SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON-FREDHOLM OPERATORS RELATED TO THE DOUBLE SCALE ANOMALOUS DIFFUSION

Vitali Vougalter

Department of Mathematics, University of Toronto
Toronto, Ontario, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

Abstract: We address the solvability of certain linear nonhomogeneous elliptic equations and establish that under reasonable technical assumptions the convergence in $L^2(\mathbb{R}^d)$ of their right sides yields the existence and the convergence in $H^{2s}(\mathbb{R}^d)$ of the solutions. In the first part of the article the problem contains the sum of the two negative Laplacians raised to two distinct fractional powers. In the second part we generalize the results obtained by incorporating a shallow, short-range potential into the equation and we use the methods of the spectral and scattering theory for the non-Fredholm Schrödinger type operators.

AMS Subject Classification: 35J05, 35J10, 35P10, 47F05
Key words: solvability conditions, non-Fredholm operators, anomalous diffusion

1. Introduction

Consider the equation

$$(-\Delta + V(x))u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and $V(x)$ is a function decaying to 0 at infinity. If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$ corresponding to the left side of equation (1.1) contains the origin. Consequently, such operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article is devoted to the studies of certain properties of the sums of the operators of this kind raised to fractional powers. Let us recall that elliptic equations with non-Fredholm operators were treated extensively in recent years (see [9], [21], [24], [25], [22], [26], [27], [28], [32], also [5]) along with their potential applications to the theory of reaction-diffusion equations (see [7], [8]). In the particular situation when $a = 0$ the operator A satisfies the Fredholm property in certain properly chosen weighted spaces (see [1], [2], [3], [4], [5]). However, the case with $a \neq 0$ is considerably different and the method developed in these works cannot be applied.

One of the important questions concerning the equations with non-Fredholm operators is their solvability. Let us address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, so that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. We denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Since the operator A fails to satisfy the Fredholm property, the sequence u_n may not be convergent. Let us call a sequence u_n so that $Au_n \to f$ a solution in the sense of sequences of problem $Au = f$ (see [20]). If such sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this equation. The solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of the non-Fredholm operators, this convergence may not hold or it can occur in some weaker sense. In such case, the solution in the sense of sequences may not imply the existence of the usual solution. In the present work we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on the sequences f_n under which the corresponding sequences u_n are strongly convergent. The solvability in the sense of sequences for the equations involving the second order differential non-Fredholm operators raised to fractional powers was studied in [32]. The present article our modest attempt to generalize such results by dealing with the solvability of the generalized Poisson type equations containing in their left sides the sums of such second order differential operators without Fredholm property raised to the two distinct fractional powers, which is relevant to the understanding of the *double scale anomalous diffusion* (see e.g. [11]). Note that a fractional power of the negative Laplacian or a Schrödinger type operator be defined via the spectral calculus.

Let us first consider the problem

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]u = f(x), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}, \quad 0 < s_1 < s_2 < 1 \qquad (1.2)$$

with a square integrable right side. The operator $(-\Delta)^s$ is actively used, for example in the studies of the anomalous diffusion problems (see e.g. [29], [30], [32] and the references therein). The probabilistic realization of the anomalous diffusion was discussed in [17]. The equation analogous to (1.2) but with the single standard Laplace operator in the context of the solvability in the sense of sequences was studied in [23]. The situation when the power of the single negative Laplacian $s = \frac{1}{2}$ was considered recently in [31]. The article [15] is devoted to the establishing of the imbedding theorems and the studies of the spectrum of a certain pseudodifferential operator. The form boundedness criterion for the relativistic Schrödinger operator was proved in [16]. Clearly, for the operator in the left side of our equation

(1.2)

$$l := (-\Delta)^{s_1} + (-\Delta)^{s_2} : H^{2s_2}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

the essential spectrum fills the semi-axis $[0, \infty)$, so that its inverse from $L^2(\mathbb{R}^d)$ to $H^{2s_2}(\mathbb{R}^d)$ is not bounded.

We write down the corresponding sequence of the approximate equations with $n \in \mathbb{N}$ as

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]u_n = f_n(x), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}, \quad 0 < s_1 < s_2 < 1 \tag{1.3}$$

with the right sides converging to the right side of (1.2) in $L^2(\mathbb{R}^d)$ as $n \to \infty$. The inner product of two functions

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \tag{1.4}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x) \in L^\infty(\mathbb{R}^d)$, then obviously the integral in the right side of (1.4) makes sense, like for example in the cases of the functions involved in the orthogonality conditions of our Theorems below. We use the space $H^{2s}(\mathbb{R}^d)$ equipped with the norm

$$\|u\|^2_{H^{2s}(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^s u\|^2_{L^2(\mathbb{R}^d)}, \quad 0 < s < 1. \tag{1.5}$$

First of all, we formulate the solvability conditions for equation (1.2).

Theorem 1.1. *Let* $f(x) : \mathbb{R}^d \to \mathbb{R}, \ d \in \mathbb{N}, \ f(x) \in L^2(\mathbb{R}^d)$ *and* $0 < s_1 < s_2 < 1.$
a) Let $d = 1$. *If* $s_1 \in \left(0, \frac{1}{4}\right)$ *and in addition* $f(x) \in L^1(\mathbb{R})$*, then problem (1.2) admits a unique solution* $u(x) \in H^{2s_2}(\mathbb{R})$.
Suppose that $s_1 \in \left[\frac{1}{4}, \frac{3}{4}\right)$ *and in addition* $xf(x) \in L^1(\mathbb{R})$. *Then equation (1.2) possesses a unique solution* $u(x) \in H^{2s_2}(\mathbb{R})$ *if and only if the orthogonality condition*

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{1.6}$$

is valid.
Suppose that $s_1 \in \left[\frac{3}{4}, 1\right)$ *and additionally* $x^2 f(x) \in L^1(\mathbb{R})$. *Then problem (1.2) admits a unique solution* $u(x) \in H^{2s_2}(\mathbb{R})$ *if and only if orthogonality relations (1.6) and*

$$(f(x), x)_{L^2(\mathbb{R})} = 0 \tag{1.7}$$

hold.
b) Let $d = 2$. *Then when* $s_1 \in \left(0, \frac{1}{2}\right)$ *and additionally* $f(x) \in L^1(\mathbb{R}^2)$*, problem (1.2) has a unique solution* $u(x) \in H^{2s_2}(\mathbb{R}^2)$.

Suppose that $s_1 \in \left[\frac{1}{2}, 1\right)$ *and in addition* $xf(x) \in L^1(\mathbb{R}^2)$. *Then problem (1.2) admits a unique solution* $u(x) \in H^{2s_2}(\mathbb{R}^2)$ *if and only if the orthogonality condition*

$$(f(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{1.8}$$

is valid.
c) Let $d = 3$. *If* $s_1 \in \left(0, \frac{3}{4}\right)$ *and in addition* $f(x) \in L^1(\mathbb{R}^3)$, *then equation (1.2) possesses a unique solution* $u(x) \in H^{2s_2}(\mathbb{R}^3)$.
Suppose that $s_1 \in \left[\frac{3}{4}, 1\right)$ *and additionally* $xf(x) \in L^1(\mathbb{R}^3)$. *Then problem (1.2) possesses a unique solution* $u(x) \in H^{2s_2}(\mathbb{R}^3)$ *if and only if the orthogonality relation*

$$(f(x), 1)_{L^2(\mathbb{R}^3)} = 0 \tag{1.9}$$

is valid.
d) If $d \geq 4$ *with* $s_1 \in (0, 1)$ *and in addition* $f(x) \in L^1(\mathbb{R}^d)$, *then equation (1.2) admits a unique solution* $u(x) \in H^{2s_2}(\mathbb{R}^d)$.

Let us turn our attention to the issue of the solvability in the sense of sequences for our equation.

Theorem 1.2. *Let* $n \in \mathbb{N}$ *and* $f_n(x) : \mathbb{R}^d \to \mathbb{R}, \ f_n(x) \in L^2(\mathbb{R}^d), \ d \in \mathbb{N}$, *so that* $f_n(x) \to f(x)$ *in* $L^2(\mathbb{R}^d)$ *as* $n \to \infty$.
a) Let $d = 1$. *If* $s_1 \in \left(0, \frac{1}{4}\right)$ *and in addition* $f_n(x) \in L^1(\mathbb{R}), \ n \in \mathbb{N}$, *so that* $f_n(x) \to f(x)$ *in* $L^1(\mathbb{R})$ *as* $n \to \infty$, *then problems (1.2) and (1.3) have unique solutions* $u(x) \in H^{2s_2}(\mathbb{R})$ *and* $u_n(x) \in H^{2s_2}(\mathbb{R})$ *respectively, so that* $u_n(x) \to u(x)$ *in* $H^{2s_2}(\mathbb{R})$ *as* $n \to \infty$.
Suppose that $s_1 \in \left[\frac{1}{4}, \frac{3}{4}\right)$. *Let in addition* $xf_n(x) \in L^1(\mathbb{R}), \ n \in \mathbb{N}$, *so that* $xf_n(x) \to xf(x)$ *in* $L^1(\mathbb{R})$ *as* $n \to \infty$ *and the orthogonality relations*

$$(f_n(x), 1)_{L^2(\mathbb{R})} = 0 \tag{1.10}$$

are valid for all $n \in \mathbb{N}$. *Then problems (1.2) and (1.3) have unique solutions* $u(x) \in H^{2s_2}(\mathbb{R})$ *and* $u_n(x) \in H^{2s_2}(\mathbb{R})$ *respectively, so that* $u_n(x) \to u(x)$ *in* $H^{2s_2}(\mathbb{R})$ *as* $n \to \infty$.
Suppose that $s_1 \in \left[\frac{3}{4}, 1\right)$. *Let in addition* $x^2 f_n(x) \in L^1(\mathbb{R}), \ n \in \mathbb{N}$, *so that* $x^2 f_n(x) \to x^2 f(x)$ *in* $L^1(\mathbb{R})$ *as* $n \to \infty$ *and the orthogonality relations*

$$(f_n(x), 1)_{L^2(\mathbb{R})} = 0, \quad (f_n(x), x)_{L^2(\mathbb{R})} = 0 \tag{1.11}$$

are valid for all $n \in \mathbb{N}$. *Then problems (1.2) and (1.3) possess unique solutions* $u(x) \in H^{2s_2}(\mathbb{R})$ *and* $u_n(x) \in H^{2s_2}(\mathbb{R})$ *respectively, so that* $u_n(x) \to u(x)$ *in* $H^{2s_2}(\mathbb{R})$ *as* $n \to \infty$.

b) Let $d = 2$. If $s_1 \in \left(0, \dfrac{1}{2}\right)$ and additionally $f_n(x) \in L^1(\mathbb{R}^2),\ n \in \mathbb{N}$, so that $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^2)$ as $n \to \infty$, then problems (1.2) and (1.3) admit unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^2)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^2)$ respectively, so that $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^2)$ as $n \to \infty$.

Suppose that $s_1 \in \left[\dfrac{1}{2}, 1\right)$. Let in addition $xf_n(x) \in L^1(\mathbb{R}^2),\ n \in \mathbb{N}$, so that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^2)$ as $n \to \infty$ and the orthogonality conditions

$$(f_n(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{1.12}$$

are valid for all $n \in \mathbb{N}$. Then problems (1.2) and (1.3) have unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^2)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^2)$ respectively, so that $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^2)$ as $n \to \infty$.

c) Let $d = 3$. Suppose that $s_1 \in \left(0, \dfrac{3}{4}\right)$ and additionally $f_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, so that $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$. Then equations (1.2) and (1.3) admit unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^3)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^3)$ respectively, so that $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^3)$ as $n \to \infty$.

Suppose that $s_1 \in \left[\dfrac{3}{4}, 1\right)$. Let in addition $xf_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, so that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$ and the orthogonality conditions

$$(f_n(x), 1)_{L^2(\mathbb{R}^3)} = 0 \tag{1.13}$$

are valid for all $n \in \mathbb{N}$. Then equations (1.2) and (1.3) possess unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^3)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^3)$ respectively, so that $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^3)$ as $n \to \infty$.

d) Let $d \geq 4$ with $s_1 \in (0, 1)$ and additionally $f_n(x) \in L^1(\mathbb{R}^d),\ n \in \mathbb{N}$, so that $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Then problems (1.2) and (1.3) admit unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^d)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^d)$ respectively, so that $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^d)$ as $n \to \infty$.

Note that in the theorems above each of the cases $a) - d)$ contains the situation when the orthogonality relations are not required.

Let us use the hat symbol to denote the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{-ipx} dx, \quad p \in \mathbb{R}^d, \quad d \in \mathbb{N}, \tag{1.14}$$

so that the inequality

$$\|\widehat{f}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|f(x)\|_{L^1(\mathbb{R}^d)} \tag{1.15}$$

holds. The second part of the article is devoted to the studies of the equation

$$\Big\{[-\Delta + V(x)]^{s_1} + [-\Delta + V(x)]^{s_2}\Big\} u = f(x), \quad x \in \mathbb{R}^3, \quad 0 < s_1 < s_2 < 1 \tag{1.16}$$

with a square integrable right side. The corresponding sequence of approximate equations for $n \in \mathbb{N}$ will be

$$\Big\{[-\Delta + V(x)]^{s_1} + [-\Delta + V(x)]^{s_2}\Big\}u_n = f_n(x), \quad x \in \mathbb{R}^3 \tag{1.17}$$

with $0 < s_1 < s_2 < 1$. Their square integrable right sides converge to the right side of (1.16) in $L^2(\mathbb{R}^3)$ as $n \to \infty$. We make the following technical assumptions on the scalar potential involved in the problems above. Note that the conditions on $V(x)$, which is shallow and short-range will be analogous to those given in Assumption 1.1 of [25] (see also [24], [26]). The essential spectrum of such a Schrödinger operator $-\Delta + V(x)$ fills the nonnegative semi-axis (see e.g. [12]).

Assumption 1.3. *The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the estimate*

$$|V(x)| \le \frac{C}{1 + |x|^{3.5+\delta}}$$

with some $\delta > 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|^{\frac{1}{9}}_{L^\infty(\mathbb{R}^3)}\|V\|^{\frac{8}{9}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} < 1 \quad and \quad \sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi. \tag{1.18}$$

Here C denotes a finite positive constant and c_{HLS} given on p.98 of [14] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\Big| \le c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

By means of Lemma 2.3 of [25], under Assumption 1.3 above on the potential function, the operator $-\Delta + V(x)$ on $L^2(\mathbb{R}^3)$ is self-adjoint and unitarily equivalent to $-\Delta$ via the wave operators (see [13], [19])

$$\Omega^{\pm} := s - \lim_{t\to\mp\infty}e^{it(-\Delta+V)}e^{it\Delta},$$

where the limit is understood in the strong L^2 sense (see e.g. [18] p.34, [6] p.90). Therefore, the operator

$$L = [-\Delta + V(x)]^{s_1} + [-\Delta + V(x)]^{s_2} \tag{1.19}$$

in the left sides of equations (1.16) and (1.17) considered on $L^2(\mathbb{R}^3)$ defined via the spectral calculus has only the essential spectrum

$$\sigma_{ess}(L) = [0, \infty)$$

and no nontrivial $L^2(\mathbb{R}^3)$ eigenfunctions. By virtue of the spectral theorem, its functions of the continuous spectrum satisfy

$$L\varphi_k(x) = (|k|^{2s_1} + |k|^{2s_2})\varphi_k(x), \quad k \in \mathbb{R}^3, \tag{1.20}$$

in the integral formulation the Lippmann-Schwinger equation for the perturbed plane waves (see e.g. [18] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{1.21}$$

and the orthogonality conditions

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3. \tag{1.22}$$

In particular, when the vector $k = 0$, we have $\varphi_0(x)$. We denote the generalized Fourier transform with respect to these functions using the tilde symbol as

$$\tilde{f}(k) := (f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}, \ k \in \mathbb{R}^3. \tag{1.23}$$

(1.23) is a unitary transform on $L^2(\mathbb{R}^3)$. The integral operator involved in (1.21) is being designated as

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi \in L^\infty(\mathbb{R}^3).$$

We consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. Under Assumption 1.3, via Lemma 2.1 of [25] the operator norm $\|Q\|_\infty$ is bounded above by the expression $I(V)$, which is the left side of the first inequality in (1.18), so that $I(V) < 1$. Corollary 2.2 of [25] under our conditions yields the bound

$$|\tilde{f}(k)| \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f(x)\|_{L^1(\mathbb{R}^3)}. \tag{1.24}$$

Our result concerning the solvability of problem (1.16) is as follows.

Theorem 1.4. *Let Assumption 1.3 hold, the powers* $0 < s_1 < s_2 < 1$ *and* $f(x) \in L^2(\mathbb{R}^3)$.
1) Let $s_1 \in \left(0, \frac{3}{4}\right)$ *and additionally* $f(x) \in L^1(\mathbb{R}^3)$. *Then equation (1.16) has a unique solution* $u(x) \in L^2(\mathbb{R}^3)$.
2) Let $s_1 \in \left[\frac{3}{4}, 1\right)$ *and in addition* $xf(x) \in L^1(\mathbb{R}^3)$. *Then problem (1.16) possesses a unique solution* $u(x) \in L^2(\mathbb{R}^3)$ *if and only if the orthogonality relation*

$$(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0 \tag{1.25}$$

holds.

Our final main result is devoted to the solvability in the sense of sequences of equation (1.16).

Theorem 1.5. *Let Assumption 1.3 hold, $n \in \mathbb{N}$, the powers $0 < s_1 < s_2 < 1$ and $f_n(x) \in L^2(\mathbb{R}^3)$, such that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$.*
1) If $s_1 \in \left(0, \dfrac{3}{4}\right)$ and additionally $f_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, such that $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$, then equations (1.16) and (1.17) admit unique solutions $u(x) \in L^2(\mathbb{R}^3)$ and $u_n(x) \in L^2(\mathbb{R}^3)$ respectively, such that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$.
2) If $s_1 \in \left[\dfrac{3}{4}, 1\right)$ and in addition $xf_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, such that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$ and

$$(f_n(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0 \tag{1.26}$$

holds for all $n \in \mathbb{N}$, then equations (1.16) and (1.17) have unique solutions $u(x) \in L^2(\mathbb{R}^3)$ and $u_n(x) \in L^2(\mathbb{R}^3)$ respectively, such that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$.

Note that (1.25) and (1.26) are the orthogonality conditions to the function of the continuous spectrum of our Schrödinger operator, as distinct from the Limiting Absorption Principle in which one needs to orthogonalize to the standard Fourier harmonics (see e.g. Lemma 2.3 and Proposition 2.4 of [10]).

2. Solvability in the sense of sequences in the no potential case

Proof of Theorem 1.1. Evidently, if $u(x) \in L^2(\mathbb{R}^d)$ is a solution of equation (1.2) with a square integrable right side, it belongs to $H^{2s_2}(\mathbb{R}^d)$ as well. Indeed, if we apply the standard Fourier transform (1.14) to both sides of (1.2), we arrive at

$$(|p|^{2s_1} + |p|^{2s_2})\widehat{u}(p) = \widehat{f}(p) \in L^2(\mathbb{R}^d),$$

such that

$$\int_{\mathbb{R}^d} [|p|^{2s_1} + |p|^{2s_2}]^2 |\widehat{u}(p)|^2 dp < \infty.$$

Using the trivial identity

$$\|(-\Delta)^{s_2} u\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |p|^{4s_2} |\widehat{u}(p)|^2 dp < \infty,$$

we easily deduce that $(-\Delta)^{s_2} u(x) \in L^2(\mathbb{R}^d)$, so that via norm definition (1.5) we obtain that $u(x) \in H^{2s_2}(\mathbb{R}^d)$ as well.

To establish the uniqueness of solutions for our problem, let us suppose that (1.2) admits two solutions $u_1(x),\ u_2(x) \in H^{2s_2}(\mathbb{R}^d)$. Then their difference $w(x) := u_1(x) - u_2(x) \in H^{2s_2}(\mathbb{R}^d)$ as well. Evidently, it satisfies the equation

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]w = 0.$$

Because the operator $l : H^{2s_2}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ does not have any nontrivial zero modes, $w(x)$ vanishes in $\mathbb{R}^d$.

Let us apply the standard Fourier transform (1.14) to both sides of equation (1.2). This yields

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| \leq 1\}} + \frac{\widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| > 1\}}. \tag{2.27}$$

Here and throughout the article χ_A will stand for the characteristic function of a set $A \subseteq \mathbb{R}^d$. Evidently, the second term in the right side of (2.27) can be estimated from above in the absolute value by $\frac{|\widehat{f}(p)|}{2} \in L^2(\mathbb{R}^d)$ via the one of our assumptions.

Let us first consider the case a) of our theorem when the dimension of the problem $d = 1$. We easily obtain the upper bound on the first term in the right side of (2.27) in the absolute value using (1.15) by $\frac{\|f(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|p|^{2s_1}} \chi_{\{|p| \leq 1\}}$. It can be easily verified that this expression belongs to $L^2(\mathbb{R})$ if $s_1 \in \left(0, \frac{1}{4}\right)$.

To treat our problem in the situation when $s_1 \in \left[\frac{1}{4}, \frac{3}{4}\right)$, we use that

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^p \frac{df(q)}{dq} dq.$$

This allows us to express the first term in the right side of (2.27) as

$$\frac{\widehat{f}(0)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| \leq 1\}} + \frac{\int_0^p \frac{d\widehat{f}(q)}{dq} dq}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| \leq 1\}}. \tag{2.28}$$

By virtue of the definition of the standard Fourier transform (1.14), we easily derive that

$$\left| \frac{d\widehat{f}(p)}{dp} \right| \leq \frac{1}{\sqrt{2\pi}} \|xf(x)\|_{L^1(\mathbb{R})} \tag{2.29}$$

and similarly for the space of an arbitrary dimension $d \in \mathbb{N},\ d \geq 2$

$$\left| \frac{\partial \widehat{f}(p)}{\partial |p|} \right| \leq \frac{\|xf(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}}. \tag{2.30}$$

Hence, the second term in (2.28) can be estimated from above in the absolute value by

$$\frac{\|xf(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}} |p|^{1-2s_1} \chi_{\{|p| \leq 1\}} \in L^2(\mathbb{R}).$$

It can be easily checked that the first term in (2.28) is square integrable if and only if $\widehat{f}(0) = 0$. This is equivalent to orthogonality condition (1.6).

Finally, for the dimension of the problem $d = 1$, it remains to investigate the situation when $s_1 \in \left[\frac{3}{4}, 1\right)$. For that purpose, we represent

$$\widehat{f}(p) = \widehat{f}(0) + p\frac{d\widehat{f}}{dp}(0) + \int_0^p \left(\int_0^r \frac{d^2\widehat{f}(q)}{dq^2} dq \right) dr.$$

This enables us to write the first term in the right side of (2.27) as

$$\left[\frac{\widehat{f}(0)}{|p|^{2s_1} + |p|^{2s_2}} + \frac{p\frac{d\widehat{f}}{dp}(0)}{|p|^{2s_1} + |p|^{2s_2}} + \frac{\int_0^p \left(\int_0^r \frac{d^2\widehat{f}(q)}{dq^2} dq \right) dr}{|p|^{2s_1} + |p|^{2s_2}} \right] \chi_{\{|p| \leq 1\}}. \tag{2.31}$$

Definition (1.14) gives us

$$\left| \frac{d^2\widehat{f}(p)}{dp^2} \right| \leq \frac{1}{\sqrt{2\pi}} \|x^2 f(x)\|_{L^1(\mathbb{R})} < \infty$$

as assumed. This allows us to estimate

$$\left| \frac{\int_0^p \left(\int_0^r \frac{d^2\widehat{f}(q)}{dq^2} dq \right) dr}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| \leq 1\}} \right| \leq \frac{1}{2\sqrt{2\pi}} \|x^2 f(x)\|_{L^1(\mathbb{R})} |p|^{2-2s_1} \chi_{\{|p| \leq 1\}} \in L^2(\mathbb{R}).$$

By means of formula (1.14), we have

$$\widehat{f}(0) = \frac{1}{\sqrt{2\pi}} (f(x), 1)_{L^2(\mathbb{R})}, \quad \frac{d\widehat{f}}{dp}(0) = -\frac{i}{\sqrt{2\pi}} (f(x), x)_{L^2(\mathbb{R})},$$

such that the sum of the first two terms in (2.31) can be written as

$$\left[\frac{(f(x), 1)_{L^2(\mathbb{R})}}{\sqrt{2\pi}(|p|^{2s_1} + |p|^{2s_2})} - \frac{ip(f(x), x)_{L^2(\mathbb{R})}}{\sqrt{2\pi}(|p|^{2s_1} + |p|^{2s_2})} \right] \chi_{\{|p| \leq 1\}}. \tag{2.32}$$

It can be easily verified that expression (2.32) belongs to $L^2(\mathbb{R})$ if and only if orthogonality conditions (1.6) and (1.7) hold.

Then we consider the case b) of our theorem when the dimension of the problem $d = 2$. We easily estimate the first term in the right side of (2.27) from above in the absolute value using (1.15) by $\frac{\|f(x)\|_{L^1(\mathbb{R}^2)}}{2\pi|p|^{2s_1}} \chi_{\{|p| \leq 1\}} \in L^2(\mathbb{R}^2)$ for $s_1 \in \left(0, \frac{1}{2}\right)$.

To treat the situation when $s_1 \in \left[\frac{1}{2}, 1\right)$, we use the identity

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^{|p|} \frac{\partial \widehat{f}(q, \sigma)}{\partial q} dq. \tag{2.33}$$

Here and further down σ will stand for the angle variables on the sphere. This allows us to express the first term in the right side of (2.27) as

$$\frac{\widehat{f}(0)}{|p|^{2s_1}+|p|^{2s_2}}\chi_{\{|p|\le 1\}}+\frac{\int_0^{|p|}\frac{\partial \widehat{f}(q,\sigma)}{\partial q}dq}{|p|^{2s_1}+|p|^{2s_2}}\chi_{\{|p|\le 1\}}. \tag{2.34}$$

The second term in (2.34) can be easily bounded from above in the absolutely value using inequality (2.30) by

$$\frac{\|xf(x)\|_{L^1(\mathbb{R}^2)}}{2\pi}|p|^{1-2s_1}\chi_{\{|p|\le 1\}}\in L^2(\mathbb{R}^2).$$

It can be checked that the first term in (2.34) belongs to $L^2(\mathbb{R}^2)$ if and only if $\widehat{f}(0)=0$. This is equivalent to orthogonality condition (1.8).

Let us turn our attention to the case c) of the theorem. We estimate the first term in the right side of (2.27) from above in the absolute value via (1.15) by $\dfrac{\|f(x)\|_{L^1(\mathbb{R}^3)}}{(2\pi)^{\frac{3}{2}}|p|^{2s_1}}\chi_{\{|p|\le 1\}}$. It can be easily verified that this expression is square integrable in $\mathbb{R}^3$ for $s_1\in\left(0,\dfrac{3}{4}\right)$.

If $s_1\in\left[\dfrac{3}{4},1\right)$, we will use the analog of formula (2.33) in the space of three dimensions, such that the first term in the right side of (2.27) is given by the analog of (2.34). By means of (2.30), we derive

$$\left|\frac{\int_0^{|p|}\frac{\partial \widehat{f}(q,\sigma)}{\partial q}dq}{|p|^{2s_1}+|p|^{2s_2}}\chi_{\{|p|\le 1\}}\right|\le\frac{\|xf(x)\|_{L^1(\mathbb{R}^3)}}{(2\pi)^{\frac{3}{2}}}|p|^{1-2s_1}\chi_{\{|p|\le 1\}}\in L^2(\mathbb{R}^3).$$

It turns out that

$$\frac{\widehat{f}(0)}{|p|^{2s_1}+|p|^{2s_2}}\chi_{\{|p|\le 1\}}\in L^2(\mathbb{R}^3)$$

if and only if $\widehat{f}(0)$ vanishes. This is equivalent to orthogonality relation (1.9).

We conclude the proof of the theorem by considering the case d) when the dimension of the problem $d\ge 4$. Let us obtain the upper bound on the first term in the right side of (2.27) in the absolute value using (1.15) by

$$\frac{\|f(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}|p|^{2s_1}}\chi_{\{|p|\le 1\}}\in L^2(\mathbb{R}^d)$$

for $s_1\in(0,1)$. ∎

We proceed to establishing the solvability in the sense of sequences for our problem in the no potential case.

Proof of Theorem 1.2. Suppose $u(x)$ and $u_n(x),\ n \in \mathbb{N}$ are the unique solutions of problems (1.2) and (1.3) in $H^{2s_2}(\mathbb{R}^d),\ d \in \mathbb{N}$ respectively, $0 < s_1 < s_2 < 1$ and it is known that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Then $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^d)$ as $n \to \infty$ as well. Indeed,

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}](u_n(x) - u(x)) = f_n(x) - f(x).$$

Using the standard Fourier transform (1.14), we easily obtain

$$\|(-\Delta)^{s_2}(u_n(x) - u(x))\|_{L^2(\mathbb{R}^d)} \le \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)} \to 0, \quad n \to \infty$$

as assumed. Norm definition (1.5) implies that $u_n(x) \to u(x)$ in $H^{2s_2}(\mathbb{R}^d)$ as $n \to \infty$.

If $u(x)$ and $u_n(x),\ n \in \mathbb{N}$ are the unique solutions of equations (1.2) and (1.3) in $H^{2s_2}(\mathbb{R}^d),\ d \in \mathbb{N}$ respectively, by applying the standard Fourier transform (1.14) we easily obtain

$$\widehat{u}_n(p) - \widehat{u}(p) = \frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| \le 1\}} + \frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| > 1\}}. \tag{2.35}$$

Obviously, the second term in the right side of identity (2.35) can be bounded from above in the absolute value in the space of any dimension by $\dfrac{|\widehat{f}_n(p) - \widehat{f}(p)|}{2}$. Hence

$$\left\| \frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| > 1\}} \right\|_{L^2(\mathbb{R}^d)} \le \frac{\|f_n(x) - f(x)\|_{L^2(\mathbb{R}^d)}}{2} \to 0, \quad n \to \infty$$

via the one of our assumptions.

Let us first consider the case a) of our theorem when the dimension of the problem $d = 1$. Then, if $s_1 \in \left(0, \dfrac{1}{4}\right)$ by means of the part a) of Theorem 1.1, problem (1.2) and each of equations (1.3) have unique solutions $u(x) \in H^{2s_2}(\mathbb{R})$ and $u_n(x) \in H^{2s_2}(\mathbb{R}),\quad n \in \mathbb{N}$ respectively. Evidently, the first term in the right side of (2.35) can be estimated from above in the absolute value using (1.15) by $\dfrac{1}{\sqrt{2\pi}} \|f_n(x) - f(x)\|_{L^1(\mathbb{R})} \dfrac{\chi_{\{|p| \le 1\}}}{|p|^{2s_1}}$, so that its $L^2(\mathbb{R})$ norm can be bounded from above by

$$\frac{1}{\sqrt{\pi}} \|f_n(x) - f(x)\|_{L^1(\mathbb{R})} \frac{1}{\sqrt{1 - 4s_1}} \to 0, \quad n \to \infty$$

as assumed if $s_1 \in \left(0, \dfrac{1}{4}\right)$. Therefore, in this case $u_n(x) \to u(x)$ in $L^2(\mathbb{R})$ as $n \to \infty$.

Then we turn our attention to the situation when $s_1 \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right)$ in dimension $d = 1$. Note that by virtue of the parts a) and b) of Lemma 4.1 of [32], under the given conditions, we have $f_n(x) \in L^1(\mathbb{R}),\ n \in \mathbb{N}$, so that $f_n(x) \to f(x)$ in $L^1(\mathbb{R})$ as $n \to \infty$. Then, by means of (1.10) we derive

$$|(f(x), 1)_{L^2(\mathbb{R})}| = |(f(x) - f_n(x), 1)_{L^2(\mathbb{R})}| \leq \|f_n(x) - f(x)\|_{L^1(\mathbb{R})} \to 0$$

as $n \to \infty$. Hence,

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{2.36}$$

is valid. By virtue of the part a) of Theorem 1.1, when $s_1 \in \Big[\dfrac{1}{4}, \dfrac{3}{4}\Big)$, problems (1.2) and (1.3) have unique solutions $u(x), u_n(x) \in H^{2s_2}(\mathbb{R}),\ n \in \mathbb{N}$ respectively. Orthogonality conditions (2.36) and (1.10) imply that

$$\widehat{f}(0) = 0, \quad \widehat{f}_n(0) = 0, \quad n \in \mathbb{N}$$

in our case. This enables us to express

$$\widehat{f}(p) = \int_0^p \frac{d\widehat{f}(q)}{dq} dq, \quad \widehat{f}_n(p) = \int_0^p \frac{d\widehat{f}_n(q)}{dq} dq, \quad n \in \mathbb{N},$$

which allows us to write the first term in the right side of formula (2.35) as

$$\frac{\int_0^p \left(\frac{d\widehat{f}_n(q)}{dq} - \frac{d\widehat{f}(q)}{dq} \right) dq}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| \leq 1\}}. \tag{2.37}$$

Using (2.29), we obtain the inequality

$$\left| \frac{d\widehat{f}_n(p)}{dp} - \frac{d\widehat{f}(p)}{dp} \right| \leq \frac{1}{\sqrt{2\pi}} \|x f_n(x) - x f(x)\|_{L^1(\mathbb{R})}. \tag{2.38}$$

Then expression (2.37) can be estimated from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}} \|x f_n(x) - x f(x)\|_{L^1(\mathbb{R})} |p|^{1-2s_1} \chi_{\{|p| \leq 1\}}.$$

Thus, we derive

$$\left\| \frac{\int_0^p \left(\frac{d\widehat{f}_n(q)}{dq} - \frac{d\widehat{f}(q)}{dq} \right) dq}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| \leq 1\}} \right\|_{L^2(\mathbb{R})} \leq \frac{1}{\sqrt{\pi(3 - 4s_1)}} \|x f_n(x) - x f(x)\|_{L^1(\mathbb{R})} \to 0$$

as $n \to \infty$ as assumed. Therefore,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}), \quad n \to \infty$$

when the dimension of the problem $d = 1$ and $s_1 \in \Big[\dfrac{1}{4}, \dfrac{3}{4}\Big)$.

Let us proceed to the proof of our theorem when $s_1 \in \left[\frac{3}{4}, 1\right)$ and $d = 1$. By virtue of the parts c) and d) of Lemma 4.1 of [32] under the given conditions we have $xf_n(x) \in L^1(\mathbb{R}),\ n \in \mathbb{N}$, so that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R})$ as $n \to \infty$. Then by means of the parts a) and b) of Lemma 4.1 of [32] we have $f_n(x) \in L^1(\mathbb{R}),\ n \in \mathbb{N}$, so that $f_n(x) \to f(x)$ in $L^1(\mathbb{R})$ as $n \to \infty$. Orthogonality relation (2.36) here can be easily derived using the limiting argument as above. By virtue of the second orthogonality condition in (1.11), we arrive at

$$|(f(x), x)_{L^2(\mathbb{R})}| = |(f(x) - f_n(x), x)_{L^2(\mathbb{R})}| \leq \|xf_n(x) - xf(x)\|_{L^1(\mathbb{R})} \to 0$$

as $n \to \infty$. Thus,

$$(f(x), x)_{L^2(\mathbb{R})} = 0 \tag{2.39}$$

is valid. By means of the part a) of Theorem 1.1, if $s_1 \in \left[\frac{3}{4}, 1\right)$, problems (1.2) and (1.3) admit unique solutions $u(x), u_n(x) \in H^{2s_2}(\mathbb{R}),\ n \in \mathbb{N}$ respectively. Definition of the standard Fourier transform (1.14) along with orthogonality conditions (2.36), (1.11) and (2.39) imply that for $n \in \mathbb{N}$

$$\widehat{f}(0) = 0, \quad \widehat{f}_n(0) = 0, \quad \frac{d\widehat{f}}{dp}(0) = 0, \quad \frac{d\widehat{f}_n}{dp}(0) = 0,$$

so that

$$\widehat{f}(p) = \int_0^p \left(\int_0^r \frac{d^2\widehat{f}(q)}{dq^2}dq\right)dr, \quad \widehat{f}_n(p) = \int_0^p \left(\int_0^r \frac{d^2\widehat{f}_n(q)}{dq^2}dq\right)dr, \quad n \in \mathbb{N}.$$

From definition (1.14) we easily obtain the inequality

$$\left|\frac{d^2\widehat{f}_n(p)}{dp^2} - \frac{d^2\widehat{f}(p)}{dp^2}\right| \leq \frac{1}{\sqrt{2\pi}}\|x^2f_n(x) - x^2f(x)\|_{L^1(\mathbb{R})}.$$

This yields the upper bound

$$|\widehat{f}_n(p) - \widehat{f}(p)| \leq \frac{1}{\sqrt{2\pi}}\|x^2f_n(x) - x^2f(x)\|_{L^1(\mathbb{R})}\frac{p^2}{2},$$

which enables us to derive the estimate from above on the absolute value of the first term in the right side of equality (2.35) by

$$\frac{1}{2\sqrt{2\pi}}\|x^2f_n(x) - x^2f(x)\|_{L^1(\mathbb{R})}|p|^{2-2s_1}\chi_{\{|p|\leq 1\}}.$$

Thus,

$$\left\|\frac{\widehat{f}_n(p) - \widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}}\chi_{\{|p|\leq 1\}}\right\|_{L^2(\mathbb{R})} \leq \frac{1}{2\sqrt{\pi(5 - 4s_1)}}\|x^2f_n(x) - x^2f(x)\|_{L^1(\mathbb{R})} \to 0$$

when $n \to \infty$ as assumed. Therefore,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}), \quad n \to \infty$$

when the dimension $d = 1$ and $s_1 \in \left[\frac{3}{4}, 1\right)$.

In the situation when the dimension $d = 2$, we first treat the case of $s_1 \in \left(0, \frac{1}{2}\right)$. By means of the part b) of Theorem 1.1, equation (1.2) and each of equations (1.3) admit unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^2)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^2),\ n \in \mathbb{N}$ respectively. Evidently, the first term in the right side of (2.35) can be bounded from above in the absolute value via (1.15) by $\frac{1}{2\pi}\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^2)}\frac{\chi_{\{|p|\leq 1\}}}{|p|^{2s_1}}$, so that its $L^2(\mathbb{R}^2)$ norm can be estimated from above by

$$\frac{1}{2\sqrt{\pi(1-2s_1)}}\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad n \to \infty$$

due to the one of our assumptions in the space of two dimensions with $s_1 \in \left(0, \frac{1}{2}\right)$. Hence, in this case

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^2), \quad n \to \infty.$$

For the values of the power of the two dimensional negative Laplacian $s_1 \in \left[\frac{1}{2}, 1\right)$, the orthogonality condition

$$(f(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{2.40}$$

can be obtained via the simple limiting argument, analogously to (2.36). Note that under the given conditions we have $f_n(x) \in L^1(\mathbb{R}^2),\ n \in \mathbb{N}$ and $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^2)$ as $n \to \infty$ by virtue of the parts a) and b) of Lemma 4.1 of [32]. By means of the part b) of Theorem 1.1, equations (1.2) and (1.3) have unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^2)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^2),\ n \in \mathbb{N}$ respectively. Orthogonality conditions (2.40) and (1.12) give us

$$\widehat{f}(0) = 0, \quad \widehat{f}_n(0) = 0, \quad n \in \mathbb{N}$$

in the space of two dimensions with $s_1 \in \left[\frac{1}{2}, 1\right)$. This allows us to express

$$\widehat{f}(p) = \int_0^{|p|}\frac{\partial \widehat{f}(q,\sigma)}{\partial q}dq, \quad \widehat{f}_n(p) = \int_0^{|p|}\frac{\partial \widehat{f}_n(q,\sigma)}{\partial q}dq, \quad n \in \mathbb{N}. \tag{2.41}$$

Let us write the first term in the right side of formula (2.35) as

$$\frac{\int_0^{|p|}\left(\frac{\partial \widehat{f}_n(q,\sigma)}{\partial q} - \frac{\partial \widehat{f}(q,\sigma)}{\partial q}\right)dq}{|p|^{2s_1} + |p|^{2s_2}}\chi_{\{|p|\leq 1\}}. \tag{2.42}$$

Inequality (2.30) yields

$$\left|\frac{\partial \widehat{f_n}(p)}{\partial |p|} - \frac{\partial \widehat{f}(p)}{\partial |p|}\right| \leq \frac{1}{2\pi}\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^2)}. \tag{2.43}$$

Hence, expression (2.42) can be estimated from above in the absolute value by

$$\frac{1}{2\pi}\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^2)}|p|^{1-2s_1}\chi_{\{|p|\leq 1\}}.$$

Thus,

$$\left\|\frac{\int_0^{|p|}\left(\frac{\partial \widehat{f_n}(q,\sigma)}{\partial q} - \frac{\partial \widehat{f}(q,\sigma)}{\partial q}\right)dq}{|p|^{2s_1} + |p|^{2s_2}}\chi_{\{|p|\leq 1\}}\right\|_{L^2(\mathbb{R}^2)} \leq \frac{\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^2)}}{2\sqrt{2\pi(1-s_1)}} \to 0$$

as $n \to \infty$ by means of the one of our assumptions. Therefore,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^2), \quad n \to \infty$$

in the space of two dimensions with $s_1 \in \left[\frac{1}{2}, 1\right)$.

We proceed to the proof of the part c) of our theorem, when the dimension $d = 3$ and $s_1 \in \left(0, \frac{3}{4}\right)$. In this case, by virtue of the part c) of Theorem 1.1, equations (1.2) and (1.3) admit unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^3)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^3),\ n \in \mathbb{N}$ respectively. Using (1.15), we derive the estimate from above in the absolute value on the first term in the right side of (2.35) by

$$\frac{\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)}}{(2\pi)^{\frac{3}{2}}|p|^{2s_1}}\chi_{\{|p|\leq 1\}},$$

so that its $L^2(\mathbb{R}^3)$ norm can be bounded from above by

$$\frac{1}{\pi\sqrt{2(3-4s_1)}}\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

via the one of the given conditions. Therefore,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^3), \quad n \to \infty$$

in the situation when the dimension $d = 3$ with $s_1 \in \left(0, \frac{3}{4}\right)$.

For the higher values of the power of the three dimensional negative Laplacian $s_1 \in \left[\frac{3}{4}, 1\right)$, we have $f_n(x) \in L^1(\mathbb{R}^3),\ n \in \mathbb{N}$, such that $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$ by means of the parts a) and b) of Lemma 4.1 of [32]. Then the orthogonality relation

$$(f(x), 1)_{L^2(\mathbb{R}^3)} = 0 \tag{2.44}$$

can be derived via the simple limiting argument, similarly to (2.36). By virtue of the part c) of Theorem 1.1, problems (1.2) and (1.3) admit unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^3)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^3),\ n \in \mathbb{N}$ respectively. Orthogonality conditions (2.44) and (1.13) give us

$$\widehat{f}(0) = 0, \quad \widehat{f_n}(0) = 0, \quad n \in \mathbb{N}$$

when the dimension $d = 3$ and $s_1 \in \left[\frac{3}{4}, 1\right)$. This enables us to derive here the expressions analogous to (2.41). We use the three dimensional analog of inequality (2.43) to obtain the estimate from above on the first term in the right side of (2.35) in the absolute value by

$$\frac{\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^3)}}{(2\pi)^{\frac{3}{2}}}|p|^{1-2s_1}\chi_{\{|p|\leq 1\}},$$

so that its $L^2(\mathbb{R}^3)$ norm can be bounded from above by

$$\frac{1}{\pi\sqrt{2(5-4s_1)}}\|xf_n(x) - xf(x)\|_{L^1(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

as assumed. Therefore,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^3), \quad n \to \infty$$

in the situation when the dimension $d = 3$ and $s_1 \in \left[\frac{3}{4}, 1\right)$.

Let us turn our attention to the case d) of the theorem. By means of the part d) of Theorem 1.1 problems (1.2) and (1.3) possess unique solutions $u(x) \in H^{2s_2}(\mathbb{R}^d)$ and $u_n(x) \in H^{2s_2}(\mathbb{R}^d),\ n \in \mathbb{N}$ respectively. Using inequality (1.15), we obtain the upper bound on the first term in the right side of (2.35) in the absolute value by

$$\frac{\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}|p|^{2s_1}}\chi_{\{|p|\leq 1\}}, \quad d \geq 4,$$

so that its $L^2(\mathbb{R}^d)$ norm can be estimated from above by

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\sqrt{\frac{|S^d|}{d-4s_1}}\|f_n(x) - f(x)\|_{L^1(\mathbb{R}^d)} \to 0, \quad n \to \infty$$

by means of the one of our assumptions. Here S^d stands for the unit sphere centered at the origin in our space of d dimensions and $|S^d|$ for its Lebesgue measure. Thus,

$$u_n(x) \to u(x) \quad in \quad L^2(\mathbb{R}^d), \quad d \geq 4, \quad n \to \infty$$

with $s_1 \in (0, 1)$. ■

3. Solvability in the sense of sequences with a scalar potential

Proof of Theorem 1.4. To establish the uniqueness of solutions for our problem, we suppose that there exist both $u_1(x)$ and $u_2(x)$ which are square integrable in $\mathbb{R}^3$ and satisfy (1.16). Then their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3)$ solves the equation

$$Lw = 0.$$

The fact that the operator L defined in (1.19) has no nontrivial $L^2(\mathbb{R}^3)$ zero modes as discussed above implies that $w(x)$ vanishes a.e. in $\mathbb{R}^3$.

We apply the generalized Fourier transform (1.23) with the functions of the continuous spectrum of our Schrödinger operator to both sides of equation (1.16). This gives us

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \leq 1\}} + \frac{\tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| > 1\}}. \tag{3.45}$$

The second term in the right side of (3.45) can be easily bounded from above in the absolute value as

$$\left| \frac{\tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| > 1\}} \right| \leq \frac{|\tilde{f}(k)|}{2} \in L^2(\mathbb{R}^3)$$

due to the one of our assumptions. Let us first discuss the case when $0 < s_1 < \dfrac{3}{4}$. Then the first term in the right side of (3.45) can be estimated from above in the absolute value via inequality (1.24) as

$$\left| \frac{\tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \leq 1\}} \right| \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f(x)\|_{L^1(\mathbb{R}^3)} \frac{\chi_{\{|k| \leq 1\}}}{|k|^{2s_1}} \in L^2(\mathbb{R}^3).$$

This completes the proof of part 1) of the theorem. We conclude the argument by considering the case when the power $\dfrac{3}{4} \leq s_1 < 1$. Let us express

$$\tilde{f}(k) = \tilde{f}(0) + \int_0^{|k|} \frac{\partial \tilde{f}(q, \sigma)}{\partial q} dq.$$

Here

$$\tilde{f}(0) = (f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)}.$$

Therefore, the first term in the right side of (3.45) can be written as

$$\frac{\tilde{f}(0)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \leq 1\}} + \frac{\int_0^{|k|} \frac{\partial \tilde{f}(q,\sigma)}{\partial q} dq}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \leq 1\}}. \tag{3.46}$$

Obviously, the second term in sum (3.46) can be easily bounded above in the absolute value as

$$\left|\frac{\int_0^{|k|}\frac{\partial\tilde{f}(q,\sigma)}{\partial q}dq}{|k|^{2s_1}+|k|^{2s_2}}\chi_{\{|k|\le 1\}}\right| \le \|\nabla_q\tilde{f}(q)\|_{L^\infty(\mathbb{R}^3)}|k|^{1-2s_1}\chi_{\{|k|\le 1\}} \in L^2(\mathbb{R}^3).$$

Note that under the given assumptions $\nabla_q\tilde{f}(q)\in L^\infty(\mathbb{R}^3)$ via Lemma 2.4 of [25]. Thus, it remains to analyze the term

$$\frac{\tilde{f}(0)}{|k|^{2s_1}+|k|^{2s_2}}\chi_{\{|k|\le 1\}}. \tag{3.47}$$

It can easily checked that (3.47) is square integrable if and only if $\tilde{f}(0)$ vanishes. This is equivalent to orthogonality condition (1.25). ■

Let us turn our attention to the establishing of our final main statement dealing with the solvability in the sense of sequences.

Proof of Theorem 1.5. Evidently, each problem (1.17) admits a unique solution $u_n(x)\in L^2(\mathbb{R}^3),\ n\in\mathbb{N}$ via the result of Theorem 1.4 above. It can be easily checked that in case 2) of the theorem the limiting orthogonality condition

$$(f(x),\varphi_0(x))_{L^2(\mathbb{R}^3)}=0 \tag{3.48}$$

is valid. Indeed, by virtue of (1.26) along with inequality (1.24)

$$|(f(x),\varphi_0(x))_{L^2(\mathbb{R}^3)}| = |(f(x)-f_n(x),\varphi_0(x))_{L^2(\mathbb{R}^3)}| \le$$

$$\le \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{1-I(V)}\|f_n(x)-f(x)\|_{L^1(\mathbb{R}^3)}\to 0, \quad n\to\infty.$$

Note that via the assumptions of part 2) of our theorem we have $f_n(x)\in L^1(\mathbb{R}^3)$, so that $f_n(x)\to f(x)$ in $L^1(\mathbb{R}^3)$ as $n\to\infty$ by means of the parts a) and b) of Lemma 4.1 of [32]. Thus, in both cases of the theorem, limiting problem (1.16) has a unique solution $u(x)\in L^2(\mathbb{R}^3)$ due to the result of Theorem 1.4. We apply the generalized Fourier transform (1.23) to both sides of equation (1.17). This yields

$$\tilde{u}_n(k)=\frac{\tilde{f}_n(k)}{|k|^{2s_1}+|k|^{2s_2}}, \quad n\in\mathbb{N},$$

such that

$$\tilde{u}_n(k)-\tilde{u}(k)=\frac{\tilde{f}_n(k)-\tilde{f}(k)}{|k|^{2s_1}+|k|^{2s_2}}\chi_{\{|k|\le 1\}}+\frac{\tilde{f}_n(k)-\tilde{f}(k)}{|k|^{2s_1}+|k|^{2s_2}}\chi_{\{|k|>1\}}. \tag{3.49}$$

Evidently, the second term in the right side of (3.49) can be easily bounded from above in the absolute value by $\dfrac{|\tilde{f}_n(k)-\tilde{f}(k)|}{2}$. Thus,

$$\left\|\frac{\tilde{f}_n(k)-\tilde{f}(k)}{|k|^{2s_1}+|k|^{2s_2}}\chi_{\{|k|>1\}}\right\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{2}\|f_n(x)-f(x)\|_{L^2(\mathbb{R}^3)} \to 0, \quad n\to\infty$$

due to the one of our assumptions. First we consider the case when $0<s_1<\dfrac{3}{4}$. (1.24) gives us

$$|\tilde{f}_n(k)-\tilde{f}(k)| \leq \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{1-I(V)}\|f_n(x)-f(x)\|_{L^1(\mathbb{R}^3)}.$$

Hence, we derive the estimate from above for the first term in the right side of (3.49) in the absolute value as

$$\left|\frac{\tilde{f}_n(k)-\tilde{f}(k)}{|k|^{2s_1}+|k|^{2s_2}}\chi_{\{|k|\leq 1\}}\right| \leq \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{1-I(V)}\|f_n(x)-f(x)\|_{L^1(\mathbb{R}^3)}\frac{\chi_{\{|k|\leq 1\}}}{|k|^{2s_1}}.$$

Clearly, this implies

$$\left\|\frac{\tilde{f}_n(k)-\tilde{f}(k)}{|k|^{2s_1}+|k|^{2s_2}}\chi_{\{|k|\leq 1\}}\right\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{\sqrt{2\pi}}\frac{1}{1-I(V)}\frac{\|f_n(x)-f(x)\|_{L^1(\mathbb{R}^3)}}{\sqrt{3-4s_1}} \to 0$$

as $n\to\infty$ as assumed. Thus, $u_n(x)\to u(x)$ in $L^2(\mathbb{R}^3)$ as $n\to\infty$ in the situation when $s_1\in\left(0,\dfrac{3}{4}\right)$.

Let us turn our attention to the case when $s_1\in\left[\dfrac{3}{4},1\right)$. As discussed above, it is sufficient to consider the first term in the right side of (3.49). Orthogonality conditions (3.48) and (1.26) imply that

$$\tilde{f}(0)=0, \quad \tilde{f}_n(0)=0, \quad n\in\mathbb{N},$$

so that

$$\tilde{f}(k)=\int_0^{|k|}\frac{\partial\tilde{f}(q,\sigma)}{\partial q}dq, \quad \tilde{f}_n(k)=\int_0^{|k|}\frac{\partial\tilde{f}_n(q,\sigma)}{\partial q}dq, \quad n\in\mathbb{N}.$$

This allows us to express the first term in the right side of (3.49) as

$$\frac{\int_0^{|k|}\left[\frac{\partial\tilde{f}_n(q,\sigma)}{\partial q}-\frac{\partial\tilde{f}(q,\sigma)}{\partial q}\right]dq}{|k|^{2s_1}+|k|^{2s_2}}\chi_{\{|k|\leq 1\}},$$

which can be easily bounded from above in the absolute value by

$$\|\nabla_q[\tilde{f}_n(q) - \tilde{f}(q)]\|_{L^\infty(\mathbb{R}^3)}|k|^{1-2s_1}\chi_{\{|k|\le 1\}}.$$

Hence,

$$\left\|\frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}}\chi_{\{|k|\le 1\}}\right\|_{L^2(\mathbb{R}^3)} \le \|\nabla_q[\tilde{f}_n(q) - \tilde{f}(q)]\|_{L^\infty(\mathbb{R}^3)}\frac{2\sqrt{\pi}}{\sqrt{5-4s_1}}.$$

By virtue of the result of Lemma 3.4 of [23] under the stated assumptions we have

$$\|\nabla_q[\tilde{f}_n(q) - \tilde{f}(q)]\|_{L^\infty(\mathbb{R}^3)} \to 0, \quad n \to \infty,$$

which completes the proof of our theorem. ∎

References

[1] C. Amrouche, V. Girault, J. Giroire. *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces*, J. Math. Pures Appl. (9), **76** (1997), no. 1, 55–81.

[2] C. Amrouche, F. Bonzom. *Mixed exterior Laplace's problem*, J. Math. Anal. Appl., **338** (2008), no. 1, 124–140.

[3] P. Bolley, T.L. Pham. *Propriétés d'indice en théorie höldérienne pour des opérateurs différentiels elliptiques dans R^n*, J. Math. Pures Appl. (9), **72** (1993), no 1, 105–119.

[4] P. Bolley, T.L. Pham. *Propriété d'indice en théorie Höldérienne pour le problème extérieur de Dirichlet*, Comm. Partial Differential Equations, **26** (2001), no. 1-2, 315–334.

[5] N. Benkirane. *Propriétés d'indice en théorie höldérienne pour des opérateurs elliptiques dans R^n*, C. R. Acad. Sci. Paris Ser. I Math., **307** (1988), no. 11, 577–580.

[6] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry*. Texts and Monographs in Physics. Springer Study Edition. Springer-Verlag, Berlin (1987), 319 pp.

[7] A. Ducrot, M. Marion, V. Volpert. *Systémes de réaction-diffusion sans propriété de Fredholm*, C. R. Math. Acad. Sci. Paris, **340** (2005), no. 9, 659–664.

[8] A. Ducrot, M. Marion, V. Volpert. *Reaction-diffusion problems with non-Fredholm operators*, Adv. Differential Equations, **13** (2008), no. 11-12, 1151–1192.

[9] M. Efendiev, V. Vougalter. *Solvability in the sense of sequences for some fourth order non-Fredholm operators*, J. Differential Equations, **271** (2021), 280-300.

[10] M. Goldberg, W. Schlag. *A limiting absorption principle for the three-dimensional Schrödinger equation with L^p potentials*, Int. Math. Res. Not., (2004), no. 75, 4049–4071.

[11] N. Guerngar, E. Nane, R. Tinaztepe, S. Ulusoy, H.W. Van Wyk. *Simultaneous inversion for the fractional exponents in the space-time fractional diffusion equation $\partial_t^\beta u = -(-\Delta)^{\frac{\alpha}{2}} u - (-\Delta)^{\frac{\gamma}{2}} u$*, Fract. Calc. Appl. Anal., **24** (2021), no. 3, 818–847.

[12] B.L.G. Jonsson, M. Merkli, I.M. Sigal, F. Ting. *Applied Analysis*, In preparation.

[13] T. Kato. *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann., **162** (1965/1966), 258–279.

[14] E.H. Lieb, M. Loss. *Analysis. Graduate Studies in Mathematics*, **14**. American Mathematical Society, Providence, RI (1997), 278 pp.

[15] V.G. Maz'ja, M. Otelbaev. *Imbedding theorems and the spectrum of a certain pseudodifferential operator.* (Russian) Sibirsk. Mat. Z., **18** (1977), no. 5, 1073–1087, 1206.

[16] V.G. Maz'ya, I.E. Verbitsky. *The form boundedness criterion for the relativistic Schrödinger operator*, Ann. Inst. Fourier (Grenoble), **54** (2004), no. 2, 317–339.

[17] R. Metzler, J. Klafter. *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep., **339** (2000), no. 1, 77 pp.

[18] M. Reed, B. Simon. *Methods of modern mathematical physics. III. Scattering theory*. Academic Press, New York-London (1979), 463 pp.

[19] I. Rodnianski, W. Schlag. *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math., **155** (2004), no. 3, 451–513.

[20] V. Volpert. Elliptic partial differential equations. Volume 1: Fredholm theory of elliptic problems in unbounded domains. Monographs in Mathematics, **101**. Birkhäuser/Springer Basel AG, Basel (2011), 639 pp.

[21] V. Volpert, B. Kazmierczak, M. Massot, Z. Peradzynski. *Solvability conditions for elliptic problems with non-Fredholm operators*, Appl. Math. (Warsaw), **29** (2002), no. 2, 219–238.

[22] V. Volpert, V. Vougalter. *On the solvability conditions for a linearized Cahn-Hilliard equation,* Rend. Istit. Mat. Univ. Trieste, **43** (2011), 1–9.

[23] V. Volpert, V. Vougalter. *Solvability in the sense of sequences to some non-Fredholm operators,* Electron. J. Differential Equations, **2013**, No. 160, 16 pp.

[24] V. Vougalter, V. Volpert. *On the solvability conditions for some non Fredholm operators,* Int. J. Pure Appl. Math., **60** (2010), no. 2, 169–191.

[25] V. Vougalter, V. Volpert. *Solvability conditions for some non-Fredholm operators*, Proc. Edinb. Math. Soc. (2), **54** (2011), no. 1, 249–271.

[26] V. Vougalter, V. Volpert. *On the solvability conditions for the diffusion equation with convection terms,* Commun. Pure Appl. Anal., **11** (2012), no. 1, 365–373.

[27] V. Vougalter, V. Volpert. *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order,* Math. Model. Nat. Phenom., **7** (2012), no. 2, 146–154.

[28] V. Vougalter, V. Volpert. *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2** (2012), no. 4, 473–496.

[29] V. Vougalter, V. Volpert. *Existence of stationary solutions for some integro-differential equations with anomalous diffusion,* J. Pseudo-Differ. Oper. Appl., **6** (2015), no. 4, 487–501.

[30] V. Vougalter, V. Volpert. *On the solvability of some systems of integro-differential equations with anomalous diffusion. Pseudo- differential operators: groups, geometry and applications,* Trends Math., *Birkhäuser / Springer, Cham* (2017), 161–179.

[31] V. Vougalter, V. Volpert. *Solvability in the sense of sequences for some non Fredholm operators related to the superdiffusion*, J. Pseudo-Differ. Oper. Appl., **9** (2018), no. 1, 25–46.

[32] V. Vougalter, V. Volpert. *On the solvability in the sense of sequences for some non-Fredholm operators related to the anomalous diffusion.* Analysis of pseudo-differential operators, Trends Math., Birkhäuser/Springer, Cham (2019), 229–257.