NONCONMUTATIVE COBOUNDARY EQUATIONS OVER INTEGRABLE SYSTEMS

RAFAEL DE LA LLAVE AND MARIA SAPRYKINA

Abstract. We prove an analog of Livšic theorem for real-analytic families of cocycles over an integrable system with values in a Banach algebra $\mathcal{G}$ or a Lie group.

Namely, we consider an integrable dynamical system $f : \mathcal{M} \equiv \mathbb{T}^d \times [-1,1]^d \to \mathcal{M}$, $f(\theta, I) = (\theta + I, I)$, and a real-analytic family of cocycles $\eta_\varepsilon : \mathcal{M} \to \mathcal{G}$, indexed by a complex parameter ε in an open ball $\mathcal{E}_\rho \in \mathbb{C}$. We show that if η_ε has trivial periodic data, i.e.,

$$\eta_\varepsilon(f^{n-1}(p)) \dots \eta_\varepsilon(f(p)) \cdot \eta_\varepsilon(p) = Id$$

for each periodic point $p = f^n p$ and each $\varepsilon \in \mathcal{E}_\rho$, then there exists a real-analytic family of maps $\phi_\varepsilon : \mathcal{M} \to \mathcal{G}$ satisfying the coboundary equation

$$\eta_\varepsilon(\theta, I) = \phi_\varepsilon^{-1} \circ f(\theta, I) \cdot \phi_\varepsilon(\theta, I)$$

for all $(\theta, I) \in \mathcal{M}$ and $\varepsilon \in \mathcal{E}_{\rho/2}$.

We also show that if the coboundary equation above with an analytic left-hand side η_ε has a solution in the sense of formal power series in ε, then it has an analytic solution.

1. Introduction

On the annulus $\mathcal{M} = \mathbb{T}^d \times [-1,1]^d$ consider an integrable dynamical system $f : \mathcal{M} \to \mathcal{M}$ of the form $f(\theta, I) = (\theta + I, I)$ and a real-analytic family of maps $\eta_\varepsilon : \mathcal{M} \to \mathcal{G}$, where $\varepsilon \in \mathcal{E}_\rho$ is a complex one-dimensional parameter, and $\mathcal{G}$ is a Banach algebra (with small modifications we can let $\mathcal{G}$ be a Lie group, see the last section). Assume that $\eta_0 = \mathrm{Id}$. We ask the question of whether η_ε is a *coboundary*, i.e., whether there exists an analytic family $\phi_\varepsilon : \mathcal{M} \to \mathcal{G}$, $\varepsilon \in \mathcal{E}_{\rho/2}$, solving the following coboundary equation:

$$\eta_\varepsilon(\theta, I) = \phi_\varepsilon^{-1} \circ f(\theta, I) \cdot \phi_\varepsilon(\theta, I), \tag{1.1}$$

where the dot between two elements of $\mathcal{G}$ denotes the product in $\mathcal{G}$.

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Equation (1.1) can be expressed by saying that cocycle η_ε is conjugated (equivalent) to the identity, which coincides with η_0, and ϕ_ε should be seen as the conjugacy. In this sense, the present paper is part of the rigidity program for cocycles.

Coboundary equations of type (1.1) appear naturally in many problems in dynamics, in particular, in the study of rigidity for integrable Hamiltonian systems.

An obvious necessary condition for η_ε to be a coboundary is the following Periodic Orbit Condition (POC for short).

Definition 1. *We say that η_ε satisfies the Periodic Orbit Condition (POC) if for every $\varepsilon \in \mathcal{E}_\rho$ and for every $p \in \mathcal{M}$ such that $p = f^N(p)$ we have:*

$$\eta_\varepsilon \circ f^{N-1}(p) \cdot \cdots \cdot \eta_\varepsilon \circ f(p) \cdot \eta_\varepsilon(p) = \mathrm{Id}\,. \tag{1.2}$$

Our main result, Theorem 3, states that if this obvious necessary condition is met, then there is an analytic solution of the coboundary equation above.

As a related result, we show that if the coboundary equation (1.1) with an analytic left-hand side η_ε has a solution in the sense of C^0 formal power series, then it has an analytic solution (which may be different from the original one). This provides a very dramatic bootstrap of the regularity of the solutions to (1.1).

Both of the above statements remain valid when the family of cocycles takes values in a Lie group. See Section 4.

The results of the present paper are related to the notion of the *uniform integrability* introduced by Poincaré in [Poi99, Chapter V]. An analytic family of Hamiltonians is called uniformly integrable if it can be reduced to an integrable one by a canonical change of variables. In the cited chapter, Poincaré described certain obstructions (i.e., necessary conditions) to the existence of such families of changes of variables even in the sense of formal power series. These conditions can be expressed via certain integrals over periodic orbits. Poincaré extended slightly these conditions and verified them in concrete problems such as the 3-body problem. In [dlL96] it is proved that these conditions are also sufficient: vanishing of the above obstructions to integrability in the sense of power series implies the existence of an analytic integrating change of variables.

In the present paper we consider an analytic family of non-commutative coboundary equations (1.1), where the base dynamics, f, is an integrable system (parabolic dynamics). We prove the existence of an

analytic family of solutions to this equation under the Periodic Orbit Condition (1.2) (which is the obvious necessary condition for the existence of such solutions).

As an iterative step of the proof we solve the linearized (commutative) coboundary equation, producing solutions with tame estimates in the sense of Nash-Moser theory. Recall that, in the context of analytic spaces, an operator $\mathcal{L}$ is said to satisfy *tame estimates* in a monotonic class of domains D_ρ, when there exists $\tau > 0$ such that for all $0 < \rho \leq \rho_0$, given $\psi : D_\rho \to \mathcal{G}$, we have $\mathcal{L}\psi$ is analytic in $D_{\rho-\delta}$ for every $0 < \delta < \rho$ and

$$\|\mathcal{L}\psi\|_{\rho-\delta} \leq C\delta^{-\tau}\|\psi\|_\rho.$$

(A similar notion of tameness exists in finite regularity spaces.) Informally speaking, our KAM iterative method shows that we can pass from the tame estimates on the solutions of commutative coboundary equations to the analytic solutions of non-commutative cobundary equations for families (under the Periodic Orbit Condition).

The type of statements where the Periodic Orbit Condition implies the existence of solutions to a functional equation is often referred to as Livšic theorems due to the seminal papers [Liv71, Liv72]. There is a whole spectrum of contexts in which these results have been proved over the past decades: in particular, different classes of smoothness, types of the base dynamics, types of the sufficient conditions, types of the coboundary equation (i.e., commutative or not) and others. Let us mention just a few results that are closest to ours.

In the case when the base dynamics f is *hyperbolic*, a lot is known. The most famous works are the classical ones by Livsic, [Liv71, Liv72], where the base dynamics is transitive Anosov, and the solutions are studied in low regularity. Some generalizations and improved results on the regularity were obtained in [dlLMM86]; in particular, they show that, in appropriate spaces, there is no loss of regularity while solving the commutative coboundary equation. There are other methods [NT95, NT01, NT02, KN11] that eliminate the use of power series and rely only on periodic orbit condition.

In the analytic regularity, the solvability of the commutative equation is done in [dlL97], but the estimates presented there are not tame. Note that the notion of tame estimates depends very much on the families of domains used.The estimates presented in [dlL97] are not tame in the families of domains considered there, but it seems possible that they are tame in other families of domains.

The case of non-commutative coboundary equations over Anosov systems was clarified in [dlLW10].

For commutative cohomology equations over integrable systems (*parabolic* base dynamics) the regularity of solutions under a variant of the Periodic Orbit Condition was studied in [dlL96]. In that paper it was shown that there are solutions satisfying tame estimates.

Another case when the cohomology equation has been studied is the case of *quasi-hyperbolic* automorphisms of the torus. The paper [Vee86] established that, under the Periodic Orbit Condition, the solutions of the equation with C^∞ data are C^∞.

The assumptions of the present paper require that a whole analytic *family* satisfies the Periodic Orbit Condition. Nevertheless, there are other methods: [dlLMM86, dlLW10, NT95, NT01, NT02, KN11], that eliminate the use of families and rely only on the Periodic Orbit Condition. When the group is non-commutative, they only require that the cocyles are close to the identity. This smallness assumption was removed when $\mathcal{G}$ is finite dimensional in the remarkable paper [Kal11]. As far as we know, results under proximity to identity (without using families) or, much less, without proximity assumptions, are not known when the dynamics in the base is an integrable system as considered here.

There are other cases where the cohomology equations are solvable with estimates (e.g., the base dynamics being a rotation of the torus, interval exchange transformation, horocycle flow, twisted versions of the above, higher rank actions) but these results require other conditions than the Periodic Orbit Condition. In particular, in the case when the base dynamics is a Diophantine rotation, the convergence of formal power series expansions in many cases was considered in [Mos67].

Results of the same kind, but with a different condition, exist for partially hyperbolic systems, see [Wil13] and references therein.

1.1. **Notations and the Main Theorem.** The following notations are needed to give a precise formulation of the results. We will consider the manifold

$$\mathcal{M} = \mathbb{T}^d \times [-1,1]^d,$$

endowed with the dynamics

$$\begin{aligned} &f : \mathcal{M} \to \mathcal{M}, \\ &f(\theta, I) = (\theta + I, I). \end{aligned} \tag{1.3}$$

Note that a more general dynamics: $f(\theta, I) = (\theta + \Omega(I), I)$ where Ω is invertible, can be reduced to $\Omega(I) = I$ by changing the variable. We will refer to $\mathcal{M}$ as the *base* and to f as the *dynamics on the base*.

Since we are working with the analytic regularity, it is useful to consider complex extensions of the above. Fix $\rho > 0$; we denote:

$$\begin{aligned} &\mathbb{T}^d_\rho = \{z \in \mathbb{C}^d/\mathbb{Z}^d \,|\, |\mathrm{Im}(z_j)| \le \rho\}, \\ &\mathcal{B}_\rho = \{I \in \mathbb{C}^d \,|\, \mathrm{dist}(I, [-1,1]^d) \le \rho/10\}, \\ &\mathcal{M}_\rho = \mathbb{T}^d_\rho \times \mathcal{B}_\rho, \\ &\mathcal{E}_\rho = \{\varepsilon \in \mathbb{C} \,|\, |\varepsilon| < \rho\}. \end{aligned} \tag{1.4}$$

We let $\mathcal{G}$ be a Banach algebra (e.g., the algebra of real or complex square matrices or the Banach algebra of bounded operators in a Banach space). Later, in Section 4, we will present the minimal – typographical – modifications needed to adapt the proofs to $\mathcal{G}$ being a Lie group.

For any $\rho > 0$, we consider $\mathcal{A}_\rho$ to be the space of real-analytic functions on $\mathcal{M}_\rho$, i.e., holomorphic functions ϕ on $\mathcal{M}_\rho$, continuous up to the boundary, and satisfying the real symmetry property: $\overline{\phi(\bar\theta, \bar I)} = \phi(\theta, I)$ (where the bar stands for the complex conjugate). In a similar way, we let $\mathcal{A}^\varepsilon_\rho$ be the space of real-analytic functions on $\mathcal{E}_\rho \times \mathcal{M}_\rho$.

We endow $\mathcal{A}_\rho$ and $\mathcal{A}^\varepsilon_\rho$ with the supremum norms which make $\mathcal{A}_\rho$ and $\mathcal{A}^\varepsilon_\rho$ into a Banach spaces. We keep the same notation, $\|\cdot\|_\rho$, for the two norms.

Since we will need to deal with truncations of power series, we use the notation:

$$v_\varepsilon^{[m,M]} = \sum_{j \in [m,M]} \varepsilon^j v^j, \tag{1.5}$$

and analogously for other ranges. For example: $v_\varepsilon^{[\le M]} = \sum_{j \le M} \varepsilon^j v^j$.

Notice that $v_\varepsilon^{[m,M]}$ is a polynomial of degree M in ε, whose coefficients of order smaller than m vanish.

A small typographical confusion is that ε^j means the variable ε raised to the jth power. On the other hand v^j is the jth coefficient in the expansion. This inconsistency is very common in mathematics and we hope will not cause much confusion.

We recall the somewhat standard definition of formal power series whose coefficients are functions of another variable. Formal power series for expansions have been used in mechanics and other mathematical disciplines for a long time.

Definition 2. *Given a sequence of continuous functions $\phi^j : \mathcal{M}_\rho \to \mathcal{G}$, we say that expression*

$$\phi_\varepsilon = \mathrm{Id} + \sum_{j=1}^{\infty} \varepsilon^j \phi^j \tag{1.6}$$

satisfies (1.1) *in the sense of formal power series if for any $L \in \mathbb{N}$ we have that the truncated series $\phi_\varepsilon^{[\le L]} = \mathrm{Id} + \sum_{j=1}^{L} \varepsilon^j \phi^j$ satisfies* (1.1) *up to the terms of order ε^L:*

$$\|\eta_\varepsilon - (\phi_\varepsilon^{-1})^{[\le L]} \circ f \cdot \phi_\varepsilon^{[\le L]}\|_{C^0} \le C_L |\varepsilon|^{L+1}. \tag{1.7}$$

We emphasise that (1.6) is just a suggestive formal expression and that it is not assumed to converge in any sense. The only meaning adscribed to the infinite sum is precisely that each of the truncated sums satisfies (1.7).

Of course, we could consider other spaces rather than C^0 for the functions in the coefficients, or even spaces depending on the order (a common situation in series expansions in mechanics is that the coefficients are analytic, but the analyticity domain depends on n and even shrinks as n tends to ∞).

Here is the main result of the paper.

Theorem 3. *Assume the notation in* (1.3), (1.4), *and consider an analytic family $\eta_\varepsilon : \mathcal{M}_\rho \to \mathcal{G}$, $\varepsilon \in \mathcal{E}_\rho$, with $\eta_0 = \mathrm{Id}$.*

Then the following statements are equivalent:

- (A). *η_ε satisfies the Periodic Orbit Condition* (1.2) *for each $\varepsilon \in \mathcal{E}_\rho$.*
- (B). *There is a solution of* (1.1) *in the sense of formal power series expansions (see Definition 2).*
- (C). *There exists a real-analytic solution ϕ_ε of* (1.1). *The domain of analyticity of ϕ_ε may be smaller than the domain of η_ε.*

1.2. **Comments on the proof.** Clearly, $(C) \implies (A), (B)$. The fact that $(B) \implies (A)$ is also easy. We observe that for any periodic point p of f, the function $\eta_\varepsilon(f^{N-1}p) \cdots \eta_\varepsilon(p)$ is analytic in ε. The existence of a formal power series solution of (1.1) implies that all the terms in the power series expansion vanish.

The fact that (A) implies (C) is the main result of this paper. It will be proved in Section 3 using a Nash-Moser iterative approach. The basic tool in this proof is solving the linearized equation with estimates. Namely, by linearizing the coboundary equation (using the fact that η_ε is close to the identity), we obtain a commutative cohomology equation over f. We show that if the commutative cohomology equation has a

formal power series solution, then it has an analytic solution with good estimates (the estimates involve a loss of the domain with respect to the initial data). Note that due to the lack of uniqueness, the solution with estimates may be different from the formal power solution assumed in the hypotesis.

Using the group structure of the coboundary equation and of the hypothesis, we can use this approximate solution to improve the situation. The process can be repeated and the convergence of this accelerated procedure is established by a slight modification of the usual Nash-Moser scheme (we get an extra term due to the truncation of power series).

1.3. **Various remarks.**

Remark 4. Notice that we are not stating that the formal power series solution in the hypothesis (B) of Theorem 3 converges (indeed, such result is false!).

In fact, the solutions of the coboundary equation are far from being unique. Indeed, if $\phi_\varepsilon(\theta, I)$ satisfies coboundary equation (1.1) and $A_\varepsilon(I)$ is a function from $\mathcal{E}_\rho \times B_\rho$ to $\mathcal{G}$, then

$$\tilde{\phi}_\varepsilon(\theta, I) = A_\varepsilon(I) \cdot \phi_\varepsilon(\theta, I) \tag{1.8}$$

is also a solution of the coboundary equation.

We show that, from the existence of a formal power series solution we can obtain a (possibly different) power series that converges in some complex domain. For this proof it is important that we work with analytic one-parameter families of cocycles.

Remark 5. For simplicity of notation, we have used only one parameter, ρ, to indicate the size of the analyticity domains in ε, I, θ. Since these variables play different roles, it would have been natural, and improve slightly the estimates, to consider them as independent. For present purposes the improvement in regularity does not justify the complication of dealing with norms indexed by 3 parameters.

Remark 6. The reason to take $\mathcal{G}$ to be a Banach algebra is that it allows to apply corrections by adding terms. This simplifies the notation, but is not essential. One can consider $\mathcal{G}$ to be a Lie group, the correction functions taking values on the Lie algebra, in which case the corrections are applied using the exponential mapping rather than just adding. After the proof of Theorem 3 is presented, we collect in Section 4 the (mostly typographical) changes needed to obtain the result for $\mathcal{G}$ being a (Banach) Lie Group.

Remark 7. Note that f does not map the complex domain $\mathcal{M}_\rho$ into itself and this will be a source of – mainly notational – problems. We have that $f(\mathcal{M}_\rho) \subset \mathcal{M}_{11\rho/10}$ (when I is complex, an application of f changes the imaginary part of the θ component. This is the reason for introducing the factor 10 in the definition of $\mathcal{B}_\rho$).

Remark 8. Note that (1.2) is far from being a trivial condition. The map f indeed has a lot of periodic orbits. If there exists $N \in \mathbb{N}$ such that $NI \in \mathbb{Z}^d$, then for any $\theta \in \mathbb{T}^d$, (θ, I) is periodic of period N. Note also that the periodic orbits of f are real, hence POC only provides information about the behaviour of η_ε in restriction to the real hyperplane. It appears that this information is sufficient in order to obtain an estimate of the solution in a complex neighborhood. We would call this a manifestation of the "magic" of complex analysis.

Remark 9. One can wonder if it would be possible to consider analogous results when the dimension of the angles θ in the base is bigger than the dimension of the actions I. The simple example $f(\theta_1, \theta_2, I) = (\theta_1 + \omega, \theta_2 + I)$ shows that such dynamics in the base can lack periodic orbits so that the Periodic Orbit Condition becomes vacuous (hence, trivially satisfied) but not all cocycles are coboundaries. One can ask whether, in this context, the existence of formal power solutions implies a convergent solution. Instead of the periodic orbit condition one can explore more general obstructions: invariant measures, pseudomeasures as in [Vee86] or distributions.

2. Commutative cohomology equations over integrable dynamics

The key tool we will use in the proof of Theorem 3 consists in estimates of the solutions to a commutative version of the coboundary equation in the case when the group $\mathcal{G}$ is commutative. Namely, in this section we will consider equations for $\alpha : \mathcal{M}_\varepsilon \to \mathbb{R}^n$ given $\beta : \mathcal{M}_\varepsilon \to \mathbb{R}^n$ of the form:

$$\alpha \circ f - \alpha = \beta. \tag{2.1}$$

Notice that the solution to (2.1) is not unique. For instance, any function α only depending on I (and not on θ) leads to $\beta = 0$. This lack of uniqueness of solutions to the commutative equation can be seen as a commutative version of the non-uniqueness (1.8) for the general equation.

The equation will be solved using Fourier series in the angle variable of the following form:

$$\beta(\theta, I) = \sum_{k\in\mathbb{Z}^d} \hat{\beta}_k(I) e^{2\pi i\langle k,\theta\rangle} \tag{2.2}$$

(and analogously for other functions). With this notation, under very small regularity requirements on α, we have

$$\alpha \circ f(\theta, I) = \sum_{k\in\mathbb{Z}^d} \hat{\alpha}_k(I) e^{2\pi i\langle k,I\rangle} \cdot e^{2\pi i\langle k,\theta\rangle},$$

and (2.1) is equivalent to the following set of equations (for $\{\hat{\alpha}_k(I)\}_{k\in\mathbb{Z}^d}$ given $\{\hat{\beta}_k(I)\}_{k\in\mathbb{Z}^d}$):

$$\hat{\alpha}_k(I)\left(e^{2\pi i\langle k,I\rangle} - 1\right) = \hat{\beta}_k(I). \tag{2.3}$$

The theory we will develop will consist in:

- Characterization of obstructions for solvability;
- Characterization of lack of uniqueness of solutions;
- Estimates for the solutions (when they exist).

2.1. **Obstructions CPOC and FC for the existence of solutions of** (2.1)**.** Here we present two necessary conditions for the existence of solutions of (1.1). The first one is based on the study of the dynamics, and the second one uses Fourier representations. We will show that they are actually equivalent. As we will see, the geometric representation is useful when we perform changes of variables. The Fourier representation is useful to obtain estimates of the solutions. Combination of these two points of view will allow us to make progress.

2.1.1. *Periodic Orbit Condition for the commutative equation (CPOC).* Suppose that equation (2.1) has a solution. If $p = f^N(p)$ is a periodic point for the base dynamics, then the sum of the values of β over the periodic orbit telescopes to zero:

$$\sum_{j=0}^{N-1} \beta \circ f^j(p) = 0.$$

In our case for each $N \in \mathbb{N}$ we have $f^N(\theta, I_*) = (\theta + NI_*, I_*)$. Hence, (θ, I_*) is periodic of period N if and only if $NI_* \in \mathbb{Z}^d$. This motivates the following definition.

Definition 10. *We say that a function $\beta(\theta, I)$ satisfies the Commutative Periodic Orbit Condition (CPOC) if for every $I_* \in \mathcal{B}_\rho$ satisfying*

$NI_* \in \mathbb{Z}^d$ for some $N \in \mathbb{N}$ we have:

$$\sum_{j=0}^{N-1} \beta(\theta + jI_*, I_*) = 0 \quad \textit{for all real } \theta. \tag{2.4}$$

Remark 11. Note that the I_* that give rise to periodic orbits are of the form ℓ/N with $\ell \in \mathbb{Z}^d$. In particular, such I_* are real. This will lead to some difficulties.

The following result summarises the discussion in the beginning of the section.

Proposition 12. *Let α and β be continuous functions satisfying* (2.1)*. Then β satisfies CPOC.*

2.1.2. *Fourier coefficient condition (FC).*

Definition 13. *We say that a function $\beta(\theta, I)$ satisfies Fourier Coefficient condition (FC) if for every $k \in \mathbb{Z}^d \setminus \{0\}$ and every $I_* \in \mathcal{B}_\rho$ we have:*

$$\langle k, I_* \rangle \in \mathbb{Z} \quad \textit{implies} \quad \hat{\beta}_k(I_*) = 0. \tag{2.5}$$

Looking at (2.3) we obtain immediately the following result.

Proposition 14. *A necessary condition for the existence of continuous solutions of* (2.1) *is that β satisfies FC.*

The above statement tells us that, in order for the commutative equation (2.1) to have a continuous solution α, the Fourier coefficient of every index k has to vanish for I_* lying in parallel planes corresponding to k. Note, however, that these planes involve complex values. In contrast, condition CPOC gives us information only about real values of I, see Remark 11. This makes the following result rather surprising.

2.1.3. *Equivalence of conditions FC and CPOC.* Conditions FC and CPOC have a very different nature. CPOC is geometrically natural, and it is clearly preserved under a cocycle conjugacy. FC, on the other hand, leads to very effective estimates of the solution, as we will see in Section 2.2. Below we prove that the two conditions are equivalent.

Proposition 15. *A continuous map $\beta : \mathcal{M}_\rho \to \mathbb{R}$ satisfies CPOC (condition* (2.4)*) if and only if it satisfies FC (condition* (2.5)*).*

At the first glance, CPOC may seem weaker than FC, since the it gives us information only about the real values of the argument I, while FC concerns a complex neighbourhood in B_ρ. The following uniqueness theorem from complex analysis will be used to bridge this gap.

Lemma 16. *If $f(I)$ is an analytic function on a domain $D \in \mathbb{C}^d$ that vanishes in a real neighbourhood $U^{\mathbb{R}}$ of a point $I_0 = x_0 + iy_0 \in D$, that is, on a set*

$$U^{\mathbb{R}} = \{I = x + iy \in \mathbb{C}^d \mid |x - x_0| < r,\ y = y_0\},$$

then $f(I) \equiv 0$ on D.

The idea of the proof is very simple. We note that all the (complex) derivatives of the function can be computed along the real space. The assumption implies that the real derivatives vanish. Hence, all the derivatives vanish.

Proof. Since $f(I) \equiv 0$ on $U^{\mathbb{R}}$, for any $v \in \mathbb{R}^d$, for $t \in \mathbb{R}$ and $j \in \mathbb{N}$ we have $\left(\frac{d}{dt}\right)^j f(u + tv) \mid_{t=0} = 0$. Hence,

$$\left(\frac{d}{dt}\right)^j f(u + tv) \mid_{t=0} = (\langle v, \nabla \rangle)^j f(u) = 0.$$

Since this is true for all $v \in \mathbb{R}^d$, we get $\partial^k f(u) = 0$ for all multi-indices $k \in \mathbb{N}^d$. Since f is analytic in D, this implies that f vanishes identically. □

Proof of Proposition 15. The direction of the equivalence that is used in this paper is that CPOC implies FC, so we do it first.

Suppose that CPOC (condition (2.4)) holds. Fix any $k \in \mathbb{Z}^d$, $n \in \mathbb{Z}$, and consider the complex hyperplane $P^{\mathbb{C}}_{k,n} = \{I \in \mathbb{C}^d \mid \langle k, I \rangle = n\}$. Denote $P^{\mathbb{R}}_{k,n} = \{I \in P^{\mathbb{C}}_{k,n} \mid \operatorname{Im} I = 0\}$.

Note that, since $k \in \mathbb{Z}^d$, the points of the form ℓ/N with $\ell \in \mathbb{Z}^d$ and $N \in \mathbb{N}$ are dense on $P^{\mathbb{R}}_{k,n}$. Fix any $I = \ell/N \in P^{\mathbb{R}}_{k,n}$. Using that $\langle k, t\ell \rangle \in \mathbb{Z}$, we get:

$$\begin{aligned}
\hat{\beta}_k(\ell/N) &= \int_{\mathbb{T}^d} \beta(\theta, \ell/N) e^{-2\pi i \langle \theta, k \rangle}\, d\theta \\
&= \int_{\mathbb{T}^d} \beta(\theta + \ell/N, \ell/N) e^{-2\pi i \langle \theta, k \rangle} e^{-2\pi \langle \ell/N, k \rangle}\, d\theta \\
&= \int_{\mathbb{T}^d} \beta(\theta + \ell/N, \ell/N) e^{-2\pi i \langle \theta, k \rangle}\, d\theta \\
&= \cdots \\
&= \int_{\mathbb{T}^d} \beta(\theta + j\ell/N, \ell/N) e^{-2\pi i \langle \theta, k \rangle}\, d\theta.
\end{aligned}$$

Adding the above N different expressions for $\hat{\beta}_k(\ell/N)$, we obtain:

$$N\hat{\beta}_k(\ell/N) = \int_{\mathbb{T}^d} \left(\sum_{j=0}^{N-1} \beta(\theta + j\ell/N, \ell/N) \right) e^{-2\pi i \langle \theta, k \rangle}\, d\theta. \tag{2.6}$$

Under the CPOC condition, the integrand in (2.6) is identically zero. We have thus shown that $\hat{\beta}_k(\ell/N) = 0$ for any rational point $(\ell/N) \in P^{\mathbb{R}}_{k,n}$. Since such points are dense on $P^{\mathbb{R}}_{k,n}$, and the zero-set of a continuous function is always closed, we have: $\hat{\beta}_k(I) = 0$ for any $I \in P^{\mathbb{R}}_{k,n}$. Since the restriction of $\hat{\beta}_k(I)$ onto $P^{\mathbb{C}}_{k,n}$ is analytic, the above result, combined with Lemma 16, implies that the restriction of $\hat{\beta}_k(I)$ onto $P^{\mathbb{C}}_{k,n}$ equals zero. Hence, FC condition (2.5) holds.

Now we prove that FC implies CPOC. Assume that FC holds. Let $I = \ell/N$ for some $\ell \in \mathbb{Z}^d$ and $N \in \mathbb{N}$. Then (2.6) holds. We note that the function

$$A(\theta) = \left(\sum_{j=0}^{N-1} \beta(\theta + j\ell/N, \ell/N) \right)$$

satisfies $A(\theta + \ell/N) = A(\theta)$, so it is a function defined in a reduced torus obtained by identifying the points that differ by a translation by ℓ/N. For functions with this extra periodicity, the exponentials $e^{2\pi i\langle k,\theta\rangle}$ with condition $\langle k, \ell/N\rangle \in \mathbb{Z}$ form a complete set. Therefore, from FC, we conclude that $A(\theta) \equiv 0$ and, hence, that the CPOC condition (2.4) is satisfied. □

Remark 17. The fact that FC implies CPOC is not used in the proofs below. We can get this statement also as a byproduct of the main line of argument. Indeed, in the next section we will show that for analytic functions β satisfying FC, one can construct an (analytic) solution α to (2.1). The fact that equation (2.1) with the right-hand side β has a continuous solution implies that β satisfies CPOC.

2.2. **Estimates for the commutative equation** (2.1)**.**

2.2.1. *A reminder on Cauchy estimates.* Assume the notations from Section 1.1; in particular consider the space $\mathcal{A}_\rho$ of analytic functions. The following statements are standard.

Lemma 18. *If $v \in \mathcal{A}_\rho$, $v(\theta, I) = \sum_{k\in\mathbb{Z}^d} v_k(I)e^{2\pi i\langle\theta,k\rangle}$, then there exists a constant $c = c(d)$, such that for each $I \in \mathcal{B}_\rho$ we have:*

$$\begin{aligned} &|v_k| \le \|v\|_\rho e^{-2\pi|k|\rho}, \\ &\|Dv\|_{\rho-\delta} \le c\delta^{-1}\|v\|_\rho. \end{aligned} \tag{2.7}$$

Given $v_\varepsilon \in \mathcal{A}^\varepsilon_\rho$, write it as a Taylor series in ε: $v_\varepsilon(\theta, I) = \sum_{j=0}^{\infty} \varepsilon^j v^j(\theta, I)$. Suppose that ρ satisfies $0 < \rho \le \rho_0$ for some fixed ρ_0. Then, recalling

notation (1.5)*, we can estimate:*

$$\begin{aligned}
&\|v^j\|_\rho \le \rho^{-j}\|v\|_\rho,\\
&\|v^{[m,M]}\|_{\rho-\delta} \le \rho_0\delta^{-1}\|v\|_\rho,\\
&\|v^{[\ge M]}\|_{\rho-\delta} \le \rho_0\delta^{-1}\exp(-M\delta/\rho_0)\|v\|_{\rho_0}.
\end{aligned} \tag{2.8}$$

Proof. Let us present the proof of the last estimate for completeness. First of all, since $\ln(1-x) \le -x$ for all $x \in \mathbb{R}$, we have

$$(1-\delta/\rho) \le e^{-\delta/\rho}.$$

Using $\|v^j\|_\rho \le \rho^{-j}\|v\|_\rho$ and $\varepsilon < \rho - \delta$, we have:

$$\begin{aligned}
&\|v^{[\ge M]}\|_{\rho-\delta} \le \sum_{j=M}^{\infty} \varepsilon^j \|v^{[j]}\|_\rho \le \sum_{j=M}^{\infty} (\rho-\delta)^j \rho^{-j}\|v\|_\rho\\
&\le \|v\|_\rho (1-\delta/\rho)^M \sum_{j=0}^{\infty}(1-\delta/\rho)^j \le \|v\|_\rho\, e^{-M\delta/\rho_0}\,\rho\delta^{-1}\\
&\le \rho_0\delta^{-1}e^{-M\delta/\rho_0}\|v\|_\rho.
\end{aligned}$$

□

2.2.2. *Estimates on the solutions of the coboundary equation* (2.1).

Lemma 19. *Assume that β is analytic in $\mathcal{M}_\rho = \mathbb{T}^d_\rho \times \mathcal{B}_\rho$ and satisfies the FC condition* (2.5)*.*

Then there exists α solving equation (2.1)*, i.e., $\alpha \circ f - \alpha = \beta$, such that, for a constant $c = c(d)$ and for any $0 < \delta < \rho$ we have:*

$$\|\alpha\|_{\rho-\delta} \le c\delta^{-d-1}\|\beta\|_\rho. \tag{2.9}$$

If β depends analytically (continuously) on a parameter ε ranging in a certain domain, a solution α_ε can be chosen to depend analytically (continuously) on ε in the same domain.

Remark 20. As an important corollary of this lemma, we obtain estimates of the solution in domains other than $\mathcal{M}_\rho$. Namely, using equation (2.1), we obtain (for a slightly modified constant, which we denote by c again):

$$\|\alpha \circ f\|_{\rho-\delta} \le c\delta^{-d-1}\|\beta\|_\rho. \tag{2.10}$$

Proof. Equation (2.1) is equivalent to the sequence of equations (2.3) for the partial Fourier coefficients defined in (2.2):

$$\hat{\alpha}_k(I)\left(e^{2\pi i\langle k,I\rangle} - 1\right) = \hat{\beta}_k(I). \tag{2.11}$$

For I such that $\langle k, I\rangle \notin \mathbb{Z}$ we can express $\hat{\alpha}_k(I) = \hat{\beta}_k(I)\left(e^{2\pi i\langle k,I\rangle} - 1\right)^{-1}$. The crucial remark is that the FC condition for β implies that

$$\langle k, I\rangle \in \mathbb{Z} \Rightarrow \hat{\beta}_k(I) = 0.$$

Hence, for I such that $\langle k, I\rangle \in \mathbb{Z}$, equation (2.11) is satisfied for any value of $\hat{\alpha}_k(I)$. We define $\hat{\alpha}_k(I)$ for these I by continuity. A way to do it is the following (compare with the L'Hôpital's rule). Differentiate equation (2.11) in the direction of the vector k:

$$\langle \nabla_I \hat{\alpha}_k(I), k\rangle \left(e^{2\pi i\langle k,I\rangle} - 1\right) + \hat{\alpha}_k(I) 2\pi i \langle k, k\rangle e^{2\pi i\langle k,I\rangle} = \langle \nabla_I \hat{\beta}_k(I), k\rangle.$$

If $\langle k, I\rangle \in \mathbb{Z}$, this gives us $2\pi i|k|^2 \hat{\alpha}_k(I) e^{2\pi i\langle k,I\rangle} = \langle \nabla_I \beta_k(I), k\rangle$. Summing up, we have defined a continuous function $\hat{\alpha}_k(I)$ by

$$\hat{\alpha}_k(I) = \begin{cases} \hat{\beta}_k(I)\left(e^{2\pi i\langle k,I\rangle} - 1\right)^{-1} & \text{if } \langle k, I\rangle \notin \mathbb{Z} \\ \langle \nabla_I \beta_k(I), k\rangle e^{-2\pi i\langle k,I\rangle}/(2\pi i|k|^2) & \text{if } \langle k, I\rangle \in \mathbb{Z}. \end{cases} \tag{2.12}$$

Since $\hat{\alpha}_k(I)$ is analytic in $\mathcal{B}_\rho \setminus \{\langle k, I\rangle = 0\}$ and bounded in $\mathcal{B}_\rho$, it is analytic in $\mathcal{B}_\rho$.

Now let us estimate the norm of the solution. This estimate is done in [dlLS21], Lemma 6; below we repeat the argument for completeness. Fix $0 < \delta < \rho/2$. For each fixed $k \in \mathbb{Z}^d$, we will estimate the corresponding $\hat{\alpha}_k(I)$ in two steps: first '$\delta/2$-close" to the resonant plane $\langle k, I\rangle$, and then in the rest of $\mathcal{B}_{\rho-\delta}$.

For the first step, let $\Pi_\delta = \{\langle k, I\rangle = 0\} \cap \mathcal{B}_{\rho-\delta}$ be the part of the resonant plane falling into $\mathcal{B}_{\rho-\delta}$. Notice that the orthogonal complement to this plane is formed by the vectors $\gamma e^{2\pi i\theta} k$, $\gamma \geq 0$, $\theta \in [0, 1)$. Let

$$\Delta = \left\{ I = \gamma \frac{k}{|k|} e^{2\pi i\theta} \;\middle|\; \gamma < \delta/2,\ \theta \in [0, 1) \right\}$$

be the complex disk of radius $\delta/2$ centered at zero and orthogonal to Π_δ. Note that the restrictions of $\hat{\alpha}_k(I)$ and $\hat{\beta}_k(I)$ to this disc are analytic. Consider the $\delta/2$-neighbourhood O_δ of Π_δ: $O_\delta = \bigcup_{I_0 \in \Pi_\delta} (I_0 + \Delta)$. Then $O_\delta \subset \mathcal{B}_{\rho-\delta}$.

For each fixed $I \in O_\delta$ there exists $I_0 \in \Pi_\delta$ such that $I \in I_0 + \Delta$. We can estimate $|\alpha_k(I)|$ by the maximum modulus principle on the disk $I_0 + \Delta$. Namely, for I lying on the boundary of this disk we have: $|\langle k, I\rangle| = |\langle k, I_0\rangle + \langle k, \delta k/(2|k|)\rangle| = |k|\delta/2$. Hence, for such I we have

$$|\hat{\alpha}_k(I)| \leq \frac{2\|\hat{\beta}_k\|_\rho}{4\pi\delta|k|} < \frac{\|\hat{\beta}_k\|_\rho}{\delta|k|}.$$

As the second step in this estimate, consider $I \in \mathcal{B}_{\rho-\delta} \setminus O_\delta$. Here $|\langle k, I\rangle| \geq |k|\delta/2$, so $|\hat{\alpha}_k(I)|$ satisfies the same estimate as above. This proves the desired estimate for an individual parameter value.

In the case that the data depend analytically (continuously) on a parameter ε, we obtain that the Fourier coefficients depend analytically (continuously) on ε. The uniform bounds on the Fourier coefficients imply that the sum depends analytically (continuously) on ε.

□

3. Proof of Theorem 3

Theorem 3 is the equivalence between the existence of an analytic solution to the coboundary equation, (C), and two formal conditions: Periodic Orbit Condition (A), and the existence of formal power series solutions, (B). As indicated in Section 1.2, the heart of the problem is to show that (A) implies (C).

3.1. **Overview of the proof.** The main part of the proof consists in estimating the results of an iterative step.

For each $n \in \mathbb{Z}$, we let $L_n = L_0 2^n$, where L_0 is an appropriate constant. The initial input of the iterative step will be an $\eta_\varepsilon^n = \mathrm{Id} + O(\varepsilon^{L_n})$ satisfying the POC condition (1.2). The iterative procedure will produce an "almost solution" $\phi_\varepsilon^n = \mathrm{Id} + O(\varepsilon^{L_n})$ such that

$$\eta_\varepsilon^{n+1} := (\phi_\varepsilon^n \circ f)^{-1} \cdot \eta_\varepsilon^n \cdot \phi_\varepsilon^n \tag{3.1}$$

satisfies $\eta_\varepsilon^{n+1} = \mathrm{Id} + O(\varepsilon^{2L_n})$. Note that, because of the construction of (3.1), we obtain that η_ε^{n+1} also satisfies the POC condition, and the iterative procedure can be applied again.

We will show that the iterative procedure leads to the estimates

$$\|\eta_\varepsilon^{n+1} - \mathrm{Id}\,\|_{\rho_n - \delta_n} \leq c\delta^{-2(d+1)} \|\eta_\varepsilon^n - \mathrm{Id}\,\|_{\rho_n}^2 + c\delta_n^{-1} \|\eta_\varepsilon^n - \mathrm{Id}\,\|_{\rho_n} e^{-L_n \frac{\delta_n}{\rho_0}},$$

as well as estimates for ϕ_ε^n. From these estimates, using standard arguments in Nash-Moser theory (presented in Section 3.4), we will conclude that the limit $\lim_{n\to\infty} \phi_\varepsilon^1 \cdot \phi_\varepsilon^2 \cdots \phi^n = \phi_\varepsilon^\infty$ exists in a domain, and that

$$(\phi_\varepsilon^\infty \circ f)^{-1} \cdot \eta_\varepsilon \cdot \phi_\varepsilon^\infty = \mathrm{Id}\,.$$

3.2. **Formal construction of the iterative step.** In this section, we will describe the formal procedure leading to an improved solution; the estimates are provided in the next section. Given an analytic function $\eta_\varepsilon(\theta, I)$ and $L \in \mathbb{N}$, we use notation (1.5) to write

$$\eta_\varepsilon = \mathrm{Id} + \eta_\varepsilon^{[L,2L-1]} + O(\varepsilon^{2L}), \tag{3.2}$$

so that $\eta_\varepsilon^{[L,2L-1]}$ is a polynomial of degre $2L-1$ in ε, but its coefficients of order smaller than L vanish.

Proposition 21. *Suppose that η_ε has the form* (3.2) *and satisfies the POC condition* (1.2)*. Then the truncated function, $\eta_\varepsilon^{[L,2L-1]}$, satisfies the CPOC condition* (2.4)*.*

Proof. Given η_ε as above and a periodic orbit $p = f^N(p)$, note that

$$\eta_\varepsilon(f^{N-1}(p)) \cdots \eta_\varepsilon(p) = \mathrm{Id} + \sum_{j=0}^{N-1} \eta_\varepsilon^{[L,2L-1]}(f^j(p)) + O(\varepsilon^{2L}). \tag{3.3}$$

By POC condition, the left hand side equals Id for all ε. Note that $\eta_\varepsilon^{[L,2L-1]}$ is a polynomial in ε (of degree $2L-1$). Therefore, the sum in the right hand side equals zero, which is precisely the CPOC condition for $\eta_\varepsilon^{[L,2L-1]}$. □

Remark 22. Notice that Proposition 21 depends crucially on the fact that we are using families of maps, and assume the Periodic Orbit Condition for all values in ε. In fact, this is the main reason for the use of families in this paper. Note also that the approximation given in (3.3) is very non-uniform in N.

By Proposition 15, condition CPOC for $\eta_\varepsilon^{[L,2L-1]}$ implies condition FC for $\eta_\varepsilon^{[L,2L-1]}$.

Under the FC condition for $\eta_\varepsilon^{[L,2L-1]}$, Lemma 19 gives us a $\phi^{[L,2L-1]}$ solving

$$\phi_\varepsilon^{[L,2L-1]} \circ f - \phi_\varepsilon^{[L,2L-1]} = \eta_\varepsilon^{[L,2L-1]}. \tag{3.4}$$

It is easy to see that, defining $\phi_\varepsilon = \mathrm{Id} + \phi_\varepsilon^{[L,2L-1]}$, we have $\phi_\varepsilon^{-1} = \mathrm{Id} - \phi_\varepsilon^{[L,2L-1]} + O(\varepsilon^{2L})$. For $\eta_\varepsilon = \mathrm{Id} + \eta_\varepsilon^{[\geq L]}$ we have:

$$\begin{aligned}
\tilde{\eta}_\varepsilon - \mathrm{Id} :=& \phi_\varepsilon^{-1} \circ f \cdot \eta_\varepsilon \cdot \phi_\varepsilon - \mathrm{Id} = \\
=& (\mathrm{Id} + \phi_\varepsilon^{[L,2L-1]} \circ f)^{-1} \cdot (\mathrm{Id} + \eta_\varepsilon^{[\geq L]}) \cdot (\mathrm{Id} + \phi_\varepsilon^{[L,2L-1]}) - \mathrm{Id} \\
=& A + (\mathrm{Id} - \phi_\varepsilon^{[L,2L-1]} \circ f) \cdot (\mathrm{Id} + \eta_\varepsilon^{[\geq L]}) \cdot (\mathrm{Id} + \phi_\varepsilon^{[L,2L-1]}) - \mathrm{Id} \\
=& A + B + \eta_\varepsilon^{[\geq L]} + \phi_\varepsilon^{[L,2L-1]} - \phi_\varepsilon^{[L,2L-1]} \circ f \\
=& A + B + C + \eta_\varepsilon^{[L,2L-1]} + \phi_\varepsilon^{[L,2L-1]} - \phi_\varepsilon^{[L,2L-1]} \circ f \\
=& A + B + C,
\end{aligned} \tag{3.5}$$

where

$\tilde{A} = (\mathrm{Id} + \phi_\varepsilon^{[L,2L-1]} \circ f)^{-1} - (\mathrm{Id} - \phi_\varepsilon^{[L,2L-1]} \circ f)$,

$A = \tilde{A} \cdot \eta_\varepsilon \cdot (\mathrm{Id} + \phi_\varepsilon^{[L,2L-1]})$,

$B = \eta_\varepsilon^{[\geq L]} \cdot \phi_\varepsilon^{[L,2L-1]} - \phi_\varepsilon^{[L,2L-1]} \circ f \cdot (\eta_\varepsilon^{[\geq L]} + \phi_\varepsilon^{[L,2L-1]} + \eta_\varepsilon^{[\geq L]} \cdot \phi_\varepsilon^{[L,2L-1]})$,

$C = \eta_\varepsilon^{[\geq 2L]}$,

and the last equality in (3.5) holds since $\phi_\varepsilon^{[L,2L-1]}$ is the solution of equation (3.4). □

Remark 23. Note that expression above readily implies that

$$\tilde{\eta}_\varepsilon = \phi_\varepsilon^{-1} \circ f \cdot \eta_\varepsilon \cdot \phi_\varepsilon = \mathrm{Id} + O(\varepsilon^{2L}).$$

Hence, the formal argument developed here shows that the POC condition (i.e., condition (A) in Theorem 3) implies the existence of a formal power series solution (B) in Theorem 3).

To prove that the resulting limit function ϕ_ε^∞ is a convergent power series, we need to develop detailed estimates that establish the convergence of the iterative procedure. The key step is to show that the error in the procedure decreases faster than exponentially. This will show that the corrections applied to ϕ are summable, so that there is a limit.

The estimates are somewhat delicate because step (3.4) involves a loss of domain with singular estimates. So, to control the convergence we will use the arguments from KAM theory.

3.3. **Estimates on the iterative step.** In this section we repeat the iterative step described in the previous section, while keeping track of the sizes of all the objects involved. The main technical tool is Lemma 19. We prove the following.

Proposition 24. *Given numbers $0 < \rho \leq \rho_0$, $L \in \mathbb{N}$ and $\sigma > 0$, let $\eta_\varepsilon \in \mathcal{A}_\rho^\varepsilon$ be an analytic family taking values in $\mathcal{G}$, such that $\eta_0 = \mathrm{Id}$, and for each $\varepsilon \in \mathcal{E}_\rho$ we have:*

- $\eta_\varepsilon = \mathrm{Id} + \eta_\varepsilon^{[L,2L-1]} + \eta_\varepsilon^{[\geq 2L]}$,
- *η_ε satisfies the POC condition* (1.2),
- $\|\eta_\varepsilon - \mathrm{Id}\|_\rho \leq \sigma$.

Then we can find $\phi_\varepsilon = \mathrm{Id} + \phi_\varepsilon^{[L,2L-1]}$, for $s = d+1$ and a certain constant $c = c(d)$ satisfying estimates

$$\|\phi_\varepsilon^{[L,2L-1]}\|_{\rho-\delta} \leq c\delta^{-s}\sigma, \quad \|\phi_\varepsilon^{[L,2L-1]} \circ f\|_{\rho-\delta} \leq c\delta^{-s}\sigma,$$

such that the following holds. Define

$$\tilde{\eta}_\varepsilon := \phi_\varepsilon^{-1} \circ f \cdot \eta_\varepsilon \cdot \phi_\varepsilon.$$

Then

- $\tilde{\eta}_\varepsilon = \mathrm{Id} + \tilde{\eta}_\varepsilon^{[\geq 2L]}$,
- $\tilde{\eta}_\varepsilon$ *satisfies the POC condition* (1.2).
- *Assuming furthermore that, for a certain constant* $c' = c'(d)$, δ *satisfies*

$$c'\delta^{-s}\sigma \leq 1/100, \tag{3.6}$$

we have that $\tilde{\eta}_\varepsilon$ *satisfies:*

$$\|\tilde{\eta}_\varepsilon - \mathrm{Id}\,\|_{\rho-\delta} \leq c\delta^{-2s}\sigma^2 + c\delta^{-1}\sigma \exp\left(-\delta L/\rho_0\right). \tag{3.7}$$

We note that the application of the iterative step has two formal conditions (the order of tangency to the identity, and the Periodic Orbit Condition). There is also a quantitative condition (3.6), telling us that the domain loss cannot be too small with respect to the error.

To show that the iterative step can indeed be repeated, we will need to recover the conditions. The formal conditions are recovered automatically, but (3.6) and (3.7) will require to specify the sequence of domain losses and show that, under appropriate assumptions, the error decreases fast enough so that (3.6) is maintained through the iteration. This is very standard in KAM theory.

Proof. Consider $\eta_\varepsilon = \mathrm{Id} + \sum_{j\geq L} \eta_\varepsilon^j \varepsilon^j$, and define ϕ_ε ans $\tilde{\eta}_\varepsilon$ as in Section 3.2. By definition, since η_ε satisfies the POC condition, $\tilde{\eta}_\varepsilon$ does so as well. The fact that $\tilde{\eta}_\varepsilon = \mathrm{Id} + \tilde{\eta}_\varepsilon^{[\geq 2L]}$ was proved in Remark 23. To prove estimate (3.7), we will use formula (3.5) and estimate the terms A, B and C in it.

- By the Cauchy estimates recalled in Lemma 18, there exists a constant $c_1 = c_1(d, \rho_0)$, such that in formula (3.5) we have:

$$\|C\|_{\rho-\delta} = \|\eta_\varepsilon^{[\geq 2L]}\|_{\rho-\delta} \leq c_1\delta^{-1}\sigma \exp\left(-\delta L/\rho_0\right).$$

  In the same way,

$$\|\eta_\varepsilon^{[L,2L-1]}\|_{\rho-\delta} \leq c_1\delta^{-1}\sigma.$$

- Since η_ε satisfies the POC condition, Proposition 21 implies that $\eta_\varepsilon^{[L,2L-1]}$ satisfies the CPOC condition. By Proposition 15, condition CPOC implies condition FC for $\eta_\varepsilon^{[L,2L-1]}$. Under the FC condition for $\eta_\varepsilon^{[L,2L-1]}$, Lemma 19 permits us to estimate:

$$\|\phi_\varepsilon^{[L,2L-1]}\|_{\rho-\delta} \leq c_2\delta^{-s}\|\eta_\varepsilon^{[L,2L-1]}\|_{\rho-\delta} \leq c_2\delta^{-s}\sigma,$$

  where c_2 only depends on the dimension.

- By (3.4), we can express $\phi_\varepsilon^{[L,2L-1]} \circ f = \eta_\varepsilon^{[L,2L-1]} + \phi_\varepsilon^{[L,2L-1]}$. This permits us to estimate:
$$\begin{aligned}\|\phi_\varepsilon^{[L,2L-1]} \circ f\|_{\rho-\delta} \leq & \|\eta_\varepsilon^{[L,2L-1]}\|_{\rho-\delta} + \|\phi_\varepsilon^{[L,2L-1]}\|_{\rho-\delta} \\ \leq & c_1\delta^{-1}\sigma + c_2\delta^{-s}\sigma \leq c_3\delta^{-s}\sigma.\end{aligned}$$
Note that c_3 only depends on the dimension.
- In the formulation of the proposition choose $c' = c_3$. Assuming $c_3\delta^{-s}\sigma \leq 1/100$, we can use Taylor's formula for $(1+x)^{-1} = 1 - x + O(x^2)$ applied to a matrix argument. Together with the boundedness of $\|\operatorname{Id} + \phi_\varepsilon^{[L,2L-1]}\|_{\rho-\delta}$ and $\|\eta_\varepsilon\|_{\rho-\delta}$, this implies:
$$\begin{aligned}\|A\|_{\rho-\delta} \leq & c_4 \| \left(\operatorname{Id} + \phi_\varepsilon^{[L,2L-1]} \circ f\right)^{-1} - \left(\operatorname{Id} - \phi_\varepsilon^{[L,2L-1]} \circ f\right) \|_{\rho-\delta} \\ \leq & c_5 \|\phi_\varepsilon^{[L,2L-1]} \circ f\|_{\rho-\delta}^2 \leq c_6\delta^{-2s}\sigma^2.\end{aligned}$$
- Using the estimates for ϕ_ε and η_ε above, we get
$$\|B\|_{\rho-\delta} \leq c_6\delta^{-2s}\sigma^2.$$
- Now the upper bound of $\tilde{\eta}_\varepsilon - \operatorname{Id}$ in the formula (3.5) is just a sum of the upper bounds above. We define c to be the maximum of the relevant constants $c_1, \dots c_6$.

□

3.4. **Convergence of the iterative procedure.** In this section, we show that, starting from a sufficiently small error, we can repeat on applying the iterative step and the accumulated transformation converges. The following calculation will be used in the construction.

Lemma 25. *Let $a \geq 0$, $b > 1$ and $c > 0$ be fixed. Given γ_0, define a sequence of numbers $(\gamma_n)_n$, $n \in \mathbb{N}$, by a recursive formula*
$$\gamma_{n+1} = c\gamma_n^2 \exp(an) + c\gamma_n \exp(-b^n).$$
For any choice of (λ, p) such that $\lambda > 0$ and $p \in (1, b]$, $p < 2$, one can find $\Gamma_0 = \Gamma_0(\lambda, p, a, b, c)$ so that for any $\gamma_0 \leq \Gamma_0$ the corresponding sequence satisfies:
$$\gamma_n \leq \lambda \exp(-p^n - an) \quad \textit{for all } n \in \mathbb{N}. \tag{3.8}$$

Proof: Fix (λ, p) as above. Since $p < 2$, there exists an N_0 such that for all $n \geq N_0$ we have:
$$c(\lambda + 1)\exp(-2p^n) < \exp(-p^{n+1} - a).$$
Pick $\Gamma_0 = \Gamma_0(\lambda, p, a, b, c)$ so that (3.8) holds for all $n \leq N_0$ and for all $\gamma_0 \leq \Gamma_0$. This is possible since γ_0 is a multiple in the recurrence relation.

The statement is proved by induction in n with $n = N_0$ being the base. Suppose that (3.8) holds for some $n \geq N_0$. Then

$$\begin{aligned}\gamma_{n+1} &= c\gamma_n^2 \exp(an) + c\gamma_n \exp(-b^n)\\ &\leq c\lambda^2 \exp(-2p^n - 2an) + c\lambda \exp(-p^n - b^n - an)\\ &\leq c(\lambda^2 + \lambda)\exp(-2p^n - an)\\ &\leq \lambda \exp(-p^{n+1} - a(n+1)).\end{aligned}$$

Hence, (3.8) holds for $n+1$, and thus for all $n \in \mathbb{N}$.

□

Remark 26. We start with $\rho = \rho_0$, $\eta_\varepsilon \in \mathcal{A}_\rho^\varepsilon$ and $\sigma > 0$ such that

$$\|\eta_\varepsilon - \mathrm{Id}\,\|_\rho = \sigma.$$

To start the recursion, we need σ to be small. This is reached by rescaling the parameter. Namely, for a real $\lambda > 0$, consider the family $\hat{\eta}_\varepsilon = \eta_{\lambda\varepsilon}$. The analyticity domain of the rescaled family $\hat{\eta}_\varepsilon$ in (θ, I) is the same as that of η_ε; also, $\hat{\eta}_\varepsilon$ satisfies the Periodic Orbit Condition if and only if η_ε does.

At the same time, since $\eta_0 = \hat{\eta}_0 = \mathrm{Id}$, we have

$$\|\hat{\eta} - \mathrm{Id}\,\|_\rho \leq C\lambda^{-1}.$$

Hence, by choosing λ large enough, we can make $\|\hat{\eta} - \mathrm{Id}\,\|_\rho$ as small as desired.

Below we prove that, if the initial error is small enough, the iterative method converges, and thus $\hat{\eta}_\varepsilon$ is a coboundary with an analytic conjugacy in a domain defined by $\rho' < \rho$. If $\hat{\eta}_\varepsilon$ satisfies (1.1) with a conjugacy $\hat{\phi}_\varepsilon$, we see that η_ε is a coboundary with a conjugacy $\phi_\varepsilon = \hat{\phi}_{\lambda^{-1}\varepsilon}$. This is enough to prove the conclusions of Theorem 3.

Note that the domain of convergence of ϕ_ε in the ε variable can be significantly smaller than that for η_ε.

Basic notations and assumptions

- Let c be the constant from Proposition 24, and let $s = d + 1$.
- Choose $\delta_0 \leq \rho_0/8$, and define $\delta_n = \delta_0(3/4)^n$,

$$\rho_n = \rho_0 - \sum_{j=1}^{n-1} \delta_{j-1}, \quad \rho_\infty = \rho_0 - \sum_{j=1}^{\infty} \delta_{j-1}.$$

  Clearly,

$$\rho_\infty \geq \rho_0/2.$$

- Define L_0 so that $(\delta_0 L_0/\rho_0) \geq 1$, and let

$$L_n = L_0 2^n.$$

Note that $(\delta_n L_n/\rho_0) = (\delta_0 L_0/\rho_0)(3/2)^n \geq (3/2)^n$.

- In Lemma 25, let a be such that
$$\exp(na) = \delta_n^{-2s}$$
(in this case a is close to $2s \ln 4/3$), take $p = b = 3/2$, c from Proposition 24 and $\lambda = 1/100$. Let $\Gamma_0(\lambda, p, a, b, c)$ be the constant given by Lemma 25. Fix $\gamma_0 = \Gamma_0(\lambda, p, a, b, c)$, and let (γ_n) be the corresponding sequence from Lemma 25. Then (3.8) reads as
$$\gamma_n \leq \frac{1}{100}\delta_n^{2s} \exp(-(3/2)^n).$$

Proof of the theorem.

Let us start the iterative procedure. We will drop the subscript ε in the notation of η_ε and ϕ_ε for notational simplicity. We add a subindex n to indicate the estimates on the n-th iterative step. In particular, the starting function is η_0 satisfying POC condition, and
$$\|\eta_0 - \mathrm{Id}\,\|_{\rho_0} = \sigma_0.$$
By Remark 26 we can assume that $\sigma_0 \leq \gamma_0$ defined above.

At the n-th step, given η_n satisfying POC, we construct ϕ_n by Proposition 24. Then $\eta_{n+1} := \phi_n \circ f \cdot \eta_n \cdot \phi_n$ satisfies POC condition.

Suppose that at each step estimate (3.6) is satisfied. Note that estimate (3.7) coinsides with the iterative relation defining the sequence (γ_n). This implies that for all n we have:
$$\sigma_n := \|\eta_n - \mathrm{Id}\,\|_{\rho_n} \leq \gamma_n.$$
For our choice of σ_0, Lemma 25 guarantees that
$$\sigma_n \leq \gamma_n \leq \frac{1}{100}\delta_n^{2s} \exp(-(3/2)^n).$$
This implies, in particular, that $c\sigma_n\delta_n^{-s} \leq \frac{1}{100}$ for each n, i.e., (3.6) holds, and the inductive step can indeed be repeated.

Moreover, the same estimate, combined with Proposition 24, implies that, for an appropriate constant c_1, we have:
$$\|\phi_n\|_{\rho_n - \delta_n} \leq c\delta_n^{-s}\sigma_n < \frac{1}{100}\delta_n^s < c_1\left(\frac{3}{4}\right)^{ns}.$$

Let
$$\Phi_n = \phi_0 \cdot \phi_1 \ldots \phi_n,$$
in which case
$$\Phi_n^{-1} \circ f = \phi_n^{-1} \circ f \ldots \phi_1^{-1} \circ f \cdot \phi_0^{-1} \circ f,$$
and let
$$\eta_n = \Phi_n^{-1} \circ f \cdot \eta \cdot \Phi_n.$$

Using formulas

$$\|\phi_n - \mathrm{Id}\,\|_{\rho_n-\delta_n} = \|\phi^{[L_n,2L_n-1]}\|_{\rho_n-\delta_n} \le c_1 \left(\frac{3}{4}\right)^{ns}$$

and

$$\|(\phi_n)^{-1}\circ f - \mathrm{Id}\,\|_{\rho_n-\delta_n} \le c_2 \left(\frac{3}{4}\right)^{ns},$$

we estimate:

$$\|\Phi_n\|_{\rho_0/2} \le \prod_{j=1}^{n} \|\phi_j\|_{\rho_0/2} \le \exp(\sum_{j=1}^{n} \|\phi_j - \mathrm{Id}\,\|_{\rho_0/2}) \le \exp(\sum_{j=1}^{n} \left(\frac{3}{4}\right)^{ns}).$$

The latter is bounded uniformly in n. By a similar argument, $\|\Phi_n^{-1}\circ f\|_{\rho_0/2}$ is bounded uniformly in n. We conclude that: $\Phi_n \in \mathcal{A}^{\varepsilon}_{\frac{1}{2}\rho_0}$, $\Phi_n^{-1}\circ f \in \mathcal{A}^{\varepsilon}_{\frac{1}{2}\rho_0}$, and the following limits exist:

$$\phi^\infty := \lim_{n\to\infty} \Phi_n \in \mathcal{A}^{\varepsilon}_{\frac{1}{2}\rho_0}, \quad (\phi^\infty)^{-1}\circ f = \lim_{n\to\infty} (\Phi_n)^{-1}\circ f \in \mathcal{A}^{\varepsilon}_{\frac{1}{2}\rho_0}.$$

Moreover,

$$\|(\Phi_n\circ f)^{-1}\cdot\eta\cdot\Phi_n - \mathrm{Id}\,\|_{\rho/2} \le \sigma_n,$$

which goes to zero when $n\to\infty$. Hence,

$$(\phi^\infty\circ f)^{-1}\cdot\eta\cdot\phi^\infty = \mathrm{Id}\,.$$

4. Adapting the proof to the case where $\mathcal{G}$ is a Lie group

In this section we explain (mostly notational) changes that permit us to adapt the above result to the case when $\mathcal{G}$ is a Lie group (rather than an algebra of operators). An important example in applications is $\mathcal{G} = Sp(n,\mathbb{C})$.

Since our results involve assumptions that the group elements are close to identity, it is natural to use the exponential mapping which provides an analytic diffeomorphism from a neighbohood of zero in the Lie algebra to a nieghborhood of the identity in the group. We write

$$\begin{aligned} \eta_\varepsilon &= \exp(\tilde\eta_\varepsilon) \equiv \exp\left(\sum_{j=1}^{\infty} \tilde\eta_j \varepsilon^n\right), \\ \phi_\varepsilon &= \exp(\tilde\phi_\varepsilon) \equiv \exp\left(\sum_{j=1}^{\infty} \tilde\phi_j\right)\varepsilon^n. \end{aligned} \tag{4.1}$$

We note that since exp is an analytic diffeomorphism, the family η_ε is analytic if and only if $\tilde\eta_\varepsilon$ is convergent. Similarly, $\tilde\phi_\varepsilon$ is a formal power series if and only if ϕ_ε is.

We will refer to the elements of the Lie algebra corresponding to the elements in the Lie group as the logarithms, and denote them by $\tilde{}$.

As it is well known, the group multiplication in the Lie group becomes (approximately) the sum in the Lie algebra. In our case the error is quadratic:

$$\exp(a) \cdot \exp(b) = \exp(a+b) + O((|a|+|b|)^2).$$

Indeed, there are asymptotic formulas for the error in terms of commutators (Campbell-Hausdorff formulas).

The key observation for our problem is that if $\tilde{\eta}_\varepsilon = O(\varepsilon^N)$ and $\tilde{\gamma}_\varepsilon = O(\varepsilon^N)$, we have:

$$\eta_\varepsilon \gamma_\varepsilon = \exp(\tilde{\eta}_\varepsilon + \tilde{\gamma}_\varepsilon + E), \tag{4.2}$$

where $E = O(\varepsilon^{2N})$ and its analytic norm can be bounded by the sums of the squares of the norms of η_ε and γ_ε.

With these observations, the proof of an analogue of Theorem 3 goes through with only minimal changes. We inductively consider $\tilde{\eta}_\varepsilon$ vanishing to order L in ε, such that η_ε satisfies the Periodic Orbit Condition. Matching powers in the product, we again obtain that $\tilde{\eta}_\varepsilon^{[L,2L-1]}$ satisfies the Commutative Periodic Orbit Condition. Hence, we can determine $\tilde{\phi}_\varepsilon^{[L,2L-1]}$ solving the commutative cohomology equation.

We see, proceeding as in the proof of Theorem 3, that the new η_ε is quadratically small with respect to the one at the previous step. The terms that appear in the estimates are the same as those considered there, supplemented by a few terms coming from the E in (4.2). Since the terms coming from (4.2) satisfy uniform quadratic estimates, they do not affect the estimates of the inductive step, and the convergence of the Nash-Moser method remains exactly the same.

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School of Mathematics, Georgia Institute of Technology, 686 Cherry St. Atlanta GA. 30332-1160, USA

Email address: `rafael.delallave@math.gatech.edu`

Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden

Email address: `masha@kth.se`