(x, y, T) \in \Omega \times \Omega \times (0, +\infty)$, satisfying $x \neq y$, the first Dirichlet functional $\mathcal{E}^{x,y}_{1,D}(s, T)$ *attains its infimum at* $s = 0$. Moreover, we show that the Dirichlet functionals in (1.6) admit a finite limit for $s \searrow 0$, as far as $x \neq y$.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected.*

Then, for every $(x, y, T) \in \Omega \times \Omega \times (0, +\infty)$ with $x \neq y$, it holds that

$$\inf_{s \in (0,1)} \mathcal{E}^{x,y}_{1,D}(s, T) = \lim_{s \searrow 0} \mathcal{E}^{x,y}_{1,D}(s, T) = 0. \tag{1.14}$$

Moreover, we have that

$$\lim_{s \searrow 0} \mathcal{E}^{x,y}_{2,D}(s, T) \in (0, +\infty) \quad \textit{and} \quad \lim_{s \searrow 0} \mathcal{E}^{x,y}_{3,D}(s, T) \in (0, +\infty). \tag{1.15}$$

From Theorem 1.6 we evince that we can extend by continuity the Dirichlet functionals in (1.6) to the whole compact interval $[0, 1]$. Hence, from now on, we will adopt the notation

$$\mathcal{E}^{x,y}_{j,D}(0, T) := \lim_{s \searrow 0} \mathcal{E}^{x,y}_{j,D}(s, T), \tag{1.16}$$

for all $(x, y, T) \in \Omega \times \Omega \times (0, +\infty)$, with $x \neq y$ and $j \in \{1, 2, 3\}$.

The following two theorems are the most important results of this section. We state that *if the forager starting position $y \in \Omega$ is close enough to the prey location $x \in \Omega$, then the best search strategy for the efficiency functionals in* (1.6) *will be in some small neighborhood of $s = 0$.*

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected and $(y, T) \in \Omega \times (0, +\infty)$.*

Then, for each $\varepsilon \in (0, 1)$ there exists some $\delta = \delta_{\varepsilon,y,T,\Omega} \in (0, +\infty)$ such that for each $x \in B_\delta(y) \setminus \{y\}$ it holds that

$$\sup_{s \in (0,1)} \mathcal{E}^{x,y}_{1,D}(s, T) = \mathcal{E}^{x,y}_{1,D}\left(s^{(1)}_{x,y,T}, T\right) \quad \textit{with} \quad s^{(1)}_{x,y,T} \in (0, \varepsilon). \tag{1.17}$$

Moreover, for each $j \in \{2, 3\}$ it holds that

$$\mathcal{E}^{x,y}_{j,D}(0, T) \geqslant \sup_{s \in (\varepsilon,1)} \mathcal{E}^{x,y}_{j,D}(s, T). \tag{1.18}$$

We stress that the situation $x \neq y$ treated in Theorem 1.7 is conceptually quite different from the case $x = y$ presented in Theorem 1.3: indeed, when the initial location of the predator is different from the position of the target, the efficiency functionals are finite for all $s \in (0, 1]$ independently from the dimension, as stated in Proposition 1.5.

The result in Theorem 1.7 is general enough to include different Dirichlet efficiency functionals and detects a somewhat *"universal" qualitative behavior.*

Moreover, an analogous situation holds true also for the Neumann functionals in (1.6):

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected and $(y, T) \in \Omega \times (0, +\infty)$.*

Then, for each for each $\varepsilon \in (0, 1)$ there exists some $\delta = \delta_{\varepsilon,y,T,\Omega} \in (0, +\infty)$ such that for each $x \in B_\delta(y) \setminus \{y\}$ and for all $j \in \{1, 2, 3\}$ it holds that

$$\sup_{s \in (0,1)} \mathcal{E}^{x,y}_{j,N}(s, T) = \mathcal{E}^{x,y}_{j,N}\left(s^{(j)}_{x,y,T}, T\right) \quad \textit{with} \quad s^{(j)}_{x,y,T} \in (0, \varepsilon). \tag{1.19}$$

Therefore, from Theorems 1.7 and 1.8 we deduce that *if the initial position of the forager approaches the position of the target, the fractional parameter $s \in (0, 1)$ maximizing the functionals in* (1.6) *converges to* 0. Thus, *in the regime of close proximity of seeker starting position and prey location, the above functionals are maximized by a search strategy with a very fat tail.*

It is interesting to notice that the maximizer $s^{(1)}_{x,y,T}$ of $\mathcal{E}_{1,D}$ given by Theorem 1.7 may turn out to be unreliable in practice, differently from the other two maximizers of the Dirichlet functionals, according to the following remark.

Remark 1.9. On the one hand, Theorem 1.6 establishes that $s = 0$ is a global minimizer for $\mathcal{E}_{1,D}$. On the other hand, if $s^{(1)}_{x,y,T}$ is a maximizer of $\mathcal{E}^{x,y}_{1,D}(\cdot, T)$, then from Theorem 1.7 we evince that

$$\lim_{x\to y} s^{(1)}_{x,y,T} = 0.$$

This means that as x approaches y, *the maximizer of the functional* $\mathcal{E}^{x,y}_{1,D}$ *converges to* $s = 0$*, which is a global minimizer*. Therefore, a small perturbation of $s^{(1)}_{x,y,T}$ can lead to very small values for $\mathcal{E}_{1,D}$, *making such choice of the most rewarding fractional exponent quite unreliable*. Therefore, in an environmental scenario where the forager starts its search in proximity of the target and the efficiency functional modelling the energy to maximize is given by $\mathcal{E}_{1,D}$, the "most rewarding" search strategy is to be considered "unreliable".

Things turn out to be different for $\mathcal{E}^{x,y}_{2,D}$ and $\mathcal{E}^{x,y}_{3,D}$. Indeed, if $s^{(j)}_{x,y,T}$ is a maximizer of the functional $\mathcal{E}^{x,y}_{j,D}(\cdot, T)$ with $j \in \{2, 3\}$, then, according to Theorem 1.7, one still has the limit

$$\lim_{x\to y} s^{(j)}_{x,y,T} = 0. \tag{1.20}$$

Nevertheless, in contrast with the case $j = 1$, now $s = 0$ is not necessarily a global minimum. Actually, see equation (1.18), for each $\varepsilon \in (0, 1)$, if x and y are close enough, then

$$\mathcal{E}^{x,y}_{j,D}(0, T) \geqslant \sup_{s\in(\varepsilon,1)} \mathcal{E}^{x,y}_{j,D}(s, T),$$

so that $s = 0$ in these two cases is "almost" a maximizer. Roughly speaking, we can say that the functionals $\mathcal{E}_{2,D}$ and $\mathcal{E}_{3,D}$ present more reliable optimal configurations than $\mathcal{E}_{1,D}$, since the maximizing fractional exponent is "separated" from the minimizers, whence the most rewarding strategy appears to be safer.

Remark 1.10. It has been observed in [WMH+14] that the case $s = 0$ occurs when some marine predators, such as anglers and blonde skates, specifically aim at a type of prey with a high energy content. It is therefore natural to relate the high-energy content of the prey and the high-risk/high-reward strategy related to $s = 0$: namely a high gain prospected by the energy content of the prey may serve as a mitigation of the chance of failure entailed by searching mode selected and as an indirect encouragement towards a potentially very beneficial, but intrinsically very risky, strategy.

Remark 1.11. One may wonder whether the unreliability of the most rewarding strategies and the corresponding high-risk/high-reward searching mode are specific of the situation considered in this paper, i.e. of a forager confined in a bounded region and a nearby prey. This is not the case, in fact in the paper [DGV22b] we will show that the same pattern persists, for instance, for a predator diffusing in the whole space and also for a prey located arbitrarily far from the predator.

The case that will be addressed in [DGV22b] is technically different from the one here, since the spectral analysis cannot be performed in unbounded domains and we will have to rely on singular integral calculations instead.

In what follows we observe a phenomenon which arises in the one-dimensional framework as a consequence of Theorem 1.4. In particular, under the same geometric assumptions of Theorem 1.4 on the domain Ω, we show that if the target location $x \in \Omega$ is sufficiently close to the forager initial position $y \in \Omega$, then *there exists a local maximizer* $s^*_{x,y,T}$ *for* $\mathcal{E}^{x,y}_{1,D}$ *and* $\mathcal{E}^{x,y}_{1,N}$ *in a neighborhood of the Brownian strategy* $s = 1$.

Corollary 1.12. *Let $a \in (0,+\infty)$, $\Omega = (0,a)$ and $T \in (0,+\infty)$.*

Then, for every $\nu \in \left(0,\frac{1}{2}\right)$ and $\varepsilon \in (0,1)$ there exists $a_\nu \in (\pi,+\infty)$ such that if $a \in (a_\nu,+\infty)$ then, for every $T \in [\nu a^{2s},+\infty)$ and $y \in (\nu a,(1-\nu)a)$, there exists some $\delta = \delta_{\nu,\varepsilon,y,T,\Omega} \in (0,+\infty)$ such that if $x \in B_\delta(y) \setminus \{y\}$ then

$$\sup_{s\in\left(\frac{1+\nu}{2},1\right)} \mathcal{E}^{x,y}_{1,D}(s,T) = \mathcal{E}^{x,y}_{1,D}\left(s^*_{x,y,T},T\right) \quad \textit{with} \quad s^*_{x,y,T} \in (1-\varepsilon,1]. \tag{1.21}$$

Also, for every $\nu \in \left(0,\frac{1}{2}\right)$ and $\varepsilon \in (0,1)$ there exists $a_\nu \in (\pi,+\infty)$ such that if $a \in (a_\nu,+\infty)$ then, for every $T \in [\nu a^{2s},+\infty)$ and $y \in \left(0,\frac{(1-\nu)a}{2}\right) \cup \left(\frac{(1+\nu)a}{2},a\right)$, there exists some $\delta = \delta_{\nu,\varepsilon,y,T,\Omega} \in (0,+\infty)$ such that if $x \in B_\delta(y) \setminus \{y\}$ then

$$\sup_{s\in\left(\frac{1+\nu}{2},1\right)} \mathcal{E}^{x,y}_{1,N}(s,T) = \mathcal{E}^{x,y}_{1,N}\left(\widehat{s}_{x,y,T},T\right) \quad \textit{with} \quad \widehat{s}_{x,y,T} \in (1-\varepsilon,1]. \tag{1.22}$$

It is interesting to compare this result with Remark 1.9 on the unreliability of the most rewarding search strategy for $\mathcal{E}^{x,y}_{1,D}$. Indeed, as a consequence of Theorem 1.7 and Corollary 1.12, we have that for each $\nu \in \left(0,\frac{1}{2}\right)$ and $\varepsilon \in (0,1)$ there exists some $a_\nu \in (\pi,+\infty)$ such that for every a, T and y given as in the statement of Corollary 1.12, there exists some $\delta^* = \delta^*_{\nu,\varepsilon,y,T,\Omega} \in (0,+\infty)$ such that, for every $x \in B_{\delta^*}(y) \setminus \{y\}$,

$$\sup_{s\in(0,1)} \mathcal{E}^{x,y}_{1,D}(s,T) = \mathcal{E}^{x,y}_{1,D}\left(s^{(1)}_{x,y,T},T\right) \quad \text{with} \quad s^{(1)}_{x,y,T} \in (0,\varepsilon)$$
$$\text{and} \qquad \sup_{s\in\left(\frac{1+\nu}{2},1\right)} \mathcal{E}^{x,y}_{1,D}(s,T) = \mathcal{E}^{x,y}_{1,D}\left(s^*_{x,y,T},T\right) \quad \text{with} \quad s^*_{x,y,T} \in (1-\varepsilon,1].$$

From this, we deduce that in this framework *there exist a global and a local maximizer*. The global maximizer $s^{(1)}_{x,y,T}$ seems to be the most rewarding option for the forager performing the search. Nevertheless, thanks to Remark 1.9, we also know that it is extremely *unreliable* for practical purposes. Indeed, a small deviation from $s^{(1)}_{x,y,T}$ can lead to the unique global minimizer $s = 0$, that makes the functional vanish.

On the other hand, even though the local maximizer $s^*_{x,y,T}$ is not optimal, it could be a better choice due to its stability. As a matter of fact, as stated in Proposition 1.5, the functional $\mathcal{E}^{x,y}_{1,D}$ vanishes nowhere near the Brownian strategy $s = 1$. Therefore, by choosing $s^*_{x,y,T}$, even under the presence of a positive error in the choice of the strategy, the outcome would not be heavily affected, as it could be for the most rewarding, but unreliable, strategy $s^{(1)}_{x,y,T}$.

This observation highlights how the definition of "best search strategy" is arguable, and how in some contexts it could not coincide with the classical notion of maximizer of a given energy: after all, what does "best" mean, is it "most rewarding" or "safest"? Thus, it may be appropriate to define new efficiency functionals that, rather than depending on an "exact choice" of the fractional exponent $s \in (0,1)$, take into account a probability measure in $(0,1)$ that allows the existence of an error range for the forager. This new approach will be investigated by the authors in a forthcoming work.

1.4. **Foragers and targets uniformly distributed in some regions.** Now we focus our attention to the study of the functionals in equation (1.9). In this case, the forager starting position and the prey location are replaced by uniform densities in disjoint subsets $\Omega_1, \Omega_2 \subset \Omega$. We begin by analyzing the continuity of these functionals with respect to the fractional exponent $s \in (0,1]$.

Proposition 1.13. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. For every $T \in (0,+\infty)$ and measurable sets $\Omega_1, \Omega_2 \subset \Omega$, let us denote by $\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}$ any of the efficiency functionals in* (1.9).

Then, $\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}(s,T) \in (0,+\infty)$ for all $s \in (0,1]$ and $\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}(\cdot,T) \in C((0,1])$.

The following result can be considered as the set-dependent counterpart of Theorem 1.6.

Theorem 1.14. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected.*
Then, for all $T \in (0,+\infty)$ and smooth and disjoint sets $\Omega_1, \Omega_2 \subset \Omega$, it holds that

$$\inf_{s\in(0,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s,T) = \lim_{s\searrow 0} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s,T) = 0. \tag{1.23}$$

Moreover, we have that

$$\lim_{s\searrow 0} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{2,D}(s,T) \in (0,+\infty) \quad \text{and} \quad \lim_{s\searrow 0} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{3,D}(s,T) \in (0,+\infty). \tag{1.24}$$

From Theorem 1.14 we deduce that we can extend by continuity also the Dirichlet functionals in (1.9) to the whole compact interval $[0,1]$. From now on, for $j \in \{1,2,3\}$, we will adopt the notation

$$\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,D}(0,T) := \lim_{s\searrow 0} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,D}(s,T), \tag{1.25}$$

for all $T \in (0,+\infty)$ and $\Omega_1, \Omega_2 \subset \Omega$ satisfying the hypothesis of Theorem 1.14.

In Theorems 1.7 and 1.8 we have established that the Neumann and Dirichlet functionals in (1.6) have a common feature. Indeed, if the prey location $x \in \Omega$ is in a sufficiently small neighborhood of the forager starting position $y \in \Omega$, then $\mathcal{E}^{x,y}_{j,D}$ and $\mathcal{E}^{x,y}_{j,N}$ attain their maximum for some value close to $s = 0$.

This characteristic is somewhat preserved if we consider the set-dependent functionals in (1.9). Indeed, we can show that if Ω_1, Ω_2 are close enough (in a sense that will be made precise later), then also for the functionals in (1.9) a strongly nonlocal search strategy will be preferred.

Before stating the precise results we fix some notation. For each $B \subset \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $r \in (0,+\infty)$ we denote

$$r_y B := \{ r(x-y) + y \quad \text{s.t. } x \in B \}. \tag{1.26}$$

Theorem 1.15. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected and $(y,T) \in \Omega \times (0,+\infty)$.*
Then, for each $\varepsilon \in (0,1)$ there exists some $r = r_{\varepsilon,y,T,\Omega} \in (0,+\infty)$ such that for any smooth and disjoint sets $\Omega_1, \Omega_2 \subset B_r(y)$ it holds that

$$\sup_{s\in(0,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s,T) = \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}\left(s^{(1)}_{\Omega_1,\Omega_2,T}, T\right) \quad \text{with} \quad s^{(1)}_{\Omega_1,\Omega_2,T} \in (0,\varepsilon). \tag{1.27}$$

Moreover, let $K \Subset \Omega$ be star-shaped with respect to some $y \in K$. Then, for all $j \in \{2,3\}$ and $\varepsilon \in (0,1)$, there exists some $r = r_{\varepsilon,K,T,\Omega}$ such that if $\Omega_1, \Omega_2 \subset r_y K$ are smooth and disjoint it holds that

$$\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,D}(0,T) \geqslant \sup_{s\in(\varepsilon,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,D}(s,T). \tag{1.28}$$

As a consequence of Theorems 1.14 and 1.15 we can deduce that the most rewarding strategy may not be the safest, similarly to what happens for the functional $\mathcal{E}_{1,D}$ (recall Remark 1.9). Also, a result analogous to Theorem 1.15 holds true when considering the Neumann functionals in (1.9).

Theorem 1.16. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected and $(y,T) \in \Omega \times (0,+\infty)$.*
Then, for each $\varepsilon \in (0,1)$ there exists some $r = r_{\varepsilon,y,T,\Omega} \in (0,+\infty)$ such that for any smooth and disjoint sets $\Omega_1, \Omega_2 \subset B_r(y)$ and for each $j \in \{1,2,3\}$ it holds that

$$\sup_{s\in(0,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,N}(s,T) = \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,N}\left(s^{(j)}_{\Omega_1,\Omega_2,T}, T\right) \quad \text{with} \quad s^{(j)}_{\Omega_1,\Omega_2,T} \in (0,\varepsilon). \tag{1.29}$$

2. Mathematical framework for the efficiency functionals

In this section we establish some technical results regarding the efficiency functionals in (1.5), (1.6) and (1.9). These are the main analytical tools that we will use to prove the results stated in the introduction.

In Section 2.1 we provide some estimates for the functionals in (1.5) and (1.7). This is the content of Lemma 2.7, Theorem 2.9 and Corollary 2.11. These results will be employed in Section 3.2 in order to discuss the environmental scenario where the prey is in proximity of the forager starting location, and thus to prove Theorems 1.7, 1.8, 1.15 and 1.16. Moreover, we establish the limits of the Dirichlet functionals in (1.6) and (1.9) as $s \searrow 0$ as stated in Lemma 2.13. These asymptotics will be used to prove Theorems 1.6, 1.14, 1.7 and 1.15.

To conclude, in Theorem 2.15 and Corollary 2.16 we show that the Neumann functionals in (1.3), (1.4), (1.5), (1.7) and (1.8) do not vanish for $s \searrow 0$, and we provide upper and lower bounds for their lim inf and lim sup. These results will be used in the proofs of Theorem 1.8 and 1.16.

To prove these results, it is useful to recall some properties regarding the fractional heat kernels r^s_D and r^s_N. It is well-known that for each $s \in (0,1)$ these two kernels can be written for each $(t,x,y) \in (0,+\infty)\times\Omega\times\Omega$ as

$$r^s_D(t,x,y) = \int_0^{+\infty} p^\Omega_D(l,x,y)\mu^s_t(l)\,dl \qquad \text{and} \qquad r^s_N(t,x,y) = \int_0^{+\infty} p^\Omega_N(l,x,y)\mu^s_t(l)\,dl, \tag{2.1}$$

where p^Ω_D and p^Ω_N are the classical Dirichlet and Neumann heat kernels in Ω, while μ^s_t is the density of a s-stable subordinator in $(0,+\infty)$ (see e.g. Definition 2.4 in [DGV22a]). For a proof of this latter fact see for instance Propositions 2.8 and 3.5 in [DGV22a].

If $s=1$, the kernels r^1_N and r^1_D coincide respectively with the classical kernels p^Ω_N and p^Ω_D. Furthermore, we also know that the density μ^s_t admits the explicit representation formula

$$\mu^s_t(l) = \frac{1}{\pi}\int_0^{+\infty} e^{-lu-tu^s\cos(\pi s)}\sin(tu^s\sin(\pi s))\,du \qquad \text{for all } (l,s)\in(0,+\infty)\times(0,1), \tag{2.2}$$

see Proposition 3.1 in [KV18].

Moreover, we also recall the following fact on the spectral representation of r^s_D and r^s_N. In what follows we denote by $\{\zeta_{D,k}\}_k$ and $\{\zeta_{N,k}\}_k$ two orthonormal basis of $L^2(\Omega)$ satisfying

$$\begin{cases} -\Delta\zeta_{D,k} = \beta_{D,k}\zeta_{D,k} & \text{in} \quad \Omega,\\ \zeta_{D,k}\in H^1_0(\Omega), \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta\zeta_{N,k} = \beta_{N,k}\zeta_{N,k} & \text{in} \quad \Omega,\\ \dfrac{\partial\zeta_{N,k}}{\partial\nu} = 0 \quad \text{on} & \partial\Omega, \end{cases} \tag{2.3}$$

where $0<\beta_{D,1}<\beta_{D,2}\dots$ and $0=\beta_{N,0}<\beta_{N,1}<\dots$ are respectively the eigenvalues of the Laplace operator with homogeneous Dirichlet and homogeneous Neumann boundary conditions.

Thus, thanks to Theorems 1.8 and 1.20 in [DGV22a], we can rewrite the Dirichlet and Neumann kernels r^s_D and r^s_N as

$$\begin{aligned} & r^s_D(t,x,y) = \sum_{k=1}^{+\infty}\zeta_{D,k}(x)\zeta_{D,k}(y)\exp(-t\beta^s_{D,k}) \\ \text{and} \qquad & r^s_N(t,x,y) = \sum_{k=0}^{+\infty}\zeta_{N,k}(x)\zeta_{N,k}(y)\exp(-t\beta^s_{N,k}), \end{aligned} \tag{2.4}$$

for all $s\in(0,1]$ and $(t,x,y)\in(0,+\infty)\times\Omega\times\Omega$.

Now, we establish some results on μ^s_t. In what follows we recall a scaling property for the density μ^s_t of the s-stable subordinator. For the convenience of the reader the statement is proved.

Lemma 2.1. *Let* $l \in (0, +\infty)$, $t \in (0, +\infty)$ *and* $s \in (0, 1)$.

Then, we have that

$$\mu_t^s(l) = \frac{1}{t^{\frac{1}{s}}} \mu_1^s \left(\frac{l}{t^{\frac{1}{s}}} \right). \tag{2.5}$$

Proof. Let $\alpha := u^s \cos(\pi s)$, $\beta := u^s \sin(\pi s)$ and

$$g(\alpha, \beta) := e^{-t\alpha} \sin(t\beta).$$

With this notation, we integrate by parts the expression on the right-hand side of (2.2) and obtain that

$$\begin{aligned} \mu_t^s(l) &= -\frac{1}{l\pi} e^{-lu} g(\alpha, \beta) \Big|_0^{+\infty} + \frac{1}{\pi l} \int_0^{+\infty} e^{-lu} \frac{d}{du} g(\alpha, \beta)\, du \\ &= 0 + \frac{1}{\pi l} \int_0^{+\infty} e^{-lu} e^{-t\alpha} stu^{s-1} (-\cos(\pi s)\sin(t\beta) + \sin(\pi s)\cos(t\beta))\, du \\ &= \frac{s\,t}{\pi l} \int_0^{+\infty} e^{-lu} e^{-tu^s \cos(\pi s)} u^{s-1} \sin(\pi s - tu^s \sin(\pi s))\, du. \end{aligned} \tag{2.6}$$

We employ the change of variable $v = ut^{\frac{1}{s}}$ and infer from the last identity that

$$\begin{aligned} \mu_t^s(l) &= \frac{s}{\pi l} \int_0^{+\infty} e^{-\frac{l}{t^{\frac{1}{s}}} v} e^{-v^s \cos(\pi s)} v^{s-1} \sin(\pi s - v^s \sin(\pi s))\, dv \\ &= \frac{1}{t^{\frac{1}{s}}} \frac{st^{\frac{1}{s}}}{\pi l} \int_0^{+\infty} e^{-\frac{l}{t^{\frac{1}{s}}} v} e^{-v^s \cos(\pi s)} v^{s-1} \sin(\pi s - v^s \sin(\pi s))\, dv \\ &= \frac{1}{t^{\frac{1}{s}}} \mu_1^s \left(\frac{l}{t^{\frac{1}{s}}} \right). \end{aligned}$$

□

Now, we discuss some asymptotic estimates for the the density $\mu_t^s(l)$ in l. As it is recalled in [BBK+09] by R. Song and proved by Skorohod in [Sko61], one has that

$$\mu_1^s(l) \sim 2\pi\Gamma(1+s) \sin\left(\frac{\pi\, s}{2}\right) \frac{1}{l^{1+s}}, \quad \text{for} \quad l \to +\infty. \tag{2.7}$$

Using this estimate and Lemma 2.1 on the time-scaling property of μ_t^s one obtains an interesting asymptotic expansion in the forthcoming Lemma 2.2. As a side comment, we point out that the asymptotic properties of this type of distributions are relevant to understand how the tail of μ_t^s changes by varying the fractional parameter s, which in turn provides some important information about the optimization problem that we analyze in this paper.

Lemma 2.2. *Let* $s \in (0, 1)$ *and* $t \in (0, +\infty)$.

Then, we have that

$$\mu_t^s(t) \sim 2\pi\Gamma(1+s) \sin\left(\frac{\pi\, s}{2}\right) \frac{t}{l^{1+s}}, \quad \textit{for} \quad l \to +\infty. \tag{2.8}$$

Proof. Thanks to Lemma 2.1, we know that for each $s \in (0, 1)$, $l \in (0, +\infty)$ and $t \in (0, +\infty)$ one has that

$$\mu_t^s(l) = \frac{1}{t^{\frac{1}{s}}} \mu_1^s \left(\frac{l}{t^{\frac{1}{s}}} \right).$$

Thus, using this identity and the estimate in (2.7) one readily obtains that

$$\begin{aligned} \mu_t^s(l) &\sim \frac{1}{t^{\frac{1}{s}}} 2\pi\Gamma(1+s) \sin\left(\frac{\pi s}{2}\right) \frac{t^{\frac{1+s}{s}}}{l^{1+s}} \\ &= 2\pi\Gamma(1+s) \sin\left(\frac{\pi s}{2}\right) \frac{t}{l^{1+s}}, \end{aligned}$$

for $l \to +\infty$. $\square$

The following theorem provides similar estimates to the one given in (2.8) in the range $s \in \left(0, \frac{1}{2}\right)$. Here, the constants involved are less accurate than the one appearing in (2.8), but on the other hand we gain some important information. In particular, while the estimate in (2.8) holds true for $l \to +\infty$, the ones that we prove below are true for each $l \in (t^{\frac{1}{s}}, +\infty)$. This additional information will be used several times.

Theorem 2.3. *Let $s \in \left(0, \frac{1}{2}\right)$ and $t \in (0, +\infty)$.*

Then, there exists some constant $C_1 \in (0, +\infty)$, independent of s and l, such that

$$\begin{aligned} &\frac{s\,t\,C_1}{\pi l^{1+s}} \leqslant \mu_t^s(l) \qquad \text{for all } \ l \in \left(t^{\frac{1}{s}}, +\infty\right) \\ \text{and} \qquad &\mu_t^s(l) \leqslant \frac{s\,t\Gamma(1+s)}{l^{1+s}} \qquad \text{for all } \ l \in (0, +\infty). \end{aligned} \tag{2.9}$$

Proof. Thanks to the scaling property proved in Lemma 2.1, it is enough to show the result for $t = 1$. Indeed, if for $t = 1$ the inequalities in (2.9) hold true, then if $t > 1$ and $l \geqslant t^{\frac{1}{s}}$, we have in view of (2.5) that

$$\mu_t^s(l) = \frac{1}{t^{\frac{1}{s}}} \mu_1^s \left(\frac{l}{t^{\frac{1}{s}}}\right) \geqslant \frac{sC_1 t}{\pi l^{1+s}}.$$

The second inequality in (2.9) is proved similarly. For this reason, we focus our attention on the case $t = 1$.

We will first prove the second inequality in (2.9). If $s \in \left(0, \frac{1}{2}\right)$, from (2.2) we notice that

$$\mu_1^s(l) \leqslant \frac{\sin(\pi s)}{\pi} \int_0^{+\infty} e^{-lu} t u^s \, du \leqslant \frac{s}{\pi l^{1+s}} \Gamma(1+s),$$

which conclude the proof of the second inequality in (2.9).

Now we focus on the proof of the first inequality. To do so, we observe that thanks to equation (2.6) one has that

$$\mu_1^s(l) = \frac{s}{\pi l} \int_0^{+\infty} e^{-lu} e^{-u^s \cos(\pi s)} u^{s-1} \sin(\pi s - u^s \sin(\pi s)) \, du.$$

We perform the change of variable $lu = \theta$ and obtain that

$$\begin{aligned} \mu_s^1(l) &= \frac{s}{\pi l^{1+s}} \int_0^{+\infty} e^{-\theta} e^{-\frac{\theta^s}{l^s} \cos(\pi s)} \theta^{s-1} \sin\left(\pi s - \frac{\theta^s}{l^s} \sin(\pi s)\right) d\theta \\ &=: \frac{s}{\pi l^{1+s}} f(s, l), \end{aligned} \tag{2.10}$$

where by construction $f(s, l) > 0$ for each $l \in (0, +\infty)$ and $s \in (0, 1)$.

Now we observe that, for each $\theta \geqslant 1$ and $s \in \left(0, \frac{1}{2}\right]$,

$$\left| e^{-\theta} e^{-\frac{\theta^s}{l^s} \cos(\pi s)} \theta^{s-1} \sin\left(\pi s - \frac{\theta^s}{l^s} \sin(\pi s)\right) \right| \leqslant e^{-\theta}. \tag{2.11}$$

Thus, by the Dominated Convergence Theorem we obtain that

$$\lim_{s \searrow 0} \int_1^{+\infty} e^{-\theta} e^{-\frac{\theta^s}{l^s} \cos(\pi s)} \theta^{s-1} \sin\left(\pi s - \frac{\theta^s}{l^s} \sin(\pi s)\right) d\theta = 0.$$

Also, by using the change of variable $\theta^s = l^s z$ we deduce that

$$\begin{aligned}&\int_0^1 e^{-\theta} e^{-\frac{\theta^s}{l^s}\cos(\pi s)}\theta^{s-1}\sin\left(\pi s - \frac{\theta^s}{l^s}\sin(\pi s)\right)\,d\theta\\ =&\frac{l^s}{s}\int_0^{\frac{1}{l^s}} e^{-lz^{\frac{1}{s}}} e^{-z\cos(\pi s)}\sin(\pi s - z\sin(\pi s))\,dz\\ =&l^s\int_0^{+\infty} \chi_{[0,\,l^{-s}]} e^{-lz^{\frac{1}{s}}} e^{-z\cos(\pi s)}\frac{\sin(\pi s - z\sin(\pi s))}{s}\,dz.\end{aligned}$$

If $s \in \left(0, \frac{1}{3}\right)$ we also notice that

$$\left|\chi_{[0,\,l^{-s}]} e^{-lz^{\frac{1}{s}}} e^{-z\cos(\pi s)}\frac{\sin(\pi s - z\sin(\pi s))}{s}\right| \leqslant \pi e^{-\frac{z}{2}}(1+z),$$

and therefore, since $l \geqslant 1$, by the Dominated Convergence Theorem we obtain that

$$\lim_{s\searrow 0}\int_0^1 e^{-\theta} e^{-\frac{\theta^s}{l^s}\cos(\pi s)}\theta^{s-1}\sin\left(\pi s - \frac{\theta^s}{l^s}\sin(\pi s)\right)\,d\theta = \pi\int_0^1 e^{-z}(1-z)\,dz.$$

Consequently, for each $l \geqslant 1$

$$\lim_{s\searrow 0} f(s,l) = \pi\int_0^1 e^{-z}(1-z)\,dz = \frac{\pi}{e}. \tag{2.12}$$

We also observe that, if $s \in \left(0, \frac{1}{2}\right]$,

$$\left|e^{-\theta} e^{-\frac{\theta^s}{l^s}\cos(\pi s)}\theta^{s-1}\sin\left(\pi s - \frac{\theta^s}{l^s}\sin(\pi s)\right)\right| \leqslant e^{-\theta}\theta^{s-1},$$

for all $\theta \in (0, +\infty)$.

As a consequence, by the Dominated Convergence Theorem we evince that

$$\lim_{l\to+\infty} f(s,l) = \sin(\pi s)\Gamma(s) > 0, \tag{2.13}$$

for all $s \in \left(0, \frac{1}{2}\right]$.

Besides, by the definition of $f(s,l)$, we have that $f \in C\left(\left(0, \frac{1}{2}\right) \times (1, +\infty)\right)$ and

$$f(s,l) > 0 \quad \text{for all} \quad (s,l) \in \left(0, \frac{1}{2}\right] \times [1, +\infty). \tag{2.14}$$

Therefore, using (2.12), (2.13) and (2.14) we deduce that there exists some $C_1 \in (0, +\infty)$ such that

$$C_1 \leqslant f(s,l) \quad \text{for all} \quad (s,l) \in \left(0, \frac{1}{2}\right) \times [1, +\infty).$$

In light of this observation and equation (2.10) we deduce that

$$\frac{C_1 s}{\pi l^{1+s}} \leqslant \mu_1^s(l). \qquad \square$$

2.1. **Structural results for the efficiency functionals.** This section is devoted to the study of the efficiency functionals in equations (1.5), (1.6) and (1.9). In particular, here we develop the main technical tools that will be employed in the proofs of the results contained in Sections 1.2 and 1.3.

In what follows we adopt the subscript $*$ to refer to the fact that the functional considered can be the one associated with both the Dirichlet and the Neumann case.

We begin by recalling here the following estimates for the classical Dirichlet heat kernel in relation to the classical heat kernel. Using the Weak Maximum Principle for the heat equation one can show that

$$p_D^\Omega(t,x,y) \leqslant \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) \qquad \text{for all } (t,x,y) \in (0,+\infty)\times\Omega\times\Omega. \tag{2.15}$$

On compact subsets of Ω and for finite time spans, one can prove the following lower bound for $p_D^\Omega(t,x,y)$.

Lemma 2.4 (See Lemma 2.1 in [Zha02])**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected.*

Then, there exists a constant $T_\Omega \in (0,+\infty)$ such that for each $K \Subset \Omega$, if we define

$$T_{K,\Omega} := \min\left\{T_\Omega,\ \min_{x\in K} \frac{d^2(x,\partial\Omega)}{2}\right\}, \tag{2.16}$$

then there exist two constants $c_1, c_2 \in (0,+\infty)$, depending on K and Ω, such that

$$p_D^\Omega(t,x,y) \geqslant \frac{c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|x-y|^2}{t}\right) \qquad \text{for all } (t,x,y) \in (0,T_{K,\Omega}]\times K\times K. \tag{2.17}$$

Using the Weak Maximum Principle, it is also possible to compare the Neumann heat kernel with the Dirichlet one, as better specified in the following result.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected and $K' \Subset \Omega$.*

Then, for each $s \in (0,1]$ we have that

$$r_D^s(t,x,y) \leqslant r_N^s(t,x,y) \qquad \text{for all } (t,x,y) \in (0,+\infty)\times\Omega\times\Omega. \tag{2.18}$$

Furthermore, if $K \subseteq K' \Subset \Omega$ is star-shaped with respect to some $x_0 \in K$, there exist some constants $C_{K',\Omega}, c_{K',\Omega} \in (0,+\infty)$ and $\varepsilon_0 \in (0,1)$, depending on K' and Ω, such that

$$r_N^s(t,x,y) \leqslant C_{K',\Omega} r_D^s(t,x_\varepsilon,y_\varepsilon) + c_{K',\Omega} \qquad \text{for all } (t,x,y) \in (0,+\infty)\times K\times K, \tag{2.19}$$

for each $\varepsilon \in (0,\varepsilon_0)$, where $(x_\varepsilon,y_\varepsilon) := (\varepsilon x + (1-\varepsilon)x_0, \varepsilon y + (1-\varepsilon)x_0)$.

Proof. We begin by proving the lower bound in (2.18). To do so, we observe that thanks to the Maximum Principle for the heat equation, one has that

$$p_D^\Omega(t,x,y) \leqslant p_N^\Omega(t,x,y) \qquad \text{for all } (t,x,y) \in (0,+\infty)\times\Omega\times\Omega.$$

Therefore, using (2.1) and the latter inequality, we obtain that

$$r_D^s(t,x,y) = \int_0^{+\infty} p_D^\Omega(l,x,y)\mu_t^s(l)\,dl \leqslant \int_0^{+\infty} p_N^\Omega(l,x,y)\mu_t^s(l)\,dl = r_N^s(t,x,y),$$

for each $(t,x,y) \in (0,+\infty)\times\Omega\times\Omega$. This concludes the proof of (2.18).

Now we show (2.19). Thanks to Theorem 3.2.9 in [Dav89], we have that there exists some constant c_Ω such that

$$p_N^\Omega(t,x,y) \leqslant c_\Omega \max\left\{1,\ \frac{1}{t^{\frac{n}{2}}}\right\} \exp\left(-\frac{|x-y|^2}{6t}\right) \qquad \text{for all } (t,x,y) \in (0,+\infty)\times\Omega\times\Omega. \tag{2.20}$$

Furthermore, if $K \subseteq K' \Subset \Omega$, thanks to Lemma 2.4 we obtain that

$$p_D^\Omega(t,x,y) \geqslant \frac{c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|x-y|^2}{t}\right) \qquad \text{for all } (t,x,y) \in (0,T_{K',\Omega}]\times K'\times K', \tag{2.21}$$

where $T_{K',\Omega}$ is introduced in (2.16) and c_1, c_2 depends on K' and Ω.

Up to a translation we can assume that K is star-shaped with respect to $x_0 = 0$. Now we observe that there exists two constants $C_{K',\Omega} \in (0,+\infty)$ and $\varepsilon_0 \in (0,1)$, such that

$$C_{K',\Omega} c_1 \geqslant c_\Omega \qquad \text{and} \qquad c_2\varepsilon^2 \leqslant \frac{1}{6} \quad \text{for all } \varepsilon \in (0,\varepsilon_0).$$

As a consequence, if for each $\varepsilon \in (0,\varepsilon_0)$ we call $(x_\varepsilon, y_\varepsilon) = \varepsilon(x,y)$, then from (2.20) and (2.21) we obtain that

$$\begin{aligned} C_{K',\Omega} p^\Omega_D(t,x_\varepsilon,y_\varepsilon) - p^\Omega_N(t,x,y) &\geqslant C_{K',\Omega}\frac{c_1}{t^{\frac{n}{2}}}\exp\left(-\frac{c_2|x_\varepsilon-y_\varepsilon|^2}{t}\right) - \frac{c_\Omega}{t^{\frac{n}{2}}}\exp\left(-\frac{|x-y|^2}{6t}\right)\\ &= \frac{c_\Omega}{t^{\frac{n}{2}}}\left(C_{K',\Omega}\frac{c_1}{c_\Omega}\exp\left(-\varepsilon^2 c_2\frac{|x-y|^2}{t}\right) - \exp\left(-\frac{|x-y|^2}{6t}\right)\right)\\ &\geqslant \frac{c_\Omega}{t^{\frac{n}{2}}}\exp\left(-\frac{|x-y|^2}{6t}\right)\left(\exp\left(-\left(\varepsilon^2 c_2 - \frac{1}{6}\right)\frac{|x-y|^2}{t}\right)-1\right)\\ &\geqslant 0, \end{aligned} \tag{2.22}$$

for each $(t,x,y) \in (0, T_{K',\Omega}] \times K \times K$.

Thus, using equation (2.1) and the relation in (2.22) we obtain that

$$\begin{aligned} r^s_N(t,x,y) &= \int_0^{+\infty} p^\Omega_N(l,x,y)\mu^s_t(l)\,dl\\ &= \int_0^{T_{K',\Omega}} p^\Omega_N(l,x,y)\mu^s_t(l)\,dl + \int_{T_{K',\Omega}}^{+\infty} p^\Omega_N(l,x,y)\mu^s_t(l)\,dl\\ &\leqslant C_{K',\Omega}\int_0^{T_{K',\Omega}} p^\Omega_D(l,x_\varepsilon,y_\varepsilon)\mu^s_t(l)\,dl + c_\Omega\int_{T_{K',\Omega}}^{+\infty}\max\left\{1,\,\frac{1}{l^{\frac{n}{2}}}\right\}\exp\left(-\frac{|x-y|^2}{6l}\right)\mu^s_t(l)\,dl\\ &\leqslant C_{K',\Omega}\int_0^{+\infty} p^\Omega_D(l,x_\varepsilon,y_\varepsilon)\mu^s_t(l)\,dl + c_{K',\Omega}\\ &= C_{K',\Omega} r^s_D(t,x_\varepsilon,y_\varepsilon) + c_{K',\Omega}, \end{aligned}$$

for each $(t,x,y) \in (0,+\infty) \times K \times K$, where we defined

$$c_{K',\Omega} := \max_{x,y\in K'}\ \max_{l\in[T_{K,\Omega},+\infty)} c_\Omega \max\left\{1,\,\frac{1}{l^{\frac{n}{2}}}\right\}\exp\left(-\frac{|x-y|^2}{6l}\right). \tag{2.23}$$ □

As a useful consequence of Theorem 2.5, we obtain that it is possible to compare the Neumann functional Φ_N with the Dirichlet one Φ_D. The result goes as follows:

Corollary 2.6. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected and $K' \Subset \Omega$.*

Then, for each $s \in (0,1]$ and $T \in (0,+\infty)$ it holds that

$$\Phi^{x,y}_D(s,T) \leqslant \Phi^{x,y}_N(s,T) \qquad \textit{for all } (x,y) \in \mathcal{C}, \tag{2.24}$$

where $\mathcal{C}$ has been given in (2.35).

Furthermore, for each $K \subseteq K' \Subset \Omega$ star-shaped with respect to some $x_0 \in K$, $s \in (0,1)$ and $T \in (0,+\infty)$, there exists some $\varepsilon_0 \in (0,1)$ such that

$$\Phi^{x,y}_N(s,T) \leqslant C_{K',\Omega}\,\Phi^{x_\varepsilon,y_\varepsilon}_D(s,T) + c_{K',\Omega}\,T \qquad \textit{for all } (x,y) \in \mathcal{C}\cap(K\times K), \tag{2.25}$$

for each $\varepsilon \in (0,\varepsilon_0)$, where $(x_\varepsilon,y_\varepsilon) := (\varepsilon x + (1-\varepsilon)x_0, \varepsilon y + (1-\varepsilon)x_0)$ and $C_{K',\Omega}, c_{K',\Omega} \in (0,+\infty)$ are given in Theorem 2.5.

Proof. Inequalities (2.24) and (2.25) are respectively obtained by integrating over the time t in $(0,T)$ both sides of (2.18) and (2.19). □

In Lemma 2.7 below we establish a lower bound for $\Phi^{x,y}_*(s,T)$, for $x \in \Omega$ in a sufficiently small neighborhood of $y \in \Omega$.

This estimate is pivotal to determine the asymptotic behavior of the functionals in (1.6) when x approaches y, providing some information on the best search strategy in the environmental scenario addressed in Section 1.3, namely where the forager starts its search in proximity of the target.

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. If $(y,T) \in \Omega \times (0,+\infty)$ and $s \in (0,1)$, then there exists some $\widehat{\delta} = \widehat{\delta}_{s,y,T,\Omega} \in (0,+\infty)$ such that, for each $x,z \in B_{\widehat{\delta}}(y)$ satisfying $x \neq z$,*

$$\Phi^{x,z}_*(s,T) \geqslant \frac{C_{s,y,\Omega}}{|x-z|^{n-2s}}, \tag{2.26}$$

for some constant $C_{s,y,\Omega} \in (0,+\infty)$.

Proof. In virtue of inequality (2.24) it is enough to show the result for Φ_D.

Let $y \in \Omega$ and let us denote $d_y := \frac{d(y,\partial\Omega)}{2}$, where

$$d(y,\partial\Omega) := \inf_{x\in\partial\Omega} |x-y|.$$

With this notation we set $B_y := B_{d_y}(y)$.

Now, by (2.16) and (2.17) (used here with $K := B_y$),

$$p^{\Omega}_D(t,x,z) \geqslant \frac{c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|x-z|^2}{t}\right) \qquad \text{for all } (t,x,z) \in (0, T_{B_y,\Omega}] \times B_y \times B_y. \tag{2.27}$$

We also observe that for each $x,z \in \mathbb{R}^n$ such that $x \neq z$, the function

$$g(t) := \frac{c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|x-z|^2}{t}\right)$$

has a maximum in $\varepsilon_{x,z} := \frac{2c_2}{n}|x-z|^2$ and it is increasing in $(0,\varepsilon_{x,z})$ and decreasing in $(\varepsilon_{x,z},+\infty)$.

We set

$$l_{s,y,T} := \min\left\{T_{B_y,\Omega},\, T^{\frac{1}{s}}\right\}$$

and we choose $\widehat{\delta} = \widehat{\delta}_{s,y,T,\Omega}$ such that

$$\widehat{\delta}_{s,y,T,\Omega} := \min\left\{\left(\frac{n l_{s,y,T}}{2c_2}\right)^{\frac{1}{2}}, d_y\right\}. \tag{2.28}$$

It follows that if $x,z \in B_{\widehat{\delta}}(y)$ with $x \neq z$, then $\varepsilon_{x,z} \leqslant l_{s,y,T}$ and $x,z \in B_y$.

To simplify the notation, we simply write $\varepsilon = \varepsilon_{x,z}$. In this way, by (2.1) and (2.27), if $x,z \in B_{\widehat{\delta}}(y)$ and $x \neq z$ we have that

$$\begin{aligned}
\Phi^{x,z}_D(s,T) &= \int_0^T \int_0^{+\infty} p^{\Omega}_D(l,x,z)\, \mu^s_t(l)\, dl\, dt \\
&\geqslant \int_0^{\varepsilon^s} \int_{\frac{\varepsilon}{2}}^{\varepsilon} p^{\Omega}_D(l,x,z)\, \mu^s_t(l)\, dl\, dt \\
&\geqslant \int_0^{\varepsilon^s} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{c_1}{l^{\frac{n}{2}}} \exp\left(-\frac{c_2|x-z|^2}{l}\right) \mu^s_t(l)\, dl\, dt \\
&\geqslant \frac{C}{\varepsilon^{\frac{n}{2}}} \int_0^{\varepsilon^s} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \mu^s_t(l)\, dl\, dt
\end{aligned} \tag{2.29}$$

where we set $C := c_1 2^{\frac{n}{2}} e^{-n}$.

Now we substitute μ^s_t in (2.29) with the expression in (2.2) and obtain that

$$\begin{aligned}
\Phi^{x,z}_D(s,T) &\geqslant \frac{C}{\pi\varepsilon^{\frac{n}{2}}} \int_0^{\varepsilon^s} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_0^{+\infty} e^{-lu-tu^s\cos(\pi s)} \sin(tu^s \sin(\pi s))\, du\, dl\, dt \\
&= \frac{C}{\pi\varepsilon^{\frac{n}{2}}} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_0^{+\infty} \int_0^{\varepsilon^s} e^{-lu-tu^s\cos(\pi s)} \sin(tu^s \sin(\pi s))\, dt\, du\, dl. \\
&=: \mathcal{L}.
\end{aligned} \tag{2.30}$$

Setting $F(t) := e^{-t\alpha}\sin(t\beta)$, with $\alpha := u^s\cos(\pi s)$ and $\beta := u^s\sin(\pi s)$, for each $T \in (0,+\infty)$ we integrate by parts and see that

$$\begin{aligned}\int_0^T F(t)\,dt &= -\frac{1}{\alpha}e^{-t\alpha}\sin(t\beta)\Big|_0^T + \frac{\beta}{\alpha}\int_0^T e^{-t\alpha}\cos(t\beta)\,dt\\ &= -\frac{1}{\alpha}e^{-T\alpha}\sin(T\beta) - \frac{\beta}{\alpha^2}e^{-t\alpha}\cos(t\beta)\Big|_0^T - \frac{\beta^2}{\alpha^2}\int_0^T e^{-t\alpha}\sin(t\beta)\,dt\\ &= -\frac{1}{\alpha}e^{-T\alpha}\sin(T\beta) - \frac{\beta}{\alpha^2}e^{-T\alpha}\cos(T\beta) + \frac{\beta}{\alpha^2} - \frac{\beta^2}{\alpha^2}\int_0^T F(t)\,dt.\end{aligned}$$

Therefore, by replacing α and β with their corresponding values, one obtains that

$$\begin{aligned}&\int_0^T F(t)\,dt\\ &= -\frac{\cos(\pi s)}{u^s}e^{-Tu^s\cos(\pi s)}\sin(Tu^s\sin(\pi s)) - \frac{\sin(\pi s)}{u^s}e^{-Tu^s\cos(\pi s)}\cos(Tu^s\sin(\pi s)) + \frac{\sin(\pi s)}{u^s}\\ &= \frac{1}{u^s}\left(\sin(\pi s) - e^{-Tu^s\cos(\pi s)}\sin(Tu^s\sin(\pi s)+\pi s)\right).\end{aligned}\tag{2.31}$$

By (2.30), (2.31) and the change of variables $(U,L) = \left(u\varepsilon, \frac{l}{\varepsilon}\right)$ one obtains that

$$\begin{aligned}\mathcal{L} &= \frac{C}{\pi\varepsilon^{\frac{n}{2}}}\int_{\frac{\varepsilon}{2}}^{\varepsilon}\int_0^{+\infty}\frac{e^{-lu}}{u^s}\left(\sin(\pi s) - e^{-\varepsilon^s u^s\cos(\pi s)}\sin(\pi s + \varepsilon^s u^s\sin(\pi s))\right)du\,dl\\ &= \frac{C}{\pi\varepsilon^{\frac{n}{2}-s}}\int_{\frac{1}{2}}^{1}\int_0^{+\infty}\frac{e^{-LU}}{U^s}\left(\sin(\pi s) - e^{-U^s\cos(\pi s)}\sin(\pi s + U^s\sin(\pi s))\right)dU\,dL\\ &=: \frac{C}{\pi\varepsilon^{\frac{n}{2}-s}}\mathcal{J}_s,\end{aligned}\tag{2.32}$$

where $\mathcal{J}_s$ does not depend on ε and is defined by

$$\mathcal{J}_s := \int_{\frac{1}{2}}^{1}\int_0^{+\infty}\frac{e^{-LU}}{U^s}\left(\sin(\pi s) - e^{-U^s\cos(\pi s)}\sin(\pi s + U^s\sin(\pi s))\right)dU\,dL.\tag{2.33}$$

Note also that by construction

$$\mathcal{J}_s = \frac{1}{\varepsilon^s}\int_{\frac{\varepsilon}{2}}^{\varepsilon}\int_0^{\varepsilon^s}\mu_t^s(l)\,dt\,dl,$$

which means that $\mathcal{J}_s \in (0,+\infty)$, since $\mu_t^s(l) \in (0,+\infty)$ for each s, t and l.

Accordingly, from (2.30) and (2.32),

$$\begin{aligned}\Phi_D^{x,z}(s,T) &\geqslant \frac{C\pi^{-1}}{\varepsilon^{n-2s}}\mathcal{J}_s\\ &= \frac{2^s\,e^{-n}\,c_1\,n^{\frac{n}{2}-s}}{\pi c_2^{\frac{n}{2}-s}|x-z|^{n-2s}}\mathcal{J}_s\\ &\geqslant \frac{C_{y,\Omega}}{|x-z|^{n-2s}}\mathcal{J}_s\\ &= \frac{C_{s,y,\Omega}}{|x-z|^{n-2s}},\end{aligned}$$

where we have defined

$$C_{y,\Omega} := \min_{s\in(0,1)}\frac{2^s\,e^{-n}\,c_1\,n^{\frac{n}{2}-s}}{\pi\,c_2^{\frac{n}{2}-s}} \qquad\text{and}\qquad C_{s,y,\Omega} := C_{y,\Omega}\mathcal{J}_s.\tag{2.34}$$

□

As a consequence of Lemma 2.7, we have that if x approaches y, then the functional $\Phi_*^{x,y}(s,T)$ diverges to infinity as far as $n>2s$.

In the following result we make this statement precise. In particular, we show that divergence holds true as far as $n\geqslant 2s$.

Before stating the result, we define

$$\mathcal{C} := (\Omega\times\Omega)\setminus\{(p,p)\ \text{ s.t. }\ p\in\Omega\} = \big\{(p,q)\in\Omega\times\Omega\ \text{ s.t. }\ p\neq q\big\}. \tag{2.35}$$

Theorem 2.8. *Let $\Omega\subset\mathbb{R}^n$ be bounded, smooth and connected and $T\in(0,+\infty)$. If $n\geqslant 2$ or $n=1$ and $s\in\left(0,\frac12\right]$ we have that*

$$\lim_{(x,y)\to(z,z)} \Phi_*^{x,y}(s,T) = +\infty \tag{2.36}$$

$$\text{and}\qquad \Phi_*^{z,z}(s,T) = +\infty, \tag{2.37}$$

for each $z\in\Omega$.

Proof. We will prove only the Dirichlet case, since the Neumann one follows easily from the Dirichlet one and (2.18).

We first focus on the proof of (2.37). Using the identity (2.1), equations (2.16) and (2.17) together with the formula in (2.2) we deduce that if

$$\delta_{s,x,T} := \min\left\{T_{x,\Omega}, T^{\frac1s}\right\},$$

where $T_{x,\Omega}$ is given in (2.16), then for each $\delta\in(0,\delta_{s,x,T})$ it holds that

$$\begin{aligned}
\Phi_D^{x,x}(s,T) &= \int_0^T\int_0^{+\infty} p_D^\Omega(l,x,x)\,\mu_t^s(l)\,dl\,dt\\
&= \int_0^{+\infty} p_D^\Omega(l,x,x)\int_0^T \mu_t^s(l)\,dt\,dl\\
&\geqslant \int_0^\delta \frac{c_1}{l^{\frac n2}}\int_0^{\delta^s}\mu_t^s(l)\,dt\,dl\\
&= \frac1\pi\int_0^\delta \frac{c_1}{l^{\frac n2}}\int_0^{\delta^s}\int_0^{+\infty} e^{-lu-tu^s\cos(\pi s)}\sin\left(tu^s\sin(\pi s)\right)\,du\,dt\,dl\\
&\geqslant \frac{c_1}{\pi\delta^{\frac n2}}\int_0^\delta\int_0^{+\infty}\int_0^{\delta^s} e^{-lu-tu^s\cos(\pi s)}\sin(tu^s\sin(\pi s))\,dt\,du\,dl,
\end{aligned}\tag{2.38}$$

where c_1 is introduced in (2.17).

Now, equations (2.31) and (2.38) together with the change of variables $(L,U)=\left(\frac l\delta, u\delta\right)$ yield

$$\begin{aligned}
\Phi_D^{x,x}(s,T) &\geqslant \frac{c_1}{\pi\delta^{\frac n2}}\int_0^\delta\int_0^{+\infty}\int_0^{\delta^s} e^{-lu-tu^s\cos(\pi s)}\sin(tu^s\sin(\pi s))\,dt\,du\,dl\\
&= \frac{c_1}{\pi\delta^{\frac n2}}\int_0^\delta\int_0^{+\infty} e^{-lu}\frac{1}{u^s}\left(\sin(\pi s)-e^{-\delta^s u^s\cos(\pi s)}\sin(\delta^s u^s\sin(\pi s)+\pi s)\right)du\,dl\\
&= \frac{c_1}{\pi\delta^{\frac n2-s}}\int_0^1\int_0^{+\infty} e^{-LU}\frac{1}{U^s}\left(\sin(\pi s)-e^{-U^s\cos(\pi s)}\sin(U^s\sin(\pi s)+\pi s)\right)dU\,dL\\
&=: \frac{c_1}{\pi\delta^{\frac n2-s}}\mathcal{G}_s.
\end{aligned}\tag{2.39}$$

We also observe that $\mathcal{G}_s$ does not depend on δ and by construction

$$\mathcal{G}_s = \frac{1}{\delta^s}\int_0^\delta\int_0^{\delta^s}\mu_t^s(l)\,dt\,dl,$$

which means that $\mathcal{G}_s \in (0,+\infty)$, since $\mu_t^s(l) \in (0,+\infty)$ for each $s \in (0,1)$, $t \in (0,+\infty)$ and $l \in (0,+\infty)$.

Therefore, recalling equation (2.39) we deduce that

$$\Phi_D^{x,x}(s,T) \geqslant \lim_{\delta \searrow 0} \frac{c_1}{\pi \delta^{\frac{n}{2}-s}} \mathcal{G}_s = +\infty.$$

if either $n \geqslant 2$ or $n = 1$ and $s \in \left(0, \frac{1}{2}\right)$.

Hence, to complete the proof of (2.37), it is left to consider the case $n = 1$ and $s = \frac{1}{2}$. When $s = \frac{1}{2}$ equation (2.2) boils down to

$$\mu_t^{\frac{1}{2}}(l) = \frac{1}{\pi} \int_0^{+\infty} e^{-lu} \sin\left(tu^{\frac{1}{2}}\right) du. \tag{2.40}$$

Therefore, using the latter identity, (2.16) and (2.17) we obtain that there exists $T_{x,\Omega} \in (0,+\infty)$ such that if $\delta \in (0, T_{x,\Omega})$ then

$$\begin{aligned}
\Phi_D^{x,x}\left(\frac{1}{2}, T\right) &= \int_0^T \int_0^{+\infty} p_D^{\Omega}(l,x,x)\, \mu_t^{\frac{1}{2}}(l)\, dl\, dt \\
&\geqslant \int_0^{\delta} \int_0^T p_D^{\Omega}(l,x,x)\, \mu_t^{\frac{1}{2}}(l)\, dt\, dl \\
&\geqslant c_1 \int_0^{\delta} \int_0^T \frac{1}{l^{\frac{1}{2}}} \mu_t^{\frac{1}{2}}(l)\, dt\, dl \\
&= \frac{c_1}{\pi} \int_0^{\delta} \int_0^T \int_0^{+\infty} \frac{1}{l^{\frac{1}{2}}} e^{-lu} \sin\left(tu^{\frac{1}{2}}\right) du\, dt\, dl \\
&= \frac{c_1}{\pi} \int_0^{\delta} \int_0^{+\infty} \int_0^T \frac{1}{l^{\frac{1}{2}}} e^{-lu} \sin\left(tu^{\frac{1}{2}}\right) dt\, du\, dl \\
&= \frac{c_1}{\pi} \int_0^{\delta} \int_0^{+\infty} \frac{1}{l^{\frac{1}{2}}} e^{-lu} \left(1 - \cos\left(Tu^{\frac{1}{2}}\right)\right) \frac{1}{u^{\frac{1}{2}}}\, du\, dl,
\end{aligned}$$

where $c_1 \in (0,+\infty)$ has been introduced in (2.17).

Furthermore, by making the change of variable $lu = a$ in the l variable we deduce that

$$\begin{aligned}
\Phi_D^{x,x}\left(\frac{1}{2}, T\right) &\geqslant \frac{c_1}{\pi} \int_0^{+\infty} \int_0^{\delta u} \frac{e^{-a}}{a^{\frac{1}{2}}} \left(1 - \cos\left(Tu^{\frac{1}{2}}\right)\right) \frac{1}{u}\, da\, du \\
&\geqslant \frac{c_1}{\pi} \int_{\frac{1}{\delta}}^{+\infty} \int_0^{\delta u} \frac{e^{-a}}{a^{\frac{1}{2}}} \left(1 - \cos\left(Tu^{\frac{1}{2}}\right)\right) \frac{1}{u}\, da\, du \\
&=: \mathcal{I}.
\end{aligned} \tag{2.41}$$

We also observe that for each $u \geqslant \frac{1}{\delta}$ one has that

$$0 < c := \int_0^1 \frac{e^{-a}}{a^{\frac{1}{2}}}\, da \leqslant \int_0^{\delta u} \frac{e^{-a}}{a^{\frac{1}{2}}}\, da \leqslant \Gamma\left(\frac{1}{2}\right).$$

Moreover, defining

$$\widetilde{k} := \min\left\{ k \in \mathbb{N} \quad \text{s.t. } k \geqslant -\frac{1}{4} + \frac{T}{2\pi \delta^{\frac{1}{2}}} \right\},$$

we find that $\left(\frac{\pi}{2T}+\frac{2\pi\widetilde{k}}{T}\right)^2>\frac{1}{\delta}$, and thus we deduce from (2.41) that

$$\begin{aligned}\mathcal{I}&\geqslant\frac{cc_1}{\pi}\int_{\frac{1}{\delta}}^{+\infty}\frac{1-\cos\left(Tu^{\frac{1}{2}}\right)}{u}\,du\\&\geqslant\frac{cc_1}{\pi}\sum_{k=\widetilde{k}}^{+\infty}\int_{\left(\frac{\pi}{2T}+\frac{2\pi k}{T}\right)^2}^{\left(\frac{3\pi}{2T}+\frac{2\pi k}{T}\right)^2}\frac{du}{u}\\&=\frac{2cc_1}{\pi}\sum_{k=\widetilde{k}}^{+\infty}\ln\left(\frac{\frac{3\pi}{2k}+2\pi}{\frac{\pi}{2k}+2\pi}\right)\\&=:\mathcal{II}.\end{aligned}$$

Therefore, using Taylor's expansion we infer that there exists some $\widetilde{K}\in\mathbb{N}$ with $\widetilde{K}\geqslant\widetilde{k}$ such that

$$\mathcal{II}\geqslant\frac{2cc_1}{\pi}\sum_{k=\widetilde{K}}^{+\infty}\frac{1}{2k}+o\left(\frac{1}{k^2}\right)=+\infty. \tag{2.42}$$

This concludes to proof of (2.37).

Now we prove (2.36). If $n>2s$, equation (2.36) is a direct consequence of inequality (2.26). Therefore, to conclude the proof of (2.36) it is left to show the case $n=1$ and $s=\frac{1}{2}$. In order to achieve this, we observe that if $V_x\Subset\Omega$ is some neighborhood of x in Ω, then there exists some $k_0\in\mathbb{N}$ such that for each $k\geqslant k_0$ it holds that $(x_k,y_k)\in V_x\times V_x$.

Thus, if we define $\varepsilon_k:=|x_k-y_k|^2$, in view of (2.16) and (2.17), and recalling (2.40), we obtain that, if $k\geqslant k_0$,

$$\begin{aligned}\Phi_D^{x_k,y_k}\left(\frac{1}{2},T\right)&=\int_0^T\int_0^{+\infty}p_D^\Omega(l,x_k,y_k)\mu_t^{\frac{1}{2}}(l)\,dl\,dt\\&=\int_0^{+\infty}\int_0^T p_D^\Omega(l,x_k,y_k)\mu_t^{\frac{1}{2}}(l)\,dt\,dl\\&\geqslant\frac{1}{\pi}\int_0^{T_{V_x,\Omega}}\int_0^T\int_0^{+\infty}\frac{c_1}{l^{\frac{1}{2}}}\exp\left(-\frac{c_2\varepsilon_k}{l}\right)e^{-lu}\sin\left(tu^{\frac{1}{2}}\right)du\,dt\,dl,\\&=\frac{1}{\pi}\int_0^{T_{V_x,\Omega}}\int_0^{+\infty}\frac{c_1}{l^{\frac{1}{2}}}\exp\left(-\frac{c_2\varepsilon_k}{l}\right)e^{-lu}\frac{\left(1-\cos\left(Tu^{\frac{1}{2}}\right)\right)}{u^{\frac{1}{2}}}\,du\,dl\\&\geqslant\frac{c_1}{\pi}\int_{\frac{2}{T_{V_x,\Omega}}}^{+\infty}\int_{\frac{1}{u}}^{\frac{2}{u}}\frac{1}{l^{\frac{1}{2}}}\exp\left(-\frac{c_2\varepsilon_k}{l}\right)e^{-lu}\frac{\left(1-\cos\left(Tu^{\frac{1}{2}}\right)\right)}{u^{\frac{1}{2}}}\,dl\,du\end{aligned} \tag{2.43}$$

where $T_{V_x,\Omega}\in(0,+\infty)$ and $c_1,c_2\in(0,+\infty)$ are given respectively in (2.16) and (2.17).

Now we choose $\widetilde{j},j(\varepsilon_k)\in\mathbb{N}$ such that

$$\widetilde{j}:=\min\left\{j\in\mathbb{N}\quad\text{s.t. }j\geqslant\frac{T}{2\pi}\left(\frac{2}{T_{V_x,\Omega}}\right)^{\frac{1}{2}}-\frac{1}{4}\right\}$$

$$\text{and}\qquad j(\varepsilon_k):=\max\left\{j\in\mathbb{N}\quad\text{s.t. }j\leqslant\frac{T}{2\pi\varepsilon_k^{\frac{1}{2}}}-\frac{3}{4}\right\}.$$

Note that if ε_k is chosen small enough, then $\widetilde{j}<j(\varepsilon_k)$.

With this choices one has that

$$\left(\frac{\pi}{2T}+\frac{2\pi\widetilde{j}}{T}\right)^2 \geqslant \frac{2}{T_{V_x,\Omega}} \qquad \text{and} \qquad \left(\frac{3\pi}{2T}+\frac{2\pi j(\varepsilon_k)}{T}\right)^2 \leqslant \frac{1}{\varepsilon_k}. \tag{2.44}$$

Therefore, with this latter notation we obtain from (2.43) that

$$\begin{aligned}\Phi_D^{x_k,y_k}\left(\frac{1}{2},T\right) &\geqslant \frac{c_1}{\pi}\sum_{j=\widetilde{j}}^{j(\varepsilon_k)}\int_{\left(\frac{\pi}{2T}+\frac{2\pi j}{T}\right)^2}^{\left(\frac{3\pi}{2T}+\frac{2\pi j}{T}\right)^2}\int_{\frac{1}{u}}^{\frac{2}{u}}\frac{1}{l^{\frac{1}{2}}}\exp\left(-\frac{c_2\varepsilon_k}{l}\right)e^{-lu}\frac{1}{u^{\frac{1}{2}}}\,dl\,du\\ &\geqslant \frac{c_1}{\pi}\sum_{j=\widetilde{j}}^{j(\varepsilon_k)}\int_{\left(\frac{\pi}{2T}+\frac{2\pi j}{T}\right)^2}^{\left(\frac{3\pi}{2T}+\frac{2\pi j}{T}\right)^2}\int_{\frac{1}{u}}^{\frac{2}{u}}\exp\left(-c_2u\varepsilon_k\right)e^{-2}\,dl\,du\\ &= \frac{c_1e^{-2}}{\pi}\sum_{j=\widetilde{j}}^{j(\varepsilon_k)}\int_{\left(\frac{\pi}{2T}+\frac{2\pi j}{T}\right)^2}^{\left(\frac{3\pi}{2T}+\frac{2\pi j}{T}\right)^2}\frac{1}{u}\exp\left(-c_2u\varepsilon_k\right)\,du.\end{aligned}$$

Now, we deduce from (2.44) that since $u \leqslant \left(\frac{3\pi}{2T}+\frac{2\pi j(\varepsilon_k)}{T}\right)^2$, then $u\varepsilon_k \leqslant 1$, and thus from the latter computations we obtain that

$$\begin{aligned}\Phi_D^{x_k,y_k}\left(\frac{1}{2},T\right) &\geqslant \frac{c_1e^{-c_2}}{\pi e^2}\sum_{j=\widetilde{j}}^{j(\varepsilon_k)}\int_{\left(\frac{\pi}{2T}+\frac{2\pi j}{T}\right)^2}^{\left(\frac{3\pi}{2T}+\frac{2\pi j}{T}\right)^2}\frac{1}{u}\,du\\ &= \frac{2c_1e^{-c_2}}{\pi e^2}\sum_{j=\widetilde{j}}^{j(\varepsilon_k)}\ln\left(\frac{\frac{3\pi}{2k}+2\pi}{\frac{\pi}{2k}+2\pi}\right).\end{aligned} \tag{2.45}$$

As we observed in (2.42), one has that

$$\sum_{k=1}^{+\infty}\ln\left(\frac{\frac{3\pi}{2k}+2\pi}{\frac{\pi}{2k}+2\pi}\right) = +\infty.$$

With reference to that, from (2.45) and the latter observation we obtain that

$$\lim_{k\to+\infty}\Phi_D^{x_k,y_k}\left(\frac{1}{2},T\right) \geqslant \lim_{k\to+\infty}\frac{2c_1e^{-c_2}}{\pi e^2}\sum_{j=\widetilde{j}}^{j(\varepsilon_k)}\ln\left(\frac{\frac{3\pi}{2k}+2\pi}{\frac{\pi}{2k}+2\pi}\right) = \frac{2c_1e^{-c_2}}{\pi e^2}\sum_{j=\widetilde{j}}^{+\infty}\ln\left(\frac{\frac{3\pi}{2k}+2\pi}{\frac{\pi}{2k}+2\pi}\right) = +\infty.$$

This completes the proof of (2.36). □

In the following result we give some upper bounds for the functional $\Phi_*^{x,y}(s,T)$. These estimates, together with the lower bound in (2.26), will turn out to be pivotal in order to determine the most rewarding search strategy in a regime where the initial position of the forager is close to the one of the prey, and thus prove Theorems 1.7 and 1.8.

In the Dirichlet framework, the behavior of the functional $\Phi_D^{x,y}(s,T)$ for x approaching y could be deduced from the already known estimates on the Green function $G_D^\Omega(x,y)$ of the Dirichlet spectral fractional Laplacian, see Theorem 5.4 in [SV03].

Indeed, the Green function is given by

$$G_D^\Omega(x,y) := \int_0^{+\infty} r_D^s(t,x,y)\,dt,$$

for $x \neq y$, and therefore

$$\Phi_D^{x,y}(s,T) \leqslant G_D^\Omega(x,y),$$

for each $(x,y,T) \in \Omega\times\Omega\times(0,+\infty)$ with $x \neq y$ and $s \in (0,1)$.

Nevertheless, for our optimization purposes we need upper bounds where the dependence of the constants on the fractional exponent $s \in (0,1)$ is known. In this sense, the inequalities provided in the following result are more suitable in this context than the ones available in the literature for G_D^Ω.

Before stating and proving the theorem, we fix the following notation. For each $n \in \mathbb{N}$ and $s \in (0,1)$ we define the set

$$A_{n,s} := \left(0, 1+\frac{n}{2}-s\right) \cap \left[\frac{n}{2}-s, 1+\frac{n}{2}-s\right), \tag{2.46}$$

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. Moreover, let $K \Subset \Omega$ be star-shaped with respect to some $x_0 \in K$.*

Then, for each $s \in (0,1)$ and $T \in (0,+\infty)$, there exists some $C_{*,K,T,\Omega} \in (0,+\infty)$ such that if $n \geqslant 3$ then

$$\Phi_*^{x,y}(s,T) \leqslant \frac{C_{*,K,T,\Omega}}{|x-y|^{n-2s}} \qquad \text{for all } (x,y) \in \mathcal{C} \cap (K \times K), \tag{2.47}$$

where $\mathcal{C}$ is given in (2.35).

Furthermore, if $n \leqslant 2$, $s \in (0,1)$ and $\mu \in A_{n,s}$ there exists some $C_{*,\mu,K,T,\Omega} \in (0,+\infty)$ such that

$$\Phi_*^{x,y}(s,T) \leqslant \frac{C_{*,\mu,K,T,\Omega}}{|x-y|^{2\mu}} \qquad \text{for all } (x,y) \in \mathcal{C} \cap (K \times K), \tag{2.48}$$

where $A_{n,s}$ is defined (2.46).

Proof. We will first show the result for the Dirichlet case. To this aim, we recall the following identity

$$\int_0^{+\infty} r_D^s(t,x,y)\,dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} p_D^\Omega(t,x,y) t^{s-1}\,dt \quad \text{for all } (x,y) \in \mathcal{C}, \tag{2.49}$$

see for instance equation (2.4) in [SV03]. For the convenience of the reader we give a proof of it in the appendix, see Proposition A.1.

We first prove (2.47). If $((x,y),T) \in \mathcal{C} \times (0,+\infty)$, thanks to the identity in (2.49) we have that

$$\Phi_D^{x,y}(s,T) = \int_0^T r_D^s(t,x,y)\,dt \leqslant \int_0^{+\infty} r_D^s(t,x,y)\,dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} p_D^\Omega(t,x,y) t^{s-1}\,dt.$$

Using inequality (2.15) and the change of variable $a = \frac{|x-y|^2}{4t}$ we obtain that

$$\begin{aligned} \Phi_D^{x,y}(s,T) &\leqslant \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) dt \\ &= \frac{4^{-s}}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{1}{|x-y|^{n-2s}} \int_0^{+\infty} a^{\frac{n}{2}-1-s} e^{-a}\,da \\ &= \frac{4^{-s}}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{\Gamma\left(\frac{n}{2}-s\right)}{|x-y|^{n-2s}}. \end{aligned} \tag{2.50}$$

Thus, by defining the constant

$$C_D := \sup_{s\in(0,1)} \frac{4^{-s}}{\pi^{\frac{n}{2}}\Gamma(s)} \Gamma\left(\frac{n}{2}-s\right), \tag{2.51}$$

we conclude the proof of (2.47) for the Dirichlet case.

Now, we prove (2.48). To this end, we observe that there exists some constant $c_3 \in (0,+\infty)$, depending on Ω, such that for all $\gamma \in [0,1)$ it holds that

$$p_D^\Omega(t,x,y) \leqslant \frac{c_3}{t^{\frac{n}{2}+\gamma}} \exp\left(-\frac{|x-y|^2}{6t}\right) \qquad \text{for all } (t,x,y) \in (0,+\infty) \times \Omega \times \Omega, \tag{2.52}$$

see for instance Theorem 4.6.9 in [Dav89].

Accordingly, using the identity given in equation (2.49) and the inequality in equation (2.52), we deduce that

$$\begin{aligned}
\Phi^{x,y}_D(s,T) &= \int_0^T r^s_D(t,x,y)\,dt \\
&\leqslant \int_0^{+\infty} r^s_D(t,x,y)\,dt \\
&= \frac{1}{\Gamma(s)}\int_0^{+\infty} p^\Omega_D(t,x,y)t^{s-1}\,dt \\
&\leqslant \frac{c_3}{\Gamma(s)}\int_0^{+\infty} \frac{1}{t^{\frac{n}{2}+\gamma-s+1}}\exp\left(-\frac{|x-y|^2}{6t}\right)dt \\
&= \frac{c_3}{\Gamma(s)}\int_0^{+\infty} \frac{(6\theta)^{\frac{n}{2}+\gamma-s-1}}{|x-y|^{n-2(s-\gamma)}}e^{-\theta}\,d\theta \\
&= \frac{C_{s,\gamma,n,\Omega}}{\Gamma(s)}\frac{1}{|x-y|^{n-2(s-\gamma)}},
\end{aligned} \tag{2.53}$$

where we applied the change of variable $\theta = \frac{|x-y|^2}{4t}$ and we defined

$$C_{s,\gamma,n,\Omega} := c_3 6^{\frac{n}{2}+\gamma-s-1}\Gamma\left(\frac{n}{2}+\gamma-s\right),$$

for all $\gamma \in \left(s-\frac{n}{2},1\right)\cap[0,1)$.

Now, we observe that if we define $\mu := \frac{n}{2}+\gamma-s$, then $\mu \in \left(0, 1+\frac{n}{2}-s\right)\cap\left[\frac{n}{2}-s, 1+\frac{n}{2}-s\right)$, and equation (2.53) becomes

$$\Phi^{x,y}_D(s,T) \leqslant \frac{c_3}{\Gamma(s)}6^{\mu-1}\frac{\Gamma(\mu)}{|x-y|^{2\mu}} \leqslant \frac{C_{D,\mu}}{|x-y|^{2\mu}}, \tag{2.54}$$

where we defined

$$C_{D,\mu} := \sup_{s\in(0,1)} \frac{c_3}{\Gamma(s)}6^{\mu-1}\Gamma(\mu). \tag{2.55}$$

This concludes the proof of (2.48) for the Dirichlet case.

Employing the result in Corollary 2.6 we prove now (2.47) and (2.48) for the Neumann case. Let $K \Subset \Omega$ and, up to a translation, let us assume that it is star-shaped with respect to $x_0 = 0$.

Then, if $T \in (0,+\infty)$, $n \geqslant 3$ and $s \in (0,1)$ using equations (2.50) and (2.25) with $K' = K$ we obtain the existence of some $c_{K,\Omega}, C_{K,\Omega} \in (0,+\infty)$ and $\varepsilon_0 \in (0,1)$, depending on K and Ω, such that

$$\Phi^{x,y}_N(s,T) \leqslant C_{K,\Omega}\Phi^{x_\varepsilon,y_\varepsilon}_D(s,T) + c_{K,\Omega}T \leqslant C_{K,\Omega}\frac{C_D\varepsilon^{2s-n}}{|x-y|^{n-2s}} + c_{K,\Omega}T,$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $(x,y) \in \mathcal{C}\cap(K\times K)$.

Consequently, if in the last equation we choose $\varepsilon_1 \in (0,\varepsilon_0)$ such that

$$\varepsilon_1 \leqslant \inf_{s\in(0,1)}\left(C_{K,\Omega}C_D\frac{d_K^{2s-n}}{c_{K,\Omega}T}\right)^{\frac{1}{n-2s}}, \tag{2.56}$$

which depends on K,Ω and T, we obtain that for all $(x,y)\in\mathcal{C}\cap(K\times K)$ and $s\in(0,1)$ it holds that

$$\Phi^{x,y}_N(s,T) \leqslant \frac{C_{K,T,\Omega}}{|x-y|^{n-2s}},$$

with

$$C_{K,T,\Omega} := 2\sup_{s\in(0,1)} C_n\varepsilon_1^{2s-n}C_{K,\Omega}. \tag{2.57}$$

Analogously, if $n \leqslant 2$, $s \in (0,1)$ and $\mu \in A_{n,s}$, then one deduces from (2.54) and (2.25) that

$$\Phi_N^{x,y}(s,T) \leqslant C_{K,\Omega}\Phi_D^{x_\varepsilon,y_\varepsilon}(s,T) + c_{K,\Omega}T \leqslant C_{K,\Omega}\frac{C_{D,\mu}\varepsilon^{-2\mu}}{|x-y|^{2\mu}} + c_{K,\Omega}T,$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $(x,y) \in \mathcal{C} \cap (K \times K)$.

As a result, if we choose some $\varepsilon_2 \in (0,\varepsilon_0)$ satisfying

$$\varepsilon_2^{2\mu} \leqslant C_{K,\Omega}C_{D,\mu}\frac{d_K^{-2\mu}}{c_{K,\Omega}T}, \tag{2.58}$$

which depends on μ, K, Ω and T, we obtain that for all $(x,y) \in \mathcal{K}$ and $s \in (0,1)$ it holds that

$$\Phi_N^{x,y}(s,T) \leqslant \frac{C_{\mu,K,T,\Omega}}{|x-y|^{2\mu}},$$

where we set

$$C_{\mu,K,T,\Omega} := 2C_{K,\Omega}C_{\mu,\Omega}\varepsilon_2^{-2\mu}. \tag{2.59}$$

□

Remark 2.10. *We note that for the Dirichlet case we obtained that the constants in equation* (2.47) *and* (2.48) *can be chosen independently from K and T. In particular, we have proved that if $n \geqslant 3$, then*

$$\Phi_D^{x,y}(s,T) \leqslant \frac{4^{-s}}{\pi^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{2}-s\right)}{|x-y|^{n-2s}} \tag{2.60}$$

for all $(x,y) \in \mathcal{C}$. If $n \leqslant 2$ and $\mu \in A_{n,s}$, where $A_{n,s}$ is given in (2.46), *then*

$$\Phi_D^{x,y}(s,T) \leqslant \frac{6^{\mu-1}c_3}{\Gamma(s)}\frac{\Gamma(\mu)}{|x-y|^{2\mu}}, \tag{2.61}$$

for all $(x,y) \in \mathcal{C}$.

We now turn our attention to the functional $\widetilde{\Phi}_*^{\Omega_1,\Omega_2}$ defined in (1.7). For this, it is convenient, for every bounded and measurable sets $\Omega_1,\Omega_2 \subset \mathbb{R}^n$ and each $s \in (0,1)$, to define

$$F^{\Omega_1,\Omega_2}(s) := \int_{\Omega_1\times\Omega_2}\frac{1}{|x-y|^{n-2s}}\,dx\,dy. \tag{2.62}$$

As a direct consequence of Lemma 2.7 and Theorem 2.9 we obtain the following upper and lower bounds for $\widetilde{\Phi}_*^{\Omega_1,\Omega_2}$. These bounds will play a crucial role in proving Theorems 1.15 and 1.16.

Corollary 2.11. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected, $K \Subset \Omega$ be star-shaped with respect to some $x_0 \in K$ and $\varepsilon_0 \in (0,1)$ be given as in Theorem 2.6.*

Then, for each $s \in (0,1)$ and $T \in (0,+\infty)$, if $n \geqslant 3$, we have that

$$\widetilde{\Phi}_*^{\Omega_1,\Omega_2}(s,T) \leqslant \frac{C_{*,K,T,\Omega}}{|\Omega_1||\Omega_2|}F^{\Omega_1,\Omega_2}(s) \qquad \text{for all } \Omega_1,\Omega_2 \subset K, \tag{2.63}$$

where F^{Ω_1,Ω_2} is given in (2.62).

Furthermore, if $n \leqslant 2$, $s \in (0,1)$ and $\mu \in A_{n,s}$, where $A_{n,s}$ is given in (2.46), *one has that*

$$\widetilde{\Phi}_*^{\Omega_1,\Omega_2}(s,T) \leqslant \frac{C_{*,\mu,K,T,\Omega}}{|\Omega_1||\Omega_2|}F^{\Omega_1,\Omega_2}\left(\frac{n-2\mu}{2}\right) \qquad \text{for all } \Omega_1,\Omega_2 \subset K. \tag{2.64}$$

Moreover, for all $s \in (0,1]$ we have that

$$\widetilde{\Phi}_*^{\Omega_1,\Omega_2}(s,T) \geqslant \frac{C_{s,y,\Omega}}{|\Omega_1||\Omega_2|}F^{\Omega_1,\Omega_2}(s), \tag{2.65}$$

with $C_{s,y,\Omega}$ defined in (2.34).

The following result is devoted to the proof of the continuity of the functionals Φ_*, l_* and $\mathcal{A}_*$ with respect to the space, time and fractional variables. Also, we show that if $n < 2s$ then the limit in (2.36) is finite, and similarly $\Phi_*^{z,z}(s,T) < +\infty$ for each $z \in \Omega$.

Proposition 2.12. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and connected.*

Then, $\Phi_*^{x,y}(s,T) \in (0,+\infty)$ for all $(s,(x,y),T) \in (0,1] \times \mathcal{C} \times (0,+\infty)$ and $\Phi_*^{x,y}(s,T) \in C((0,1] \times \mathcal{C} \times (0,+\infty))$, where $\mathcal{C}$ is given in (2.35).

Also, if $n=1$, then $\Phi_*^{x,y}(s,T) \in C\left(\left(\frac{1}{2},1\right] \times \Omega \times \Omega \times (0,+\infty)\right)$.

Moreover, for each $T \in (0,+\infty)$ there exists some $M \in (0,+\infty)$ such that $l_N^y(s,T)$, $\mathcal{A}_N^y(s,T) \in (0,M)$ for all $(s,y) \in (0,1] \times \Omega$ and $l_*^y(s,T)$, $\mathcal{A}_*^y(s,T) \in C\left((0,1] \times \Omega \times (0,+\infty)\right)$.

Furthermore, there exists some $M \in (0,+\infty)$ such that $l_D^y(s,T)$, $\mathcal{A}_D^y(s,T) \in (0,M)$ for all $(s,y,T) \in (0,1] \times \Omega \times (0,+\infty)$

Proof. The positivity of the functionals follows from (1.3), (1.4) and the fact that $r_*^s(t,x,y)$ is strictly positive for all $(t,x,y) \in (0,+\infty) \times \Omega \times \Omega$, see for instance Corollaries 2.15 and 3.12 in [DGV22a].

Now we establish the continuity statement. Thanks to equation (2.4) we have that

$$r_*^s(t,x,y) = \sum_{k=0}^{+\infty} \zeta_{*,k}(x)\zeta_{*,k}(y)\exp(-t\beta_{*,k}^s),$$

and each term of the series is continuous in $(s,t,x,y) \in (0,1] \times (0,+\infty) \times \Omega \times \Omega$.

Furthermore, thanks to Proposition A.1 and Lemma A.2 in [DGV22a], we have the existence of some $M \in \mathbb{N}$ such that for each $\varepsilon \in (0,1]$ and $\delta \in (0,+\infty)$ it holds that

$$\begin{aligned}
&\sum_{k=0}^{+\infty} \|\zeta_{*,k}(x)\zeta_{*,k}(y)\exp\left(-t\beta_{*,k}^s\right)\|_{C^0(\Omega\times\Omega\times(\varepsilon,1)\times(\delta,+\infty))} \\
&\leqslant \sum_{k=0}^{M} \|\zeta_{*,k}(x)\zeta_{*,k}(y)\exp\left(-t\beta_{*,k}^s\right)\|_{C^0(\Omega\times\Omega\times(\varepsilon,1)\times(\delta,+\infty))} + C_{*,m_0,\Omega,0}^2 \sum_{k=M}^{+\infty} \beta_{*,k}^{2\alpha(m_0)} \exp\left(-\delta\beta_{*,k}^{\varepsilon}\right) \\
&< +\infty,
\end{aligned}$$

where $C_{*,m_0,\Omega,0}$ and $\alpha(m_0)$ are positive constants given in Proposition A.1.

Consequently, $r_*^s(t,x,y)$ is continuous for all $(s,t,x,y) \in (0,1] \times (0,+\infty) \times \Omega \times \Omega$.

Suppose now that $(\xi,y) \in \mathcal{C}$, and that $\{(s_k,T_k,y_k)\}_k \subset (0,1] \times (0,+\infty) \times \Omega$ satisfies $(s_k,T_k,y_k) \to (s,T,y) \in (0,1] \times (0,+\infty) \times \Omega$. Then, since $r_*^s(t,\xi,y)$ is continuous for all $(s,y) \in (0,1] \times \Omega$, we have that

$$r_*^{s_k}(t,\xi,y_k)\chi_{(0,T_k)}(t) \to r_*^s(t,\xi,y)\chi_{(0,T)}(t),$$

for almost every $t \in (0,+\infty)$.

Moreover, if $\widetilde{T} := \sup_{k\in\mathbb{N}} T_k$, then using equations (2.1), (2.15) and (2.20) we obtain that

$$\begin{aligned}
\chi_{T_k}(t) r_*^{s_k}(t,\xi,y_k) &\leqslant \chi_{\widetilde{T}}(t) \int_0^{+\infty} p_*^{\Omega}(l,\xi,y_k)\mu_t^{s_k}(l)\,dl \\
&\leqslant \chi_{\widetilde{T}}(t) \int_0^{+\infty} C_{*,l} \exp\left(-\frac{|\xi-y_k|^2}{4l}\right) \mu_t^{s_k}(l)\,dl \\
&\leqslant \chi_{\widetilde{T}}(t) M_* \int_0^{+\infty} \mu_t^{s_k}(l)\,dl \\
&= \chi_{\widetilde{T}}(t) M_*,
\end{aligned} \tag{2.66}$$

where we defined

$$M_* := \sup_{l\in(0,+\infty)} \sup_{k\in\mathbb{N}} C_{*,l} \exp\left(-\frac{|\xi-y_k|^2}{4l}\right).$$

The last function in (2.66) is in $L^1((0,+\infty))$, and thus by the Dominated Convergence Theorem we obtain that $\Phi_*^{\xi,y}(s,T)$ is continuous for all $(s,y,T)\in(0,1]\times(\Omega\setminus\{\xi\})\times(0,+\infty)$, and since it is symmetric with respect to the space variables, we deduce the continuity for all $(s,(\xi,y),T)\in(0,1]\times\mathcal{C}\times(0,+\infty)$.

If $n=1$ the eigenfunctions $\zeta_{*,k}$'s are uniformly bounded in $L^\infty(\Omega)$ and the eigenvalues $\beta_{*,k}$'s are proportional to k^2, for each $k\geqslant 1$. More precisely, there exist two positive constants $C_*, c_*>0$ such that

$$c_* k^2 \leqslant \beta_{*,k} \leqslant C_* k^2,$$

for each $k\geqslant 1$, see for instance [Pro87].

Therefore, we have that

$$\begin{aligned} r_*^s(t,x,y) &= \sum_{k=0}^{+\infty} \zeta_{*,k}(x)\,\zeta_{*,k}(y)\,\exp(-t\beta_{*,k}^s)\\ &\leqslant \sum_{k=0}^{+\infty} \|\zeta_{*,k}\|^2_{L^\infty(\Omega)}\exp(-t\beta_{*,k}^s)\\ &\leqslant M_*^2\sum_{k=0}^{+\infty}\exp(-tc_*^s k^{2s})\\ &=: f_{*,s}(t), \end{aligned} \tag{2.67}$$

where $\|\zeta_{*,k}\|_{L^\infty(\Omega)}\leqslant M_*$ for some $M_*\in(0,+\infty)$ and we adopted the convention $\zeta_{D,0}=0=\beta_{D,0}$.

Thus, if $s, s_k\in\left(\frac12,1\right]$, then we can choose also $(\xi,y)\in\Omega\times\Omega$, indeed thanks to (2.67) we have that

$$\chi_{T_k}(t) r_*^{s_k}(t,\xi,y_k)\leqslant \chi_{\widetilde{T}}(t)\inf_{k\in\mathbb{N}} f_{*,s_k}(t),$$

and the right-hand side is $L^1(0,+\infty)$.

Repeating the above reasoning, if $n=1$, we obtain that $\Phi_*^{x,y}(s,T)\in C\left(\left(\frac12,1\right]\times\Omega\times\Omega\times(0,+\infty)\right)$.

Now, we observe that

$$\begin{aligned} l_*^y(s,T) &= \int_0^T\int_\Omega |\xi-y| r_*^s(t,\xi,y)\,d\xi\,dt\\ &= \int_\Omega |\xi-y|\int_0^T r_*^s(t,\xi,y)\,dt\,d\xi\\ &= \int_\Omega |\xi-y|\Phi_*^{\xi,y}(s,T). \end{aligned} \tag{2.68}$$

Using this identity, the continuity of Φ_* and the estimates in Theorem 2.9, we conclude using the Dominated Convergence Theorem. The proof of the continuity of $\mathcal{A}_*^y(s,T)$ is analogous

Also, if $n\geqslant 3$, from (2.68) and (2.60) we have that

$$l_D^y(s,T)\leqslant \frac{C_n}{\Gamma(s)}\int_\Omega |\xi-y|^{n-1}\,d\xi,$$

for some suitable C_n, which proves that l_D^y is uniformly bounded in $(0,1]\times\Omega\times(0,+\infty)$ if $n\geqslant 3$.

The proof of the uniform boundedness in the case $n\leqslant 2$ is done similarly replacing (2.60) in the above equation with (2.61).

Finally, using (3.10) in [DGV22a] we obtain that

$$\begin{aligned} l_N^y(s,T) &= \int_0^T \int_\Omega |\xi - y| r_N^s(t,\xi,y)\,d\xi\,dt \\ &\leqslant d_\Omega \int_0^T \int_\Omega r_N^s(t,\xi,y)\,d\xi\,dt \\ &= d_\Omega T. \end{aligned}$$

The proof of the boundedness of $\mathcal{A}_*$ is analogous. □

In the following two lemmas we establish the limits as $s \searrow 0$ of the Dirichlet efficiency functionals given in (1.6) and (1.9).

We will show that Φ_D, l_D, $\mathcal{A}_D$, $\widetilde{\Phi}_D$, $\widetilde{l}_D$ and $\widetilde{\mathcal{A}}_D$ all go to 0 linearly in s. Moreover, we will also determine the value of the limit for $\mathcal{E}_{2,D}$, $\mathcal{E}_{3,D}$, $\widetilde{\mathcal{E}}_{2,D}$ and $\widetilde{\mathcal{E}}_{3,D}$ so that we will be able to extend them by continuity in $[0,1]$.

This asymptotic analysis is a fundamental tool in order to establish Theorems 1.6 and 1.14 and the claims in (1.18) and (1.28).

Lemma 2.13. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected.*

Then, for all $((x,y),T) \in \mathcal{C} \times (0,+\infty)$, it holds that

$$\lim_{s\searrow 0} \mathcal{E}_{1,D}^{x,y}(s,T) = 0, \tag{2.69}$$

$$\lim_{s\searrow 0} \mathcal{E}_{2,D}^{x,y}(s,T) = \frac{F_D(x,y)}{\int_\Omega |\xi - y| F_D(\xi,y)\,d\xi}, \tag{2.70}$$

$$\lim_{s\searrow 0} \mathcal{E}_{3,D}^{x,y}(s,T) = \frac{F_D(x,y)}{\int_\Omega |\xi - y|^2 F_D(\xi,y)\,d\xi}, \tag{2.71}$$

where we have defined

$$F_D(x,y) := \int_0^{+\infty} \frac{p_D^\Omega(l,x,y)}{l}\,dl \qquad \text{for all } (x,y) \in \mathcal{C}. \tag{2.72}$$

Proof. Equation (2.69) is a direct consequence of (2.60) and (2.61), since $\Gamma(s) \to +\infty$ for $s \searrow 0$.

Now we focus on the proof of (2.70). For this, we claim that

$$\lim_{s\searrow 0} \frac{\Phi_D^{x,y}(s,T)}{s} = \left(1 - e^{-T}(T+1)\right) F_D(x,y) \qquad \text{for all } ((x,y),T) \in \mathcal{C} \times (0,+\infty). \tag{2.73}$$

Thanks to (2.9) and (2.15), if $s \in \left(0, \frac{1}{2}\right)$ we have that

$$\frac{1}{s}\left| p_D^\Omega(l,x,y)\mu_t^s(l) \right| \leqslant \frac{t}{(4\pi l)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4l}\right) \frac{\Gamma(1+s)}{l^{1+s}}. \tag{2.74}$$

This bound together with (2.1), (D.1) and the Dominated Convergence Theorem yields to

$$\lim_{s\searrow 0} \frac{r_D^s(t,x,y)}{s} = te^{-t} F_D(x,y), \tag{2.75}$$

for all $(t,(x,y)) \in (0,+\infty) \times \mathcal{C}$. Therefore, if $s \in \left(0, \frac{1}{2}\right)$, from (2.1) and (2.74) we obtain that

$$\begin{aligned} \frac{r_D^s(t,x,y)}{s} &\leqslant \int_0^{+\infty} \frac{t\Gamma(1+s)}{(4\pi)^{\frac{n}{2}} l^{\frac{n}{2}+s+1}} \exp\left(-\frac{|x-y|^2}{4l}\right) dl \\ &\leqslant \frac{C_0 t}{|x-y|^{n+2s}} \\ &=: f_{x,y}(t), \end{aligned} \tag{2.76}$$

where we defined

$$C_0 := \sup_{s\in\left(0,\frac{1}{2}\right)} \frac{4^s\Gamma(1+s)}{\pi^{\frac{n}{2}}}\Gamma\left(\frac{n}{2}+s\right).$$

Now, clearly we have that

$$f_{x,y} \in L^1((0,T)), \tag{2.77}$$

and thus from (2.75), (2.76) and (2.77) we can apply the Dominated Convergence Theorem to obtain that

$$\lim_{s\searrow 0} \frac{\Phi^{x,y}_D(s,T)}{s} = \int_0^T te^{-t}F_D(x,y)\,dt = \left(1-e^{-T}(T+1)\right)F_D(x,y). \tag{2.78}$$

This concludes the proof of (2.73).

Note that using (2.60), (2.61) and (2.73), we obtain that

$$\begin{aligned}\lim_{s\searrow 0}\frac{l^y_D(s,T)}{s} &= \lim_{s\searrow 0}\frac{1}{s}\int_0^T\int_\Omega |\xi-y| r^s_D(t,\xi,y)\,d\xi\,dt\\ &= \lim_{s\searrow 0}\int_\Omega |\xi-y|\frac{\Phi^{\xi,y}_D(s,T)}{s}\,d\xi\\ &= \left(1-e^{-T}(T+1)\right)\int_\Omega |\xi-y|F_D(\xi,y)\,d\xi,\end{aligned} \tag{2.79}$$

by means of the Dominated Convergence Theorem. Finally, from (2.73) and (2.79) we deduce that

$$\begin{aligned}\lim_{s\searrow 0}\frac{\Phi^{x,y}_D(s,T)}{l^y_D(s,T)} &= \lim_{s\searrow 0}\frac{\int_0^T r^s_D(t,x,y)\,dt}{\int_0^T\int_\Omega |\xi-y| r^s_D(t,\xi,y)\,d\xi\,dt}\frac{s}{s}\\ &= \frac{F_D(x,y)}{\int_\Omega |\xi-y|F_D(\xi,y)\,d\xi},\end{aligned} \tag{2.80}$$

which concludes the proof of (2.70).

It is left to show (2.71). To do so, we observe that applying the same reasoning we used to show (2.79), one can easily prove that

$$\lim_{s\searrow 0}\frac{\mathcal{A}^y_D(s,T)}{s} = \left(1-e^{-T}(T+1)\right)\int_\Omega |\xi-y|^2F_D(\xi,y)\,d\xi, \tag{2.81}$$

for all $(y,T)\in\Omega\times(0,+\infty)$. From this identity and (2.73) it is immediate to deduce (2.71). □

The following result can be considered as the set functional version of Lemma 2.13.

Lemma 2.14. *Let Ω be bounded, smooth and connected and $\Omega_1,\Omega_2\subset\Omega$ be smooth and disjoint.*

Then, for all $T\in(0,+\infty)$, it holds that

$$\lim_{s\searrow 0}\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s,T) = 0, \tag{2.82}$$

$$\lim_{s\searrow 0}\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{2,D}(s,T) = \frac{|\Omega_2|\widetilde{F}_D(\Omega_1,\Omega_2)}{\int_{\Omega_2\times\Omega}|\xi-y|F_D(\xi,y)\,d\xi\,dy} \tag{2.83}$$

and

$$\lim_{s\searrow 0}\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{3,D}(s,T) = \frac{|\Omega_2|\widetilde{F}_D(\Omega_1,\Omega_2)}{\int_{\Omega_2\times\Omega}|\xi-y|^2F_D(\xi,y)\,d\xi\,dy}, \tag{2.84}$$

where

$$\widetilde{F}_D(\Omega_1,\Omega_2) := \frac{1}{|\Omega_1||\Omega_2|}\int_{\Omega_1\times\Omega_2}F_D(x,y)\,dx\,dy, \tag{2.85}$$

and F_D is given in equation (2.72).

Proof. We begin by proving (2.82). We have that

$$\frac{\widetilde{\Phi}_D^{\Omega_1,\Omega_2}(s,T)}{s} = \frac{1}{|\Omega_1||\Omega_2|}\int_{\Omega_1\times\Omega_2} \frac{\Phi_D^{x,y}(s,T)}{s}\,dx\,dy.$$

Thanks to equations (2.60) and (2.61), if $s \in \left(0, \frac{1}{2}\right)$, there exists some constant $\widehat{C}_n$ depending on n such that

$$\frac{\Phi_D^{x,y}(s,T)}{s} \leqslant \frac{\widehat{C}_n}{s\Gamma(s)}\frac{1}{|x-y|^{n-2s}} \leqslant \frac{C_3}{|x-y|^n} =: g(x,y), \tag{2.86}$$

where C_3 depends only on Ω. If Ω_1, Ω_2 are smooth and disjoint, then $g \in L^1(\Omega_1 \times \Omega_2)$. Therefore, under these assumptions we can apply the Dominated Convergence Theorem, which together with (2.73) yields to

$$\lim_{s\searrow 0}\frac{\widetilde{\Phi}_D^{\Omega_1,\Omega_2}(s,T)}{s} = \left(1-e^{-T}(T+1)\right)\widetilde{F}_D(\Omega_1,\Omega_2). \tag{2.87}$$

Also, thanks to Lemma D.4 and the hypothesis on Ω_1, Ω_2 we have that $\tilde{F}_D(\Omega_1, \Omega_2)$ is finite. From this observation and (2.87) one readily deduces (2.82).

Now, we show (2.83). To do so, we claim that

$$\lim_{s\searrow 0}\frac{\widetilde{l}_D^{\Omega_2}(s,T)}{s} = \frac{\left(1-e^{-T}(T+1)\right)}{|\Omega_2|}\int_{\Omega_2\times\Omega}|\xi-y|F_D(\xi,y)\,d\xi\,dy, \tag{2.88}$$

for all $T \in (0, +\infty)$. As a matter of fact

$$\frac{\widetilde{l}_D^{\Omega_2}(s,T)}{s} := \frac{1}{|\Omega_2|}\int_{\Omega_2}\frac{l_D^y(s,T)}{s}\,dy.$$

Hence, from (2.86) and the definition of $l_D^y(s,T)$ we infer the existence of some $C_4 \in (0, +\infty)$ such that

$$\frac{l_D^y(s,T)}{s} \leqslant C_4,$$

for all $s \in \left(0, \frac{1}{2}\right)$. Therefore, by the Dominated Convergence Theorem we can conclude the proof of (2.88). The limit in equation (2.83) follows easily from (2.87) and (2.88).

Following the same procedure adopted to prove (2.88), one obtains that

$$\lim_{s\searrow 0}\frac{\widetilde{\mathcal{A}}_D^{\Omega_2}(s,T)}{s} = \frac{\left(1-e^{-T}(T+1)\right)}{|\Omega_2|}\int_{\Omega_2\times\Omega}|\xi-y|^2F_D(\xi,y)\,d\xi\,dy, \tag{2.89}$$

for all $T \in (0, +\infty)$. Thereby, the limit in equation (2.84) follows easily from (2.87) and (2.89). □

In the following lemma we study the asymptotic behavior of the Neumann functional $\Phi_N^{x,y}(s,T)$ for $s \searrow 0$. In particular, we observe that the limit substantially differs from the one of $\Phi_D^{x,y}$, which was indeed vanishing, see Lemma 2.13. With this result we establish also that the lim inf and lim sup of $\Phi_N^{x,y}(s,T)$ for $s \searrow 0$ are controlled by some quantities that do not depend on $x, y \in \Omega$. This feature will let us prove that if the forager starting position and target location are close enough, then the most rewarding search strategy for the Neumann functionals in equation (1.6) is not $s = 0$.

Lemma 2.15. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected.*

Then, there exist $h_1, h_2 \in C([0, +\infty))$ such that for each $T \in (0, +\infty)$ it holds that

$$\frac{h_1(T)}{T} \leqslant \liminf_{s\searrow 0}\mathcal{E}_{1,N}^{x,y}(s,T) \leqslant \limsup_{s\searrow 0}\mathcal{E}_{1,N}^{x,y}(s,T) \leqslant \frac{h_2(T)}{T}, \tag{2.90}$$

$$\frac{h_1(T)}{h_2(T)M(y)} \leqslant \liminf_{s\searrow 0}\mathcal{E}_{2,N}^{x,y}(s,T) \leqslant \limsup_{s\searrow 0}\mathcal{E}_{2,N}^{x,y}(s,T) \leqslant \frac{h_2(T)}{M(y)h_1(T)} \tag{2.91}$$

$$\text{(2.92)} \qquad \textit{and} \qquad \frac{h_1(T)}{h_2(T)\widetilde{M}(y)} \leqslant \liminf_{s\searrow 0} \mathcal{E}^{x,y}_{3,N}(s,T) \leqslant \limsup_{s\searrow 0} \mathcal{E}^{x,y}_{3,N}(s,T) \leqslant \frac{h_2(T)}{\widetilde{M}(y)h_1(T)},$$

for all $(x,y)\in\mathcal{C}$, *where we set*

$$\text{(2.93)} \qquad M(y) := \int_\Omega |\xi - y|\,d\xi \qquad \textit{and} \qquad \widetilde{M}(y) := \int_\Omega |\xi-y|^2\,d\xi.$$

Proof. Let $(x,y)\in\mathcal{C}$. Notice that if $t\in(0,+\infty)$ we can write

$$p^\Omega_N(t,x,y) = \frac{1}{|\Omega|} + \sum_{k=1}^{+\infty} \zeta_{N,k}(x)\zeta_{N,k}(y)\exp\left(-t\,\beta_{N,k}\right),$$

where $\zeta_{N,k}$'s and $\beta_{N,k}$'s are given in (2.3). Now, thanks to Proposition A.1 and Lemma A.3 in [DGV22a], together with Weyl's law on the asymptotic behavior of the eigenvalues $\beta_{N,k}$'s (see for instance [Pro87]), we have that

$$\lim_{t\to+\infty}\sum_{k=1}^{+\infty} \zeta_{N,k}(x)\zeta_{N,k}(y)\exp\left(-t\,\beta_{N,k}\right) = 0,$$

from which we deduce that

$$\lim_{l\to+\infty} p^\Omega_N(l,x,y) = \frac{1}{|\Omega|}.$$

Therefore, there exists some $t_0\in(1,+\infty)$ such that

$$\frac{1}{2|\Omega|} \leqslant p^\Omega_N(t,x,y) \qquad \text{for all } \ t\in[t_0,+\infty).$$

Thus, using (2.1), we have that if $t_{1,s} = \max\left\{t_0, T^{\frac{1}{s}}\right\}$, we can apply Theorem 2.3 and obtain that

$$\begin{aligned}
\Phi^{x,y}_N(s,T) &= \int_0^T\int_0^{+\infty} p^\Omega_N(l,x,y)\mu^s_t(l)\,dl\,dt \\
&\geqslant \frac{1}{2|\Omega|}\int_0^T\int_{t_{1,s}}^{+\infty} \mu^s_t(l)\,dl\,dt \\
&\geqslant \frac{C_1}{2\pi|\Omega|}\int_0^T\int_{t_{1,s}}^{+\infty} \frac{s\,t}{l^{1+s}}\,dl\,dt \\
&\geqslant \frac{C_1}{2\pi|\Omega|}\int_0^T \frac{t}{t^s_{1,s}}\,dt \\
&= \frac{C_1}{4\pi|\Omega|}\frac{T^2}{t^s_{1,s}}.
\end{aligned}$$

Therefore, if $T\in(1,+\infty)$ from the above inequality we obtain

$$\text{(2.94)} \qquad \limsup_{s\searrow 0}\Phi^{x,y}_N(s,T) \geqslant \liminf_{s\searrow 0}\Phi^{x,y}_N(s,T) \geqslant \frac{C_1}{4\pi|\Omega|}T,$$

while if $T\in(0,1]$ we have that

$$\text{(2.95)} \qquad \limsup_{s\searrow 0}\Phi^{x,y}_N(s,T) \geqslant \liminf_{s\searrow 0}\Phi^{x,y}_N(s,T) \geqslant \frac{C_1}{4\pi|\Omega|}T^2.$$

Hence, we have just proved the left-hand side inequality in (2.90) with

$$h_1(T) := \begin{cases} \dfrac{C_1}{4\pi|\Omega|}T^2 & \text{if} \quad T \in (0,1], \\ \dfrac{C_1}{4\pi|\Omega|}T & \text{if} \quad T \in (1,+\infty). \end{cases} \tag{2.96}$$

Now we show the right-hand side inequality of (2.90). Using (2.20), we obtain that

$$\begin{aligned} \Phi_N^{x,y}(s,T) &= \int_0^T \int_0^{+\infty} p_N^\Omega(l,x,y)\mu_t^s(l)\,dl \\ &\leqslant \int_0^T \int_0^1 \frac{c_\Omega}{l^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{6l}\right)\mu_t^s(l)\,dl\,dt + \int_0^T \int_1^{+\infty} c_\Omega \mu_t^s(l)\,dl\,dt \\ &\leqslant \int_0^T \int_0^1 \frac{c_\Omega}{l^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{6l}\right)\mu_t^s(l)\,dl\,dt + c_\Omega T. \end{aligned} \tag{2.97}$$

Now, in view of (2.9) we have that

$$\left|\frac{c_\Omega}{l^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{6l}\right)\mu_t^s(l)\right| \leqslant \frac{tc_\Omega\Gamma(1+s)}{l^{\frac{n}{2}+1+s}} \exp\left(-\frac{|x-y|^2}{6l}\right),$$

and the function on the right-hand side in the above equation is in $L^1((0,T)\times(0,1))$.

Therefore, using also (2.9) we can apply the Dominated Convergence Theorem and obtain the limit

$$\lim_{s\searrow 0} \int_0^T \int_0^1 \frac{c_\Omega}{l^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{6l}\right)\mu_t^s(l)\,dl\,dt = 0. \tag{2.98}$$

From this equation and (2.97), we can infer that if $T \in (1,+\infty)$

$$\liminf_{s\searrow 0} \Phi_N^{x,y}(s,T) \leqslant \limsup_{s\searrow 0} \Phi_N^{x,y}(s,T) \leqslant c_\Omega T. \tag{2.99}$$

Also, assuming that $T \in (0,1]$, from (2.9) we obtain that

$$\int_0^T \int_1^{+\infty} c_\Omega \mu_t^s(l)\,dl\,dt \leqslant c_\Omega \int_0^T \int_1^{+\infty} \frac{st\Gamma(1+s)}{l^{1+s}}\,dl\,dt = \frac{c_\Omega\Gamma(1+s)}{2}T^2.$$

Thus, from this latter observation, the limit in (2.98) and equation (2.97) we deduce that

$$\liminf_{s\searrow 0} \Phi_N^{x,y}(s,T) \leqslant \limsup_{s\searrow 0} \Phi_N^{x,y}(s,T) \leqslant \frac{c_\Omega}{2}T^2. \tag{2.100}$$

In light of (2.99) and (2.100), and defining

$$h_2(T) := \begin{cases} c_\Omega T^2 & \text{if} \quad T \in (0,1], \\ c_\Omega T & \text{if} \quad T \in (1,+\infty), \end{cases} \tag{2.101}$$

we conclude the proof of the right-hand side inequality of (2.90).

Now, we prove (2.91). To do so, we claim that

$$h_1(T)M(y) \leqslant \liminf_{s\searrow 0} l_N^y(s,T) \leqslant \limsup_{s\searrow 0} l_N^y(s,T) \leqslant h_2(T)M(y). \tag{2.102}$$

We recall that

$$l_N^y(s,T) = \int_\Omega |\xi - y|\Phi_N^{\xi,y}(s,T)\,d\xi,$$

with $(y,T) \in \Omega \times (0,+\infty)$. Then, using (2.94), (2.95) and Fatou's Lemma we prove the left-hand side inequality of (2.91).

Now, we focus on the proof of the right-hand side inequality . Let $K \Subset \Omega$ be any compact such that it is star-shaped with respect to y and $y \in K^{\circ}$, and $d_K \leqslant 1$. Then, in view of (2.47), (2.48) and Proposition D.2 with $E = \Omega \setminus K$ and $F = y$, we evince the existence of some $u \in L^1(\Omega)$ such that

$$|\xi - y|\Phi_N^{\xi,y}(s,T) \leqslant u(\xi),$$

for all $\xi \in \Omega$. Thus, thanks to Fatou's Lemma and (2.99) we obtain the right-hand side inequality of (2.91). Note that from (2.90) and (2.102) one evinces (2.91).

It is left to show (2.92). Reasoning analogously to the proof of claim (2.102), one obtains that

$$h_1(T)\widetilde{M}(y) \leqslant \liminf_{s\searrow 0} \mathcal{A}_N^y(s,T) \leqslant \limsup_{s\searrow 0} \mathcal{A}_N^y(s,T) \leqslant h_2(T)\widetilde{M}(y)$$

Making use of this two sided inequality and (2.90) we conclude the proof of (2.92). □

The following result is the Neumann counterpart of Lemma 2.14.

Lemma 2.16. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected.*

Then, for all $T \in (0, +\infty)$ and $\Omega_1, \Omega_2 \Subset \Omega$ smooth and disjoint, it holds that

$$\frac{h_1(T)}{T} \leqslant \liminf_{s\searrow 0} \widetilde{\mathcal{E}}_{1,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \limsup_{s\searrow 0} \widetilde{\mathcal{E}}_{1,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \frac{h_2(T)}{T}, \tag{2.103}$$

$$\frac{h_1(T)}{h_2(T)P(\Omega_2)} \leqslant \liminf_{s\searrow 0} \widetilde{\mathcal{E}}_{2,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \limsup_{s\searrow 0} \widetilde{\mathcal{E}}_{2,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \frac{h_2(T)}{h_1(T)P(\Omega_2)} \tag{2.104}$$

$$\text{and} \quad \frac{h_1(T)}{h_2(T)\widetilde{P}(\Omega_2)} \leqslant \liminf_{s\searrow 0} \widetilde{\mathcal{E}}_{3,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \limsup_{s\searrow 0} \widetilde{\mathcal{E}}_{3,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \frac{h_2(T)}{h_1(T)\widetilde{P}(\Omega_2)}, \tag{2.105}$$

where h_1 and h_2 are given respectively in (2.96) and (2.101), and we set

$$P(\Omega_2) := \frac{\|M\|_{L^1(\Omega_2)}}{|\Omega_2|} \qquad \text{and} \qquad \widetilde{P}(\Omega_2) := \frac{\|\widetilde{M}\|_{L^1(\Omega_2)}}{|\Omega_2|}, \tag{2.106}$$

where M and $\widetilde{M}$ are defined in (2.93).

Proof. We begin by proving (2.103). To do so, we notice that by definition we have

$$\widetilde{\Phi}_N^{\Omega_1,\Omega_2}(s,T) = \frac{1}{|\Omega_1||\Omega_2|} \int_{\Omega_1\times\Omega_2} \Phi_N^{x,y}(s,T)\,dx\,dy. \tag{2.107}$$

From Proposition 2.12 we know that $\Phi_N^{x,y}(s,T) \geqslant 0$. Thus, by Fatou's Lemma, (2.94) and (2.95) we conclude the proof of the left-hand side inequality of (2.103).

Now, if $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \varnothing$, thanks to Proposition D.2 with $\Omega_1 = E$ and $\Omega_2 = F$, we easily obtain the right-hand side inequality of (2.103) using Fatou's Lemma.

We assume now that $\overline{\Omega}_1 \cap \overline{\Omega}_2 \neq \varnothing$. We claim that there exists some $z \in L^1(\Omega_1 \times \Omega_2)$ such that for all $s \in \left(0, \frac{1}{2}\right)$ it holds that

$$\Phi_N^{x,y}(s,T) \leqslant z(x,y) \quad \text{for all} \quad (x,y) \in \Omega_1 \times \Omega_2. \tag{2.108}$$

We prove claim (2.108). Thanks to the assumption $\overline{\Omega}_1 \cap \overline{\Omega}_2 \neq \varnothing$, the set $A := \partial\Omega_1 \cap \partial\Omega_2$ is nonempty.

Since $\Omega_1, \Omega_2 \Subset \Omega$ and A is compact, then we can choose $r > 0$ and $P_i \in A$ with $i \in \{1, \ldots, N\}$ such that

$$A \subset B := \bigcup_{i=1}^N \overline{B_r(P_i)} \Subset \Omega.$$

If for some $i, j \in \{1, \dots, N\}$ it holds that $\overline{B_r(P_i)} \cap \overline{B_r(P_j)} \neq \varnothing$, then we can choose $K_{i,j} = \overline{B_r(P_i)} \cup \overline{B_r(P_j)}$ in (2.47) and (2.48) and deduce that

$$\Phi_N^{x,y}(s,T) \leqslant \frac{C}{|x-y|^n}, \tag{2.109}$$

for all $(x, y) \in K_{i,j} \times K_{i,j}$ with i, j such that $\overline{B_r(P_i)} \cap \overline{B_r(P_j)} \neq \varnothing$, where C depends on B, T, Ω. Moreover, we define the constant

$$\widetilde{C}_N := \max\left\{ C_{B_r(P_i),B_r(P_j)} \quad \text{s.t.} \quad \overline{B_r(P_i)} \cap \overline{B_r(P_j)} = \varnothing \right\}, \tag{2.110}$$

where $C_{B_r(P_i),B_r(P_j)}$ is given in (D.4) with $E = B_r(P_i)$ and $F = B_r(P_j)$. Therefore, if $x \in \Omega_1 \cap \overline{B_r(P_i)}$ and $y \in \Omega_2 \cap \overline{B_r(P_j)}$, such that $\overline{B_r(P_i)} \cap \overline{B_r(P_j)} = \varnothing$, then by (D.3) and (2.110) we see that

$$\Phi_N^{x,y}(s,T) \leqslant \widetilde{C}_N T. \tag{2.111}$$

Finally, if we set

$$\widehat{C}_{\Omega_1,\Omega_2} := \max\left\{ C_{\Omega_1\cap K',\Omega_2\setminus K'}, C_{\Omega_1\setminus K',\Omega_2\cap K'}, C_{\Omega_1\setminus K',\Omega_2\setminus K'} \right\},$$

thanks to Proposition D.2, we obtain that

$$\Phi_N^{x,y}(s,T) \leqslant \widehat{C}_{\Omega_1,\Omega_2} T, \tag{2.112}$$

for all $(x, y) \in ((\Omega_1 \cap K') \times (\Omega_2 \setminus K')) \cup ((\Omega_1 \setminus K') \times (\Omega_2 \cap K)) \cup ((\Omega_1 \setminus K') \times (\Omega_2 \setminus K'))$.

Thanks to (2.109), (2.111) and (2.112) we conclude the proof of claim (2.108).

By that means, we can apply Fatou's Lemma and using (2.90) we prove the right-hand side inequality in (2.103).

Now, we focus our attention to the proof of (2.104). In order to do so, we claim that

$$h_1(T)P(\Omega_2) \leqslant \liminf_{s\searrow 0} \widetilde{l}_N^{\Omega_2}(s,T) \leqslant \limsup_{s\searrow 0} \widetilde{l}_N^{\Omega_2}(s,T) \leqslant h_2(T)P(\Omega_2) \tag{2.113}$$

We observe that

$$\widetilde{l}_N^{\Omega_2} := \frac{1}{|\Omega_2|} \int_{\Omega_2} l_N^y(s,T)\, dy$$

and, since $l_N^y(s,T) \geqslant 0$, see Proposition 2.12, using Fatou's Lemma and (2.102) we prove the left-hand side inequality of (2.113).

Furthermore, we discussed in Proposition 2.12 that $l_N^y(s,T)$ is uniformly bounded in $(s, y) \in (0, 1) \times \Omega$. Thus, we can apply again Fatou's Lemma together with (2.102) and conclude the proof of the right-hand side inequality of (2.113). The inequalities in (2.103) and (2.113) yields to (2.104).

It is left to show (2.105). To do so, it is enough to show that

$$h_1(T)\widetilde{P}(\Omega_2) \leqslant \liminf_{s\searrow 0} \widetilde{\mathcal{A}}_N^{\Omega_2}(s,T) \leqslant \limsup_{s\searrow 0} \widetilde{\mathcal{A}}_N^{\Omega_2}(s,T) \leqslant h_2(T)\widetilde{P}(\Omega_2). \tag{2.114}$$

From this and (2.103) it is easy to deduce (2.105). The proof of (2.114) is analogous to the one of (2.113), and thus it is omitted. □

3. Proof of the main results

This section is devoted to the proofs of the main results discussed in the introduction. It is divided into two main parts.

In Section 3.1 we prove the results stated in Section 1.2. Namely, we analyze the environmental scenario where the target location coincides with the forager starting point.

In Section 3.2 we instead discuss the best search strategy when the prey is in a small neighborhood of the seeker initial position. In particular, we prove all the results contained in Sections 1.3 and 1.4.

3.1. **Proof of the results in Section 1.2.** To prove the results presented in Section 1.2, we consider $\Omega = (0, a)$ for some $a \in (0, +\infty)$. The normalized eigenfunctions of the Laplacian in $(0, a)$ with Dirichlet datum as defined in (2.3) are

$$\zeta_{D,k}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi k x}{a}\right) \tag{3.1}$$

and the corresponding eigenvalues are

$$\beta_{D,k} = \left(\frac{\pi k}{a}\right)^2. \tag{3.2}$$

As a consequence, recalling (2.4), the Dirichlet spectral fractional heat kernel reads as

$$r_D^s(t, x, y) = \frac{2}{a} \sum_{k=1}^{+\infty} \sin\left(\frac{\pi k y}{a}\right) \sin\left(\frac{\pi k x}{a}\right) \exp\left(-t\left(\frac{\pi k}{a}\right)^{2s}\right). \tag{3.3}$$

This and (1.5) lead to

$$\Phi_D^{x,y}(s, T) = \frac{2}{a} \int_0^T \sum_{k=1}^{+\infty} \sin\left(\frac{\pi k y}{a}\right) \sin\left(\frac{\pi k x}{a}\right) \exp\left(-t\left(\frac{\pi k}{a}\right)^{2s}\right) dt \tag{3.4}$$

and accordingly, if $s \in \left(\frac{1}{2}, 1\right)$,

$$\Phi_D^{x,y}(s, T) = \frac{2a^{2s-1}}{\pi^{2s}} \sum_{k=1}^{+\infty} \frac{1}{k^{2s}} \sin\left(\frac{\pi k y}{a}\right) \sin\left(\frac{\pi k x}{a}\right) \left[1 - \exp\left(-T\left(\frac{\pi k}{a}\right)^{2s}\right)\right]. \tag{3.5}$$

We can also compute explicitly the average distance $l_D^y(s, T)$ and the mean square displacement $\mathcal{A}_D^y(s, T)$ as a series, as showed in detail in Appendix B.

The normalized eigenfunctions of the Laplacian in $(0, a)$ under Neumann conditions as defined in (2.3) take the form

$$\begin{cases} \zeta_{N,k}(x) = \sqrt{\frac{2}{a}} \cos\left(\frac{\pi k x}{a}\right) & \text{if } k \in \{1, 2, 3, \dots\}, \\ \zeta_{N,0}(x) = \frac{1}{\sqrt{a}} & \text{if } k = 0, \end{cases} \tag{3.6}$$

and the corresponding eigenvalues are

$$\beta_{N,k} = \left(\frac{\pi k}{a}\right)^2 \qquad \text{if } k \in \{0, 1, 2, 3, \dots\}. \tag{3.7}$$

Therefore, in view of (2.4), the Neumann spectral fractional heat kernel reads as

$$r_N^s(t, x, y) = \frac{1}{a} + \frac{2}{a} \sum_{k=1}^{+\infty} \cos\left(\frac{\pi k x}{a}\right) \cos\left(\frac{\pi k y}{a}\right) \exp\left(-t\left(\frac{\pi k}{a}\right)^{2s}\right).$$

Hence, by (1.5),

$$\Phi_N^{x,y}(s, T) = \frac{T}{a} + \frac{2}{a} \sum_{k=1}^{+\infty} \int_0^T \cos\left(\frac{\pi k x}{a}\right) \cos\left(\frac{\pi k y}{a}\right) \exp\left(-t\left(\frac{\pi k}{a}\right)^{2s}\right) dt \tag{3.8}$$

and, as a result, when $s \in \left(\frac{1}{2}, 1\right)$,

$$\Phi_N^{x,y}(s, T) = \frac{T}{a} + \frac{2a^{2s-1}}{\pi^{2s}} \sum_{k=1}^{+\infty} \frac{1}{k^{2s}} \cos\left(\frac{\pi k x}{a}\right) \cos\left(\frac{\pi k y}{a}\right) \left[1 - \exp\left(-T\left(\frac{\pi k}{a}\right)^{2s}\right)\right]. \tag{3.9}$$

Thanks to these preliminary observations, we are now in the position of proving the results presented in Section 1.2. We begin by showing Proposition 1.1.

We recall that we adopt the subscript $*$ every time that a functional refers to both the Dirichlet and the Neumann case.

Proof of Proposition 1.1. Let $x \in \Omega$ and $T \in (0,+\infty)$. Then, thanks to Theorem 2.8 we know that if either $n \geqslant 2$ or $n = 1$ and $s \in \left(0, \frac{1}{2}\right]$ it holds that

$$\Phi_*^{x,x}(s,T) = +\infty. \tag{3.10}$$

Furthermore, from Proposition 2.12 we have that

$$l_*^x(s,T) \in (0,+\infty) \qquad \text{and} \qquad \mathcal{A}_*^x(s,T) \in (0,+\infty), \tag{3.11}$$

for all $s \in (0,1]$. Therefore, as a direct consequence of (3.10) and (3.11) we obtain the desired claim. □

The proof of Proposition 1.2 that we present here below is a consequence of Theorem 2.8. For the sake of completeness, in Appendix C we also provide an alternative proof of Proposition 1.2 which employs directly the spectral structure of the efficiency functionals.

Proof of Proposition 1.2. Let $x \in \Omega = (0,a)$, for some $a \in (0,+\infty)$. Then, thanks to Theorem 2.8 we have that for each $s \in \left(0, \frac{1}{2}\right]$ and $T \in (0,+\infty)$ the statement in (3.10) holds true. Also, if $s \in \left(\frac{1}{2}, 1\right]$, in view of Proposition 2.12 one has that

$$\Phi_*^{x,x}(s,T) \in (0,+\infty). \tag{3.12}$$

Furthermore, from Proposition 2.12 we know that for each $s \in (0,1]$ and $T \in (0,+\infty)$ the statement in (3.11) holds true as well. Therefore, using equation (3.12), in the notation of Proposition 1.2, we conclude that

$$\mathcal{E}_{*,j}(s,T) \in (0,+\infty) \quad \text{for all} \quad s \in \left(\frac{1}{2}, 1\right],$$

for all $j \in \{1,2,3\}$.

Hence, to complete the proof of Proposition 1.2, it is only left to show the continuity statement. Thanks to Proposition 2.12 we have that, for each $x \in \Omega$ and $T \in (0,+\infty)$, the functional $\Phi_*^{x,x}(\cdot,T)$ is continuous with respect to $s \in \left(\frac{1}{2}, 1\right]$. Also, the continuity with respect to $s \in (0,1]$ of the functionals $\mathcal{A}_*^x(s,T)$ and $l_*^x(s,T)$ was already established in Proposition 2.12.

As a consequence, recalling (3.11) we conclude that the functionals in (1.6) are continuous in $s \in \left(\frac{1}{2}, 1\right]$ for $x = y$. □

Now we prove Theorem 1.3. Here we establish that $s = \frac{1}{2}$ is the best search strategy in $\left(\frac{1}{2}, 1\right]$ when the forager initial point coincide with the target location.

Proof of Theorem 1.3. We point out that, in order to prove Theorem 1.3, it suffices to establish (1.10). Indeed, once (1.10) is proved, we already know from Proposition 1.2 that $\mathcal{E}_{*,j}(s,T) \in (0,+\infty)$ for all $s \in \left(\frac{1}{2}, 1\right]$ and $j \in \{1,2,3\}$ and accordingly the supremum over $s \in \left(\frac{1}{2}, 1\right)$ of $\mathcal{E}_{j,*}$ is attained at $s = 1/2$.

Furthermore, thanks to (C.1) it is enough to show (1.10) for $a := 1$. To prove it, we observe that all the denominators in (1.6) satisfy (3.11). Consequently, the claim in (1.10) is equivalent to

$$\lim_{s \searrow \frac{1}{2}} \Phi_*^{x,x}(s,T) = +\infty \tag{3.13}$$

Thus, from now on we focus on the proof of the claims in (3.13). We establish the claim for the Dirichlet case, since the Neumann one follows from the Dirichlet one and (2.24).

For this, we recall (C.7) and we see that there exist K_0, $N \geqslant 1$ such that, for every $\overline{N} \in \mathbb{N}$,

$$\Phi_D^{x,x}(s,T) \;\geqslant\; \frac{1}{\pi^{2s}} \sum_{\ell=0}^{\overline{N}-1} \frac{\varepsilon_0}{\big(N+3\ell K_0\big)^{2s}}.$$

We now pick $L>0$, to be taken as large as we wish in what follows, such that $e^L \in \mathbb{N}$, and we choose $\overline{N} := e^{2L}+1$. In this way, we find that

$$\begin{aligned}
\Phi_D^{x,x}(s,T) &\geqslant \frac{1}{\pi^{2s}} \sum_{\ell=\exp(L)+1}^{\exp(2L)} \frac{\varepsilon_0}{\big(N+3\ell K_0\big)^{2s}} \geqslant \frac{1}{\pi^{2s}} \sum_{\ell=\exp(L)+1}^{\exp(2L)} \frac{\varepsilon_0}{\big(4\ell K_0\big)^{2s}}\\
&= \frac{\varepsilon_0}{(4\pi K_0)^{2s}} \sum_{j=1}^{L} \sum_{\ell=\exp(L+j-1)+1}^{\exp(L+j)} \frac{1}{\ell^{2s}} \geqslant \frac{\varepsilon_0}{(4\pi K_0)^{2s}} \sum_{j=1}^{L} \sum_{\ell=\exp(L+j-1)+1}^{\exp(L+j)} \frac{1}{\exp(2s(L+j))}\\
&= \frac{\varepsilon_0\,(e-1)}{(4\pi K_0)^{2s}} \sum_{j=1}^{L} \frac{\exp(L+j-1)}{\exp(2s(L+j))} = \frac{\varepsilon_0\,(e-1)}{(4\pi K_0)^{2s}\exp((2s-1)L+1)} \sum_{j=1}^{L} \frac{1}{\exp((2s-1)j)}\\
&= \frac{\varepsilon_0\,(e-1)}{(4\pi K_0)^{2s}\exp((2s-1)L+1)} \times \frac{\exp(1-2s)\big(1-\exp((1-2s)L)\big)}{1-\exp(1-2s)}\\
&= \frac{\varepsilon_0\,(e-1)}{(4\pi e K_0)^{2s}\exp(2(2s-1)L)} \times \frac{\big(\exp((2s-1)L)-1\big)}{1-\exp(1-2s)}.
\end{aligned}$$

In particular, we can choose $L \in \left[\frac{1}{2s-1}, \frac{2}{2s-1}\right]$ such that $e^L \in \mathbb{N}$ and deduce from the above estimate that

$$\Phi_D^{x,x}(s,T) \geqslant \frac{\varepsilon_0\,(e-1)^2}{(4\pi e K_0)^{2s}\,e^4} \times \frac{1}{1-\exp(1-2s)}.$$

Sending now $s \searrow \frac{1}{2}$ we see that

$$\lim_{s \searrow \frac{1}{2}} \Phi_D^{x,x}(s,T) = +\infty,$$

proving the claim in (3.13) for the Dirichlet case, as desired. □

Finally, we prove Theorem 1.4. In this result, we discuss the impact of some geometrical properties of the domain, such as the size of it, on the monotonicity of the efficiency functionals in (1.5) with respect to the fractional exponent.

Proof of Theorem 1.4. We prove the monotonicity properties of Φ_*. To this end, in the Dirichlet case, when $a \in (0,\pi]$ the first eigenvalue of the Laplacian is less than or equal to 1, thanks to (3.2): hence, we capitalize on Theorem 1.10 in [DGV22a] and we conclude that, for all $s_0 \in (0,1)$ and $s_1 \in (s_0,1)$, we have that, for every $x \in (0,a)$,

$$r_D^{s_0}(t,x,x) > r_D^{s_1}(t,x,x). \tag{3.14}$$

Similarly, in the Neumann case, when $a \in (0,\pi]$ the first nontrivial eigenvalue of the Laplacian is less than or equal to 1, due to (3.7). This allows us to use Theorem 1.22 in [DGV22a] and obtain that, for all $s_0 \in (0,1)$, $s_1 \in (s_0,1)$ and $x \in (0,a)$,

$$r_N^{s_0}(t,x,x) > r_N^{s_1}(t,x,x). \tag{3.15}$$

Now, from (1.5), (3.14) and (3.15) it follows that, for all $s_0 \in (0,1)$, $s_1 \in (s_0,1)$ and $x \in (0,a)$,

$$\Phi_*^{x,x}(s_0,T) > \Phi_*^{x,x}(s_1,T). \tag{3.16}$$

From (3.16) we obtain the desired monotonicity property when $a \in (0,\pi]$, as stated in formula (1.11) of Theorem 1.4.

Now we deal with the case in which a is sufficiently large and we prove (1.12) and (1.13). To this end, we start with the Dirichlet case, utilize (3.5) with the notation $\alpha := \frac{a}{\pi}$ and deduce that, for every $T \in (0,+\infty)$ and $x \in (0,a)$,

$$\begin{aligned}
\frac{a}{2}\partial_s \Phi_D^{x,x}(s,T) &= \frac{\partial}{\partial s}\left[\alpha^{2s}\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]\right]\\
&= 2\alpha^{2s}\ln\alpha\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]\\
&\qquad - 2\alpha^{2s}\sum_{k=1}^{+\infty}\frac{\ln k}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]\\
&\qquad + 2T\alpha^{2s}\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(\frac{k}{\alpha}\right)^{2s}\ln\left(\frac{k}{\alpha}\right)\\
&= 2\alpha^{2s}\sum_{k=1}^{+\infty}\frac{\ln\alpha-\ln k}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]\\
&\qquad - 2T\alpha^{2s}\sum_{k=1}^{+\infty}\frac{\ln\alpha-\ln k}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(\frac{k}{\alpha}\right)^{2s}\\
&= 2\alpha^{2s}\sum_{k=1}^{+\infty}\frac{\ln\alpha-\ln k}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(1+T\left(\frac{k}{\alpha}\right)^{2s}\right)\right].
\end{aligned}\tag{3.17}$$

These observations lead to

$$\begin{aligned}
\frac{a}{4\alpha^{2s}}\partial_s \Phi_D^{x,x}(s,T) \;=\;& \ln\alpha\sin^2\left(\frac{x}{\alpha}\right)\left[1-\exp\left(-\frac{T}{\alpha^{2s}}\right)\left(1+\frac{T}{\alpha^{2s}}\right)\right]\\
&+\sum_{k=2}^{+\infty}\frac{\ln\alpha-\ln k}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(1+T\left(\frac{k}{\alpha}\right)^{2s}\right)\right].
\end{aligned}$$

We also observe that, if $f(\tau) := 1-e^{-\tau}(1+\tau)$, we have that $f'(\tau)=\tau e^{-\tau}>0$ for all $\tau>0$. Accordingly, we see that $1-e^{-\tau}(1+\tau)>f(0)=0$ for all $\tau>0$. In addition, we have that $f(\tau)\leqslant 1$ for all $\tau>0$. As a result,

$$\begin{aligned}
&\sum_{k=2}^{+\infty}\frac{\ln\alpha-\ln k}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(1+T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]\\
\geqslant\;& \sum_{k\in\mathbb{N}\cap(\alpha,+\infty)}\frac{\ln\alpha-\ln k}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(1+T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]\\
\geqslant\;& -\sum_{k\in\mathbb{N}\cap(\alpha,+\infty)}\frac{\ln k}{k^{2s}}.
\end{aligned}$$

From these remarks, we arrive at

$$\frac{a}{4\alpha^{2s}}\partial_s \Phi_D^{x,x}(s,T) \;\geqslant\; \ln\alpha\sin^2\left(\frac{x}{\alpha}\right)\left[1-\exp\left(-\frac{T}{\alpha^{2s}}\right)\left(1+\frac{T}{\alpha^{2s}}\right)\right]-\sum_{k\in\mathbb{N}\cap(\alpha,+\infty)}\frac{\ln k}{k^{2s}}.$$

Now, if $T \in [\nu a^{2s}, +\infty) = [\nu \pi^{2s}\alpha^{2s}, +\infty)$ then

$$1 - \exp\left(-\frac{T}{\alpha^{2s}}\right)\left(1 + \frac{T}{\alpha^{2s}}\right) = f\left(\frac{T}{\alpha^{2s}}\right) \geqslant f(\nu\pi^{2s}) \geqslant f(\nu).$$

Hence, in this situation,

$$\frac{a}{4\alpha^{2s}}\partial_s \Phi_D^{x,x}(s,T) \;\geqslant\; \ln\alpha \sin^2\left(\frac{x}{\alpha}\right) f(\nu) - \sum_{k\in\mathbb{N}\cap(\alpha,+\infty)} \frac{\ln k}{k^{2s}}.$$

We also recall that

$$\int_{\alpha-2}^{+\infty} \frac{\ln\tau}{\tau^{2s}}\,d\tau = \frac{1+(2s-1)\ln(\alpha-2)}{(2s-1)^2(\alpha-2)^{2s-1}}$$

and therefore, if α is large enough,

$$\sum_{k\in\mathbb{N}\cap(\alpha,+\infty)} \frac{\ln k}{k^{2s}} \leqslant \frac{1+(2s-1)\ln(\alpha-2)}{(2s-1)^2(\alpha-2)^{2s-1}} \leqslant \frac{2s\ln(\alpha-2)}{(2s-1)^2(\alpha-2)^{2s-1}} \leqslant \frac{2s\ln\alpha}{(2s-1)^2(\alpha-2)^{2s-1}}.$$

Besides, if $x \in \big(\nu a, (1-\nu)a\big) = (\nu\alpha\pi, (1-\nu)\alpha\pi)$ we have that

$$\left|\sin\left(\frac{x}{\alpha}\right)\right| > \sin(\varepsilon\pi). \tag{3.18}$$

These observations lead to

$$\begin{aligned}\frac{a}{4\alpha^{2s}\ln\alpha}\partial_s\Phi_D^{x,x}(s,T) &\geqslant \sin^2(\nu\pi)f(\nu) - \frac{2s}{(2s-1)^2(\alpha-2)^{2s-1}}\\ &\geqslant \sin^2(\nu\pi)f(\nu) - \frac{2}{\nu^2(\alpha-2)^{\nu}} > 0,\end{aligned}$$

as long as α (whence a) is sufficiently large, possibly in dependence of ν.

This establishes (1.12) in the Dirichlet case and we now focus on the proof of (1.13) in the Neumann case. In this situation, recalling (3.9),

$$\frac{a}{2}\Phi_N^{x,x}(s,T) \;=\; \frac{T}{2} + \alpha^{2s}\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\cos^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]$$

and therefore

$$\begin{aligned}\frac{a}{4\alpha^{2s}}\partial_s\Phi_N^{x,x}(s,T) \;=\;& \ln\alpha\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\cos^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]\\ &-\sum_{k=1}^{+\infty}\frac{\ln k}{k^{2s}}\cos^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]\\ &-T\sum_{k=1}^{+\infty}\frac{\ln\alpha-\ln k}{k^{2s}}\cos^2\left(\frac{kx}{\alpha}\right)\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(\frac{k}{\alpha}\right)^{2s}.\end{aligned}$$

This puts us in the same position as in (3.17), but with the sine replaced by the cosine. Hence, in this case, we only need to detect the analog of (3.18). For this, we observe that if $x \in \left(0, \frac{(1-\nu)a}{2}\right) \cup \left(\frac{(1+\nu)a}{2}, a\right) = \left(0, \frac{(1-\nu)\alpha\pi}{2}\right) \cup \left(\frac{(1+\nu)\alpha\pi}{2}, \alpha\pi\right)$ we have that

$$\left|\cos\left(\frac{x}{\alpha}\right)\right| > \cos\left(\frac{(1-\nu)\pi}{2}\right).$$

Thus, the same argument as in the Dirichlet case leads to (1.13). □

3.2. **Proof of the results in Sections 1.3 and 1.4.** In this section we prove the results stated in Section 1.3. Here we discuss the optimal search strategy when the forager starting position $y \in \Omega$ is sufficiently close to the prey location $x \in \Omega$, but does not coincide with it.

We recall that we adopt the subscript $*$ every time that we refer to both the Dirichlet and the Neumann case.

We start this section by showing that all the functionals defined in (1.6) are continuous with respect to $s \in (0,1]$.

Proof of Proposition 1.5. Let $(x,y,T) \in \Omega \times \Omega \times (0,+\infty)$ be such that $x \neq y$. Then, thanks to Proposition 2.12 we have that

$$\Phi_*^{x,y}(s,T) \in (0,+\infty),$$

and also

$$l_*^y(s,T) \in (0,+\infty) \quad \text{and} \quad \mathcal{A}_*^y(s,T) \in (0,+\infty), \tag{3.19}$$

for each $s \in (0,1]$. These considerations give that $\mathcal{E}_{j,*}^{x,y}(s,T)$ for all $s \in (0,1]$.

Now, from Proposition 2.12 and (3.19), we deduce that the functionals $\mathcal{E}_{j,*}^{x,y}(\cdot,T)$ are continuous with respect to $s \in (0,1]$. ☐

Now we prove Theorem 1.6. We show that $s = 0$ is a global minimizer for $\mathcal{E}_{1,D}^{x,y}(\cdot,T)$ in $(0,1)$ for each $x, y \in \Omega$ such that $x \neq y$ and for all $T \in (0,+\infty)$. Moreover, we discuss the existence of the limit for $s \searrow 0$ of $\mathcal{E}_{2,D}$ and $\mathcal{E}_{3,D}$.

Proof of Theorem 1.6. Let $x, y \in \Omega$ such that $x \neq y$ and $T \in (0,+\infty)$. Then, thanks to Lemma 2.13 we have that

$$\lim_{s \searrow 0} \mathcal{E}_{1,D}^{x,y}(s,T) = 0. \tag{3.20}$$

Since $\Phi_D^{x,y}(s,T) \in (0,+\infty)$ for each $s \in (0,1]$, see Proposition 2.12, we establish (1.14). We point out that the existence of the limits in (1.15) was already obtained in Lemma 2.13.

Besides, making use of the Maximum Principle for the heat equation, we see that

$$F_D(z,w) > 0 \qquad \text{for all } z, w \in \Omega, \tag{3.21}$$

and so the right-hand sides of the expressions in (2.70) and (2.71) are non negative. Also, using (D.6) and (3.21) we deduce that the limits in (2.70) and (2.71) are also positive and finite. ☐

We prove now Theorems 1.7 and 1.8. We recall that this result states that if the forager starting position is close enough to the target location, then the optimal search strategy for the functionals in equation (1.6) is in a small neighborhood of $s = 0$.

Proof of Theorems 1.7 and 1.8. Let $(y,T) \in \Omega \times (0,+\infty)$. We recall the limit in (2.69) and we observe that

$$\sup_{s \in (0,1)} \mathcal{E}_{1,D}^{x,y}(s,T) = \mathcal{E}_{1,D}^{x,y}\left(s_{x,y,T}^{(1)}, T\right) \quad \text{with} \quad s_{x,y,T}^{(1)} \in (0,1], \tag{3.22}$$

for each $x \in \Omega \setminus \{y\}$.

Also, From Lemma 2.15 and equation (2.26), we evince that if $s_0 \in \left(0, \frac{1}{2}\right)$ there exists some $\beta = \beta_{s_0,y,T,\Omega} \in (0, \widehat{\delta})$ such that, if $x \in B_\beta(y) \setminus \{y\}$, then

$$\limsup_{s \searrow 0} \mathcal{E}_{j,N}^{x,y}(s,T) \leqslant \mathcal{E}_{j,N}^{x,y}(s_0,T),$$

for all $j \in \{1,2,3\}$, where $\widehat{\delta}$ is provided in (2.28).

Thus, we deduce that there exists some $\widehat{\beta} = \widehat{\beta}_{y,T,\Omega}$ such that if $x \in B_{\widehat{\beta}}(y) \setminus \{y\}$, then

$$\sup_{s \in (0,1)} \mathcal{E}_{j,N}^{x,y}(s,T) = \mathcal{E}_{1,D}^{x,y}\left(s_{x,y,T}^{(j)}, T\right) \quad \text{with} \quad s_{x,y,T}^{(j)} \in (0,1], \tag{3.23}$$

for all $j \in \{1,2,3\}$.

Let us first study the case $n \leqslant 2$. We recall that thanks to Lemma 2.7, for each $s_0 \in \left(0, \frac{1}{2}\right)$ we have the existence of some $\widehat{\delta} = \widehat{\delta}_{s_0,y,T,\Omega}$, given in (2.28), such that, for each $x \in B_{\widehat{\delta}}(y) \setminus \{y\}$, one has that

$$\Phi_*^{x,y}(s_0,T) \geqslant \frac{C_{s_0,y,\Omega}}{|x-y|^{n-2s_0}}, \tag{3.24}$$

where $C_{s_0,y,\Omega}$ is provided in (2.34). Also, for each $s \in (0,1)$ and $\mu \in A_{n,s}$, where $A_{n,s}$ is given in (2.46), thanks to (2.48) we have the existence of some constant $C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}$ such that

$$\Phi_*^{x,y}(s,T) \leqslant \frac{C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}}{|x-y|^{2\mu}}, \tag{3.25}$$

for each $x \in B_{\widehat{\delta}}(y) \setminus \{y\}$.

Consequently, from the last two inequalities we obtain that if $s_0 \in \left(0, \frac{1}{2}\right)$, $s_1 \in (s_0, 1)$ and $\mu \in A_{n,s_1}$, then

$$\frac{\mathcal{E}_{1,*}^{x,y}(s_0,T)}{\mathcal{E}_{1,*}^{x,y}(s_1,T)} = \frac{\Phi_*^{x,y}(s_0,T)}{\Phi_*^{x,y}(s_1,T)} \geqslant \frac{C_{*,s_0,y,\mu,B_{\widehat{\delta}}(y),T,\Omega}}{|x-y|^{n-2s_0-2\mu}}, \tag{3.26}$$

for all $x \in B_{\widehat{\delta}}(y) \setminus \{y\}$, where we set

$$C_{*,s_0,y,\mu,B_{\widehat{\delta}}(y),T,\Omega} := \frac{C_{s_0,y,\Omega}}{C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}}. \tag{3.27}$$

As a result, for each $\varepsilon \in (0,1)$, by choosing $s_0 := \frac{\varepsilon}{4}$, $s_1 \in (\varepsilon, 1)$ and $\mu := (n-\varepsilon)/2$ in (3.26), and recalling also (3.22) and (3.23), we infer the existence of some $\delta^{(1)} = \delta^{(1)}_{\varepsilon,y,T,\Omega} \in (0, \widehat{\delta})$ such that for each $x \in B_{\delta^{(1)}}(y) \setminus \{y\}$ it holds that

$$\sup_{s\in(0,1)} \mathcal{E}_{1,*}^{x,y}(s,T) = \mathcal{E}_{1,*}^{x,y}\left(s^{(1)}_{*,x,y,T}, T\right) \quad \text{with} \quad s^{(1)}_{*,x,y,T} \in (0,\varepsilon).$$

This concludes the proof of (1.17) and (1.19) with $j = 1$.

Let us now prove (1.18) for the functional $\mathcal{E}_{2,D}$. To this end, let $d_y := \frac{d(y,\partial\Omega)}{2}$ and $B_y := B_{d_y}(y)$. Then, thanks to equation (D.5) in Lemma D.4 we have that there exists a constant $\widetilde{c}_{B_y,\Omega}$ such that for each $x \in B_y \setminus \{y\}$ it holds that

$$\mathcal{E}_{2,D}^{x,y}(0,T) = \frac{F_D(x,y)}{\int_\Omega |\xi-y| F_D(\xi,y)\,d\xi} \geqslant \frac{\widetilde{c}_{B_y,\Omega}}{\int_\Omega |\xi-y| F_D(\xi,y)\,d\xi} \frac{1}{|x-y|^n}. \tag{3.28}$$

Therefore, using (3.25) and the estimates in (3.28), if $s \in (0,1)$, $x \in B_{\widehat{\delta}}(y) \setminus \{y\}$ and μ is given as in (2.46), we obtain that

$$\begin{aligned} \frac{\mathcal{E}_{2,D}^{x,y}(0,T)}{\mathcal{E}_{2,D}^{x,y}(s,T)} &= \frac{F_D(x,y)}{\int_\Omega |\xi-y| F_D(\xi,y)\,d\xi} \frac{l_D^y(s,T)}{\Phi_D^{x,y}(s,T)} \\ &\geqslant \frac{\Gamma(s) l_D^y(s,T)}{\int_\Omega |\xi-y| F_D(\xi,y)\,d\xi} \frac{\widetilde{c}_{B_y,\Omega}}{C_{D,\mu,B_{\widehat{\delta}}(y),T,\Omega}} \frac{1}{|x-y|^{n-2\mu}}. \end{aligned} \tag{3.29}$$

Now, using (2.79) and the limit

$$\lim_{s\searrow 0} s\,\Gamma(s) = 1,$$

we obtain that

$$\lim_{s\searrow 0} \frac{\Gamma(s) l_D^y(s,T)}{\int_\Omega |\xi-y| F_D(\xi,y)\,d\xi} = 1 - e^{-T}(T+1).$$

Thanks to this observation and Proposition 2.12, we can define the positive constant

$$C_{\mu,y,B_{\widehat{\delta}}(y),T,\Omega} := \inf_{s\in(0,1)} \frac{\Gamma(s) l_D^y(s,T)}{\int_\Omega |\xi-y| F_D(\xi,y)\,d\xi} \frac{\widetilde{c}_{B_y,\Omega}}{C_{D,\mu,B_{\widehat{\delta}}(y),T,\Omega}} > 0. \tag{3.30}$$

Then, we obtain from (3.29) that for each $x\in B_y\setminus\{y\}$, $s\in(0,1)$ and μ as in (2.46) it holds that

$$\frac{\mathcal{E}_{2,D}^{x,y}(0,T)}{\mathcal{E}_{2,D}^{x,y}(s,T)} \geqslant \frac{C_{\mu,y,B_{\widehat{\delta}}(y),T,\Omega}}{|x-y|^{n-2\mu}}. \tag{3.31}$$

Therefore, for each $\varepsilon\in(0,1)$ by taking $s\in(\varepsilon,1)$ and choosing $\mu:=(n-\varepsilon)/2$ in (3.31), we deduce that there exists some $\delta^{(2)}=\delta^{(2)}_{\varepsilon,y,T,\Omega}$ such that for each $x\in B_{\delta^{(2)}}(y)\setminus\{y\}$ it holds that

$$\mathcal{E}_{2,D}^{x,y}(0,T) \geqslant \sup_{s\in(\varepsilon,1)} \mathcal{E}_{2,D}^{x,y}(s,T).$$

The proof of (1.18) for $\mathcal{E}_{3,D}$ is analogous to the one for $\mathcal{E}_{2,D}$ and therefore it will be omitted. This last step concludes the proof of Theorem 1.7 for $n\leqslant 2$.

Now we show (1.19) when $n\leqslant 2$ for $\mathcal{E}_{2,N}$. To do so, thanks to Proposition 2.12 and (2.102) we can define the positive constant

$$\widetilde{C}_{y,T,\Omega} := \inf_{\substack{s_0\in(0,1)\\ s_1\in(0,1)}} \frac{l_N^y(s_1,T)}{l_N^y(s_0,T)} > 0. \tag{3.32}$$

Then, if $s_0\in\left(0,\frac{1}{2}\right)$, $s_1\in(s_0,1)$ and $\mu\in A_{n,s_1}$, thanks to equations (3.24) and (3.25) we have that

$$\frac{\mathcal{E}_{2,N}^{x,y}(s_0,T)}{\mathcal{E}_{2,N}^{x,y}(s_1,T)} = \frac{\Phi_N^{x,y}(s_0,T)}{\Phi_N^{x,y}(s_1,T)} \frac{l_N^y(s_1,T)}{l_N^y(s_0,T)} \geqslant \frac{\widetilde{C}_{\mu,s_0,y,K,T,\Omega}}{|x-y|^{n-2s_0-2\mu}}, \tag{3.33}$$

for all $x\in B_{\widehat{\delta}}(y)\setminus\{y\}$, where we defined

$$\widetilde{C}_{\mu,s_0,y,K,T,\Omega} := C_{N,s_0,y,\mu,B_{\widehat{\delta}}(y),T,\Omega}\widetilde{C}_{y,T,\Omega}. \tag{3.34}$$

Therefore, for each $\varepsilon\in(0,1)$, by choosing $s_0:=\frac{\varepsilon}{4}$, $s_1\in(\varepsilon,1)$ and $\mu:=(n-\varepsilon)/2$ in (3.33), and recalling (3.23), we deduce the existence of some $\delta^{(2)}=\delta^{(2)}_{\varepsilon,y,T,\Omega}\in(0,\widehat{\beta})$ such that for each $x\in B_{\delta^{(2)}}(y)\setminus\{y\}$ it holds that

$$\sup_{s\in(0,1)} \mathcal{E}_{2,N}^{x,y}(s,T) = \mathcal{E}_{2,N}^{x,y}\left(s^{(2)}_{x,y,T},T\right) \quad \text{with} \quad s^{(2)}_{x,y,T}\in(0,\varepsilon).$$

This concludes the proof of (1.19) for $\mathcal{E}_{2,N}$. The proof of (1.19) for $\mathcal{E}_{3,N}$ is analogous to the one for $\mathcal{E}_{2,N}$.

It is left to prove Theorems 1.7 and 1.8 when $n\geqslant 3$.

If $n\geqslant 3$, we just have to replace the inequality (3.25) with the one in (2.47). Thus, repeating the above procedure with this change, the inequalities in (3.26) and (3.33) become

$$\frac{\mathcal{E}_{*}^{x,y}(s_0,T)}{\mathcal{E}_{*}^{x,y}(s_1,T)} \geqslant \frac{C_{*,s_0,y,B_{\widehat{\delta}}(y),T,\Omega}}{|x-y|^{2(s_1-s_0)}}, \tag{3.35}$$

for all $s_0\in\left(0,\frac{1}{2}\right)$, $s_1\in(s_0,1)$ and $x\in B_{\widehat{\delta}}(y)\setminus\{y\}$, where we denoted by $\mathcal{E}_*$ any of the functionals $\mathcal{E}_{1,D}$, $\mathcal{E}_{1,N}$ and $\mathcal{E}_{2,N}$.

The constant $C_{*,s_0,y,B_{\widehat{\delta}}(y),T,\Omega}$ is obtained substituting the constant $C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}$ with $C_{*,B_{\widehat{\delta}}(y),T,\Omega}$ in (3.27) for $\mathcal{E}_{1,D}$ and $\mathcal{E}_{1,N}$, and in (3.34) for $\mathcal{E}_{2,N}$.

Analogously, equation (3.31) becomes

$$\frac{\mathcal{E}_{2,D}^{x,y}(0,T)}{\mathcal{E}_{2,D}^{x,y}(s_1,T)} \geqslant \frac{C^{(1)}_{y,B_{\widehat{\delta}}(y),T,\Omega}}{|x-y|^{2s_1}} \tag{3.36}$$

for all $s \in (0,1)$ and $x \in B_{\widehat{\delta}}(y) \setminus \{y\}$, where we defined

$$C^{(1)}_{y,B_{\widehat{\delta}}(y),T,\Omega} := \inf_{s\in(0,1)} \frac{\Gamma(s) l^y_D(s,T)}{\int_\Omega |\xi - y| F_D(\xi,y)\, d\xi} \frac{\widetilde{c}_{B_y,\Omega}}{C_{D,B_{\widehat{\delta}}(y),T,\Omega}}.$$

Therefore, for each $\varepsilon \in (0,1)$, by choosing $s_0 := \frac{\varepsilon}{2}$ in (3.35) and $s_1 \in (\varepsilon, 1)$ in (3.35) and (3.36), we obtain (1.17), (1.18) for $\mathcal{E}_{2,D}$ and (1.19) for both $\mathcal{E}_{1,N}$ and $\mathcal{E}_{2,N}$ when $n \geqslant 3$.

The proof of (1.18) and (1.19) respectively for $\mathcal{E}_{3,D}$ and $\mathcal{E}_{3,N}$ are analogous to the one of $\mathcal{E}_{2,D}$ and $\mathcal{E}_{2,N}$ when $n \geqslant 3$ and are therefore omitted. □

Now, we prove Corollary 1.12. Namely, we establish in the one-dimensional framework, and under suitable geometric assumptions on the domain, that if the target location $x \in \Omega$ is close enough to the forager starting position $y \in \Omega$, then there exists a local maximizer for the functionals $\mathcal{E}^{x,y}_{1,D}$ and $\mathcal{E}^{x,y}_{1,N}$ in a neighborhood of the local Brownian strategy $s = 1$.

Proof of Corollary 1.12. We will only prove (1.21), since the proof of (1.22) is analogous. For this, let $\nu \in \left(0, \frac{1}{2}\right)$. Then, thanks to Theorem 1.4, we have that there exists some $a_\nu \in (\pi, +\infty)$ such that, for all $a \in (a_\nu, +\infty)$, $T \in [\nu a^{2s}, +\infty)$ and $y \in (\nu a, \nu(1-a))$, it holds that

$$\Phi^{y,y}_D(s_0,T) < \Phi^{y,y}_D(s_1,T), \tag{3.37}$$

for all $s_0 \in \left(\frac{1+\nu}{2}, 1\right]$ and $s_1 \in (s_0, 1]$.

Now, for any $\varepsilon \in \left(\frac{1+\nu}{2}, 1\right)$ we define the positive quantity

$$\widetilde{\delta} = \widetilde{\delta}_{\varepsilon,\nu,y,T} := \Phi^{y,y}_D(1,T) - \Phi^{y,y}_D(1-\varepsilon,T).$$

Also, thanks to the continuity of $\Phi^{x,y}_D(s,T)$ with respect to $(s,x,y) \in \left(\frac{1}{2}, 1\right] \times \Omega \times \Omega$ stated in Proposition 2.12, we can define $\delta_{\varepsilon,\nu\, y,T,\Omega} \in (0,+\infty)$, such that

$$|\Phi^{x,y}_D(s,T) - \Phi^{y,y}_D(s,T)| \leqslant \frac{\widetilde{\delta}}{4} \qquad \text{for all } s \in \left(\frac{1+\nu}{2}, 1\right].$$

Thus, using the monotonicity of $\Phi^{y,y}_D$ in (3.37), we obtain that, for each $x \in \Omega$ and $s_0 \in \left(\frac{1+\nu}{2}, 1-\varepsilon\right)$,

$$\begin{aligned}
&\Phi^{x,y}_D(1,T) - \Phi^{x,y}_D(s_0,T) \\
&= \Phi^{x,y}_D(1,T) - \Phi^{y,y}_D(1,T) + \Phi^{y,y}_D(1,T) - \Phi^{y,y}_D(s_0,T) + \Phi^{y,y}_D(s_0,T) - \Phi^{x,y}_D(s_0,T) \\
&> \Phi^{x,y}_D(1,T) - \Phi^{y,y}_D(1,T) + \widetilde{\delta} + \Phi^{y,y}_D(s_0,T) - \Phi^{x,y}_D(s_0,T) \\
&\geqslant -\frac{\widetilde{\delta}}{4} + \widetilde{\delta} - \frac{\widetilde{\delta}}{4} \\
&= \frac{\widetilde{\delta}}{2}.
\end{aligned}$$

From this, we infer that

$$\sup_{s\in\left(\frac{1+\nu}{2},1\right)} \mathcal{E}^{x,y}_{1,D}(s,T) = \mathcal{E}^{x,y}_{1,D}\left(s^*_{x,y,T}, T\right) \quad \text{with} \quad s^*_{x,y,T} \in (1-\varepsilon, 1],$$

which proves (1.21). □

We now prove Proposition 1.13 and establish the continuity with respect to the fractional exponent of the set functionls in (1.9).

Proof of Proposition 1.13. Since the proof for the Dirichlet and Neumann case are analogous, we focus on the Dirichlet framework.

We already established in Proposition 2.12 that for all $y \in \Omega$ and $s \in (0,1)$ one has that (3.19) holds, and also the functionals in (1.3) and (1.4) are uniformly bounded in $(0,1] \times \Omega$.

Therefore, by definition we obtain that, for all $\Omega_2 \subset \Omega$,

$$\widetilde{l}_D^{\Omega_2}(s,T) \in (0,+\infty) \qquad \text{and} \qquad \widetilde{\mathcal{A}}_D^{\Omega_2}(s,T) \in (0,+\infty), \tag{3.38}$$

for all $s \in (0,1]$ and $T \in (0,+\infty)$.

Besides, thanks to Proposition 2.12 we know that $l_D^y(\cdot,T)$ and $\mathcal{A}_D^y(\cdot,T)$ are continuous in $(0,1]$. Thus, by the Dominated Convergence Theorem we obtain that $\widetilde{l}_D^{\Omega_2}(\cdot,T)$ and $\widetilde{\mathcal{A}}_D^{\Omega_2}(\cdot,T)$ are continuous in $(0,1]$.

Now, we observe that

$$\widetilde{\Phi}_D^{\Omega_1,\Omega_2}(s,T) = \frac{1}{|\Omega_1||\Omega_2|}\int_0^T \int_{\Omega_1\times\Omega_2} r_D^s(t,x,y)\,dx\,dy\,dt.$$

Therefore, thanks to Theorem 1.9 in [DGV22a] we obtain that

$$\widetilde{\Phi}_D^{\Omega_1,\Omega_2}(s,T) \in (0,+\infty). \tag{3.39}$$

Also, $r_D^s(t,x,y)$ is continuous for $s \in (0,1]$ for all $(t,x,y) \in (0,+\infty)\times\Omega\times\Omega$, see e.g. Theorem 1.8 in [DGV22a]. Thanks to Proposition A.1 and Lemma A.2 in [DGV22a], we have that, for each $t \in (0,+\infty)$ and $\varepsilon \in (0,1)$, the kernel $r_D^s(t,x,y)$ is uniformly bounded in $(s,x,y) \in (\varepsilon,1]\times\Omega\times\Omega$. Thus, as a consequence of the Dominated Convergence Theorem we obtain that

$$f(s,t) := \int_{\Omega_1\times\Omega_2} r_D^s(t,x,y)\,dx\,dy$$

is continuous in $s \in (0,1]$.

Additionally, in view of Theorem 1.9 in [DGV22a], we see that

$$|f(s,t)| \leqslant |\Omega_2| \qquad \text{for all } (s,t) \in (0,1)\times(0,+\infty),$$

and therefore by the Dominated Convergence Theorem we obtain the continuity of $\widetilde{\Phi}_D^{\Omega_1,\Omega_2}$ for $s \in (0,1]$.

Finally, the continuity of the functionals in (1.9) with respect to $s \in (0,1]$ follows from (3.38) and the fact that $\widetilde{\Phi}_D^{\Omega_1,\Omega_2}(\cdot,T) \in C((0,1])$ and $\widetilde{l}_D^{\Omega_2}(\cdot,T), \widetilde{\mathcal{A}}_D^{\Omega_2}(\cdot,T) \in C((0,1])$. □

Now we prove Theorem 1.14. In this result we show that $s = 0$ is a minimizer for the functional $\widetilde{\mathcal{E}}_{1,D}^{\Omega_1,\Omega_2}$, where Ω_1 and Ω_2 are disjoint and smooth. Also, we show that $\widetilde{\mathcal{E}}_{2,D}^{\Omega_1,\Omega_2}$ and $\widetilde{\mathcal{E}}_{3,D}^{\Omega_1,\Omega_2}$ admit a positive and finite limit for $s \searrow 0$.

Proof of Theorem 1.14. Let $T \in (0,+\infty)$ and $\Omega_1, \Omega_2 \Subset \Omega$ be disjoint and smooth. Then, thanks to Lemma 2.14, we obtain that

$$\lim_{s\searrow 0} \widetilde{\mathcal{E}}_{1,D}^{\Omega_1,\Omega_2}(s,T) = 0.$$

Furthermore, thanks to (3.39), we see that $\mathcal{E}_{1,D}^{\Omega_1,\Omega_2}(s,T) \in (0,+\infty)$ for all $s \in (0,1]$. This latter observation together with the above limit lead to (1.23).

Now we prove (1.24). The existence of the limits in (1.24) was already established in Lemma 2.14. Using the fact that Ω_1 and Ω_2 are disjoint and smooth, together with the inequality in (D.6) and also (3.21), we evince that

$$\widetilde{F}_D(\Omega_1,\Omega_2) \in (0,+\infty), \qquad \int_{\Omega\times\Omega_2} |\xi-y| F_D(\xi,y)\,d\xi\,dy \in (0,+\infty)$$

$$\text{and} \qquad \int_{\Omega\times\Omega_2} |\xi-y|^2 F_D(\xi,y)\,d\xi\,dy \in (0,+\infty),$$

where F_D and $\widetilde{F}_D$ are given respectively in (2.72) and (2.85). Therefore, from (2.83), (2.84) and these considerations we conclude the proof of (1.24).

□

Now we focus our attention to Theorems 1.15 and 1.16. To prove these results, it is useful to state and prove the following proposition regarding a monotonicity property with respect to s and a scaling property for the functional F^{Ω_1,Ω_2} introduced in (2.62). In what follows we denote by d_B the diameter of B for each bounded set $B \subset \mathbb{R}^n$.

Proposition 3.1. *Let $K \subset \mathbb{R}^n$ be a compact set and $\Omega_1, \Omega_2 \subset K$ be measurable sets such that $\Omega_1 \cap \Omega_2 = \varnothing$.*

Then, if $d_K \leqslant 1$, we have that

$$\frac{d}{ds} F^{\Omega_1,\Omega_2}(s) \leqslant 0 \qquad \text{for all } s \in (0,1). \tag{3.40}$$

Moreover, for each $r \in (0,+\infty)$ and $y \in \mathbb{R}^n$, it holds that

$$F^{r_y\Omega_1, r_y\Omega_2}(s) = r^{n+2s} F^{\Omega_1,\Omega_2}(s). \tag{3.41}$$

Proof. We observe that, thanks to the Dominated Convergence Theorem,

$$\frac{d}{ds} F^{\Omega_1,\Omega_2}(s) = 2 \int_{\Omega_1\times\Omega_2} \frac{\ln|x-y|}{|x-y|^{n-2s}}\, dx\, dy.$$

Hence, if $d_K \leqslant 1$, then

$$\frac{d}{ds} F^{\Omega_1,\Omega_2}(s) \leqslant 0,$$

which proves (3.40).

Now we show the scaling property in (3.41). Let $r \in (0,+\infty)$ and, up to a translation, assume that $y = 0$. Then, applying the change of variable $(x,y) = (rX, rY)$ we obtain that

$$\begin{aligned} F^{r\Omega_1, r\Omega_2}(s) &= \int_{r\Omega_1\times r\Omega_2} \frac{1}{|x-y|^{n-2s}}\, dx\, dy \\ &= \int_{\Omega_1\times\Omega_2} \frac{r^{2n}}{r^{n-2s}|X-Y|^{n-2s}}\, dX\, dY \\ &= r^{n+2s} F^{\Omega_1,\Omega_2}(s), \end{aligned}$$

which completes the proof. ∎

With this preliminary work, we can now prove Theorems 1.15 and 1.16. We recall that the aim of this result is to show that if $\Omega_1, \Omega_2 \subset \Omega$ are disjoint, smooth and close enough, then the best search strategy for the set efficiency functionals provided in (1.9) is in a small neighborhood of $s = 0$.

Proof of Theorems 1.15 and 1.16. Let $(y,T) \in \Omega \times (0,+\infty)$. If $\Omega_1, \Omega_2 \subset \Omega$ are smooth and disjoint, then thanks to Theorem 1.14 we have that

$$\sup_{s\in(0,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s,T) = \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}\left(s^{(1)}_{\Omega_1,\Omega_2,T}, T\right) \qquad \text{with } s^{(1)}_{\Omega_1,\Omega_2,T} \in (0,1]. \tag{3.42}$$

Moreover, If P and $\widetilde{P}$ are given as in (2.106), we observe that

$$\inf_{\Omega_2\subset\Omega} P(\Omega_2) \in (0,+\infty) \qquad \text{and} \qquad \inf_{\Omega_2\subset\Omega} \widetilde{P}(\Omega_2) \in (0,+\infty).$$

Now, using (2.65) we have that, for $s_0 \in \left(0,\frac{1}{2}\right)$ and $r \in (0,\widehat{\delta})$, where $\widehat{\delta} = \widehat{\delta}_{s_0,y,T,\Omega}$ has been given in (2.28), then

$$\widetilde{\Phi}^{\Omega_1,\Omega_2}_N(s_0,T) \geqslant \frac{C_{s_0,y,\Omega}}{(2r)^{n-2s_0}},$$

for all $\Omega_1, \Omega_2 \subset B_r(y)$, where $C_{s_0,y,\Omega}$ is given (2.34).

Consequently, using also (2.103), (2.104) and (2.105), we deduce that there exists some $\beta=\beta_{y,T,\Omega}\in(0,1)$ such that if $\Omega_1,\Omega_2\subset B_\beta(y)$ are smooth and disjoint then

$$\sup_{s\in(0,1)}\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,N}(s,T)=\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,N}\left(s^{(j)}_{\Omega_1,\Omega_2,T},T\right)\qquad\text{with } s^{(j)}_{\Omega_1,\Omega_2,T}\in(0,1],\tag{3.43}$$

for all $j\in\{1,2,3\}$.

We will first prove the results for $n\leqslant 2$.

We recall that, by Corollary 2.11, if $s_1\in(0,1)$ and $\mu\in A_{n,s_1}$, where A_{n,s_1} is given in (2.46), then

$$\widetilde{\Phi}^{\Omega_1,\Omega_2}_{*}(s_1,T)\leqslant\frac{C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}}{|\Omega_1||\Omega_2|}F^{\Omega_1,\Omega_2}\left(\frac{n-2\mu}{2}\right),\tag{3.44}$$

for all $\Omega_1,\Omega_2\Subset B_{\widehat{\delta}}(y)$, where $C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}$ is introduced in Theorem 2.9.

Also, in light of (2.65) we deduce that if $s_0\in\left(0,\frac{1}{2}\right)$ and $\Omega_1,\Omega_2\Subset B_{\widehat{\delta}}(y)$, then

$$\widetilde{\Phi}^{\Omega_1,\Omega_2}_{*}(s_0,T)\geqslant\frac{C_{s_0,y,\Omega}}{|\Omega_1||\Omega_2|}F^{\Omega_1,\Omega_2}(s_0).\tag{3.45}$$

Now, we define

$$\delta_0:=\min\left\{\widehat{\delta},\frac{1}{2}\right\},$$

and we consider $\Omega_1,\Omega_2\subset B_{\delta_0}(y)$ smooth and such that $\Omega_1\cap\Omega_2=\varnothing$. Thus, from (1.9), (3.40), (3.41), (3.44) and (3.45) we deduce that if $r\in(0,1)$, $s_0\in\left(0,\frac{1}{2}\right)$, $s_1\in(s_0,1)$ and $\mu\in\left(0,\frac{n}{2}-s_0\right)\cap A_{n,s_1}$, where A_{n,s_1} is given as in (2.46), it holds that

$$\begin{aligned}\frac{\widetilde{\mathcal{E}}^{r_y\Omega_1,r_y\Omega_2}_{1,*}(s_0,T)}{\widetilde{\mathcal{E}}^{r_y\Omega_1,r_y\Omega_2}_{1,*}(s_1,T)}&=\frac{\widetilde{\Phi}^{r_y\Omega_1,r_y\Omega_2}_{*}(s_0,T)}{\widetilde{\Phi}^{r_y\Omega_1,r_y\Omega_2}_{*}(s_1,T)}\\&\geqslant C^{(1)}_{s_0,*,\mu,B_{\widehat{\delta}}(y),y,T,\Omega}\frac{F^{r_y\Omega_1,r_y\Omega_2}(s_0)}{F^{r_y\Omega_1,r_y\Omega_2}\left(\frac{n-2\mu}{2}\right)}\\&\geqslant C^{(1)}_{s_0,*,\mu,B_{\widehat{\delta}}(y),y,T,\Omega}\frac{r^{n+2s_0}F^{\Omega_1,\Omega_2}(s_0)}{r^{2n-2\mu}F^{\Omega_1,\Omega_2}\left(\frac{n-2\mu}{2}\right)}\\&\geqslant\frac{C^{(1)}_{s_0,*,\mu,B_{\widehat{\delta}}(y),y,T,\Omega}}{r^{n-2s_0-2\mu}},\end{aligned}\tag{3.46}$$

where we defined

$$C^{(1)}_{s_0,*,\mu,B_{\widehat{\delta}}(y),y,T,\Omega}:=\frac{C_{s_0,y,\Omega}}{C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}}.$$

We recall that in writing $r_y\Omega_1$ and $r_y\Omega_2$ we adopted the notation in (1.26).

As a result, for all $\varepsilon\in(0,1)$, by choosing for instance $s_0:=\frac{\varepsilon}{4}$ and $\mu:=(n-\varepsilon)/2$ in (3.46), and using also (3.42) and (3.43), we infer that there exists some $r^{(1)}=r^{(1)}_{\varepsilon,y,T,\Omega}$ such that if $\Omega_1,\Omega_2\Subset B_{r^{(1)}\delta_0}(y)$ are smooth and satisfy $\Omega_1\cap\Omega_2=\varnothing$, then

$$\sup_{s\in(0,1)}\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,*}(s,T)=\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,*}\left(s^{(1)}_{*,\Omega_1,\Omega_2,T},T\right)\qquad\text{with } s^{(1)}_{*,\Omega_1,\Omega_2,T}\in(0,\varepsilon).$$

We now focus on the proof of (1.28) for $\widetilde{\mathcal{E}}_{2,D}$. Let $K\Subset\Omega$ and assume that $\Omega_1,\Omega_2\subset K$. Then, thanks to equations (2.83) and (D.5) we have that

$$\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{2,D}(0,T)\geqslant\frac{\widetilde{c}_{K,\Omega}}{\int_{\Omega\times\Omega_2}|\xi-y|F_D(\xi,y)\,d\xi\,dy}\frac{F^{\Omega_1,\Omega_2}(0)}{|\Omega_1|},\tag{3.47}$$

where F_D and $\widetilde{c}_{K,\Omega}$ are given respectively in (2.72) and (D.8). Then, in light of (2.64) and (3.44), if $s\in(0,1)$ and $\mu\in A_{n,s}\cap(0,1)$, where $A_{n,s}$ is given in (2.46), we have that, for each $r\in(0,1)$,

$$\begin{aligned}\frac{\widetilde{\mathcal{E}}_{2,D}^{r_y\Omega_1,r_y\Omega_2}(0,T)}{\widetilde{\mathcal{E}}_{2,D}^{r_y\Omega_1,r_y\Omega_2}(s,T)}&=\frac{|r_y\Omega_2|\widetilde{F}_D(r_y\Omega_1,r_y\Omega_2)}{\int_{\Omega\times r_y\Omega_2}|\xi-y|F_D(\xi,y)\,d\xi\,dy}\frac{\widetilde{l}_D^{r_y\Omega_2}(s,T)}{\widetilde{\Phi}_D^{r_y\Omega_1,r_y\Omega_2}(s,T)}\\&\geqslant\frac{|r_y\Omega_2|\widetilde{c}_{K,\Omega}\Gamma(s)}{\int_{\Omega\times r_y\Omega_2}|\xi-y|F_D(\xi,y)\,d\xi\,dy}\frac{\widetilde{l}_D^{r_y\Omega_2}(s,T)}{C_{D,\mu,K,T,\Omega}}\frac{F^{r_y\Omega_1,r_y\Omega_2}(0)}{F^{r_y\Omega_1,r_y\Omega_2}\left(\frac{n-2\mu}{2}\right)},\end{aligned}\tag{3.48}$$

where $C_{D,\mu,K,T,\Omega}$ was introduced in Theorem 2.9.

Now, we observe that thanks to the limit in equation (2.88) one has that

$$\begin{aligned}\lim_{s\searrow0}\widetilde{l}_D^{r_y\Omega_2}(s,T)\Gamma(s)&=\lim_{s\searrow0}\frac{\widetilde{l}_D^{r_y\Omega_2}(s,T)}{s}\Gamma(s)s\\&=\frac{(1-e^{-T}(T+1))}{|r_y\Omega_2|}\int_{\Omega\times r_y\Omega_2}|\xi-y|F_D(\xi,y)\,d\xi\,dy.\end{aligned}\tag{3.49}$$

Let us set the notation

$$C_{\mu,K,T,r_y\Omega_2,\Omega}:=\inf_{s\in(0,1)}\frac{|r_y\Omega_2|\widetilde{c}_{K,\Omega}\Gamma(s)}{\int_{\Omega\times r_y\Omega_2}|\xi-y|F_D(\xi,y)\,d\xi\,dy}\frac{\widetilde{l}_D^{r_y\Omega_2}(s,T)}{C_{D,\mu,K,T,\Omega}}.\tag{3.50}$$

In view of (3.49), we see that if such infimum is attained at $s=0$, then it does not depend on $r_y\Omega_2$.

If the infimum is attained for some $\widehat{s}\in(0,1]$, then using Proposition 2.12 and Lemma D.3 with

$$f(y):=l_D^y(\widehat{s},T)\qquad\text{and}\qquad g(y)=\int_\Omega|\xi-y|F_D(\xi,y)\,d\xi,\tag{3.51}$$

we obtain that

$$C_{\mu,K,T,\Omega}:=\inf_{\substack{r\in(0,1)\\\Omega_2\subset K}}C_{\mu,K,T,r_y\Omega_2,\Omega}>0.$$

As a result, using equation (3.48) and Proposition 3.1, we deduce that if $d_K\leqslant1$, then

$$\frac{\widetilde{\mathcal{E}}_{2,D}^{r_y\Omega_1,r_y\Omega_2}(0,T)}{\widetilde{\mathcal{E}}_{2,D}^{r_y\Omega_1,r_y\Omega_2}(s,T)}\geqslant C_{\mu,K,T,\Omega}\frac{F^{r_y\Omega_1,r_y\Omega_2}(0)}{F^{r_y\Omega_1,r_y\Omega_2}\left(\frac{n-2\mu}{2}\right)}\geqslant\frac{C_{\mu,K,T,\Omega}}{r^{n-2\mu}}.\tag{3.52}$$

Therefore, for all $\varepsilon\in(0,1)$ and $K\Subset\Omega$ that are start-shaped with respect to $y\in K$, by choosing $s\in(\varepsilon,1)$ and $\mu:=(n-\varepsilon)/2$ in (3.52), we deduce the existence of some $r^{(2)}=r^{(2)}_{\varepsilon,K,T,\Omega}$ such that if $\Omega_1,\Omega_2\subset r_y^{(1)}K$ satisfy $\Omega_1\cap\Omega_2=\varnothing$ and are smooth, then

$$\widetilde{\mathcal{E}}_{2,D}^{\Omega_1,\Omega_2}(0,T)\geqslant\sup_{s\in(\varepsilon,1)}\widetilde{\mathcal{E}}_{2,D}^{\Omega_1,\Omega_2}(s,T).$$

This concludes the proof of (1.28) for $\widetilde{\mathcal{E}}_{2,D}$. The proof of (1.28) for $\widetilde{\mathcal{E}}_{3,D}$ will be omitted, being analogous to the one for $\widetilde{\mathcal{E}}_{2,D}$.

We now prove (1.29) for $\widetilde{\mathcal{E}}_{2,N}$. To do so, we fix some $s_0\in\left(0,\frac12\right)$, and, in light of Proposition 2.12, we define the positive constant

$$C_{s_0,T,\Omega_2,\Omega}:=\inf_{s_1\in(s_0,1)}\frac{\widetilde{l}_N^{\Omega_2}(s_1,T)}{\widetilde{l}_N^{\Omega_2}(s_0,T)}.$$

Also, if the above infimum is attained for some $\widehat{s}\in[s_0,1]$, using Lemma D.3 with

$$f(y)=l_N^y(\widehat{s},T)\qquad\text{and}\qquad g(y)=l_N^y(s_0,T),$$

we set

$$C_{s_0,T,\Omega} := \inf_{\Omega_2 \Subset \Omega} C_{s_0,T,\Omega_2,\Omega} > 0. \tag{3.53}$$

Thus, making use of equations (3.40), (3.41), (3.44) and (3.45), we deduce that if $\Omega_1, \Omega_2 \Subset B_{\delta_0}(y)$, and $K \supset B_{\delta_0}(y)$, $r \in (0,1)$, $s_0 \in \left(0, \frac{1}{2}\right)$, $s_1 \in (s_0, 1)$ and $\mu \in A_{n,s_1} \cap \left(0, \frac{n}{2} - s_0\right)$, we have that

$$\begin{aligned}
\frac{\widetilde{\mathcal{E}}_{2,N}^{r_y\Omega_1,r_y\Omega_2}(s_0,T)}{\widetilde{\mathcal{E}}_{2,N}^{r_y\Omega_1,r_y\Omega_2}(s_1,T)} &= \frac{\widetilde{\Phi}_N^{r_y\Omega_1,r_y\Omega_2}(s_0,T)}{\widetilde{l}_N^{r_y\Omega_2}(s_0,T)} \frac{\widetilde{l}_N^{r_y\Omega_2}(s_1,T)}{\widetilde{\Phi}_N^{r_y\Omega_1,r_y\Omega_2}(s_1,T)} \\
&\geqslant C^{(1)}_{\mu,s_0,y,K,T,\Omega} \frac{F^{r_y\Omega_1,r_y\Omega_2}(s_0)}{F^{r_y\Omega_1,r_y\Omega_2}\left(\frac{n-2\mu}{2}\right)} \\
&= C^{(1)}_{\mu,s_0,y,K,T,\Omega} \frac{r^{n+2s_0} F^{\Omega_1,\Omega_2}(s_0)}{r^{2n-2\mu} F^{\Omega_1,\Omega_2}\left(\frac{n-2\mu}{2}\right)} \\
&\geqslant \frac{C^{(1)}_{\mu,s_0,y,K,T,\Omega}}{r^{n-2s_0-2\mu}},
\end{aligned} \tag{3.54}$$

where we defined

$$C^{(1)}_{\mu,s_0,y,K,T,\Omega} := \frac{C_{s_0,y,\Omega}}{C_{N,\mu,B_{\widehat{\delta}}(y),T,\Omega}} C_{s_0,T,\Omega}.$$

Therefore, for each $\varepsilon \in (0,1)$, by choosing for instance $s_0 := \frac{\varepsilon}{4}$, $s_1 \in (\varepsilon, 1)$ and $\mu := (n-\varepsilon)/2$ in (3.54), and also thanks to (3.43), we deduce that there exists some $r^{(2)} = r^{(2)}_{\varepsilon,y,T,\Omega} \in (0,\beta)$ such that, for each $\Omega_1, \Omega_2 \subset B_{r^{(2)}\delta_0}(y)$ smooth and disjoint,

$$\sup_{s\in(0,1)} \widetilde{\mathcal{E}}_{2,N}^{\Omega_1,\Omega_2}(s,T) = \widetilde{\mathcal{E}}_{2,N}^{\Omega_1,\Omega_2}\left(s^{(2)}_{\Omega_1,\Omega_2,T}, T\right) \quad \text{with} \quad s^{(2)}_{\Omega_1,\Omega_2,T} \in (0,\varepsilon).$$

This concludes the proof of (1.29) for $\widetilde{\mathcal{E}}_{2,N}$. The proof of (1.29) for $\widetilde{\mathcal{E}}_{3,N}$ is analogous to the one for $\widetilde{\mathcal{E}}_{2,N}$ just concluded and therefore it will be omitted.

This concludes the proof of Theorems 1.15 and 1.16 for $n \leqslant 2$.

Few changes are in order to show Theorems 1.15 and 1.16 also for $n \geqslant 3$. In particular, we have to repeat the above arguments by replacing (3.44) with the inequality in (2.64). The procedure will determine changes only on the constants involved, in the same fashion of the proof of Theorems 1.7 and 1.8 for $n \geqslant 3$.

□

Appendix A. Green Function for the Dirichlet spectral fractional Laplacian

Here we give a proof of a well-known identity for the Green function $G_D^s(x,y)$ of the Diriclet spectral fractional Laplacian. The Green function is given by

$$G_D^s(x,y) = \frac{1}{\Gamma(s)} \int_0^{+\infty} p_D^\Omega(t,x,y) t^{s-1}\, dt,$$

see also [AD17]. Before we state the following result, let us recall the notation

$$\mathcal{C} = \{(x,y) \in \Omega \times \Omega \quad \text{s.t. } x \neq y\}.$$

Proposition A.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected.*

Then, for each $(x,y) \in \mathcal{C}$ it holds that

$$\int_0^{+\infty} r_D^s(t,x,y)\, dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} p_D^\Omega(t,x,y) t^{s-1}\, dt. \tag{A.1}$$

Proof. Given $x, y \in \mathcal{C}$, we let

$$\mathcal{I}(x,y) := \int_0^{+\infty} r_D^s(t,x,y)\,dt$$

and

$$\mathcal{J}(x,y) := \frac{1}{\Gamma(s)} \int_0^{+\infty} p_D^{\Omega}(t,x,y) t^{s-1}\,dt,$$

Now, let $\{\phi_k\}_k$ be an orthonormal basis of $L^2(\Omega)$ made of eigenfunctions of the Laplacian with Dirichlet boundary conditions, ordered such that if λ_k's are the corresponding eigenvalues, then $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$ (see for instance [Eva10]). In view of Theorem 1.8 in [DGV22a] we know that

$$r_D^s(t,x,y) = \sum_{k=1}^{+\infty} \phi_k(x)\phi_k(y)\exp(-t\lambda_k^s),$$

for each $(t,x,y) \in (0,+\infty) \times \Omega \times \Omega$. In order to prove (A.1), we first show that $\mathcal{I}(x,y)$ and $\mathcal{J}(x,y)$ are both continuous in $\mathcal{C}$. Thanks to Theorem 2.8 we know that

$$\int_0^T r_D^s(t,x,y)\,dy < +\infty$$

for each $T \in (0,+\infty)$, $s \in (0,1]$ and $x \neq y$. Moreover, thanks to Proposition A.1 in [DGV22a] we observe that for each $t > T$ and $s \in (0,1]$ it holds that

$$\begin{aligned} r_D^s(t,x,y) &= \exp(-t\lambda_1^s) \sum_{k=1}^{+\infty} \phi_k(x)\phi_k(y)\exp(-t(\lambda_k^s - \lambda_1^s)) \\ &\leqslant c_{m_0,\Omega,0} \exp(-t\lambda_1^s) \sum_{k=1}^{+\infty} \lambda_k^{2\alpha(m_0)} \exp(-T(\lambda_k^s - \lambda_1^s)) \\ &\leqslant C_{T,s,\Omega} \exp(-t\lambda_1^s), \end{aligned} \tag{A.2}$$

where the last inequality is a consequence of Lemma A.3 in [DGV22a], and $C_{T,s,\Omega} > 0$ is a constant depending on $T > 0$, $s \in (0,1]$ and Ω. The constants $\alpha(m_0)$ and $c_{m_0,\Omega,0}$ have been explicitly defined in Proposition A.1 in [DGV22a]. Therefore, if we call

$$g_D^s(t,x,y) := \begin{cases} r_D^s(t,x,y) & \text{for all } (t,x,y) \in (0,T] \times \mathcal{C}, \\ C_{T,s,\Omega}\exp(-t\lambda_1^s) & \text{for all } (t,x,y) \in (T,+\infty) \times \mathcal{C}, \end{cases}$$

we obtain that $g_D^s(t,x,y) \in L^1(0,+\infty)$ for each $(x,y) \in \mathcal{C}$, and also

$$r_D^s(t,x,y) \leqslant g_D^s(t,x,y),$$

for each $(t,x,y) \in (0,+\infty) \times \Omega \times \Omega$ and $s \in (0,1]$. Therefore, thanks to the continuity of the kernel r_D^s discussed in Lemma 2.12 in [DGV22a], we conclude by the Dominated Convergence Theorem that $\mathcal{I}(\cdot,\cdot)$ is continuous in $\mathcal{C}$.

Furthermore, thanks to the inequalities in (2.15) and (A.2) we have that if we define

$$f_D(t,x,y) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\dfrac{|x-y|^2}{4t}\right) & \text{for all } (t,x,y) \in (0,T] \times \mathcal{C}, \\ C_{T,1,\Omega}\exp(-t\lambda_1) & \text{for all } (t,x,y) \in (T,+\infty) \times \mathcal{C}, \end{cases}$$

then we get that $f_D(t,x,y)t^{s-1} \in L^1(0,+\infty)$ for each $(x,y) \in \mathcal{C}$, and also

$$p_D^{\Omega}(t,x,y)t^{s-1} \leqslant f_D(t,x,y)t^{s-1}$$

for each $(t,x,y) \in (0,+\infty) \times \mathcal{C}$. Thanks to the continuity of p_D^{Ω} (see for instance Lemma 2.12 in [DGV22a]) and the last observations we can apply the Dominated Convergence Theorem and conclude that $\mathcal{J}(\cdot,\cdot) \in C(\mathcal{C})$.

Now let $f \in C_c^\infty(\Omega)$ such that $f \geqslant 0$. Then, for each $x \in \Omega$ we compute

$$\begin{aligned}\int_\Omega \mathcal{I}(x,y) f(y)\,dy &= \int_\Omega \int_0^{+\infty} r_D^s(t,x,y) f(y)\,dt\,dy \\ &= \int_0^{+\infty} \int_\Omega r_D^s(t,x,y) f(y)\,dy\,dt \\ &= \int_0^{+\infty} \sum_{k=1}^{+\infty} f_k \phi_k(x) \exp(-t\lambda_k^s)\,dt \\ &= \sum_{k=1}^{+\infty} \frac{f_k \phi_k(x)}{\lambda_k^s}.\end{aligned} \tag{A.3}$$

In the above computation we denoted

$$f_k := \int_\Omega f(y) \phi_k(y)\,dy,$$

and the identity between the first and the second line, as well as between the second and the third, are due to Lemma A.2 in [DGV22a]; in addition, the estimates on the coefficients f_k given in Proposition A.4 in [DGV22a].

Similarly, we also observe that

$$\begin{aligned}\int_\Omega \mathcal{J}(x,y) f(y)\,dy &= \frac{1}{\Gamma(s)} \int_\Omega \int_0^{+\infty} p_D^\Omega(t,x,y) t^{s-1} f(y)\,dt\,dy \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \int_\Omega p_D^\Omega(t,x,y) f(y)\,dy\,dt \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \sum_{k=1}^{+\infty} f_k \phi_k(x) \exp(-t\lambda_k) t^{s-1}\,dt \\ &= \frac{1}{\Gamma(s)} \sum_{k=1}^{+\infty} f_k \phi_k(x) \int_0^{+\infty} \exp(-t\lambda_k) t^{s-1}\,dt \\ &= \frac{1}{\Gamma(s)} \sum_{k=1}^{+\infty} f_k \phi_k(x) \frac{\Gamma(s)}{\lambda_k^s} \\ &= \sum_{k=1}^{+\infty} \frac{f_k \phi_k(x)}{\lambda_k^s}.\end{aligned} \tag{A.4}$$

Therefore, from equations (A.3) and (A.4) we deduce that for each $x \in \Omega$ and $f \in C_c^\infty(\Omega)$ such that $f \geqslant 0$ it holds

$$\int_\Omega (\mathcal{I}(x,y) - \mathcal{J}(x,y)) f(y)\,dy = 0.$$

Thanks to this latter identity and the fact that $\mathcal{J}, \mathcal{I} \in C(\mathcal{C})$ we conclude the proof of (A.1). $\square$

Appendix B. Some explicit formula for the 1-dimensional functionals

Lemma B.1. *We have that*

$$l_D^y(s,T) = 2a^{1+2s}\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{y}{\pi ka}+\frac{y(-1)^k}{\pi ka}-\frac{2}{(\pi k)^2}\sin\left(\frac{\pi ky}{a}\right)\right)\times\sin\left(\frac{\pi ky}{a}\right)\left(\frac{1-\exp\left(-T\left(\frac{\pi k}{a}\right)^{2s}\right)}{(\pi k)^{2s}}\right) \tag{B.1}$$

and

$$\mathcal{A}_D^y(s,T) = 2a^{2+2s}\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{2(-1)^k}{(\pi k)^3}-\frac{2}{(\pi k)^3}-\frac{y^2(-1)^k}{\pi ka^2}+\frac{y^2}{\pi ka^2}-\frac{2y(-1)^k}{\pi ka}\right)\times\sin\left(\frac{\pi ky}{a}\right)\left(\frac{1-\exp\left(-T\left(\frac{\pi k}{a}\right)^{2s}\right)}{(\pi k)^{2s}}\right). \tag{B.2}$$

Proof. The gist to obtain explicit formulas for the average distance $l_D^y(s,T)$ and the mean square displacement $\mathcal{A}_D^y(s,T)$ is to compute the $L^2((0,a))$ components of the decomposition in eigenfunctions of the functions $|x-y|$ and $(x-y)^2$. For this, it is first useful to consider the case $a:=1$ and then to reduce to it via a scaling argument. Thus, we first suppose that $a=1$ and note that

$$\int_0^1|x-y|\sin(\pi kx)dx = \frac{(-1)^{k+1}}{\pi k}+\frac{y}{\pi k}+\frac{y(-1)^k}{\pi k}-2\frac{\sin(\pi ky)}{(\pi k)^2}$$

$$\text{and}\quad \int_0^1(x-y)^2\sin(\pi kx)dx = \frac{(-1)^{k+1}}{\pi k}+\frac{2(-1)^k}{(\pi k)^3}-\frac{2}{(\pi k)^3}-\frac{y^2(-1)^k}{\pi k}+\frac{y^2}{\pi k}-\frac{2y(-1)^k}{\pi k}.$$

Therefore $l_D^y(s,T)$ and $\mathcal{A}_D^y(s,T)$ take the form

$$\begin{aligned} l_D^y(s,T) &= 2\int_0^T\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{y}{\pi k}+\frac{y(-1)^k}{\pi k}-2\frac{\sin(\pi ky)}{(\pi k)^2}\right)\sin(\pi ky)\exp(-(\pi k)^{2s}t)dt\\ &= 2\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{y}{\pi k}+\frac{y(-1)^k}{\pi k}-2\frac{\sin(\pi ky)}{(\pi k)^2}\right)\sin(\pi ky)\left(\frac{1-\exp(-T(\pi k)^{2s})}{(\pi k)^{2s}}\right)\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}\mathcal{A}_D^y(s,T) &= 2\int_0^1\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{2(-1)^k}{(\pi k)^3}-\frac{2}{(\pi k)^3}-\frac{y^2(-1)^k}{\pi k}+\frac{y^2}{\pi k}-\frac{2y(-1)^k}{\pi k}\right)\\ &\qquad\times\sin(\pi ky)\exp(-(\pi k)^{2s}t)dt\\ &= 2\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{2(-1)^k}{(\pi k)^3}-\frac{2}{(\pi k)^3}-\frac{y^2(-1)^k}{\pi k}+\frac{y^2}{\pi k}-\frac{2y(-1)^k}{\pi k}\right)\\ &\qquad\times\sin(\pi ky)\left(\frac{1-\exp(-T(\pi k)^{2s})}{(\pi k)^{2s}}\right),\end{aligned} \tag{B.4}$$

which is the desired result for $a=1$.

Now we address the case of a general $a>0$. To this end, we denote with an additional subscript a the quantities related to the interval $(0,a)$ (and, consistently, with an additional subscript 1 the quantities related to the interval $(0,1)$). With this notation, we infer from (3.1), (3.2) and (3.3) that

$$\phi_{k,a}(x)=\frac{1}{\sqrt{a}}\phi_{k,1}\left(\frac{x}{a}\right),\qquad \lambda_{k,a}=\frac{\lambda_{k,1}}{a^2}\qquad\text{and}\qquad r^s_{D,a}(t,x,y)=\frac{1}{a}\,r^s_{D,1}\left(\frac{t}{a^{2s}},\frac{x}{a},\frac{y}{a}\right).$$

As a consequence, by (1.3),

$$\begin{aligned} l^y_{D,a}(s,T) &= \int_0^T\int_0^a |\zeta-y|\,r^s_{D,a}(t,\zeta,y)\,d\zeta\,dt\\ &= \int_0^T\int_0^a \left|\frac{\zeta}{a}-\frac{y}{a}\right|\,r^s_{D,1}\left(\frac{t}{a^{2s}},\frac{\zeta}{a},\frac{y}{a}\right)\,d\zeta\,dt\\ &= a^{1+2s}\int_0^{T/a^{2s}}\int_0^1 \left|\widetilde{\zeta}-\frac{y}{a}\right|\,r^s_{D,1}\left(\widetilde{t},\widetilde{\zeta},\frac{y}{a}\right)\,d\widetilde{\zeta}\,d\widetilde{t}\\ &= a^{1+2s}\,l^{y/a}_{D,1}\left(s,\frac{T}{a^{2s}}\right). \end{aligned}\tag{B.5}$$

This and (B.3) yield that

$$\begin{aligned} l^y_{D,a}(s,T) \;=\; 2a^{1+2s}\sum_{k=1}^{+\infty}&\left(\frac{(-1)^{k+1}}{\pi k}+\frac{y}{\pi ka}+\frac{y(-1)^k}{\pi ka}-\frac{2}{(\pi k)^2}\sin\left(\frac{\pi ky}{a}\right)\right)\\ &\times\sin\left(\frac{\pi ky}{a}\right)\left(\frac{1-\exp\left(-T\left(\frac{\pi k}{a}\right)^{2s}\right)}{(\pi k)^{2s}}\right) \end{aligned}$$

and this gives (B.1), as desired.

Furthermore, by (1.4),

$$\begin{aligned} \mathcal{A}^y_{D,a}(s,T) &= \int_0^T\int_0^a |\zeta-y|^2\,r^s_{D,a}(t,\zeta,y)\,d\zeta\,dt\\ &= \frac{1}{a}\int_0^T\int_0^a |\zeta-y|^2\,r^s_{D,1}\left(\frac{t}{a^{2s}},\frac{\zeta}{a},\frac{y}{a}\right)\,d\zeta\,dt\\ &= a^{2+2s}\int_0^{T/a^{2s}}\int_0^1 \left|\widetilde{x}-\frac{y}{a}\right|^2\,r^s_{D,1}\left(\widetilde{t},\widetilde{x},\frac{y}{a}\right)\,d\widetilde{x}\,d\widetilde{t}\\ &= a^{2+2s}\,\mathcal{A}^{y/a}_{D,1}\left(s,\frac{T}{a^{2s}}\right). \end{aligned}\tag{B.6}$$

Thus, recalling (B.4),

$$\begin{aligned} \mathcal{A}^y_{D,a}(s,T) \;=\; 2a^{2+2s}\sum_{k=1}^{+\infty}&\left(\frac{(-1)^{k+1}}{\pi k}+\frac{2(-1)^k}{(\pi k)^3}-\frac{2}{(\pi k)^3}-\frac{y^2(-1)^k}{\pi ka^2}+\frac{y^2}{\pi ka^2}-\frac{2y(-1)^k}{\pi ka}\right)\\ &\times\sin\left(\frac{\pi ky}{a}\right)\left(\frac{1-\exp\left(-T\left(\frac{\pi k}{a}\right)^{2s}\right)}{(\pi k)^{2s}}\right), \end{aligned}$$

which proves (B.2), as desired. □

Additionally, the Neumann counterpart of Lemma B.1 reads

Lemma B.2. *We have that*

$$l_N^y(s,T) = aT\left(\frac{1}{2}+\frac{y^2}{a^2}-\frac{y}{a}\right) + 2a^{1+2s}\sum_{k=1}^{+\infty}\left(\frac{1}{(\pi k)^2}-\frac{2}{(\pi k)^2}\cos\left(\frac{\pi k y}{a}\right)+\frac{(-1)^k}{(\pi k)^2}\right)\cos\left(\frac{\pi k y}{a}\right) \times \frac{1-\exp\left(-T\left(\frac{\pi(2k+1)}{a}\right)^{2s}\right)}{(\pi(2k+1))^{2s}},$$

and $$\mathcal{A}_N^y(s,T) = a^2T\left(\frac{1}{3}+\frac{y^2}{a^2}-\frac{y}{a}\right) + 2a^{2+2s}\sum_{k=1}^{+\infty}\left(\frac{2}{(\pi k)^2}-\frac{2y}{a}\left(\frac{(-1)^k}{(\pi k)^2}-\frac{1}{(\pi k)^2}\right)\right)\cos\left(\frac{\pi k y}{a}\right)\frac{1-\exp\left(-T\left(\frac{\pi k}{a}\right)^{2s}\right)}{(\pi k)^{2s}}.$$

Proof. As in the proof of Lemma B.1, we can focus on the case $a := 1$, since the general case then would follow from scaling. Thus, we consider the coefficients of the $L^2((0,1))$ expansion of the functions $|x-y|$ and $(x-y)^2$ in terms of the Neumann eigenfunctions, thus finding that

$$\int_0^1 |x-y|\cos(\pi k x)\,dx = \begin{cases} \frac{1}{2}+y^2-y & \text{if } k=0,\\ \frac{1}{(\pi k)^2}-2\frac{\cos(\pi k y)}{(\pi k)^2}+\frac{(-1)^k}{(\pi k)^2} & \text{if } k\neq 0\end{cases}$$

and

$$\int_0^1 (x-y)^2\cos(\pi k x)\,dx = \begin{cases} \frac{1}{3}-y+y^2 & \text{if } k=0,\\ \frac{2}{(\pi k)^2}-2y\left(\frac{(-1)^k}{(\pi k)^2}-\frac{1}{(\pi k)^2}\right) & \text{if } k\neq 0.\end{cases}$$

Therefore $l_N^y(s,T)$ and $\mathcal{A}_N^y(s,T)$ take the form

$$\begin{aligned} l_N^y(s,T) &= \int_0^T \left(\frac{1}{2}+y^2-y\right) + 2\sum_{k=1}^{+\infty}\left(\frac{1}{(\pi k)^2}-2\frac{\cos(\pi k y)}{(\pi k)^2}+\frac{(-1)^k}{(\pi k)^2}\right)\cos(\pi k y)\exp(-t(\pi(2k+1))^{2s})\,dt\\ &= T\left(\frac{1}{2}+y^2-y\right)\\ &\quad + 2\sum_{k=1}^{+\infty}\left(\frac{1}{(\pi k)^2}-2\frac{\cos(\pi k y)}{(\pi k)^2}+\frac{(-1)^k}{(\pi k)^2}\right)\cos(\pi k y)\left(\frac{1-\exp(-T(\pi(2k+1))^{2s})}{(\pi(2k+1))^{2s}}\right)\end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_N^y(s,T) &= \int_0^T 2\left(\frac{1}{3}+y^2-y\right) + 2\sum_{k=1}^{+\infty}\left(\frac{2}{(\pi k)^2}-2y\left(\frac{(-1)^k}{(\pi k)^2}-\frac{1}{(\pi k)^2}\right)\right)\cos(\pi k y)\exp(-t(\pi k)^{2s})\,dt\\ &= T\left(\frac{1}{3}+y^2-y\right) + 2\sum_{k=1}^{+\infty}\left(\frac{2}{(\pi k)^2}-2y\left(\frac{(-1)^k}{(\pi k)^2}-\frac{1}{(\pi k)^2}\right)\right)\cos(\pi k y)\left(\frac{1-\exp(-T(\pi k)^{2s})}{(\pi k)^{2s}}\right),\end{aligned}$$

as claimed. □

Appendix C. Alternative proof of Proposition 1.2

Here we showcase an alternative proof of Proposition 1.2. The advantage of this argument is that it does not make use of the explicit formula (2.2) for the density μ_t^s of a s-stable subordinator. The details go as follows:

Proof of Proposition 1.2. As in the proof of Lemma B.1, we denote by an additional subscript a the quantities related to the interval $(0,a)$. In particular, by (3.4) and (3.8),

$$\Phi^{x,y}_{D,a}(s,T)=a^{2s-1}\Phi^{x/a,y/a}_{D,1}\left(s,\frac{T}{a^{2s}}\right)\qquad\text{and}\qquad \Phi^{x,y}_{N,a}(s,T)=a^{2s-1}\Phi^{x/a,y/a}_{N,1}\left(s,\frac{T}{a^{2s}}\right).\tag{C.1}$$

From this, (B.5) and (B.6) (and the corresponding scaling properties for the Neumann case), we deduce that it suffices to establish Proposition 1.2 for $a:=1$.

Hence, let $x=y\in\Omega=(0,1)$. We have that

$$\frac{1-\exp(-T(\pi k)^{2s})}{(\pi k)^{2s}}+\frac{1-\exp(-T(\pi(2k+1))^{2s})}{(\pi(2k+1))^{2s}}\leqslant\frac{1}{(\pi k)^{2s}}+\frac{1}{(\pi(2k+1))^{2s}}\leqslant\frac{2}{(\pi k)^{2s}}$$

and, as a result, we obtain that

$$\begin{aligned}&\text{the series in Lemmata B.1 and B.2 converge absolutely for all } s\in(0,1) \text{ and } T>0\\ &\text{and uniformly in } s \text{ in every set of the form } (s_0,1) \text{ with } s_0\in(0,1).\end{aligned}\tag{C.2}$$

Consequently, the convergence or divergence of $\mathcal{E}(s,T)$ in this case is equivalent to that of $\Phi^{x,x}_D(s,T)$ or $\Phi^{x,x}_N(s,T)$, depending on the boundary conditions considered. Hence, when $s\in(0,1/2]$, for all $M\in\mathbb{N}$, we infer from (3.4) that

$$\Phi^{x,x}_D(s,T)\geqslant 2\sum_{k=1}^M\int_0^T\sin^2(\pi kx)\exp(-t(\pi k)^{2s})\,dt=2\sum_{k=1}^M\frac{\sin^2(\pi kx)\big(1-\exp(-T(\pi k)^{2s})\big)}{(\pi k)^{2s}}\tag{C.3}$$

and from (3.8) that

$$\Phi^{x,x}_N(s,T)\geqslant 2\sum_{k=1}^M\int_0^T\cos^2(\pi kx)\exp(-t(\pi k)^{2s})\,dt=2\sum_{k=1}^M\frac{\cos^2(\pi kx)\big(1-\exp(-T(\pi k)^{2s})\big)}{(\pi k)^{2s}}.\tag{C.4}$$

We now want to check the fact that, when $s\in(0,1/2]$, the quantities in (C.3) and (C.4) are divergent as $M\to+\infty$. To this end, we need to estimate "how often" in k the functions $\sin^2(\pi kx)$ and $\cos^2(\pi kx)$ can get close to zero. This concept is formalized via the following claim: given $x\in(0,1)$,

$$\begin{aligned}&\text{there exist } \varepsilon_0>0 \text{ and } K_0\in\mathbb{N}\cap[1,+\infty) \text{ such that for every } k_0\in\mathbb{N}\\ &\text{there exists } k\in\{k_0,k_0+1,\dots,k_0+K_0\} \text{ such that } \sin^2(\pi kx)\geqslant\varepsilon_0.\end{aligned}\tag{C.5}$$

To prove this, up to exchanging x with $1-x$, we can suppose that $x\in\left(0,\frac12\right]$. Thus, we argue by contradiction and we suppose that, for some $x\in\left(0,\frac12\right]$, for every $\varepsilon>0$, as small as we wish, and every $K\in\mathbb{N}$, as large as we wish, there exists $k_{\varepsilon,K}\in\mathbb{N}$ such that for all $k\in\{k_{\varepsilon,K},k_{\varepsilon,K}+1,\dots,k_{\varepsilon,K}+K\}$ we have that $\sin^2(\pi kx)<\varepsilon$.

This means that for all $k\in\{k_{\varepsilon,K},k_{\varepsilon,K}+1,\dots,k_{\varepsilon,K}+K\}$ the angle πkx is sufficiently close to either 0 or π, modulo multiples of 2π. Hence, for concreteness, let us suppose that the angle $\pi k_{\varepsilon,K}\,x$ is sufficiently close to 0 modulo multiples of 2π, namely that

$$|\pi k_{\varepsilon,K}\,x+2\pi J|<\delta:=\arcsin\sqrt{\varepsilon},$$

for some $J\in\mathbb{N}$.

Therefore, for every $j\in\mathbb{N}$,

$$\pi(k_{\varepsilon,K}+j)\,x+2\pi J\in(-\delta+\pi jx,\delta+\pi jx).$$

We also note that, if $j \leqslant \frac{\pi-2\delta}{\pi x}$ and δ is sufficiently small, it follows that $(-\delta+\pi j x, \delta+\pi j x) \subseteq (-\delta, \pi-\delta)$. Chosing $K \geqslant 1+\frac{\pi-2\delta}{\pi x}$, we thus conclude that, for every $j \in \mathbb{N} \cap \left[0, \frac{\pi-2\delta}{\pi x}\right]$,

$$\pi(k_{\varepsilon,K}+j)\,x+2\pi J \in (-\delta,\delta).$$

Now we remark that, for sufficiently small δ, we have

$$\frac{\pi-2\delta}{\pi x} \geqslant \frac{2(\pi-2\delta)}{\pi} \geqslant \frac{3}{2}.$$

In particular, we can find $j_\star \in \mathbb{N} \cap \left[\frac{\pi-2\delta}{\pi x}-1, \frac{\pi-2\delta}{\pi x}\right]$. It thereby follows that

$$\begin{aligned}\delta &> \pi(k_{\varepsilon,K}+j_\star)\,x+2\pi J = \pi k_{\varepsilon,K}\,x+2\pi J+\pi j_\star\,x > -\delta+\pi j_\star\,x \\ &\geqslant -\delta+\pi\,x\left(\frac{\pi-2\delta}{\pi x}-1\right) = \pi-3\delta-\pi x \geqslant \frac{\pi}{2}-3\delta > \delta,\end{aligned}$$

provided that δ is sufficiently small. This is a contradiction and the claim in (C.5) is established.

Similarly, one can prove that given $x \in (0,1)$

$$\begin{aligned}&\text{there exist } \varepsilon_0>0 \text{ and } K_0 \in \mathbb{N}\cap[1,+\infty) \text{ such that for every } k_0 \in \mathbb{N} \\ &\text{there exists } k \in \{k_0, k_0+1, \dots, k_0+K_0\} \text{ such that } \cos^2(\pi k x) \geqslant \varepsilon_0.\end{aligned} \tag{C.6}$$

We now pick arbitrary integers N, $\overline{N} \in \mathbb{N}$ with $N<\overline{N}$ and take $M := \overline{N}(K_0+2)$ in (C.3). Thus, assuming N large enough such that $\exp(-T(\pi N)^{2s}) \leqslant \frac{1}{2}$ and using (C.5), we conclude that

$$\begin{aligned}\Phi_D^{x,x}(s,T) &\geqslant 2\sum_{k=N}^{\overline{N}(K_0+2)} \frac{\sin^2(\pi k x)\big(1-\exp(-T(\pi k)^{2s})\big)}{(\pi k)^{2s}} \\ &\geqslant \sum_{k=N}^{\overline{N}(K_0+2)} \frac{\sin^2(\pi k x)}{(\pi k)^{2s}} \\ &\geqslant \sum_{\ell=0}^{\overline{N}-1} \sum_{k=N+\ell K_0+\ell}^{N+(\ell+1)K_0+\ell} \frac{\sin^2(\pi k x)}{(\pi k)^{2s}} \\ &\geqslant \frac{1}{\pi^{2s}} \sum_{\ell=0}^{\overline{N}-1} \sum_{k=N+\ell K_0+\ell}^{N+(\ell+1)K_0+\ell} \frac{\sin^2(\pi k x)}{\big(N+(\ell+1)K_0+\ell\big)^{2s}} \\ &\geqslant \frac{1}{\pi^{2s}} \sum_{\ell=0}^{\overline{N}-1} \frac{\varepsilon_0}{\big(N+(\ell+1)K_0+\ell\big)^{2s}}.\end{aligned} \tag{C.7}$$

Sending now $\overline{N} \to +\infty$ we conclude that, when $s \in (0,1/2]$,

$$\Phi_D^{x,x}(s,T) \geqslant \frac{\varepsilon_0}{\pi^{2s}} \sum_{\ell=0}^{+\infty} \frac{1}{\big(N+(\ell+1)K_0+\ell\big)^{2s}} = +\infty. \tag{C.8}$$

Similarly, combining (C.4) and (C.6), we find that, when $s \in (0,1/2]$,

$$\Phi_N^{x,x}(s,T) = +\infty.$$

This and (C.8) yield that $\mathcal{E}(s,T) = +\infty$ for all $s \in (0,1/2]$, as claimed in the statement of Proposition 1.2.

We now consider the case $s \in (1/2, 1]$. In this situation, it follows from (3.5) that, for every x, $y \in (0,1)$,

$$\Phi_D^{x,y}(s,T) \leqslant 2\sum_{k=1}^{+\infty} \frac{1-\exp(-T(\pi k)^{2s})}{(\pi k)^{2s}} \leqslant \frac{2}{\pi^{2s}} \sum_{k=1}^{+\infty} \frac{1}{k^{2s}} < +\infty. \tag{C.9}$$

Similarly, using (3.9), for all $s \in (1/2, 1]$ and x, $y \in (0,1)$,

$$\Phi_N^{x,y}(s,T) < +\infty. \tag{C.10}$$

From this estimate and (C.9) we infer that $\mathcal{E}(s,T) \in (0,+\infty)$ for all $s \in (1/2,1)$, as desired. □

Appendix D. Some technical results

In this section we collect some technical results which have been used throughout the paper.

Proposition D.1. *Let* $(l,t) \in (0,+\infty) \times (0,+\infty)$. *Then,*

$$\lim_{s\searrow 0} \frac{\mu_t^s(l)}{s} = \frac{te^{-t}}{l}. \tag{D.1}$$

Proof. Thanks to (2.2), we have that

$$\begin{aligned}\lim_{s\searrow 0} \frac{\mu_t^s(l)}{s} &= \lim_{s\searrow 0} \frac{1}{\pi} \int_0^{+\infty} e^{-lu} e^{-tu^s\cos(\pi s)} tu^s \frac{\sin(tu^s \sin(\pi s))}{tu^s s}\,du \\ &= \int_0^{+\infty} \lim_{s\searrow 0} e^{-lu-tu^s\cos(\pi s)} tu^s \frac{\sin(tu^s\sin(\pi s))}{tu^s s}\,du \\ &= te^{-t}\int_0^{+\infty} e^{-lu}\,du \\ &= \frac{te^{-t}}{l},\end{aligned} \tag{D.2}$$

where we have used the fact that for each $s \in \left(0, \frac{1}{2}\right)$ it holds that

$$\left| e^{-lu-tu^s\cos(\pi s)} tu^s \frac{\sin(tu^s\sin(\pi s))}{tu^s s}\right| \leqslant te^{-lu}\left(\chi_{(0,1)}(u) + \chi_{(1,+\infty)}(u)u^{\frac{1}{2}}\right) \in L^1((0,+\infty))$$

in order to apply the Dominated Convergence Theorem in (D.2). □

Proposition D.2. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded, smooth and connected.*

Then, if $E, F \subset \Omega$ *and* $\overline{E} \cap \overline{F} = \varnothing$, *there exists some constant* $C_{E,F} \in (0,+\infty)$, *depending only on* E *and* F, *such that for all* $(s,T) \in (0,1) \times (0,+\infty)$ *it holds that*

$$\Phi_N^{x,y}(s,T) \leqslant C_{E,F}T \qquad \text{for all } (x,y) \in E \times F. \tag{D.3}$$

Proof. Thanks to the hypothesis $\overline{E} \cap \overline{F} = \varnothing$, we can define the positive constant

$$\overline{d} := \inf_{\substack{x\in E\\ y\in F}} |x-y|.$$

Then, by the definition of Φ_N and the upper bound in (2.20), we obtain that

$$\begin{aligned}\Phi_N^{x,y}(s,T) :=& \int_0^T \int_0^{+\infty} p_N^\Omega(l,x,y)\mu_t^s(l)\,dl\,dt \\ \leqslant& \int_0^T \int_0^{+\infty} c_\Omega \max\left\{\frac{1}{l^{\frac{n}{2}}},1\right\} \exp\left(-\frac{\overline{d}^2}{6l}\right)\mu_t^s(l)\,dl\,dt \\ \leqslant& \int_0^T \int_0^{+\infty} C_{E,F}\mu_t^s(l)\,dl\,dt \\ =& C_{E,F}T,\end{aligned}$$

where we set

$$C_{E,F} := \sup_{l\in(0,+\infty)} c_\Omega \max\left\{\frac{1}{l^{\frac{n}{2}}},1\right\} \exp\left(-\frac{\overline{d}^2}{6l}\right). \tag{D.4}$$

□

Lemma D.3. *Let $\Omega\subset\mathbb{R}^n$ be bounded and $f,g\in C(\overline{\Omega})$ be strictly positive in a compact set $K\Subset\Omega$.*
Then,

$$\inf_{\Omega_2\subset K}\frac{\int_{\Omega_2} f(x)\,dx}{\int_{\Omega_2} g(x)\,dx}\in(0,+\infty).$$

Proof. We set

$$m := \min_{x\in K} f(x)\in(0,+\infty) \qquad \text{and} \qquad M = \max_{x\in K} g(x)\in(0,+\infty).$$

Then,

$$\inf_{\Omega_2\subset K}\frac{\int_{\Omega_2} f(x)\,dx}{\int_{\Omega_2} g(x)\,dx}\geqslant\frac{m}{M}\in(0,+\infty).$$

□

We give some lower and upper bounds for the function $F_D(x,y)$ defined in equation (2.72). This result is applied several times, when proving Theorem 1.7.

Lemma D.4. *Let $\Omega\subset\mathbb{R}^n$ be bounded, smooth and connected.*
Then, for each $K\Subset\Omega$ there exists some constant $\widetilde{c}_{K,\Omega}\in(0,+\infty)$ such that

$$F_D(x,y)\geqslant\frac{\widetilde{c}_{K,\Omega}}{|x-y|^n} \qquad \text{for all } (x,y)\in\mathcal{C}\cap(K\times K), \tag{D.5}$$

where $\mathcal{C}$ has been defined in (2.35).
Furthermore, it holds that

$$F_D(x,y)\leqslant\frac{C_n}{|x-y|^n} \qquad \text{for all } (x,y)\in\mathcal{C}, \tag{D.6}$$

for some $C_n\in(0,+\infty)$.

Proof. We first prove (D.5). Thanks to equations (2.16) and (2.17) we observe that there exists two constants c_1,c_2 and some $T_{K,\Omega}\in(0,+\infty)$ depending on Ω and K, such that

$$p_D^\Omega(t,x,y)\geqslant\frac{c_1}{t^{\frac{n}{2}}}\exp\left(-\frac{c_2|x-y|^2}{t}\right) \quad \text{for all} \quad (t,x,y)\in(0,+\infty)\times K\times K.$$

Therefore, thanks to equation (2.72) we deduce that for each $(x,y) \in \mathcal{C} \cap (K \times K)$ it holds that

$$\begin{aligned} F_D(x,y) &= \int_0^{+\infty} \frac{p_D^\Omega(l,x,y)}{l}\, dl \\ &\geqslant \int_0^{T_{K,\Omega}} \frac{c_1}{l^{\frac{n}{2}+1}} \exp\left(-\frac{c_2|x-y|^2}{l}\right) dl \\ &= \frac{c_1 c_2^{-\frac{n}{2}}}{|x-y|^n} \int_{\frac{c_2|x-y|^2}{T_{K,\Omega}}}^{+\infty} a^{\frac{n}{2}-1} e^{-a}\, da \\ &\geqslant \frac{\widetilde{c}_{K,\Omega}}{|x-y|^n}, \end{aligned} \tag{D.7}$$

where, by calling as usual d_K the diameter of K, we defined

$$\widetilde{c}_{k,\Omega} := c_1 c_2^{-\frac{n}{2}} \int_{\frac{c_2 d_K^2}{T_{K,\Omega}}}^{+\infty} a^{\frac{n}{2}-1} e^{-a}\, da. \tag{D.8}$$

This concludes the proof of (D.5).

We now show (D.6). By equation (2.15) and the change of variable $\theta = \frac{|x-y|^2}{4l}$ we obtain that

$$\begin{aligned} F_D(x,y) &= \int_0^{+\infty} \frac{p_D^\Omega(l,x,y)}{l}\, dl \\ &\leqslant \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^{+\infty} \frac{1}{l^{\frac{n}{2}+1}} \exp\left(-\frac{|x-y|^2}{4l}\right) dl \\ &\leqslant \frac{1}{\pi^{\frac{n}{2}}|x-y|^n} \int_0^{+\infty} \theta^{\frac{n}{2}-1} e^{-\theta}\, d\theta \\ &= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \frac{1}{|x-y|^n}. \end{aligned}$$

Therefore, (D.6) is proved with $C_n := \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}}$. □

References

[AD17] Nicola Abatangelo and Louis Dupaigne, *Nonhomogeneous boundary conditions for the spectral fractional Laplacian*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), no. 2, 439–467, DOI 10.1016/j.anihpc.2016.02.001. MR3610940 ↑51

[AV19] Nicola Abatangelo and Enrico Valdinoci, *Getting acquainted with the fractional Laplacian*, Contemporary research in elliptic PDEs and related topics, Springer INdAM Ser., vol. 33, Springer, Cham, 2019, pp. 1–105. MR3967804 ↑3, 4, 5

[ARMA02] R. P. D. Atkinson, C. J. Rhodes, D. W. Macdonald, and R. M. Anderson, *Scale-free dynamics in the movement patterns of jackals*, Oikos **98** (2002), no. 1, 134–140. ↑3

[BCF+02] Frederic Bartumeus, Jordi Catalan, UL Fulco, ML Lyra, and GM Viswanathan, *Optimizing the encounter rate in biological interactions: Lévy versus Brownian strategies*, Phys. Rev. Lett. **88** (2002), no. 9, 097901. ↑3

[BBK+09] Krzysztof Bogdan, Tomasz Byczkowski, Tadeusz Kulczycki, Michal Ryznar, Renming Song, and Zoran Vondraček, *Potential analysis of stable processes and its extensions*, Lecture Notes in Mathematics, vol. 1980, Springer-Verlag, Berlin, 2009. Edited by Piotr Graczyk and Andrzej Stos. MR2569321 ↑15

[BV16] Claudia Bucur and Enrico Valdinoci, *Nonlocal diffusion and applications*, Lecture Notes of the Unione Matematica Italiana, vol. 20, Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016. MR3469920 ↑4

[BRB+21] S. V. Buldyrev, E. P. Raposo, F. Bartumeus, S. Havlin, F. R. Rusch, M. G. E. da Luz, and G. M. Viswanathan, *Comment on "Inverse square Lévy walks are not optimal search strategies for d ⩾ 2"*, Phys. Rev. Lett. **126** (2021), no. 4, Paper No. 048901, 2, DOI 10.1103/physrevlett.126.048901. MR4207853 ↑3

[CDV17] Luis Caffarelli, Serena Dipierro, and Enrico Valdinoci, *A logistic equation with nonlocal interactions*, Kinet. Relat. Models **10** (2017), no. 1, 141–170, DOI 10.3934/krm.2017006. MR3579567 ↑4

[Dav89] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989. MR990239 ↑18, 26

[DGV] Serena Dipierro, Giovanni Giacomin, and Enrico Valdinoci, *Efficiency functionals for the Lévy flight foraging hypothesis*, J. Math. Biol. ↑4

[DGV22a] ———, *Diffusive processes modeled on the spectral fractional Laplacian with Dirichlet and Neumann boundary conditions*, posted on 2022, DOI web.ma.utexas.edu/mp_arc-bin/mpa?yn=22-12. ↑4, 6, 7, 9, 14, 29, 31, 34, 40, 47, 52, 53

[DGV22b] ———, *Analysis of the Lévy flight foraging hypothesis in $\mathbb{R}^n$ and unreliability of the most rewarding strategies*, Preprint (2022). ↑2, 4, 11

[DPLV] Serena Dipierro, Edoardo Proietti Lippi, and Enrico Valdinoci, *(Non)local logistic equations with Neumann conditions*, Ann. Inst. H. Poincaré C Anal. Non Linéaire. ↑4

[DV21] Serena Dipierro and Enrico Valdinoci, *Description of an ecological niche for a mixed local/nonlocal dispersal: an evolution equation and a new Neumann condition arising from the superposition of Brownian and Lévy processes*, Phys. A **575** (2021), Paper No. 126052, 20, DOI 10.1016/j.physa.2021.126052. MR4249816 ↑4

[DGNBD17] Bartłomiej Dybiec, Ewa Gudowska-Nowak, Eli Barkai, and Alexander A. Dubkov, *Lévy flights versus Lévy walks in bounded domains*, Phys. Rev. E **95** (2017), 052102.1–052102.13, DOI 10.1103/PhysRevE.95.052102. ↑2, 5

[EPW$^+$07] Andrew M. Edwards, Richard A. Phillips, Nicholas W. Watkins, Mervyn P. Freeman, Eugene J. Murphy, Vsevolod Afanasyev, Sergey V. Buldyrev, M. G. E. da Luz, E. P. Raposo, H. Eugene Stanley, and Gandhimohan M. Viswanathan, *Revisiting Lévy flight search patterns of wandering albatrosses, bumblebees and deer*, Nature **449** (2007), 1044–1048, DOI 10.1038/nature06199. ↑3

[Eva10] Lawrence C. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010. MR2597943 ↑52

[Gad21] Raghavendra Gadagkar, *Experiments in animal behaviour: cutting-edge research at trifling cost*, e-Books of the Indian Academy of Sciences, Indian Academy of Sciences, Bangalore, 2021. ↑5

[HQD$^+$10] Nicolas E. Humphries, Nuno Queiroz, Jennifer R. M. Dyer, Nicolas G. Pade, Michael K. Musyl, Kurt M. Schaefer, Daniel W. Fuller, Juerg M. Brunnschweiler, Thomas K. Doyle, and Jonathan D. R. and others Houghton, *Environmental context explains Lévy and Brownian movement patterns of marine predators*, Nature **465** (2010), no. 7301, 1066–1069. ↑3

[HWQ$^+$12] Nicolas E. Humphries, Henri Weimerskirch, Nuno Queiroz, Emily J. Southall, and David W. Sims, *Foraging success of biological Lévy flights recorded in situ*, Proceedings of the National Academy of Sciences **109** (2012), no. 19, 7169–7174. ↑3

[HWS13] Nicolas E. Humphries, Henri Weimerskirch, and David W. Sims, *A new approach for objective identification of turns and steps in organism movement data relevant to random walk modelling*, Methods in Ecology and Evolution **4** (2013), no. 10, 930–938. ↑3

[KV18] K. K. Kataria and P. Vellaisamy, *On densities of the product, quotient and power of independent subordinators*, J. Math. Anal. Appl. **462** (2018), no. 2, 1627–1643, DOI 10.1016/j.jmaa.2018.02.059. MR3774308 ↑14

[KS05] Joseph Klafter and Igor M Sokolov, *Anomalous diffusion spreads its wings*, Physics world **18** (2005), no. 8, 29. ↑3

[LTBV20] Nicolas Levernier, Johannes Textor, Olivier Bénichou, and Raphaël Voituriez, *Inverse square Lévy walks are not optimal search strategies for $d \geqslant 2$*, Phys. Rev. Lett. **124** (2020), no. 8, 080601, 5, DOI 10.1103/physrevlett.124.080601. MR4071775 ↑3

[LTBV21] ———, *Reply to "Comment on 'Inverse square Lévy walks are not optimal search strategies for $d \geqslant 2$'"*, Phys. Rev. Lett. **126** (2021), no. 4, Paper No. 048902, 1, DOI 10.1103/physrevlett.126.048902. MR4207854 ↑3

[MPV13] Eugenio Montefusco, Benedetta Pellacci, and Gianmaria Verzini, *Fractional diffusion with Neumann boundary conditions: the logistic equation*, Discrete Contin. Dyn. Syst. Ser. B **18** (2013), no. 8, 2175–2202, DOI 10.3934/dcdsb.2013.18.2175. MR3082317 ↑4

[PV21] Gianni Pagnini and Silvia Vitali, *Should I stay or should I go? Zero-size jumps in random walks for Lévy flights*, Fract. Calc. Appl. Anal. **24** (2021), no. 1, 137–167, DOI 10.1515/fca-2021-0007. MR4225516 ↑3

[Pro87] M. H. Protter, *Can one hear the shape of a drum? revisited*, SIAM Rev. **29** (1987), no. 2, 185–197, DOI 10.1137/1029041. MR889243 ↑30, 34

[RFMM+04] Gabriel Ramos-Fernández, José L. Mateos, Octavio Miramontes, Germinal Cocho, Hernán Larralde, and Barbara Ayala-Orozco, *Lévy walk patterns in the foraging movements of spider monkeys (Ateles geoffroyi)*, Behavioral ecology and Sociobiology **55** (2004), no. 3, 223–230. ↑3

[Rey18] Andy M. Reynolds, *Current status and future directions of Lévy walk research*, Biol. Open **7** (2018), no. 1, 1–5, DOI 10.1242/bio.030106. ↑3

[SSH+08] David W. Sims, Emily J. Southall, Nicolas E. Humphries, Graeme C. Hays, Corey J. A. Bradshaw, Jonathan W. Pitchford, Alex James, Mohammed Z. Ahmed, Andrew S. Brierley, and Mark A. and others Hindell, *Scaling laws of marine predator search behaviour*, Nature **451** (2008), no. 7182, 1098–1102. ↑3

[Sko61] A. V. Skorohod, *Asymptotic formulas for stable distribution laws*, Select. Transl. Math. Statist. and Probability, Vol. 1, Inst. Math. Statist. and Amer. Math. Soc., Providence, R.I., 1961, pp. 157–161. MR0116373 ↑15

[SV03] Renming Song and Zoran Vondraček, *Potential theory of subordinate killed Brownian motion in a domain*, Probab. Theory Related Fields **125** (2003), no. 4, 578–592, DOI 10.1007/s00440-002-0251-1. MR1974415 ↑25, 26

[SV17] Jürgen Sprekels and Enrico Valdinoci, *A new type of identification problems: optimizing the fractional order in a nonlocal evolution equation*, SIAM J. Control Optim. **55** (2017), no. 1, 70–93, DOI 10.1137/16M105575X. MR3590646 ↑4

[SZATJ00] Mandyan V. Srinivasan, Shaowu Zhang, Monika Altwein, and Tautz Jürgen, *Honeybee navigation: nature and calibration of the "Odometer"*, Science **287** (2000), 851–853, DOI 10.1126/science.287.5454.851. ↑5

[SK86] D. W. Stevens and J. R. Krebs, *Foraging theory Monographs in Behavior and Ecology*, Princeton University Press, New Jersey, 1986. ↑3

[Val09] Enrico Valdinoci, *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. SeMA **49** (2009), 33–44. MR2584076 ↑4

[VAB+00] G. M. Viswanathan, V. Afanasyev, Sergey V. Buldyrev, Shlomo Havlin, M. G. E. Da Luz, E. P. Raposo, and H. Eugene Stanley, *Lévy flights in random searches*, Phys. A **282** (2000), no. 1-2, 1–12. ↑3

[VAB+96] G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphy, P. A. Prince, and H. E. Stanley, *Lévy flight search patterns of wandering albatrosses*, Nature **381** (1996), 413–415, DOI 10.1038/381413a0. ↑3

[VBB+02] G. M. Viswanathan, Frederic Bartumeus, Sergey V. Buldyrev, Jordi Catalan, U. L. Fulco, Shlomo Havlin, M. G. E. Da Luz, Marcelo Leite Lyra, E. P. Raposo, and H. Eugene Stanley, *Lévy flight random searches in biological phenomena*, Phys. A **314** (2002), no. 1-4, 208–213. ↑3

[VBH+99] Gandimohan M. Viswanathan, Sergey V. Buldyrev, Shlomo Havlin, M. G. E. Da Luz, E. P. Raposo, and H. Eugene Stanley, *Optimizing the success of random searches*, Nature **401** (1999), no. 6756, 911–914. ↑3

[WMH+14] Victoria J. Wearmouth, Matthew J. McHugh, Nicolas E. Humphries, Aurore Naegelen, Mohammed Z. Ahmed, Emily J. Southall, Andrew M. Reynolds, and David W. Sims, *Scaling laws of ambush predator 'waiting' behaviour are tuned to a common ecology*, Proc. R. Soc. B. **281** (2014), 1–9, DOI 10.1098/rspb.2013.2997. ↑6, 11

[WWW06] Matthias Wittlinger, Rüdiger Wehner, and Harald Wolf, *The ant odometer: stepping on stilts and stumps*, Science **312** (2006), 1965–1967, DOI 10.1126/science.1126912. ↑5

[Zha02] Qi S. Zhang, *The boundary behavior of heat kernels of Dirichlet Laplacians*, J. Differential Equations **182** (2002), no. 2, 416–430, DOI 10.1006/jdeq.2001.4112. MR1900329 ↑18

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