Singular continuous phase for Schrödinger operators over multicritical circle maps

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Abstract

We consider a class of Schrödinger operators — referred to as Schrödinger operators over circle maps — that generalize one-frequency quasiperiodic Schrödinger operators, with a base dynamics given by an orientation-preserving homeomorphism of a circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, instead of a circle rotation. In particular, we consider Schrödinger operators over multicritical circle maps, i.e., circle diffeomorphisms with a finite number of singular points where the derivative vanishes. We show that in a two-parameter region — determined by the geometry of dynamical partitions and α — the spectrum of Schrödinger operators over every sufficiently smooth such map, is purely singular continuous, for every α-Hölder-continuous potential V. For $\alpha = 1$, the region extends beyond the corresponding region for the almost Mathieu operator. As a corollary, we obtain that for every sufficiently smooth such map, with an invariant measure μ and with rotation number in a set $\mathcal{S}$, and μ-almost all $x \in \mathbb{T}^1$, the corresponding Schrödinger operator has a purely continuous spectrum, for every Hölder-continuous potential V.

1 Introduction

We consider a class of Schrödinger operators $H = H(T, V, x)$ on a space of square-summable sequences $\ell^2(\mathbb{Z})$, defined by

$$(Hu)_n := u_{n-1} + u_{n+1} + V(T^n x)u_n, \qquad x \in \mathbb{T}^1, \quad u \in \ell^2(\mathbb{Z}), \tag{1.1}$$

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where $V : \mathbb{T}^1 \to \mathbb{R}$ is a potential function on a circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, and $T : \mathbb{T}^1 \to \mathbb{T}^1$ is a multicritical circle map, a circle map in a particular class of orientation-preserving homeomorphisms of a circle. Schrödinger operators over more general circle maps, i.e., orientation-preserving circle homeomorphisms, have been introduced in [22]. For an overview of recent results on the spectral theory of Schrödinger operators over dynamically defined potentials the reader is directed, e.g., to [6] (see also [17]).

A map $T : \mathbb{T}^1 \to \mathbb{T}^1$ is called a multicritical circle map if it is a C^r-smooth homeomorphisms having $s > 1$ critical points $\xi_0, \dots, \xi_{s-1}$ such that, in a neighborhood of each critical point ξ_i, in a suitable coordinate system, the map takes the form $x \mapsto T(\xi_i) + x|x|^{\beta_i - 1}$, for some $\beta_i > 1$. The real number β_i is called the order of the critical point ξ_i.

When the rotation number ρ of T is irrational, Schrödinger operators over circle maps are a natural generalization of one-frequency quasiperiodic Schrödinger operators for which $T = R_\rho$, where $R_\rho : x \mapsto x+\rho \mod 1$ is the rigid rotation. When T is transitive, it is topologically conjugate to the rotation, i.e., there is a homeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, such that $T \circ \varphi = \varphi \circ R_\rho$. Hence, in that case, $T^n \circ \varphi = \varphi \circ R_\rho^n$, for every $n \in \mathbb{N}$, and we have $H(T, V, x) = H(R_\rho, V \circ \varphi, y)$, where $x = \varphi(y)$, $y \in \mathbb{T}^1$.

In some cases, the spectral properties of $H(T, V, x)$ can be deduced directly from the spectral properties of the corresponding Schrödinger operator over R_ρ, using this identity. In particular, if T is an analytic circle diffeomorphism with rotation number satisfying Yoccoz's $\mathcal{H}$ arithmetic condition [28], it follows from the theory of Herman [14] and Yoccoz [28] that φ is analytic, and the spectral properties of $H(T, V, x)$, with V analytic [16] follow directly from Avila's global theory of one-frequency quasiperiodic Schrödinger operators over rotations [1]. Although for circle diffeomorphisms T with Liouville rotation numbers the conjugacy to the corresponding rotation can even be singular, certain spectral properties of $H(T, V, x)$, with potentials of the same regularity, are still analogous to those of the one-frequency quasiperiodic Schrödinger operators over rotations with the same rotation numbers [16].

In [22], we initiated the study of Schrödinger operators over more general circle maps, in particular circle maps with a break and critical circle maps. Here, we extend the work to include Schrödinger operators over multicritical circle maps. We are interested in the spectral phase diagram of Schrödinger operators over circle maps and, in particular, the singular continuous phase. Such a phase diagram emerges in one of most studied examples — the almost Mathieu family — which corresponds to $T = R_\rho$ and $V(x) = \lambda \cos(2\pi x)$. It was conjectured by Jitomirskaya [15] (Problem 8 therein), and proved by Avila, You and Zhou [2], that the almost Mathieu operator has a purely singular continuous spectrum in the region $0 < L(E) < \beta$ of the Lyapunov exponent $L(E)$, and that $L(E) = \beta$ is the boundary between continuous and pure point spectrum, for almost all $x \in \mathbb{T}^1$. Here,

$$\beta = \beta(\rho) := \limsup_{n\to\infty} \frac{\ln k_{n+1}}{q_n}, \tag{1.2}$$

with k_n and $\frac{p_n}{q_n}$, $n \in \mathbb{N}$, being the partial quotients and rational convergents of $\rho \in (0,1)\backslash\mathbb{Q}$ (see section 2.2). It was shown in [16] that, in the same region, the spectrum is singular continuous for Schrödinger operators $H(T,V,x)$ with Lipschitz continuous potentials V over C^{1+BV}-smooth circle diffeomorphisms T, for almost all $x \in \mathbb{T}^1$, suggesting that $L(E) = \beta$ could be the boundary between continuous and pure point spectrum, in this case as well and more generally. The class of Schrödinger operators over multicritical circle maps considered here provides an example where this is not the case.

Circle maps with a singularity, i.e., smooth circle diffeomorphisms with a single singular point where the derivative vanishes (critical circle maps) or has a jump discontinuity (circle maps with a break) have played a central role in the rigidity theory of circle maps — an extension of Herman's theory on the linearization of circle diffeomorphisms [4,5,9,13,18–21] — over the last couple of decades. More recent focus has included circle maps with several critical or break points [7,8].

We begin with a few more definitions. A number $\rho \in \mathbb{R}\backslash\mathbb{Q}$ is called *Diophantine* of class $D(\delta)$, for some $\delta \geq 0$, if there exists $\mathcal{C} > 0$ such that $|\rho - p/q| > \mathcal{C}/q^{2+\delta}$, for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. The set of all Diophantine numbers is denoted by $D := \cup_{\delta\geq 0} D(\delta)$ and the complement of this set in $\mathbb{R}\backslash\mathbb{Q}$ is the set of Liouville numbers. If $\rho \in D(\delta) \cap (0,1)$, then $\limsup_{n\to\infty} \frac{\ln k_{n+1}}{\ln q_n} \leq \delta$ and, thus, $\beta(\rho) = 0$. We call a Liouville number $\rho \in (0,1)$ *exponentially Liouville* if $\beta(\rho) > 0$ and *super Liouville* if $\beta(\rho) = \infty$. The set of all super Liouville numbers will be denoted by $\mathcal{S}_{\mathcal{L}}$.

The following theorem is a corollary of the main results of this paper. Since the rotation number ρ of T is irrational, T is uniquely ergodic [11]. We will denote by μ the unique invariant probability measure of T.

Theorem 1.1 *For every C^r-smooth, $r \geq 3$, multicritical circle map T, with rotation number $\rho \in \mathcal{S}_{\mathcal{L}}$ and the invariant measure μ, and μ-almost all $x \in \mathbb{T}^1$, the corresponding Schrödinger operator $H(T,V,x)$ has a purely continuous spectrum, for every Hölder-continuous potential $V : \mathbb{T}^1 \to \mathbb{R}$.*

Remark 1 For C^{1+BV}-smooth circle diffeomorphisms and a set $\mathcal{S} = \mathcal{S}_{\mathcal{L}}$, an analogous claim was proved in [16]. A map is said to be C^{1+BV}-smooth if it is C^1-smooth with the logarithm of the derivative of bounded variation.

Ergodic Schrödinger operators are intimately related to a family of cocycles — dynamical systems associated with each eigen-equation $Hu = Eu$. In the case of Schrödinger operators over circle maps with irrational rotation numbers, the cocycle is given by

$$(T, A) : (x, y) \mapsto (Tx, A(x, E)y), \tag{1.3}$$

where $A \in \mathrm{SL}(2,\mathbb{R})$, $x \in \mathbb{T}^1$, $y \in \mathbb{R}^2$. If $u = (u_n)_{n\in\mathbb{Z}}$ is a sequence satisfying $Hu = Eu$, then

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_n(x,E) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \quad \text{where} \quad A_n(x,E) := \begin{pmatrix} E - V(T^n x) & -1 \\ 1 & 0 \end{pmatrix} \tag{1.4}$$

is the transfer matrix. Thus,

$$\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = P_n(x,E) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}, \tag{1.5}$$

where $P_n(x,E) := \prod_{i=n-1}^{0} A_i(x,E) = A_{n-1}(x,E) \dots A_0(x,E)$.

We define the Lyapunov exponent

$$L(E) := \lim_{n\to\infty} \int L_n(x,E)\, d\mu, \quad \text{where} \quad L_n(x,E) := \frac{1}{n} \ln \|P_n(x,E)\|. \tag{1.6}$$

Due to submultiplicativity of $P_n(x,E)$, $L(E)$ exists. Since T is ergodic, by Kingman's ergodic theorem, for almost every x,

$$L(E) = L(x,E) := \lim_{n\to\infty} \frac{1}{n} \ln \|P_n(x,E)\|. \tag{1.7}$$

Different components of the spectrum of an operator $H(T,V,x)$ are denoted by σ_{ac} (absolutely continuous), σ_{sc} (singular continuous) and σ_{pp} (pure point). We also denote by $S_{pp}(x)$ the set of eigenvalues of $H(T,V,x)$, with $\sigma_{pp}(x) = \overline{S_{pp}(x)}$. Finally, we set $\mathcal{H} = \ell^2(\mathbb{Z})$, $\mathcal{H}_{sc}(x)$ the corresponding singular continuous subspace, and $P_A(x)$ the operator of spectral projection on a Borel set A, corresponding to $H(T,V,x)$.

The main result of this paper is the following.

Theorem 1.2 *Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be any C^r-smooth multicritical circle map, $r \geq 3$, with a rotation number $\rho \in (0,1)\backslash\mathbb{Q}$, and an invariant measure μ. For μ-almost all $x \in \mathbb{T}^1$, and any α-Hölder-continuous potential $V : \mathbb{T}^1 \to \mathbb{R}$, $\alpha \in (0,1]$, we have*

(i) $S_{pp}(x) \cap \{E : 0 \leq L(E) < 2\alpha\beta\} = \emptyset$,

(ii) $P_{\{E: 0 < L(E) < 2\alpha\beta\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

Remark 2 The regions in the $(\beta, L(E))$ plane with purely singular continuous spectrum in Theorem 1.2 extend beyond the corresponding region in Theorem 1.5 of [16] for circle diffeomorphisms and, for $\alpha = 1$, beyond the corresponding region for the almost Mathieu family (Theorem 1.1 of [2]).

The main result of this paper can be reformulated in the following way. Let

$$\delta_{\max} := \limsup_{n\to\infty} \frac{|\ln \ell_n|}{q_n}, \tag{1.8}$$

where $\ell_n = \min_{I\in\mathcal{P}_{n+1}, I\subset\Delta_0^{(n-1)}} |\tau_n(I)|$ is the length of the smallest renormalized interval of partition $\mathcal{P}_{n+1}$ inside the fundamental interval $\Delta_0^{(n-1)}$ of partition $\mathcal{P}_n$ (see section 2.2).

Theorem 1.3 *Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be any C^r-smooth multicritical circle map, $r \geq 3$, with a rotation number $\rho \in (0,1)\backslash\mathbb{Q}$, and an invariant measure μ. For μ-almost all $x \in \mathbb{T}^1$, and any α-Hölder-continuous potential $V : \mathbb{T}^1 \to \mathbb{R}$, $\alpha \in (0,1]$, we have*

(i) $S_{pp}(x) \cap \{E : 0 \leq L(E) < \alpha\delta_{\max}\} = \emptyset$,

(ii) $P_{\{E:0<L(E)<\alpha\delta_{\max}\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

Remark 3 An anologous theorem was proved for circle maps with a single singular critical or break point in [22].

Remark 4 It seems reasonable to expect that for Schrödinger operators over sufficiently smooth circle maps, in a large class of maps including circle diffeomorphisms with singularities, for μ-almost all $x \in \mathbb{T}^1$, and sufficiently regular potentials, the boundary between the continuous and pure point spectrum is given by $L(E) = \delta_{\max}$, i.e., that the spectrum is pure point with exponentially decaying eigenfunctions for $L(E) > \delta_{\max}$.

The proofs of these theorems use tools of both the spectral theory of Schrödinger operators and one-dimensional circle dynamics. In the next section, we state a sharp version of Gordon's theorem [16], and introduce dynamical partitions of a circle and renormalizations of circle maps, which play an important role in our analysis. In section 3, we define a set of full invariant measure for critical circle maps, and prove Theorem 1.2 and Theorem 1.3.

2 Preliminaries

2.1 A criterion for the absence of eigenvalues

In this section, we state a sharp version [16] of a theorem of Gordon [12] that has been used to prove absence of point spectra of one-dimensional operators since the pioneering work of Avron and Simon [3]. Such a sharp version was used in [2] to establish the singular continuous phase for the almost Mathieu operator.

Consider a Schrödinger operator H on $\ell^2(\mathbb{Z})$ given by the action on $u \in \ell^2(\mathbb{Z})$, as

$$(Hu)_n = u_{n+1} + u_{n-1} + V(n)u_n. \tag{2.1}$$

We can define the transfer matrix $A_n(E)$ and the n-step transfer-matrix $P_n(E) = \prod_{i=n-1}^{0} A_i(E)$, as in (1.4) and (1.5), respectively. Let also $P_{-n}(E) = \prod_{i=-n}^{-1} (A_i(E))^{-1}$. Let

$$\Lambda(E) := \limsup_{|n|\to\infty} \frac{\ln \|P_n(E)\|}{n}. \tag{2.2}$$

Clearly, for bounded V, $\Lambda(E) < \infty$, for every E.

Theorem 2.1 ([16]) *Assume that there exists $\beta > 0$, and an increasing sequence of positive integers q_n diverging to infinity, such that the sequence $\{V(n)\}_{n\in Z}$ in (2.1) satisfies*

$$\max_{0\le j<q_n} |V(j) - V(j \pm q_n)| \le e^{-\beta q_n}. \tag{2.3}$$

If $\beta > \Lambda(E)$, then E is not an eigenvalue of operator (2.1).

Consider the Schrödinger operator (2.1) with $V_n = V(T^n x)$ where $V : \mathbb{T}^1 \to \mathbb{R}$ is a bounded real-valued function on the circle and T is an orientation-preserving homeomorphism of a circle with an irrational rotation number ρ. Let the Lyapunov exponent $L(E)$ be defined as in (1.6). We then have

Theorem 2.2 *Assume that for some $x \in \mathbb{T}^1$, $C > 0$ and $\bar\beta > 0$, there is a sequence of positive integers $q_n \to \infty$ such that*

$$\sup_{0\le i<q_n} |V_{i\pm q_n}(x) - V_i(x)| < Ce^{-\bar\beta q_n}. \tag{2.4}$$

If $L(E) < \bar\beta$, then E is not an eigenvalue of the Schrödinger operator $H(T, V, x)$.

Proof. In order to apply Theorem 2.1, it suffices to prove $\limsup_{|n|\to\infty} \frac{\ln \|P_n(E)\|}{n} \le L(E)$. This follows from a result of Furman [10]. **QED**

For a sequence $q_n \to \infty$, let

$$\hat\beta = \hat\beta(x) := \liminf_{n\to\infty} \frac{\ln(\sup_{0\le i<q_n} |x_i - x_{i\pm q_n}|)^{-1}}{q_n}, \tag{2.5}$$

where $x_i = T^i x$.

Let $\sigma_{pp}, P_A, \mathcal{H}, \mathcal{H}_{sc}$ be as in Theorem 1.2.

Theorem 2.3 *Let $V : \mathbb{T}^1 \to \mathbb{R}$ be a α-Hölder continuous real-valued function on the circle, with $\alpha \in (0, 1)$. Then, we have* (*i*) $S_{pp}(x) \cap \{E : 0 \leq L(E) < \alpha\hat{\beta}\} = \emptyset$,

(*ii*) $P_{\{E:0<L<\alpha\hat{\beta}\}}(x)\mathcal{H} \subset \mathcal{H}_{sc}(x)$.

Proof. It suffices to prove part (*i*) of the claim, i.e., to exclude the point spectrum. Part (*ii*) of the claim then follows from Kotani's theory [23–25], x-independence of the absolutely continuous spectrum [26], and the minimality of T, since the set $\{E : L(E) > 0\}$ does not support any absolutely continuous spectrum.

If $L < \alpha\hat{\beta}$, then $v_i = V(T^i x)$ satisfy the assumption (2.4) of Theorem 2.2 for any $\bar{\beta}$ satisfying $L < \bar{\beta} < \alpha\hat{\beta}$. The claim follows. **QED**

In order to prove Theorem 1.2, we need appropriate bounds on $\hat{\beta}(x)$.

2.2 Dynamical partitions of a circle and renormalization

The quantity $\hat{\beta}(x)$ involves information about the geometry of the dynamical partitions of a circle. These partitions are obtained by using the continued fraction expansion of the rotation number $\rho \in (0,1)$ of the circle map T. Every irrational $\rho \in (0,1)$ can be written uniquely as

$$\rho = \cfrac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \ldots}}} =: [k_1, k_2, k_3, \ldots], \tag{2.6}$$

with an infinite sequence of partial quotients $k_n \in \mathbb{N}$. Conversely, every infinite sequence of partial quotients defines uniquely an irrational number ρ as the limit of the sequence of rational convergents $p_n/q_n = [k_1, k_2, \ldots, k_n]$, obtained by the finite truncations of the continued fraction expansion (2.6). It is well-known that p_n/q_n form a sequence of best rational approximations of an irrational ρ, i.e., there are no rational numbers, with denominators smaller or equal to q_n, that are closer to ρ than p_n/q_n. The rational convergents can also be defined recursively by $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$, starting with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

To define the dynamical partitions of an orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, with an irrational rotation number ρ, we start with an arbitrary point $x_0 \in \mathbb{T}^1$, and consider the semi-orbit $x_i = T^i x_0$, with $i \in \mathbb{N}$. The subsequence $(x_{q_n})_{n\in\mathbb{N}}$, indexed by the denominators q_n of the sequence of rational convergents of the rotation number ρ, is called the sequence of dynamical convergents. It follows from the simple arithmetic properties of the rational convergents that the sequence of dynamical convergents $(x_{q_n})_{n\in\mathbb{N}}$, for the rigid rotation R_ρ has the property that its subsequence with n odd approaches x_0 from the left and the subsequence with n even approaches x_0 from the right. Since all circle homeomorphisms with the same irrational rotation number are combinatorially equivalent, the order of the dynamical convergents of T is the same.

The intervals $[x_{q_n}, x_0]$, for n odd, and $[x_0, x_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$. We also define $\Delta_i^{(n)} = T^i(\Delta_0^{(n)})$. Certain number of images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, under the iterations of a map T, cover the whole circle without intersecting each other except possibly at the end points, and form the n-th dynamical partition of the circle

$$\mathcal{P}_n := \{T^i(\Delta_0^{(n-1)}) : 0 \le i < q_n\} \cup \{T^i(\Delta_0^{(n)}) : 0 \le i < q_{n-1}\}. \tag{2.7}$$

Intervals $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ are called the fundamental intervals of $\mathcal{P}_n$. These partitions are nested, in the sense that intervals of partition $\mathcal{P}_{n+1}$ are obtained by dividing intervals of partition $\mathcal{P}_n$ into finitely many intervals.

The n-th renormalization of an orientation-preserving homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$, with rotation number ρ, with respect to partition-defining point $x_0 \in \mathbb{T}^1$, is a function $f_n : [-1, 0] \to \mathbb{R}$, obtained from the restriction of T^{q_n} to $\Delta_0^{(n-1)}$, by rescaling the coordinates. If τ_n is the affine change of coordinates that maps $x_{q_{n-1}}$ to -1 and x_0 to 0, then

$$f_n := \tau_n \circ T^{q_n} \circ \tau_n^{-1}\,. \tag{2.8}$$

If we identify x_0 with zero, then τ_n is just multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$. Here, and in what follows, $|I|$ denotes the length of an interval I on $\mathbb{T}^1$.

We use the notation $f(x) = \Theta(g(x))$, if there exist constants $\mathfrak{C}_1, \mathfrak{C}_2 > 0$, such that $\mathfrak{C}_1 g(x) \le f(x) \le \mathfrak{C}_2 g(x)$, for all x.

3 Multicritical circle maps and proof of the main theorem

3.1 Geometry of dynamical partitions

We consider a multicritical circle map T with s critical points. If ξ_0 is a critical point of T, we can form dynamical partitions of the circle $\mathcal{P}_n$ associated to it. Let s_n be the number of the critical points of T^{q_n} in $\Delta_0^{(n-1)}$.

Let $a_0^{(n)} = 0$, $a_{s_n}^{(n)} = k_{n+1}$ and let $0 < a_1^{(n)} < a_2^{(n)} < \cdots < a_{s_n-1}^{(n)} < k_{n+1}$ be the indices i of the (right-end open) intervals $\Delta_{q_{n-1}+iq_n}^{(n)}$ containing the critical points of T^{q_n}. These intervals will be referred to as critical. Clearly, $s_n \le s$.

For each $0 \le j < s_n$, let $F_j^{(n)} = \cup_{i=a_j^{(n)}+1}^{a_{j+1}^{(n)}-1} \Delta_{q_{n-1}+iq_n}^{(n)}$ be the j-th "bridge" of $\Delta_0^{(n-1)}$, consisting of $\max\{a_{j+1}^{(n)} - a_j^{(n)} - 1, 0\}$ adjacent intervals between the critical ones.

We define $F_{j,i}^{(n)} = T^i(F_j^{(n)})$. Clearly, $\Delta_i^{(n-1)} \backslash \Delta_i^{(n+1)} = \cup_{j=0}^{s_n-1} \Delta_{i+q_{n-1}+a_j^{(n)}q_n}^{(n)} \cup F_{a_j^{(n)},i}$.

To prove Theorem 1.2, we will use some properties of multicritical circle maps. The following estimates have been proven by Estevez and de Faria in [7]. There exists a constant $\varkappa > 0$ such that, for every C^3-smooth multicritical circle map T with an irrational rotation number, and sufficiently large n (depending on T), the following holds.

(a) For every two adjacent intervals $I, J \in \mathcal{P}_n$,

$$\varkappa^{-1} \leq \frac{|I|}{|J|} \leq \varkappa.$$

(b) For every non-empty bridge $F^{(n)}_{j,i}$, for all $0 \leq i < q_n$ and $0 \leq j \leq s_n$,

$$\varkappa^{-1} \leq \frac{|F^{(n)}_{j,i}|}{|\Delta^{(n-1)}_i|} \leq \varkappa.$$

(c) For all $0 \leq i < q_n$ and $0 \leq j \leq s_n$,

$$\varkappa^{-1} \leq \frac{|\Delta^{(n)}_{i+q_{n-1}+a^{(n)}_j q_n}|}{|\Delta^{(n-1)}_i|} \leq \varkappa.$$

(d) For all $0 \leq i < q_n$, $0 \leq k < s_n$ and $a^{(n)}_k < j < a^{(n)}_{k+1}$,

$$\varkappa^{-1} \frac{1}{(\min\{j - a^{(n)}_k, a^{(n)}_{k+1} - j\})^2} \leq \frac{|\Delta^{(n)}_{i+q_{n-1}+jq_n}|}{|\Delta^{(n-1)}_i|} \leq \varkappa \frac{1}{(\min\{j - a^{(n)}_k, a^{(n)}_{k+1} - j\})^2}.$$

We emphasize that constant $\varkappa$ is universal, i.e., it does not depend of the map T, for sufficiently large n, but only on the orders of the critical points. Estimate (a) (with non-universal constant $\varkappa$), reflecting the bounded geometry of these maps, follows from Swiatek's estimates [27].

3.2 Set $\mathcal{E}$ of full measure

In this section, we construct a set of full invariant measure $\mathcal{E}$ for which Theorem 1.2 holds, i.e., we have appropriate control on the distances between points of an orbit and their dynamical convergents, for multicritical circle maps.

Let σ_n, $n \in \mathbb{N}$, be any increasing subsequence of $\mathbb{N}$ such that the corresponding sequence k_{σ_n+1} of partial quotients diverges to infinity. We will assume that such a subsequence exists since if the sequence of partial quotients is bounded, then $\beta = 0$. Let $\eta_n \in (0, 1)$, $n \in \mathbb{N}$, be any sequence converging to 1 such that $(1 - \eta_n) \ln k_{\sigma_n+1}$ diverges to infinity, as $n \to \infty$. Consider partitions $\mathcal{P}_n$ defined with the partitions defining point χ_0 being a critical point x_c.

For each $n \in \mathbb{N}$, let

$$\mathcal{E}_{n,0} := \bigcup_{I \in \mathcal{J}_{n,0}} I, \quad \mathcal{J}_{n,0} := \left\{ I \in \mathcal{P}_{\sigma_n+1} | I \subset \Delta_0^{(\sigma_n-1)} \backslash \Delta_0^{(\sigma_n+1)},\ |\tau_{\sigma_n}(I)| \leq \frac{\varkappa^{-1}}{k_{\sigma_n+1}^{2\eta_n}} \right\}, \tag{3.1}$$

and let

$$\mathcal{E}_{n,i} := T^i(\mathcal{E}_{n,0}), \quad \text{for} \quad i = 1, \ldots, q_{\sigma_n} - 1. \tag{3.2}$$

We define

$$\mathcal{E}_n := \bigcup_{i=0}^{q_{\sigma_n}-1} \mathcal{E}_{n,i}, \tag{3.3}$$

and

$$\mathcal{E} := \limsup_{n\to\infty} \mathcal{E}_n = \bigcap_{n \geq 1} \bigcup_{j \geq n} \mathcal{E}_j. \tag{3.4}$$

Proposition 3.1 *For sufficiently large $n \in \mathbb{N}$, $\mu(\mathcal{E}_n) \geq (1 - 2s_n \varkappa k_{\sigma_n+1}^{\eta_n-1}) k_{\sigma_n+1} q_{\sigma_n} \mu(\Delta_0^{(\sigma_n)})$.*

Proof. For sufficiently large n, the number of the elements I of partition $\mathcal{P}_{\sigma_n+1}$ inside of $\Delta_0^{(\sigma_n-1)} \backslash \Delta_0^{(\sigma_n+1)}$ that do not belong to $\mathcal{E}_{n,0}$ is smaller than $2s_n \varkappa k_{\sigma_n+1}^{\eta_n}$. This follows from the property (d). Since the partition $\mathcal{P}_{\sigma_n}$ consists of q_{σ_n} "large" intervals $\Delta_i^{(\sigma_n-1)} = T^i(\Delta_0^{(\sigma_n-1)})$, for $i = 0, \ldots, q_{\sigma_n} - 1$, each of which has invariant measure $\mu(\Delta_0^{(\sigma_n-1)})$ and q_{σ_n-1} "small" intervals $\Delta_i^{(\sigma_n)} = T^i(\Delta_0^{(\sigma_n)})$, for $i = 0, \ldots, q_{\sigma_n-1} - 1$, each of which has invariant measure $\mu(\Delta_0^{(\sigma_n)})$, and since the interval $\Delta_0^{(\sigma_n-1)}$ consists of the union of k_{σ_n+1} disjoint (except at the end points) intervals $\Delta_{q_{\sigma_n-1}+iq_{\sigma_n}}^{(\sigma_n)} \in \mathcal{P}_{\sigma_n+1}$, for $i = 0, \ldots, k_{\sigma_n+1} - 1$, each of which has invariant measure $\mu(\Delta_0^{(\sigma_n)})$, and $\Delta_0^{(\sigma_n+1)} \subset \Delta_{q_{\sigma_n+1}}^{(\sigma_n)}$, we have that the invariant measure of the complement of $\mathcal{E}_n$ is

$$\mu(\mathcal{E}_n^c) \leq 2s_n \varkappa k_{\sigma_n+1}^{\eta_n} q_{\sigma_n} \mu(\Delta_0^{(\sigma_n)}) + q_{\sigma_n} \mu(\Delta_0^{(\sigma_n+1)}) + q_{\sigma_n-1} \mu(\Delta_0^{(\sigma_n)}), \tag{3.5}$$

and, hence,

$$\begin{aligned} \mu(\mathcal{E}_n) = 1 - \mu(E_n^c) &\geq k_{\sigma_n+1} q_{\sigma_n} \mu(\Delta_0^{(\sigma_n)}) - 2s_n \varkappa k_{\sigma_n+1}^{\eta_n} q_{\sigma_n} \mu(\Delta_0^{(\sigma_n)}) \\ &\geq (1 - 2s_n \varkappa k_{\sigma_n+1}^{\eta_n-1}) k_{\sigma_n+1} q_{\sigma_n} \mu(\Delta_0^{(\sigma_n)}). \end{aligned} \tag{3.6}$$

QED

Proposition 3.2 $\mu(\mathcal{E}) = 1$.

Proof. As $k_{\sigma_n+1} q_{\sigma_n} \mu(\Delta_0^{(\sigma_n)}) \to 1$, and $(1 - \eta_n) \ln k_{\sigma_n+1} \to \infty$, $\mu(\mathcal{E}_n) \to 1$, as $n \to \infty$. Since $\mu(\cup_{j \geq n} \mathcal{E}_j) \geq \mu(\mathcal{E}_i)$, for any $i \geq n$, and $\mu(\mathcal{E}_i) \to 1$ as $i \to \infty$, it follows that $\mu(\cup_{j \geq n} \mathcal{E}_j) = 1$, for any $n \in \mathbb{N}$. The claim follows. **QED**

3.3 Distance of dynamical convergents

We now estimate the distance between points on an orbit and their dynamical convergents for multicritical circle maps.

Let $\varepsilon > 0$ be the half-width of the neighborhood $\mathcal{I}_\varepsilon^{(i)}$ around the critical points $x_c^{(i)}$, $i = 0, \ldots, s-1$, where T' has the desired power law behavior. We assume that $\varepsilon > 0$ has been chosen sufficiently small such that no two of these neighborhoods, corresponding to different critical points, intersect. We consider partitions $\mathcal{P}_n = \mathcal{P}_n(x_c^{(0)})$ defined by one of these critical points that we will assume is, without loss of generality, $x_c^{(0)}$. Let N be large enough such that there are intervals $\mathcal{J}_N^{(i)}$ consisting of the union of at most two elements of partition $\mathcal{P}_N$ such that $x_c^{(i)} \in \mathcal{J}_N^{(i)} \subset \mathcal{I}_\varepsilon^{(i)}$. Such an N exists as the lengths of the intervals of partitions $\mathcal{P}_N$ decrease as N increases [29]. Let

$$\mathcal{J} = \cup_{i=0}^{s-1} \mathcal{J}_N^{(i)}. \tag{3.7}$$

The next proposition gives an estimate on a number of "large" intervals of partition $\mathcal{P}_n \in$ inside of $\mathcal{J}$.

Proposition 3.3 *For every $n \geq N$, the cardinality*

$$\text{card}\, \{\Delta_i^{(n-1)} \subset \mathcal{J} | i = 0, \ldots, q_n - 1\} \leq \frac{2sq_n}{q_N}. \tag{3.8}$$

Proof. The partitioning of each of the q_N intervals $\Delta_i^{(N-1)}$ by the higher level partitions follows the same pattern: a "large" interval of partition $\mathcal{P}_i$ is divided into k_{i+1} "large" intervals and a "small" interval of partition $\mathcal{P}_{i+1}$; a small interval of partition $\mathcal{P}_i$ becomes a "large" interval of partition $\mathcal{P}_{i+1}$. Therefore, for each $n > N$, the number of "large" intervals $\Delta_i^{(n-1)}$ of partition $\mathcal{P}_n$ inside of $\Delta_i^{(N-1)}$ is bounded by

$$\frac{(k_{N+1}+1)q_N q_n}{(q_{N+1}+q_N)q_N} \leq \frac{(k_{N+1}+1)q_n}{q_{N+1}+q_N} \leq \frac{q_n}{q_N}, \tag{3.9}$$

Here, we have used that $q_{N+1} = k_{N+1}q_N + q_{N-1}$. Since each of the intervals $\mathcal{J}_N^{(i)}$ consists of at most two intervals of partition $\mathcal{P}_N$, the claim follows. **QED**

Let $x_c = x_c^{(i)} \in \mathbb{T}^1$ be a critical point of order $\beta_c = \beta_i$, and $\mathcal{I}_\varepsilon = \mathcal{I}_\varepsilon^{(i)}$.

Proposition 3.4 *If $\Delta \subset \mathcal{I}_\varepsilon$, and ℓ is the distance of x_c from Δ, then*

$$\frac{|T(\Delta)|}{|\Delta|} = \Theta((|\Delta| + \ell)^{\beta_c - 1}). \tag{3.10}$$

Proof. If the interval $\Delta \subset \mathcal{I}_\varepsilon$ covers the critical point x_c of order β_c, x_c divides Δ into the union of two disjoint except at x_c intervals Δ_1 and Δ_2. Without loss of generality, we can assume that $|\Delta_| \geq |\Delta_2|$. It is not difficult to see that

$$\frac{|T(\Delta)|}{|\Delta|} = \frac{|T(\Delta_1)| + |T(\Delta_2)|}{|\Delta_1| + |\Delta_2|} = \Theta\left(\frac{|\Delta_1|^{\beta_c} + |\Delta_2|^{\beta_c}}{|\Delta_1| + |\Delta_2|}\right) = \Theta(|\Delta_1|^{\beta_c - 1}) = \Theta(|\Delta|^{\beta_c - 1}). \tag{3.11}$$

If the interval $\Delta \subset \mathcal{I}_\varepsilon$ is at a distance $\ell > 0$ from a critical point x_c of order β_c, then

$$\frac{|T(\Delta)|}{|\Delta|} = \Theta\left(\frac{(|\Delta| + \ell)^{\beta_c} - \ell^{\beta_c}}{|\Delta|}\right) = \Theta((|\Delta| + \ell)^{\beta_c - 1}). \tag{3.12}$$

The claim follows. **QED**

The following proposition holds for all "large" intervals $I_0 \subset \Delta_0^{(n-1)}$ such that $I_0 \in \mathcal{P}_{n+1}$ and the corresponding intervals $I_i = T^i(I_0)$, $i \in \mathbb{Z}$.

Let $V = V(N) := \mathrm{Var}_{\xi \in \mathbb{T}^1 \setminus Int(\mathcal{J})} \ln T'(\xi)$. Notice that $V \to \infty$, as $N \to \infty$.

Proposition 3.5 *If T is a C^3-smooth multicritical circle map with an irrational rotation number, there exist $\delta = \delta(N)$ satisfying $\delta \to 0$ as $N \to \infty$, such that*

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} \leq \frac{|I_0|}{|\Delta_0^{(n-1)}|} e^{V + \delta q_n}, \tag{3.13}$$

for all $n \in \mathbb{N}$ and all $i = 0, \ldots, q_n - 1$.

Proof. For $i = 0, \ldots, q_n - 1$, there exist $\zeta_i \in I_i \subset \Delta_i^{(n-1)}$ and $\xi_i \in \Delta_i^{(n-1)}$ such that

$$\begin{aligned} |T(I_i)| &= T'(\zeta_i)|I_i|, \\ |T(\Delta_i^{(n-1)})| &= T'(\xi_i)|\Delta_i^{(n-1)}|. \end{aligned} \tag{3.14}$$

Using these identities, we obtain that, for some $\zeta_j \in I_j$ and $\xi_j \in \Delta_j^{(n-1)}$,

$$\frac{|I_i|}{|\Delta_i^{(n-1)}|} = \frac{|I_0|}{|\Delta_0^{(n-1)}|} \prod_{j=0}^{i-1} \frac{T'(\zeta_j)}{T'(\xi_j)} = \frac{|I_0|}{|\Delta_0^{(n-1)}|} \prod_{\substack{j=0 \\ \Delta_j^{(n-1)} \not\subset \mathcal{J}}}^{i-1} \frac{T'(\zeta_j)}{T'(\xi_j)} \prod_{\substack{j=0 \\ \Delta_j^{(n-1)} \subset \mathcal{J}}}^{i-1} \frac{T'(\zeta_j)}{T'(\xi_j)}. \tag{3.15}$$

By taking the logarithm of the first product, we obtain

$$\left| \ln \prod_{\substack{j=0 \\ \Delta_j^{(n-1)} \not\subset \mathcal{J}}}^{i-1} \frac{T'(\zeta_j)}{T'(\xi_j)} \right| \leq \sum_{\substack{j=0 \\ \Delta_j^{(n-1)} \not\subset \mathcal{J}}}^{i-1} |\ln T'(\zeta_j) - \ln T'(\xi_j)| \leq V. \tag{3.16}$$

Here, we have used that for $j = 0, \ldots, q_n - 1$, the intervals $\Delta_j^{(n-1)}$ do not overlap, except possibly at the end points.

Using Proposition 3.3 and Proposition 3.4, the second product can be bounded as

$$\prod_{\substack{j=0 \\ \Delta_j^{(n-1)} \subset \mathcal{J}}}^{i-1} \frac{T'(\zeta_j)}{T'(\xi_j)} \leq e^{\delta q_n}, \tag{3.17}$$

as each term in the product is bounded from above by a positive constant by Proposition 3.4. The claim follows. **QED**

Let l_{n-1} be the length of the longest "large" interval of partition $\mathcal{P}_n$.

Lemma 3.6 *If T is a C^3-smooth multicritical circle map with an irrational rotation number $\rho \in (0,1)$, then there exist $V = V(N) > 0$ and $\delta = \delta(N) > 0$, satisfying $V \to \infty$ and $\delta \to 0$, as $N \to \infty$, such that, for all $x \in \mathcal{E}$, there are infinitely many $n \in \mathbb{N}$ such that, for all $i = -q_{\sigma_n}, \ldots, q_{\sigma_n} - 1$,*

$$|T^{q_{\sigma_n}} x_i - x_i| \leq 2 l_{\sigma_n - 1} e^{V + \delta q_{\sigma_n}} \frac{\varkappa}{k_{\sigma_n+1}^{2\eta_n}}. \tag{3.18}$$

Proof. For every $x \in \mathcal{E}$, there are infinitely many n, such that $x \in \mathcal{E}_n$. Also, for each such n, there exist $j, j' \in \mathbb{N}$, such that $\chi_{j'} = T^{j'} \chi_0 \in \mathcal{E}_{n,j}$ and an element $I_j = [\chi_{j'}, T^{q_{\sigma_n}} \chi_{j'}] \subset \mathcal{E}_{n,j} \subset \Delta_j^{(\sigma_n - 1)}$ of partition $\mathcal{P}_{\sigma_n+1}$, for some $j = 0, \ldots, q_{\sigma_n} - 1$, such that $x \in I_j$. Furthermore, it follows from the definition $\mathcal{E}_{n,0}$ and properties (c) and (d) that $I_{j-q_{\sigma_n}}, I_{j+q_{\sigma_n}}, I_{j+2q_{\sigma_n}} \subset \Delta_j^{(\sigma_n-1)}$, for sufficiently large n (such that $k_{\sigma_n+1}^{\eta_n} \geq 3$). Estimates

$$|I_{i+j}|, |I_{i+j+q_{\sigma_n}}| \leq e^{V + \delta q_{\sigma_n}} |\Delta_{i+j}^{(\sigma_n - 1)}| \frac{\varkappa}{k_{\sigma_n+1}^{2\eta_n}}. \tag{3.19}$$

on the sizes of intervals $I_{i+j}, I_{i+j+q_{\sigma_n}} \subset \Delta_{i+j}^{(\sigma_n-1)}$, for $i = -q_{\sigma_n}, \ldots, q_{\sigma_n - 1}$, follow from Proposition 3.5, definition of $\mathcal{E}_{n,j}$ (see (3.1) and (3.2)), and property (a). The claim follows from the fact that $[x_i, T^{q_{\sigma_n}} x_i] \subset I_{i+j} \cup I_{i+j+q_{\sigma_n}}$. **QED**

3.4 Proof of the main theorem

Proof of Theorem 1.2. If $L(E) < 2\alpha\beta$, then $\beta > 0$, and there is an increasing sequence σ_n, $n \in \mathbb{N}$ such that $\beta = \lim_{n \to \infty} \frac{\ln k_{\sigma_n+1}}{q_{\sigma_n}}$. Furthermore, there exist $\hat{\delta} > 0$ such that $L < \alpha(2\beta - \hat{\delta})$ as well. Let $\eta_n \in (0,1)$ be any sequence converging to 1 such that $(1 - \eta_n) \ln k_{\sigma_n+1}$ diverges to infinity, as $n \to \infty$. We use these sequences to construct the set $\mathcal{E}$, as in section 3.2. By Proposition 3.2, $\mu(\mathcal{E}) = 1$. For $N \in \mathbb{N}$ large enough, by Lemma 3.6, there exist $\delta = \delta(N) > 0$ and $V = V(N) > 0$, satisfying $\delta \to 0$ as $N \to \infty$, such that, for every $x \in \mathcal{E}$, there are infinitely many n, such that estimate (3.18) holds. We assume that $N > 0$ has been chosen large enough such that $\delta \leq \hat{\hat{\delta}}$. This implies $\hat{\beta} \geq 2\beta - \delta \geq 2\beta - \hat{\delta}$. Hence, $L(E) < \alpha\hat{\beta}$, and the claim follows from Theorem 2.3. **QED**

For C^3-smooth multicritical circle maps, Theorem 1.3 follows directly from Theorem 1.2, as property (d) implies that $\ell_n = \Theta(k_{n+1}^{-2})$ and, thus, $\delta_{\max} = 2\beta$.

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