Solvability of some systems of non-Fredholm integro-differential equations with mixed diffusion

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Abstract. We prove the existence in the sense of sequences of solutions for some system of integro-differential type equations in two dimensions containing the normal diffusion in one direction and the anomalous diffusion in the other direction in $H^2(\mathbb{R}^2, \mathbb{R}^N)$ using the fixed point technique. The system of elliptic equations contains second order differential operators without the Fredholm property. It is established that, under the reasonable technical assumptions, the convergence in $L^1(\mathbb{R}^2)$ of the integral kernels yields the existence and convergence in $H^2(\mathbb{R}^2, \mathbb{R}^N)$ of the solutions. We emphasize that the study of the systems is more difficult than of the scalar case and requires to overcome more cumbersome technicalities.

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1 Introduction

We recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. Consequently, the equation $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. Such properties of the Fredholm operators are broadly used in many methods of the linear and nonlinear analysis.
Elliptic equations in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, the proper ellipticity and the Shapiro-Lopatinskii conditions are fulfilled (see e.g. [1], [6], [20], [23]). This is the main result of the theory of

the linear elliptic problems. In the case of the unbounded domains, such conditions may not be sufficient and the Fredholm property may not be satisfied. For example, the Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ fails to satisfy the Fredholm property if considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.
Linear elliptic equations in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions given above, the limiting operators are invertible (see [24]). In certain simple cases, the limiting operators can be constructed explicitly. For example, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have the limits at the infinities,

$$a_\pm = \lim_{x\to\pm\infty} a(x), \quad b_\pm = \lim_{x\to\pm\infty} b(x), \quad c_\pm = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are:

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Because the coefficients here are constants, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ does not satisfy the Fredholm property, can be found explicitly by using the standard Fourier transform, so that

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

The invertibility of the limiting operators is equivalent to the condition that the origin does not belong to the essential spectrum.
For the general elliptic equations, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, such conditions may not be written explicitly.
In the case of the non-Fredholm operators the usual solvability relations may not be applicable and the solvability conditions are, in general, unknown. There are certain classes of operators for which the solvability relations are obtained. Let us illustrate them with the following example. Consider the equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, $d \in \mathbb{N}$, where a is a positive constant. Such operator L coincides with its limiting operators. The homogeneous problem admits a nonzero bounded solution. Thus, the Fredholm property is not satisfied. However, since the operator has the constant coefficients, we can use the standard Fourier transform and find the solution explicitly. The solvability conditions can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exists a unique solution of such equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [29]). Here $S^d_{\sqrt{a}}$ denotes the sphere in $\mathbb{R}^d$ of radius $\sqrt{a}$ centered at the origin. Hence, though our operator fails to satisfy the Fredholm property, the solvability conditions are formulated similarly. However, such similarity is only formal since the range of the operator is not closed.

In the case of the operator with a scalar potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

the standard Fourier transform is not applicable directly. Nevertheless, the solvability relations in $\mathbb{R}^3$ can be obtained by a rather sophisticated application of the theory of self-adjoint operators (see [28]). As before, the solvability conditions are formulated in terms of the orthogonality to the solutions of the homogeneous adjoint problem. There are several other examples of the linear elliptic non Fredholm operators for which the solvability relations can be derived (see [12], [14], [24], [25], [27], [29]).

The solvability conditions play a significant role in the analysis of the nonlinear elliptic equations. In the case of the non-Fredholm operators, in spite of a certain progress in understanding of the linear problems, there exist only few examples where the nonlinear non-Fredholm operators were analyzed (see [5], [9], [10], [13], [14], [15], [26], [29], [30], [31], [32]). The large time behavior of the solutions of a class of fourth-order parabolic problems defined on unbounded domains using the Kolmogorov ε-entropy as a measure was studied in [8]. The article [7] deals with the finite and infinite dimensional attractors for the evolution equations of mathematical physics. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in $\mathbb{R}^3$ was investigated in [16]. The works [17] and [22] are devoted to the understanding of the Fredholm and properness properties of quasilinear elliptic systems of second order and of the operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in the second-order quasilinear elliptic systems were discussed in [18]. The present work is devoted to another class of stationary nonlinear systems of equations, for which the Fredholm property may not be satisfied:

$$\frac{\partial^2 u_k}{\partial x_1^2} - \left(-\frac{\partial^2}{\partial x_2^2}\right)^{s_k} u_k + \int_{\mathbb{R}^2} G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy = 0, \quad 0 < s_k < 1, \tag{1.2}$$

where $1 \leq k \leq N, \ N \geq 2, \ x = (x_1, x_2) \in \mathbb{R}^2, \ y = (y_1, y_2) \in \mathbb{R}^2$. Here and further down the vector function

$$u := (u_1, u_2, ..., u_N)^T \in \mathbb{R}^N. \tag{1.3}$$

The nonlocal operators

$$L_{s_k} := -\frac{\partial^2}{\partial x_1^2} + \left(-\frac{\partial^2}{\partial x_2^2}\right)^{s_k} : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad 0 < s_k < 1, \quad 1 \leq k \leq N, \quad N \geq 2 \tag{1.4}$$

are defined via the spectral calculus. The existence of solutions of the single equation analogous to system (1.2) was covered in [13]. The novelty of the works of this kind is that in each diffusion term we add the standard negative Laplacian in the x_1 variable to the minus Laplacian in x_2 raised to a fractional power. These models are new and not much is understood about them, especially in the context of the integro-differential equations. The technical difficulty we have to overcome is that such problems become anisotropic and it is more difficult to derive the desired estimates when dealing with them. In the population dynamics in the Mathematical Biology the integro-differential equations describe models with the nonlocal consumption of resources and intra-specific competition (see e.g. [2], [3]). It is very important to study the problems of this kind in unbounded domains from the point of view of the understanding of the spread of the viral infections, since many countries have to deal with the pandemics. We use the explicit form of the solvability relations and prove the existence of solutions of such nonlinear equations. In the case of the standard Laplacian instead of (1.4), the system analogical to (1.2) was treated in [26] and [32]. The solvability of the integro-differential problems containing in the diffusion terms only the negative Laplace operator raised to a fractional power was actively discussed in recent years in the context of the anomalous diffusion (see e.g. [10], [14], [30], [31]). The anomalous diffusion can be described as a random process of particle motion characterized by the probability density distribution of jump length. The moments of such density distribution are finite in the case of the normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at the infinity of the probability density function determines the value of the power of the Laplacian (see [21]). The chapter [11] deals with the necessary condition of the preservation of the nonnegativity of the solutions of a system of parabolic equations in the case of the mixed diffusion.

2 Formulation of the results

The technical conditions of the present article will be analogous to the ones of [13], adapted to the work with vector functions. Performing the analysis in the Sobolev spaces for vector functions is more complicated. The nonlinear part of system (1.2) will satisfy the following regularity conditions.

Assumption 1. *Let $1 \le k \le N$. Functions $F_k(u,x) : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ are satisfying the Caratheodory condition (see [19]), such that*

$$\sqrt{\sum_{k=1}^{N} F_k^2(u,x)} \le K|u|_{\mathbb{R}^N} + h(x) \quad for \quad u \in \mathbb{R}^N, \quad x \in \mathbb{R}^2 \tag{2.1}$$

with a constant $K > 0$ and $h(x) : \mathbb{R}^2 \to \mathbb{R}^+$, $h(x) \in L^2(\mathbb{R}^2)$. Furthermore, they are Lipschitz continuous function, so that for any $u^{(1),(2)} \in \mathbb{R}^N$, $x \in \mathbb{R}^2$:

$$\sqrt{\sum_{k=1}^{N} (F_k(u^{(1)},x) - F_k(u^{(2)},x))^2} \le L|u^{(1)} - u^{(2)}|_{\mathbb{R}^N} \tag{2.2}$$

with a constant $L > 0$.

Here and below the norm of a vector function given by (1.3) is:

$$|u|_{\mathbb{R}^N} = \sqrt{\sum_{k=1}^{N} u_k^2}.$$

The solvability of a local elliptic problem in a bounded domain in $\mathbb{R}^N$ was discussed in [4]. The nonlinear function there was allowed to have a sublinear growth. In order to establish the existence of solutions of (1.2), we introduce the auxiliary system of equations with $1 \leq k \leq N, \ N \geq 2$, namely

$$-\frac{\partial^2 u_k}{\partial x_1^2} + \left(-\frac{\partial^2}{\partial x_2^2}\right)^{s_k} u_k = \int_{\mathbb{R}^2} G_k(x-y) F_k(v_1(y), v_2(y), ..., v_N(y), y) dy, \quad 0 < s_k < 1. \tag{2.3}$$

Let us denote

$$(f_1(x), f_2(x))_{L^2(\mathbb{R}^2)} := \int_{\mathbb{R}^2} f_1(x) \bar{f}_2(x) dx, \tag{2.4}$$

with a slight abuse of notations when these functions do not belong to $L^2(\mathbb{R}^2)$. Indeed, if $f_1(x) \in L^1(\mathbb{R}^2)$ and $f_2(x) \in L^\infty(\mathbb{R}^2)$, like for instance those involved in orthogonality condition (4.5) further down, the integral in the right side of (2.4) is well defined. In the article we consider the situation in the space of the two dimensions, so that the appropriate Sobolev space is equipped with the norm

$$\|\phi\|^2_{H^2(\mathbb{R}^2)} := \|\phi\|^2_{L^2(\mathbb{R}^2)} + \|\Delta\phi\|^2_{L^2(\mathbb{R}^2)}. \tag{2.5}$$

Then for a vector function (1.3), we have

$$\|u\|^2_{H^2(\mathbb{R}^2, \mathbb{R}^N)} := \sum_{k=1}^{N} \|u_k\|^2_{H^2(\mathbb{R}^2)} = \sum_{k=1}^{N} \{\|u_k\|^2_{L^2(\mathbb{R}^2)} + \|\Delta u_k\|^2_{L^2(\mathbb{R}^2)}\}. \tag{2.6}$$

Let us also use the norm

$$\|u\|^2_{L^2(\mathbb{R}^2, \mathbb{R}^N)} := \sum_{k=1}^{N} \|u_k\|^2_{L^2(\mathbb{R}^2)}.$$

By virtue of Assumption 1 above, we are not allowed to consider the higher powers of the nonlinearities, than the first one, which is restrictive from the point of view of the applications. But this guarantees that our nonlinear vector function is a bounded and continuous map from $L^2(\mathbb{R}^2, \mathbb{R}^N)$ to $L^2(\mathbb{R}^2, \mathbb{R}^N)$. In the system above we are dealing with the operators L_{s_k} defined in (1.4). By means of the standard Fourier transform (4.1), it can be trivially obtained that the essential spectrum of L_{s_k} is given by

$$\lambda_{s_k}(p) = p_1^2 + |p_2|^{2s_k}, \quad p = (p_1, p_2) \in \mathbb{R}^2, \quad 1 \leq k \leq N. \tag{2.7}$$

Clearly, each set (2.7) contains the origin. Hence, our operators L_{s_k} do not satisfy the Fredholm property, which is the obstacle to solve our system of equations.
The similar situations but in linear equations, containing both self- adjoint and non self-adjoint differential operators without the Fredholm property have been studied extensively in recent years (see [24], [25], [28], [29]). Our present article is related to our work [12] because we also deal with the non Fredholm operators, now involved in the system, which is not linear anymore and contains the nonlocal terms. Presently, as distinct from [12], the space dimension is restricted to $d = 2$ to avoid the extra technicalities.
In our current work we manage to demonstrate that under the reasonable technical conditions system (2.3) defines a map $T_{2,\ s} : H^2(\mathbb{R}^2, \mathbb{R}^N) \to H^2(\mathbb{R}^2, \mathbb{R}^N)$, which is a strict contraction.

Theorem 1. *Let $N \geq 2,\ 1 \leq k \leq N,\ 0 < s_k < 1,\ 1 \leq M \leq N-1$, the functions $G_k(x) : \mathbb{R}^2 \to \mathbb{R}$, such that $G_k(x) \in L^1(\mathbb{R}^2)$ and $x^2 G_k(x) \in L^1(\mathbb{R}^2)$. Furthermore, $(-\Delta)^{1-s_k} G_k(x) \in L^1(\mathbb{R}^2)$ and Assumption 1 holds.*

Let us also assume that for $1 \leq k \leq M$ we have $0 < s_k \leq \frac{1}{2}$ and orthogonality relations (4.5), (4.6) are valid. Moreover, for $M+1 \leq k \leq N$ we have $\frac{1}{2} < s_k < 1$ and orthogonality conditions (4.5), (4.6) and (4.7) hold and that $2\sqrt{2}\pi N_{2,\ s} L < 1$, where $N_{2,\ s}$ is defined in (4.4). Then the map $v \mapsto T_{2,\ s} v = u$ on $H^2(\mathbb{R}^2, \mathbb{R}^N)$ defined by system (2.3) has a unique fixed point $v_{2,\ s}$, which is the only solution of the system of equations (1.2) in $H^2(\mathbb{R}^2, \mathbb{R}^N)$.

This fixed point $v_{2,\ s}$ is nontrivial provided that for some $1 \leq k \leq N$ the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)} \cap supp\widehat{G_k}$ is a set of nonzero Lebesgue measure in $\mathbb{R}^2$.

Related to system (1.2) in the space of two dimensions, we consider the sequence of the approximate systems of equations with $m \in \mathbb{N}$, namely

$$\frac{\partial^2 u_k^{(m)}}{\partial x_1^2} - \left(-\frac{\partial^2}{\partial x_2^2}\right)^{s_k} u_k^{(m)} + \int_{\mathbb{R}^2} G_{k,m}(x-y) F_k(u_1^{(m)}(y), u_2^{(m)}(y), ..., u_N^{(m)}(y), y) dy = 0 \quad (2.8)$$

with $0 < s_k < 1,\ 1 \leq k \leq N,\ N \geq 2$. Each sequence of kernels $\{G_{k,m}(x)\}_{m=1}^{\infty}$ converges to $G_k(x)$ as $m \to \infty$ in the appropriate function spaces discussed further down. Let us prove that, under the appropriate technical assumptions, each of systems (2.8) possesses a unique solution $u^{(m)}(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$, the limiting system of equations (1.2) has a unique solution $u(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$, and $u^{(m)}(x) \to u(x)$ in $H^2(\mathbb{R}^2, \mathbb{R}^N)$ as $m \to \infty$. This is the so-called *existence of solutions in the sense of sequences.* In such case, the solvability conditions can be formulated for the iterated kernels $G_{k,m}$. They imply the convergence of the kernels in terms of the Fourier transforms (see the Appendix) and, consequently, the convergence or the solutions (Theorem 2). The analogous ideas in the context of the standard Schrödinger type operators were used in [12], [14], [27]. Our second main result is as follows.

Theorem 2. *Let $m \in \mathbb{N},\ N \geq 2,\ 1 \leq k \leq N,\ 0 < s_k < 1,\ 1 \leq M \leq N-1$, the functions $G_{k,m}(x) : \mathbb{R}^2 \to \mathbb{R},\ G_{k,m}(x) \in L^1(\mathbb{R}^2),\ x^2 G_{k,m}(x) \in L^1(\mathbb{R}^2)$ are such that $G_{k,m}(x) \to$*

$G_k(x)$, $x^2G_{k,m}(x) \to x^2G_k(x)$ in $L^1(\mathbb{R}^2)$ as $m \to \infty$. Moreover, $(-\Delta)^{1-s_k}G_{k,m}(x) \in L^1(\mathbb{R}^2)$, so that $(-\Delta)^{1-s_k}G_{k,m}(x) \to (-\Delta)^{1-s_k}G_k(x)$ in $L^1(\mathbb{R}^2)$ as $m \to \infty$ and Assumption 1 holds.

Suppose $0 < s_k \leq \frac{1}{2}$ for $1 \leq k \leq M$ and orthogonality relations (4.24), (4.25) are valid, $\frac{1}{2} < s_k < 1$ for $M+1 \leq k \leq N$ and orthogonality conditions (4.24), (4.25) and (4.26) hold. Furthermore, we assume that (4.27) is valid for all $m \in \mathbb{N}$ with some fixed $0 < \varepsilon < 1$.

Then each system of equations (2.8) has a unique solution $u^{(m)}(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$, limiting system (1.2) admits a unique solution $u(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$, so that $u^{(m)}(x) \to u(x)$ in $H^2(\mathbb{R}^2, \mathbb{R}^N)$ as $m \to \infty$.

The unique solution $u^{(m)}(x)$ of each system of equations (2.8) does not vanish identically in our space of two dimensions provided that for some $1 \leq k \leq N$ the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)} \cap supp\widehat{G_{k,m}}$ is a set of nonzero Lebesgue measure in $\mathbb{R}^2$. Analogously, the unique solution $u(x)$ of limiting system (1.2) is nontrivial if $supp\widehat{F_k(0,x)} \cap supp\widehat{G_k}$ is a set of nonzero Lebesgue measure in $\mathbb{R}^2$ for a certain $1 \leq k \leq N$.

Remark 1. *In the article we deal with the real valued vector functions due to the assumptions on $F_k(u,x), G_{k,m}(x)$ and $G_k(x)$ involved in the integral terms of the approximate and limiting systems discussed above.*

Remark 2. *The importance of Theorem 2 of our work is the continuous dependence of the solutions with respect to the integral kernels.*

3 Proofs Of The Main Results

Proof of Theorem 1. First we suppose that for some $v(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)}(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$ of system (2.3). Then their difference $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$ will satisfy the homogeneous system of equations

$$-\frac{\partial^2 w_k}{\partial x_1^2} + \left(-\frac{\partial^2}{\partial x_2^2}\right)^{s_k} w_k = 0, \quad 1 \leq k \leq N.$$

Evidently, each operator $L_{s_k} : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined in (1.4) does not possess any nontrivial zero modes. Hence, the vector function $w(x)$ is trivial in the space of two dimensions. Let us choose an arbitrary $v(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$ and apply the standard Fourier transform (4.1) to both sides of system (2.3). This yields

$$\widehat{u_k}(p) = 2\pi \frac{\widehat{G_k}(p)\widehat{f_k}(p)}{p_1^2 + |p_2|^{2s_k}}, \quad p^2\widehat{u_k}(p) = 2\pi \frac{p^2\widehat{G_k}(p)\widehat{f_k}(p)}{p_1^2 + |p_2|^{2s_k}}, \quad 1 \leq k \leq N, \tag{3.1}$$

where $\widehat{f_k}(p)$ stands for the Fourier image of $F_k(v(x), x)$. Clearly, we have the upper bounds

$$|\widehat{u_k}(p)| \le 2\pi N_{2,\ s_k} |\widehat{f_k}(p)| \quad \textit{and} \quad |p^2 \widehat{u_k}(p)| \le 2\pi N_{2,\ s_k} |\widehat{f_k}(p)|, \quad 1 \le k \le N.$$

Note that all $N_{2,\ s_k}$ here are finite by virtue of Lemma 3 of the Appendix under the stated assumptions. This allows us to derive the estimate from above on the norm

$$\|u\|^2_{H^2(\mathbb{R}^2, \mathbb{R}^N)} = \sum_{k=1}^{N} \{\|\widehat{u_k}(p)\|^2_{L^2(\mathbb{R}^2)} + \|p^2 \widehat{u_k}(p)\|^2_{L^2(\mathbb{R}^2)}\} \le$$

$$\le 8\pi^2 \sum_{k=1}^{N} N^2_{2,\ s_k} \|F_k(v(x), x)\|^2_{L^2(\mathbb{R}^2)}. \tag{3.2}$$

The right side of (3.2) is finite by means of (2.1) of Assumption 1 since $v(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$. Obviously, $v_k(x) \in H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$, $1 \le k \le N$ via the Sobolev embedding. Hence, for an arbitrary $v(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$ there exists a unique solution $u(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$ of system (2.3), so that its Fourier image is given by (3.1). Therefore, the map $T_{2,\ s} : H^2(\mathbb{R}^2, \mathbb{R}^N) \to H^2(\mathbb{R}^2, \mathbb{R}^N)$ is well defined. This enables us to choose arbitrary vector functions $v^{(1),(2)}(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$, so that their images $u^{(1),(2)} := T_{2,\ s} v^{(1),(2)} \in H^2(\mathbb{R}^2, \mathbb{R}^N)$. By virtue of (2.3), we have for $1 \le k \le N$

$$-\frac{\partial^2 u_k^{(1)}}{\partial x_1^2} + \Big(-\frac{\partial^2}{\partial x_2^2}\Big)^{s_k} u_k^{(1)} = \int_{\mathbb{R}^2} G_k(x-y) F_k(v_1^{(1)}(y), v_2^{(1)}(y), ..., v_N^{(1)}(y), y) dy, \tag{3.3}$$

$$-\frac{\partial^2 u_k^{(2)}}{\partial x_1^2} + \Big(-\frac{\partial^2}{\partial x_2^2}\Big)^{s_k} u_k^{(2)} = \int_{\mathbb{R}^2} G_k(x-y) F_k(v_1^{(2)}(y), v_2^{(2)}(y), ..., v_N^{(2)}(y), y) dy, \tag{3.4}$$

where $0 < s_k < 1$. We apply the standard Fourier transform (4.1) to both sides of systems (3.3), (3.4) above. This gives us for $1 \le k \le N$

$$\widehat{u_k^{(1)}}(p) = 2\pi \frac{\widehat{G_k}(p) \widehat{f_k^{(1)}}(p)}{p_1^2 + |p_2|^{2s_k}}, \quad p^2 \widehat{u_k^{(1)}}(p) = 2\pi \frac{p^2 \widehat{G_k}(p) \widehat{f_k^{(1)}}(p)}{p_1^2 + |p_2|^{2s_k}}, \tag{3.5}$$

$$\widehat{u_k^{(2)}}(p) = 2\pi \frac{\widehat{G_k}(p) \widehat{f_k^{(2)}}(p)}{p_1^2 + |p_2|^{2s_k}}, \quad p^2 \widehat{u_k^{(2)}}(p) = 2\pi \frac{p^2 \widehat{G_k}(p) \widehat{f_k^{(2)}}(p)}{p_1^2 + |p_2|^{2s_k}}. \tag{3.6}$$

In the formulas above $\widehat{f_k^{(1)}}(p)$ and $\widehat{f_k^{(2)}}(p)$ designate the Fourier images of $F_k(v^{(1)}(x), x)$ and $F_k(v^{(2)}(x), x)$ respectively. Using (3.5) and (3.6), we obtain the estimates from above

$$\Big|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\Big| \le 2\pi N_{2,\ s_k} \Big|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\Big|,$$

$$\Big|p^2 \widehat{u_k^{(1)}}(p) - p^2 \widehat{u_k^{(2)}}(p)\Big| \le 2\pi N_{2,\ s_k} \Big|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\Big|$$

with $1 \le k \le N$. Hence, we derive the upper bound for the norm

$$\|u^{(1)} - u^{(2)}\|^2_{H^2(\mathbb{R}^2, \mathbb{R}^N)} = \sum_{k=1}^{N} \left\{ \left\| \widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| p^2 \left[\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p) \right] \right\|^2_{L^2(\mathbb{R}^2)} \right\} \le$$

$$\le 8\pi^2 N_{2,\ s}^2 \sum_{k=1}^{N} \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\mathbb{R}^2)},$$

where $N_{2,\ s}$ is defined in (4.4). Clearly, $v_k^{(1),(2)}(x) \in H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ due to the Sobolev embedding. Condition (2.2) of Assumption 1 above yields

$$\|T_{2,\ s}v^{(1)} - T_{2,\ s}v^{(2)}\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} \le 2\sqrt{2}\pi N_{2,\ s} L \|v^{(1)} - v^{(2)}\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)}. \tag{3.7}$$

The constant in the right side of inequality (3.7) is less than one as assumed. Therefore, by virtue of the Fixed Point Theorem, there exists a unique vector function $v_{2,\ s} \in H^2(\mathbb{R}^2, \mathbb{R}^N)$, so that $T_{2,\ s}v_{2,\ s} = v_{2,\ s}$, which is the only solution of the system of equations (1.2) in $H^2(\mathbb{R}^2, \mathbb{R}^N)$. Suppose $v_{2,\ s}(x)$ is trivial in the whole space of two dimensions. This will contradict to the given condition that for a certain $1 \le k \le N$ the Fourier images of $G_k(x)$ and $F_k(0, x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}^2$. ■

We proceed to establishing the solvability in the sense of sequences for our system of integro-differential equations in the space of two dimensions.

Proof of Theorem 2. By means of the result of Theorem 1 above, each system of equations (2.8) admits a unique solution $u^{(m)}(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$, $m \in \mathbb{N}$. Limiting system (1.2) possesses a unique solution $u(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N)$ by virtue of Lemma 4 below along with Theorem 1. We apply the standard Fourier transform (4.1) to both sides of the systems of equations (1.2) and (2.8). This gives us for $1 \le k \le N, \quad m \in \mathbb{N}$

$$\widehat{u_k}(p) = 2\pi \frac{\widehat{G_k}(p)\widehat{\varphi_k}(p)}{p_1^2 + |p_2|^{2s_k}}, \quad \widehat{u_k^{(m)}}(p) = 2\pi \frac{\widehat{G_{k,m}}(p)\widehat{\varphi_{k,m}}(p)}{p_1^2 + |p_2|^{2s_k}}, \tag{3.8}$$

$$p^2\widehat{u_k}(p) = 2\pi \frac{p^2\widehat{G_k}(p)\widehat{\varphi_k}(p)}{p_1^2 + |p_2|^{2s_k}}, \quad p^2\widehat{u_k^{(m)}}(p) = 2\pi \frac{p^2\widehat{G_{k,m}}(p)\widehat{\varphi_{k,m}}(p)}{p_1^2 + |p_2|^{2s_k}}. \tag{3.9}$$

Here $\widehat{\varphi_k}(p)$ and $\widehat{\varphi_{k,m}}(p)$ designate the Fourier images of $F_k(u(x), x)$ and $F_k(u^{(m)}(x), x)$ respectively. Obviously,

$$|\widehat{u_k^{(m)}}(p) - \widehat{u_k}(p)| \le 2\pi \left\| \frac{\widehat{G_{k,m}}(p)}{p_1^2 + |p_2|^{2s_k}} - \frac{\widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}} \right\|_{L^\infty(\mathbb{R}^2)} |\widehat{\varphi_k}(p)| +$$

$$+ 2\pi \left\| \frac{\widehat{G_{k,m}}(p)}{p_1^2 + |p_2|^{2s_k}} \right\|_{L^\infty(\mathbb{R}^2)} |\widehat{\varphi_{k,m}}(p) - \widehat{\varphi_k}(p)|.$$

Thus,

$$\|u_k^{(m)}-u_k\|_{L^2(\mathbb{R}^2)}\le 2\pi\left\|\frac{\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}-\frac{\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}\|F_k(u(x),x)\|_{L^2(\mathbb{R}^2)}+$$

$$+2\pi\left\|\frac{\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}\|F_k(u^{(m)}(x),x)-F_k(u(x),x)\|_{L^2(\mathbb{R}^2)}.$$

Inequality (2.2) of Assumption 1 yields

$$\sqrt{\sum_{k=1}^N\|F_k(u^{(m)}(x),x)-F_k(u(x),x)\|^2_{L^2(\mathbb{R}^2)}}\le L\|u^{(m)}(x)-u(x)\|_{L^2(\mathbb{R}^2,\mathbb{R}^N)}. \tag{3.10}$$

Clearly, $u_k^{(m)}(x),\ u_k(x)\in H^2(\mathbb{R}^2)\subset L^\infty(\mathbb{R}^2)$ with $1\le k\le N,\ m\in\mathbb{N}$ by virtue of the Sobolev embedding. Hence, we obtain

$$\|u^{(m)}(x)-u(x)\|^2_{L^2(\mathbb{R}^2,\mathbb{R}^N)}\le 8\pi^2\sum_{k=1}^N\left\|\frac{\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}-\frac{\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\right\|^2_{L^\infty(\mathbb{R}^2)}\|F_k(u(x),x)\|^2_{L^2(\mathbb{R}^2)}+$$

$$+8\pi^2\left[N_{2,\ s}^{(m)}\right]^2L^2\|u^{(m)}(x)-u(x)\|^2_{L^2(\mathbb{R}^2,\mathbb{R}^N)},$$

where $N_{2,\ s}^{(m)}$ is defined in (4.23). Using (4.27), we derive $\|u^{(m)}(x)-u(x)\|^2_{L^2(\mathbb{R}^2,\mathbb{R}^N)}\le$

$$\le\frac{8\pi^2}{\varepsilon(2-\varepsilon)}\sum_{k=1}^N\left\|\frac{\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}-\frac{\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\right\|^2_{L^\infty(\mathbb{R}^2)}\|F_k(u(x),x)\|^2_{L^2(\mathbb{R}^2)}.$$

By means of upper bound (2.1) of Assumption 1, we have $F_k(u(x),x)\in L^2(\mathbb{R}^2),\ 1\le k\le N$ for $u(x)\in H^2(\mathbb{R}^2,\mathbb{R}^N)$. Let us recall the result of Lemma 4 below. Therefore, under the stated conditions

$$u^{(m)}(x)\to u(x),\quad m\to\infty \tag{3.11}$$

in $L^2(\mathbb{R}^2,\mathbb{R}^N)$. Using (3.9), we arrive at

$$\left|p^2\widehat{u_k^{(m)}}(p)-p^2\widehat{u_k}(p)\right|\le 2\pi\left\|\frac{p^2\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}-\frac{p^2\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}|\widehat{\varphi_k}(p)|+$$

$$+2\pi\left\|\frac{p^2\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}|\widehat{\varphi_{k,m}}(p)-\widehat{\varphi_k}(p)|.$$

Hence,

$$\|\Delta u_k^{(m)}(x)-\Delta u_k(x)\|_{L^2(\mathbb{R}^2)}\le 2\pi\left\|\frac{p^2\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}-\frac{p^2\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}\|F_k(u(x),x)\|_{L^2(\mathbb{R}^2)}+$$

$$+2\pi\left\|\frac{p^2\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}\|F_k(u^{(m)}(x),x)-F_k(u(x),x)\|_{L^2(\mathbb{R}^2)}.$$

Upper bound (3.10) allows us to derive the estimate from above

$$\|\Delta u_k^{(m)}(x)-\Delta u_k(x)\|_{L^2(\mathbb{R}^2)}\le 2\pi\left\|\frac{p^2\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}-\frac{p^2\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}\|F_k(u(x),x)\|_{L^2(\mathbb{R}^2)}+$$

$$+2\pi\left\|\frac{p^2\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}L\|u^{(m)}(x)-u(x)\|_{L^2(\mathbb{R}^2,\mathbb{R}^N)}.$$

We recall the result of Lemma 4 of the Appendix and use (3.11). This yields

$$\Delta u^{(m)}(x)\to\Delta u(x)\quad in\quad L^2(\mathbb{R}^2,\mathbb{R}^N),\quad m\to\infty.$$

By virtue of the definition (2.6) of the norm we establish that $u^{(m)}(x)\to u(x)$ in $H^2(\mathbb{R}^2,\mathbb{R}^N)$ as $m\to\infty$.
Let us suppose the unique solution $u^{(m)}(x)$ of the system of equations (2.8) discussed above is trivial in our space of two dimensions for some $m\in\mathbb{N}$. This will contradict to the stated condition that the Fourier transforms of $G_{k,m}(x)$ and $F_k(0,x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}^2$ for a certain $1\le k\le N$. The similar argument holds for the unique solution $u(x)$ of limiting system (1.2). ∎

4 Appendix

Let $G_k(x)$ be a function, $G_k(x):\mathbb{R}^2\to\mathbb{R}$. We denote its standard Fourier transform using the hat symbol as

$$\widehat{G_k}(p):=\frac{1}{2\pi}\int_{\mathbb{R}^2}G_k(x)e^{-ipx}dx,\quad p\in\mathbb{R}^2. \tag{4.1}$$

Evidently,

$$\|\widehat{G_k}(p)\|_{L^\infty(\mathbb{R}^2)}\le\frac{1}{2\pi}\|G_k(x)\|_{L^1(\mathbb{R}^2)} \tag{4.2}$$

and $G_k(x)=\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2}\widehat{G_k}(q)e^{iqx}dq,\ x\in\mathbb{R}^2$. For the technical purposes we introduce the auxiliary expressions

$$N_{2,\ s_k}:=max\left\{\left\|\frac{\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)},\quad\left\|\frac{p^2\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}\right\},\quad 0<s_k<1, \tag{4.3}$$

where $1\le k\le N,\ N\ge 2$. Under the conditions of Lemma 3 below, all the quantities (4.3) will be finite. Hence

$$N_{2,\ s}:=max_{1\le k\le N}N_{2,\ s_k}<\infty. \tag{4.4}$$

The auxiliary lemmas below are the adaptations of the ones established in [13] for the studies of the single integro-differential equation with mixed diffusion, analogous to system (1.2). We provide them for the convenience of the readers.

Lemma 3. *Let $N \geq 2$, $1 \leq k \leq N$, $0 < s_k < 1$, $1 \leq M \leq N-1$, the functions $G_k(x) : \mathbb{R}^2 \to \mathbb{R}$, so that $G_k(x) \in L^1(\mathbb{R}^2)$ and $x^2 G_k(x) \in L^1(\mathbb{R}^2)$. Let us also assume that $(-\Delta)^{1-s_k} G_k(x) \in L^1(\mathbb{R}^2)$.*

a) Suppose $0 < s_k \leq \frac{1}{2}$ for $1 \leq k \leq M$. Then $N_{2,\ s_k} < \infty$ if and only if

$$(G_k(x), 1)_{L^2(\mathbb{R}^2)} = 0, \tag{4.5}$$

$$(G_k(x), x_1)_{L^2(\mathbb{R}^2)} = 0. \tag{4.6}$$

b) Let $\frac{1}{2} < s_k < 1$ for $M+1 \leq k \leq N$. Then $N_{2,\ s_k} < \infty$ if and only if orthogonality relations (4.5) and (4.6) along with

$$(G_k(x), x_2)_{L^2(\mathbb{R}^2)} = 0 \tag{4.7}$$

are valid.

Proof. It can be easily verified that in both cases a) and b) of the lemma the boundedness of $\frac{\widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}}$ implies that $\frac{p^2 \widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}} \in L^\infty(\mathbb{R}^2)$ as well. Let us express

$$\frac{p^2 \widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}} = \frac{p^2 \widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}} \chi_{\{|p| \leq 1\}} + \frac{p^2 \widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}} \chi_{\{|p| > 1\}}. \tag{4.8}$$

Here and below χ_A will stand for the characteristic function of a set $A \subseteq \mathbb{R}^2$. Evidently, the first term in the right side of (4.8) can be bounded from above in the absolute value by

$$\left\| \frac{\widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}} \right\|_{L^\infty(\mathbb{R}^2)} < \infty$$

as assumed. Inequality (4.2) yields

$$\| |p|^{2(1-s_k)} \widehat{G_k}(p) \|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{2\pi} \| (-\Delta)^{1-s_k} G_k(x) \|_{L^1(\mathbb{R}^2)} < \infty \tag{4.9}$$

via the one of our assumptions. In the polar coordinates we have

$$p = (|p| cos\theta, |p| sin\theta) \in \mathbb{R}^2,$$

where θ designates the angle variable. Obviosly, the second term in the right side of (4.8) can be estimated from above in the absolute value as

$$\frac{|p|^{2(1-s_k)} |\widehat{G_k}(p)|}{|p|^{2(1-s_k)} cos^2\theta + |sin\theta|^{2s_k}} \chi_{\{|p|>1\}} \leq \frac{||p|^{2(1-s_k)} \widehat{G_k}(p)|}{cos^2\theta + |sin\theta|^{2s_k}} \leq C ||p|^{2(1-s_k)} \widehat{G_k}(p)|. \tag{4.10}$$

Here and further down C will denote a finite, positive constant. By virtue of (4.9), the right side of (4.10) can be bounded from above by

$$\frac{C}{2\pi}\|(-\Delta)^{1-s_k}G_k(x)\|_{L^1(\mathbb{R}^2)} < \infty.$$

Hence, $\dfrac{p^2\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}} \in L^\infty(\mathbb{R}^2)$ as well. Clearly,

$$\frac{\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}} = \frac{\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\chi_{\{|p|\le 1\}} + \frac{\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\chi_{\{|p|>1\}}. \tag{4.11}$$

The second term in the right side of (4.11) can be easily estimated from above in the absolute value via (4.2) as

$$\frac{|\widehat{G_k}(p)|\chi_{\{|p|>1\}}}{|p|^2cos^2\theta+|p|^{2s_k}|sin\theta|^{2s_k}} \le \frac{\|G_k(x)\|_{L^1(\mathbb{R}^2)}}{2\pi(cos^2\theta+|sin\theta|^{2s_k})} \le C\|G_k(x)\|_{L^1(\mathbb{R}^2)} < \infty$$

due to the one of our assumptions. We can write

$$\widehat{G_k}(p) = \widehat{G_k}(0) + |p|\frac{\partial\widehat{G_k}}{\partial|p|}(0,\theta) + \int_0^{|p|}\left(\int_0^s \frac{\partial^2\widehat{G_k}(|q|,\theta)}{\partial|q|^2}d|q|\right)ds. \tag{4.12}$$

Equality (4.12) enables us to express the first term in the right side of (4.11) as

$$\frac{\widehat{G_k}(0)}{p_1^2+|p_2|^{2s_k}}\chi_{\{|p|\le 1\}} + \frac{|p|\frac{\partial\widehat{G_k}}{\partial|p|}(0,\theta)}{p_1^2+|p_2|^{2s_k}}\chi_{\{|p|\le 1\}} + \frac{\int_0^{|p|}(\int_0^s \frac{\partial^2\widehat{G_k}(|q|,\theta)}{\partial|q|^2}d|q|)ds}{p_1^2+|p_2|^{2s_k}}\chi_{\{|p|\le 1\}}. \tag{4.13}$$

The definition of the standard Fourier transform (4.1) gives us

$$\left|\frac{\partial^2\widehat{G_k}(p)}{\partial|p|^2}\right| \le \frac{1}{2\pi}\|x^2G_k(x)\|_{L^1(\mathbb{R}^2)} < \infty \tag{4.14}$$

via the one of the given conditions. It can be easily checked that

$$\frac{p^2}{p_1^2+|p_2|^{2s_k}}\chi_{\{|p|\le 1\}} \le 1, \quad p\in\mathbb{R}^2. \tag{4.15}$$

Evidently, the left side of (4.15) can be trivially bounded from above as

$$\frac{|p|^2}{|p|^2cos^2\theta+|p|^{2s_k}|sin\theta|^{2s_k}}\chi_{\{|p|\le 1\}} = \frac{|p|^{2(1-s_k)}}{|p|^{2(1-s_k)}-|p|^{2(1-s_k)}sin^2\theta+|sin\theta|^{2s_k}}\chi_{\{|p|\le 1\}} \le$$

$$\le \frac{|p|^{2(1-s_k)}}{|p|^{2(1-s_k)}+|sin\theta|^{2s_k}-sin^2\theta}\chi_{\{|p|\le 1\}} \le 1.$$

Hence, (4.15) holds. By virtue of (4.14) along with (4.15), we derive the estimate from above in the absolute value for the third term in (4.13) as

$$\frac{\|x^2 G_k(x)\|_{L^1(\mathbb{R}^2)}|p|^2}{4\pi(p_1^2+|p_2|^{2s_k})}\chi_{\{|p|\le 1\}} \le \frac{\|x^2 G_k(x)\|_{L^1(\mathbb{R}^2)}}{4\pi} < \infty$$

as assumed. By means of definition (4.1) of the standard Fourier transform we obtain

$$\frac{\partial \widehat{G_k}}{\partial |p|}(0,\theta) = -\frac{i}{2\pi}\int_{\mathbb{R}^2} G_k(x)|x|cos\beta dx, \tag{4.16}$$

where β stands for the angle between the vectors p and x in the plane. We introduce the technical expressions

$$Q_{1,k} := \int_{\mathbb{R}^2} G_k(x)x_1 dx, \quad Q_{2,k} := \int_{\mathbb{R}^2} G_k(x)x_2 dx. \tag{4.17}$$

Obviously, under our assumptions (4.17) are well defined, because of the trivial upper bound on the norm

$$\|xG_k(x)\|_{L^1(\mathbb{R}^2)} = \int_{|x|\le 1}|x||G_k(x)|dx + \int_{|x|>1}|x||G_k(x)|dx \le$$

$$\le \|G_k(x)\|_{L^1(\mathbb{R}^2)} + \|x^2 G_k(x)\|_{L^1(\mathbb{R}^2)}, \tag{4.18}$$

which is finite. Formulas (4.16) and (4.17) give us

$$\frac{\partial \widehat{G_k}}{\partial |p|}(0,\theta) = -\frac{i}{2\pi}\{Q_{1,k}cos\theta + Q_{2,k}sin\theta\}. \tag{4.19}$$

By virtue of (4.19), the sum of the first two terms in (4.13) can be written as

$$\frac{\widehat{G_k}(0)}{|p|^2cos^2\theta + |p|^{2s_k}|sin\theta|^{2s_k}}\chi_{\{|p|\le 1\}} - \frac{i|p|\{Q_{1,k}cos\theta + Q_{2,k}sin\theta\}}{2\pi(|p|^2cos^2\theta + |p|^{2s_k}|sin\theta|^{2s_k})}\chi_{\{|p|\le 1\}}. \tag{4.20}$$

We fix the polar angle $\theta = 0$ and let $|p| \to 0$. Clearly, (4.20) will be unbounded unless $\widehat{G_k}(0) = Q_{1,k} = 0$ in both cases a) and b) of the lemma. This is equivalent to orthogonality relations (4.5) and (4.6). Thus, it remains to consider the term

$$-\frac{i|p|Q_{2,k}sin\theta}{2\pi(|p|^2cos^2\theta + |p|^{2s_k}|sin\theta|^{2s_k})}\chi_{\{|p|\le 1\}}. \tag{4.21}$$

First we discuss the situation when $\frac{1}{2} < s_k < 1$. Let us fix the polar angle $\theta = \frac{\pi}{2}$ and let $|p|$ tend to zero. Then (4.21) will be unbounded unless $Q_{2,k}$ vanishes. This is equivalent to orthogonality condition (4.7) and completes the proof of the part b) of the lemma.

Finally, we study the case when $0 < s_k \leq \frac{1}{2}$. Then (4.21) can be easily bounded from above in the absolute value as

$$\frac{|p||Q_{2,k}||sin\theta|}{2\pi(|p|^2cos^2\theta+|p|^{2s_k}|sin\theta|^{2s_k})}\chi_{\{|p|\leq 1\}} \leq \frac{1}{2\pi}|p|^{1-2s_k}|sin\theta|^{1-2s_k}|Q_{2,k}|\chi_{\{|p|\leq 1\}} \leq \frac{|Q_{2,k}|}{2\pi},$$

which is finite as discussed above. Therefore, in the case a) of our lemma no any orthogonality conditions other than (4.5) and (4.6) are needed. ∎

In order to study the systems of equations (2.8), we introduce the following auxiliary expressions

$$N_{2,\ s_k}^{(m)} := max\left\{\left\|\frac{\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}, \quad \left\|\frac{p^2\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)}\right\} \tag{4.22}$$

with $0 < s_k < 1$, $1 \leq k \leq N$, $N \geq 2$ and $m \in \mathbb{N}$. Under the assumptions of Lemma 4 below, all expressions (4.22) will be finite. This will allow us to define

$$N_{2,\ s}^{(m)} := max_{1\leq k\leq N} N_{2,\ s_k}^{(m)} < \infty, \quad m \in \mathbb{N}. \tag{4.23}$$

Our final technical statement is as follows.

Lemma 4. *Let $m \in \mathbb{N}$, $N \geq 2$, $1 \leq k \leq N$, $0 < s_k < 1$, $1 \leq M \leq N-1$, the functions $G_{k,m}(x) : \mathbb{R}^2 \to \mathbb{R}$, $G_{k,m}(x) \in L^1(\mathbb{R}^2)$, $x^2G_{k,m}(x) \in L^1(\mathbb{R}^2)$, so that $G_{k,m}(x) \to G_k(x)$, $x^2G_{k,m}(x) \to x^2G_k(x)$ in $L^1(\mathbb{R}^2)$ as $m \to \infty$. Furthermore, $(-\Delta)^{1-s_k}G_{k,m}(x) \in L^1(\mathbb{R}^2)$, such that $(-\Delta)^{1-s_k}G_{k,m}(x) \to (-\Delta)^{1-s_k}G_k(x)$ in $L^1(\mathbb{R}^2)$ as $m \to \infty$.*

a) Let $0 < s_k \leq \frac{1}{2}$ for $1 \leq k \leq M$ and

$$(G_{k,m}(x), 1)_{L^2(\mathbb{R}^2)} = 0, \quad m \in \mathbb{N}, \tag{4.24}$$

$$(G_{k,m}(x), x_1)_{L^2(\mathbb{R}^2)} = 0, \quad m \in \mathbb{N}. \tag{4.25}$$

b) Suppose $\frac{1}{2} < s_k < 1$ for $M+1 \leq k \leq N$, orthogonality relations (4.24), (4.25) along with

$$(G_{k,m}(x), x_2)_{L^2(\mathbb{R}^2)} = 0, \quad m \in \mathbb{N} \tag{4.26}$$

are valid. Let in addition

$$2\sqrt{2}\pi N_{2,\ s}^{(m)} L \leq 1-\varepsilon \tag{4.27}$$

for all $m \in \mathbb{N}$ with some fixed $0 < \varepsilon < 1$. Then, for all $1 \leq k \leq N$, we have

$$\frac{\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}} \to \frac{\widehat{G_k(p)}}{p_1^2+|p_2|^{2s_k}}, \quad m \to \infty, \tag{4.28}$$

$$\frac{p^2\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}} \to \frac{p^2\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}, \quad m \to \infty \tag{4.29}$$

in $L^{\infty}(\mathbb{R}^2)$, so that

$$\left\|\frac{\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^{\infty}(\mathbb{R}^2)} \to \left\|\frac{\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^{\infty}(\mathbb{R}^2)}, \quad m\to\infty, \tag{4.30}$$

$$\left\|\frac{p^2\widehat{G_{k,m}}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^{\infty}(\mathbb{R}^2)} \to \left\|\frac{p^2\widehat{G_k}(p)}{p_1^2+|p_2|^{2s_k}}\right\|_{L^{\infty}(\mathbb{R}^2)}, \quad m\to\infty. \tag{4.31}$$

Moreover,

$$2\sqrt{2}\pi N_{2,\ s}L \le 1-\varepsilon \tag{4.32}$$

is valid.

Proof. By virtue of inequality (4.2) along with the one of the given conditions, we derive for $1\le k\le N$ that

$$\|\widehat{G_{k,m}}(p)-\widehat{G_k}(p)\|_{L^{\infty}(\mathbb{R}^2)} \le \frac{1}{2\pi}\|G_{k,m}(x)-G_k(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad m\to\infty. \tag{4.33}$$

Clearly, under the stated assumptions by means of the result of Lemma 3 above we have $N_{2,\ s}^{(m)}<\infty$. Using (4.24), we estimate for $1\le k\le N$ that

$$|(G_k(x),1)_{L^2(\mathbb{R}^2)}| = |(G_k(x)-G_{k,m}(x),1)_{L^2(\mathbb{R}^2)}| \le \|G_{k,m}(x)-G_k(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad m\to\infty$$

as we assume. Hence, orthogonality relations (4.5) are valid in the limit with $1\le k\le N$. By virtue of (4.25) along with the given conditions, we easily arrive at for $1\le k\le N$ that

$$|(G_k(x),x_1)_{L^2(\mathbb{R}^2)}| = |(G_k(x)-G_{k,m}(x),x_1)_{L^2(\mathbb{R}^2)}| \le \int_{\mathbb{R}^2}|G_{k,m}(x)-G_k(x)||x_1|dx \le$$

$$\le \int_{|x|\le 1}|G_{k,m}(x)-G_k(x)||x|dx + \int_{|x|>1}|G_{k,m}(x)-G_k(x)||x|dx \le$$

$$\le \|G_{k,m}(x)-G_k(x)\|_{L^1(\mathbb{R}^2)} + \|x^2G_{k,m}(x)-x^2G_k(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad m\to\infty.$$

Thus, orthogonality conditions (4.6) hold in the limit with $1\le k\le N$. When $M+1\le k\le N$, by the similar argument we can easily demonstrate that orthogonality relations (4.7) are valid in the limit as well. Using the result of Lemma 3 above, we obtain that $N_{2,\ s}<\infty$.
Let us show that (4.28) yields (4.29). Evidently, we have the equality

$$\frac{p^2[\widehat{G_{k,m}}(p)-\widehat{G_k}(p)]}{p_1^2+|p_2|^{2s_k}} = \frac{p^2[\widehat{G_{k,m}}(p)-\widehat{G_k}(p)]}{p_1^2+|p_2|^{2s_k}}\chi_{\{|p|\le 1\}} + \frac{p^2[\widehat{G_{k,m}}(p)-\widehat{G_k}(p)]}{p_1^2+|p_2|^{2s_k}}\chi_{\{|p|>1\}}. \tag{4.34}$$

Obviously, the second term in the right side of (4.34) can be bounded from above in the absolute value as

$$\frac{|p|^{2(1-s_k)}|\widehat{G_{k,m}}(p)-\widehat{G_k}(p)|}{|p|^{2(1-s_k)}cos^2\theta+|sin\theta|^{2s_k}}\chi_{\{|p|>1\}} \le \frac{|p|^{2(1-s_k)}|\widehat{G_{k,m}}(p)-\widehat{G_k}(p)|}{cos^2\theta+|sin\theta|^{2s_k}} \le$$

$$\le C\| |p|^{2(1-s_k)}[\widehat{G_{k,m}}(p) - \widehat{G_k}(p)]\|_{L^\infty(\mathbb{R}^2)} \le \frac{C}{2\pi}\|(-\Delta)^{1-s_k}[G_{k,m}(x) - G_k(x)]\|_{L^1(\mathbb{R}^2)}$$

via the analog of inequality (4.9). Hence,

$$\left\|\frac{p^2[\widehat{G_{k,m}}(p) - \widehat{G_k}(p)]}{p_1^2 + |p_2|^{2s_k}}\chi_{\{|p|>1\}}\right\|_{L^\infty(\mathbb{R}^2)} \le \frac{C}{2\pi}\|(-\Delta)^{1-s_k}[G_{k,m}(x) - G_k(x)]\|_{L^1(\mathbb{R}^2)} \to 0$$

as $m \to \infty$ as we assume. Clearly, the first term in the right side of (4.34) can be estimated from above in the norm as

$$\left\|\frac{p^2[\widehat{G_{k,m}}(p) - \widehat{G_k}(p)]}{p_1^2 + |p_2|^{2s_k}}\chi_{\{|p|\le1\}}\right\|_{L^\infty(\mathbb{R}^2)} \le \left\|\frac{\widehat{G_{k,m}}(p)}{p_1^2 + |p_2|^{2s_k}} - \frac{\widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}}\right\|_{L^\infty(\mathbb{R}^2)} \to 0, \quad m \to \infty$$

assuming that (4.28) is valid. Thus, (4.29) will hold as well. Evidently,

$$\frac{\widehat{G_{k,m}}(p)}{p_1^2 + |p_2|^{2s_k}} - \frac{\widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}} = \frac{\widehat{G_{k,m}}(p) - \widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}}\chi_{\{|p|\le1\}} + \frac{\widehat{G_{k,m}}(p) - \widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}}\chi_{\{|p|>1\}}. \tag{4.35}$$

The second term in the right side of (4.35) can be bounded from above in the absolute value using (4.33) as

$$\frac{|\widehat{G_{k,m}}(p) - \widehat{G_k}(p)|}{|p|^2 cos^2\theta + |p|^{2s_k}|sin\theta|^{2s_k}}\chi_{\{|p|>1\}} \le \frac{|\widehat{G_{k,m}}(p) - \widehat{G_k}(p)|}{cos^2\theta + |sin\theta|^{2s_k}} \le C\|\widehat{G_{k,m}}(p) - \widehat{G_k}(p)\|_{L^\infty(\mathbb{R}^2)} \le$$

$$\le \frac{C}{2\pi}\|G_{k,m}(x) - G_k(x)\|_{L^1(\mathbb{R}^2)},$$

so that

$$\left\|\frac{\widehat{G_{k,m}}(p) - \widehat{G_k}(p)}{p_1^2 + |p_2|^{2s_k}}\chi_{\{|p|>1\}}\right\|_{L^\infty(\mathbb{R}^2)} \le \frac{C}{2\pi}\|G_{k,m}(x) - G_k(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad m \to \infty$$

according to our assumption. Analogously to (4.12), we write for $1 \le k \le N,\ m \in \mathbb{N}$

$$\widehat{G_{k,m}}(p) = \widehat{G_{k,m}}(0) + |p|\frac{\partial\widehat{G_{k,m}}}{\partial|p|}(0,\theta) + \int_0^{|p|}\left(\int_0^s \frac{\partial^2\widehat{G_{k,m}}(|q|,\theta)}{\partial|q|^2}d|q|\right)ds. \tag{4.36}$$

Orthogonality relations (4.5) and (4.24) imply that

$$\widehat{G_k}(0) = 0, \quad \widehat{G_{k,m}}(0) = 0, \quad 1 \le k \le N, \quad m \in \mathbb{N}. \tag{4.37}$$

By virtue of (4.36) along with (4.12) and (4.37) the first term in the right side of (4.35) is equal to

$$\frac{|p|\left[\frac{\partial\widehat{G_{k,m}}}{\partial|p|}(0,\theta) - \frac{\partial\widehat{G_k}}{\partial|p|}(0,\theta)\right]}{p_1^2 + |p_2|^{2s_k}}\chi_{\{|p|\le1\}}+$$

$$+\frac{\int_0^{|p|}\left(\int_0^s\left[\frac{\partial^2\widehat{G_{k,m}}(|q|,\theta)}{\partial|q|^2}-\frac{\partial^2\widehat{G_k}(|q|,\theta)}{\partial|q|^2}\right]d|q|\right)ds}{p_1^2+|p_2|^{2s_k}}\chi_{\{|p|\le 1\}}. \tag{4.38}$$

Using the definition of the standard Fourier transform (4.1), we easily obtain

$$\left|\frac{\partial^2\widehat{G_{k,m}}(|p|,\theta)}{\partial|p|^2}-\frac{\partial^2\widehat{G_k}(|p|,\theta)}{\partial|p|^2}\right|\le\frac{1}{2\pi}\|x^2G_{k,m}(x)-x^2G_k(x)\|_{L^1(\mathbb{R}^2)}. \tag{4.39}$$

Inequalities (4.39) and (4.15) allow us to derive the estimate from above in the absolute value for the second term in (4.38) as

$$\frac{|p|^2\|x^2G_{k,m}(x)-x^2G_k(x)\|_{L^1(\mathbb{R}^2)}}{4\pi(p_1^2+|p_2|^{2s_k})}\chi_{\{|p|\le 1\}}\le\frac{1}{4\pi}\|x^2G_{k,m}(x)-x^2G_k(x)\|_{L^1(\mathbb{R}^2)}.$$

Thus,

$$\left\|\frac{\int_0^{|p|}\left(\int_0^s\left[\frac{\partial^2\widehat{G_{k,m}}(|q|,\theta)}{\partial|q|^2}-\frac{\partial^2\widehat{G_k}(|q|,\theta)}{\partial|q|^2}\right]d|q|\right)ds}{p_1^2+|p_2|^{2s_k}}\chi_{\{|p|\le 1\}}\right\|_{L^\infty(\mathbb{R}^2)}\le\frac{1}{4\pi}\|x^2G_{k,m}(x)-x^2G_k(x)\|_{L^1(\mathbb{R}^2)},$$

which tends to zero as $m\to\infty$ as assumed. An elementary computation gives us

$$\frac{\partial\widehat{G_{k,m}}}{\partial|p|}(0,\theta)=-\frac{i}{2\pi}[(G_{k,m}(x),x_1)_{L^2(\mathbb{R}^2)}cos\theta+(G_{k,m}(x),x_2)_{L^2(\mathbb{R}^2)}sin\theta], \tag{4.40}$$

$$\frac{\partial\widehat{G_k}}{\partial|p|}(0,\theta)=-\frac{i}{2\pi}[(G_k(x),x_1)_{L^2(\mathbb{R}^2)}cos\theta+(G_k(x),x_2)_{L^2(\mathbb{R}^2)}sin\theta]. \tag{4.41}$$

Let us first discuss the case b) of our lemma. By virtue of orthogonality conditions (4.25) and (4.26) along with (4.40), we have $\frac{\partial\widehat{G_{k,m}}}{\partial|p|}(0,\theta)=0,\ M+1\le k\le N,\ m\in\mathbb{N}$. Analogously, (4.6) and (4.7) along with (4.41) yield that $\frac{\partial\widehat{G_k}}{\partial|p|}(0,\theta)=0,\ M+1\le k\le N$. Thus, in the situation b), the first term in (4.38) is trivial.

Let us turn our attention to the case a) of our lemma. We recall inequality (4.18). Since it is assumed that $G_{k,m}(x), x^2G_{k,m}(x)\in L^1(\mathbb{R}^2)$, we have $|x|G_{k,m}(x)\in L^1(\mathbb{R}^2)$. Similarly,

$$\||x|G_{k,m}(x)-|x|G_k(x)\|_{L^1(\mathbb{R}^2)}=\int_{|x|\le 1}|x||G_{k,m}(x)-G_k(x)|dx+\int_{|x|>1}|x||G_{k,m}(x)-G_k(x)|dx\le$$

$$\le\|G_{k,m}(x)-G_k(x)\|_{L^1(\mathbb{R}^2)}+\|x^2G_{k,m}(x)-x^2G_k(x)\|_{L^1(\mathbb{R}^2)}\to 0,\quad m\to\infty$$

as we assume. Hence, $|x|G_{k,m}(x)\to|x|G_k(x)$ in $L^1(\mathbb{R}^2)$ as $m\to\infty$. Orthogonality conditions (4.25) along with (4.40) give us

$$\frac{\partial\widehat{G_{k,m}}}{\partial|p|}(0,\theta)=-\frac{i}{2\pi}sin\theta\int_{\mathbb{R}^2}G_{k,m}(x)x_2dx,\quad 1\le k\le M,\quad m\in\mathbb{N}.$$

By means of (4.6) and (4.41) we obtain

$$\frac{\partial \widehat{G_k}}{\partial |p|}(0,\theta) = -\frac{i}{2\pi} sin\theta \int_{\mathbb{R}^2} G_k(x) x_2 dx, \quad 1 \le k \le M, \quad m \in \mathbb{N}.$$

This enables us to derive the upper bound on the first term in (4.38) in the absolute value as

$$\frac{|p||sin\theta| \int_{\mathbb{R}^2} |x||G_{k,m}(x) - G_k(x)|dx}{2\pi(|p|^2 cos^2\theta + |p|^{2s_k}|sin\theta|^{2s_k})} \chi_{\{|p|\le 1\}} \le$$

$$\le \frac{|p|^{1-2s_k}|sin\theta|^{1-2s_k}}{2\pi} \| |x|G_{k,m}(x) - |x|G_k(x)\|_{L^1(\mathbb{R}^2)} \chi_{\{|p|\le 1\}} \le \frac{1}{2\pi} \| |x|G_{k,m}(x) - |x|G_k(x)\|_{L^1(\mathbb{R}^2)},$$

since $0 < s_k \le \frac{1}{2}$ for $1 \le k \le M$. Therefore, in the situation a) we derive

$$\left\| \frac{|p|\left[\frac{\partial \widehat{G_{k,m}}}{\partial |p|}(0,\theta) - \frac{\partial \widehat{G_k}}{\partial |p|}(0,\theta)\right]}{p_1^2 + |p_2|^{2s_k}} \chi_{\{|p|\le 1\}} \right\|_{L^\infty(\mathbb{R}^2)} \le \frac{1}{2\pi} \| |x|G_{k,m}(x) - |x|G_k(x)\|_{L^1(\mathbb{R}^2)} \to 0$$

as $m \to \infty$ as discussed above. Thus, by means of the argument above (4.28) is valid in both cases a) and b) of our lemma. Obviously, by virtue of the standard triangle inequality (4.30) and (4.31) follow easily from (4.28) and (4.29) respectively. Inequality (4.32) holds via a trivial limiting argument, which relies on (4.30) and (4.31). ■

Remark 3. *Note that in the parts a) of Lemmas 3 and 4 above for each* $1 \le k \le M$, $m \in \mathbb{N}$ *there are only two orthogonality relations for our integral kernels required, as distinct from the second part of the Assumption 2 of [26].*

Remark 4. *The existence in the sense of sequences of the solutions of our system of equations (1.2) involving the drift terms will be discussed in our consecutive article.*

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