Linear and Nonlinear non-Fredholm Operators and their Applications

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Dedicated to Professor Norman Dancer on the occasion of his 75th birthday.

Abstract. In this survey we discuss the recent results on the existence in the sense of sequences of solutions for certain elliptic problems containing the non-Fredholm operators. First of all, we deal with the solvability in the sense of sequences for some fourth order non-Fredholm operators, such that the methods of the spectral and scattering theory for Schrödinger type operators are used for the analysis. Moreover, we present the easily verifiable necessary condition of the preservation of the nonnegativity of the solutions of a system of parabolic equations in the case of the anomalous diffusion with the negative Laplacian in a fractional power in one dimension, which imposes the necessary form of such system of equations that must be studied mathematically. This class of systems of PDEs has a wide range of applications. We conclude the survey with several new results nowhere published concerning the solvability in the sense of sequences for the generalized Poisson type equation with a scalar potential.

Keywords: solvability conditions, non-Fredholm operators, anomalous diffusion, nonnegativity of solutions
AMS subject classification: 35J05, 35K55, 35K57, 35P30

1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. Consequently, the equation $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. Such properties of Fredholm operators are actively used in many methods of linear and nonlinear analysis.

Elliptic equations in the bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Shapiro-Lopatinskii conditions are fulfilled (see e.g. [1], [10], [25], [31]). This is the main result of the theory of linear elliptic problems. In the case of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For instance, the Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ fails to satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic equations in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions stated above, the limiting operators are invertible (see [32]). In certain simple cases, the limiting operators can be explicitly constructed. For example, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_\pm = \lim_{x\to\pm\infty} a(x), \quad b_\pm = \lim_{x\to\pm\infty} b(x), \quad c_\pm = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are:

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Since the coefficients are constants, the essential spectrum of the operator, that is the set of the complex numbers λ for which the operator $L - \lambda$ does not satisfy the Fredholm property, can be explicitly found by means of the Fourier transform:

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the essential spectrum does not contain the origin, which is equivalent to that $\lambda_\pm(\xi)$ must be nonzero for any $\xi \in \mathbb{R}$.

In the case of general elliptic equations, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, such conditions may not be explicitly written.

The works [20] and [28] are important for the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of second order and of operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in second-order quasilinear elliptic systems were considered in [21]. Book [13] deals with a systematic study of a dynamical systems approach to investigating the symmetrization and stabilization properties of nonnegative solutions of nonlinear elliptic problems in asymptotically symmetric unbounded domains (see also [7], [8]). Book [11] is devoted to the finite and infinite dimensional attractors for evolution equations of mathematical physics. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in $\mathbb{R}^3$ was studied in [19].

In the case of non-Fredholm operators the usual solvability conditions may not be applicable and solvability relations are, in general, not known. There are certain classes of operators for which solvability conditions are derived. We illustrate them with the following example. Consider the equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. The operator L coincides with its limiting operators. The homogeneous problem has a nonzero bounded solution. Thus the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. Solvability conditions can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exists a solution of this equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [37]). Here and throughout the article S^d_r denotes the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Thus, though the operator fails to satisfy the Fredholm property, the solvability conditions are formulated analogously. However, such similarity is only formal since the range of the operator is not closed.

In the case of the operator with a potential,

$$Lu \equiv -\Delta u + V(x)u - au = f, \tag{1.2}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, $a \geq 0$ is a constant and the scalar potential function $V(x)$ tends to 0 at infinity, the Fourier transform is not directly applicable. Nevertheless, the solvability conditions in $\mathbb{R}^3$ in such non-Fredholm situation can be obtained by a rather sophisticated application of the theory of self-adjoint operators (see [35]). As before, the solvability relations are formulated in terms of the orthogonality to the solutions of the homogeneous adjoint problem. There are several other examples of linear elliptic non Fredholm operators for which solvability conditions can be derived (see [17], [32], [33], [34], [35], [37]).

Solvability conditions play a significant role in the analysis of the nonlinear elliptic equations. In the case of non-Fredholm operators, in spite of some progress in the understanding of the linear problems, there exist only few examples where the nonlinear non-Fredholm operators are analyzed (see [9], [15], [36], [37], [42]). The article [36] is devoted to the solvability in the appropriate H^2 spaces of the nonlinear, nonlocal equation

$$\Delta u + \int_\Omega G(x-y)F(u(y),y)dy + au = 0, \quad a \geq 0. \tag{1.3}$$

Clearly, when (1.3) is considered in the whole space or in the product of the $[0, 2\pi]$ and $\mathbb{R}^d$, $d = 1, 2$ with the periodic boundary conditions on the sides, it contains the non-Fredholm operator. In [15] the authors study the solvability in H^2 of the equation similar to (1.3), which includes the drift term, namely

$$\frac{d^2u}{dx^2} + b\frac{du}{dx} + au + \int_{-\infty}^{\infty} G(x-y)F(u(y),y)dy = 0, \quad a \geq 0, \quad b \in \mathbb{R}, \quad b \neq 0, \quad x \in \mathbb{R}. \tag{1.4}$$

The article [18] deals with the solvability in $H^2(\mathbb{R}^2)$ of the integro-differential equation involving the normal diffusion in one direction and the anomalous diffusion in the other direction.

One of the important questions about problems with non-Fredholm operators concerns their solvability. We address it in the following setting. Let $A : E \to F$ be the operator corresponding to the left side of equation (1.2). For $a \geq 0$, its essential spectrum contains the origin, so that this operator fails to satisfy the Fredholm property. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Since the operator A does not satisfy the Fredholm property, the sequence u_n may not be convergent. Let us call a sequence u_n the solution in the sense of sequences of the equation $Au = f$ if $Au_n \to f$ (see [32]). If such sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this equation. Solution in the sense of sequences is equivalent in this case to the usual solution. However, in the case of the non-Fredholm operators, this convergence may not hold or it can occur in some weaker sense. In such case, the solution in the sense of sequences may not imply the existence of the usual solution. In the our work (see [17]) we find the sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent.

In the Mathematical Biology (in particular, in the modelling of the Population Dynamics) the integro-differential equations describe the models with the intra-specific competition and nonlocal consumption of resources (see e.g. [2], [3]). It is important to study the problems of this kind in unbounded domains from the point of view of the understanding of the spread of the viral infections, since many countries have to deal with the pandemics. In our works we use the explicit form of the solvability relations and establish the existence of solutions of our nonlinear equations. In the case of the standard Laplacian in the diffusion term, the integro- differential equations were considered in [15], [36], [42]. The solvability of the nonlocal reaction-diffusion problems involving the negative Laplacian raised to a fractional power was actively studied in recent years in the context of the anomalous diffusion (see e.g. [16], [39], [40]). The probabilistic realization of the anomalous diffusion was discussed in [27]. In [26] the authors establish the imbedding theorems and study the spectrum of certain pseudodifferential operators. Let us describe the results derived in [17].

2 Solvability in the sense of sequences for some fourth order non-Fredholm operators

The large time behavior of the solutions of a class of fourth-order parabolic problems defined on unbounded domains using the Kolmogorov ε-entropy as a measure was investigated in [14]. The equations of this type appear in the studies of the bistable systems, the pattern formation, the phase transitions in the multicomponent systems, in the Statistical Mechanics. To understand their dynamics and robustness, it is useful sometimes to consider the quasi-stationary models that lead to the fourth order elliptic equations.

Solvability in the sense of sequences for the sums of non-Fredholm Schrödinger type operators was considered in [38]. In the first part of this section we discuss such operators squared, namely

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}^2 u - a^2 u = f(x,y), \quad x, y \in \mathbb{R}^3, \tag{2.1}$$

where $a > 0$ is a constant. The operator

$$H_{U,\,V} := \{-\Delta_x + V(x) - \Delta_y + U(y)\}^2 : H^4(\mathbb{R}^6) \to L^2(\mathbb{R}^6) \tag{2.2}$$

under the technical assumptions on the scalar potential functions $V(x)$ and $U(y)$ given below. The Laplacians Δ_x and Δ_y are with respect to the x and y variables respectively. Similarly for the gradients, ∇_x and ∇_y are with respect to the x and y respectively. In the physical applications the sum of the two Schrödinger type operators has the meaning of the cumulative hamiltonian of the two non-interacting quantum particles.

The scalar potentials involved in operator (2.2) are assumed to be shallow and short-range, satisfying the assumptions similar to the ones of [35]. We also add a few extra regularity conditions.

Assumption 2.1. *The potential functions $V(x), U(y) : \mathbb{R}^3 \to \mathbb{R}$ satisfy the estimates*

$$|V(x)| \le \frac{C}{1+|x|^{3.5+\varepsilon}}, \quad |U(y)| \le \frac{C}{1+|y|^{3.5+\varepsilon}} \tag{2.3}$$

with some $\varepsilon > 0$ and $x, y \in \mathbb{R}^3$ a.e. so that

$$4^{\frac{1}{9}} \frac{9}{8} (4\pi)^{-\frac{2}{3}} \|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}} \|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1, \tag{2.4}$$

$$4^{\frac{1}{9}} \frac{9}{8} (4\pi)^{-\frac{2}{3}} \|U\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}} \|U\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \tag{2.5}$$

and

$$\sqrt{c_{HLS}} \|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi, \quad \sqrt{c_{HLS}} \|U\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi. \tag{2.6}$$

Furthermore, $|\nabla_x V(x)|,\ \Delta_x V(x),\ |\nabla_y U(y)|,\ \Delta_y U(y) \in L^\infty(\mathbb{R}^3)$.

Here and below C stands for a finite positive constant and c_{HLS} given on p.98 of [24] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\right| \leq c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

The norm of a function $f_1 \in L^p(\mathbb{R}^d),\ 1 \leq p \leq \infty,\ d \in \mathbb{N}$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^d)}$.

Remark 2.2. *The function* $V(x) = \dfrac{C}{1+|x|^4}$, *where* C *is small enough satisfies Assumption 2.1.*

We denote the inner product of two functions as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \tag{2.7}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x) \in L^\infty(\mathbb{R}^d)$, like for instance the functions of the continuos spectrum of the Schrödinger operators discussed below (see Corollary 2.2 of [35]), then the integral in the right side of (2.7) is well defined. Let us use the function space $H^4(\mathbb{R}^d)$ equipped with the norm

$$\|u\|^2_{H^4(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta^2 u\|^2_{L^2(\mathbb{R}^d)} \tag{2.8}$$

respectively. We designate the sphere of radius $r > 0$ in $\mathbb{R}^d$ centered at the origin by S^d_r. By means of Lemma 2.3 of [35], under Assumption 2.1 above on the scalar potentials, operator (2.2) considered as acting in $L^2(\mathbb{R}^6)$ with domain $H^4(\mathbb{R}^6)$ is self-adjoint and is unitarily equivalent to $\{-\Delta_x - \Delta_y\}^2$ on $L^2(\mathbb{R}^6)$ via the product of the wave operators (see [23], [30])

$$\Omega^{\pm}_V := s - \lim_{t\to\mp\infty} e^{it(-\Delta_x + V(x))}e^{it\Delta_x}, \quad \Omega^{\pm}_U := s - \lim_{t\to\mp\infty} e^{it(-\Delta_y + U(y))}e^{it\Delta_y},$$

with the limits here understood in the strong L^2 sense (see e.g. [29] p.34, [4] p.90). Thus, operator (2.2) does not have any nontrivial $L^2(\mathbb{R}^6)$ eigenfunctions. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Hence, operator (2.2) fails to satisfy the Fredholm property. The functions of the continuos spectrum of the first operator involved in (2.2) are the solutions of the Schrödinger equation

$$[-\Delta_x + V(x)]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3.$$

They satisfy in the integral form the Lippmann-Schwinger equation (see e.g. [29] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{2.9}$$

for the perturbed plane waves and the orthogonality conditions

$$(\varphi_k(x), \varphi_{k_1}(x))_{L^2(\mathbb{R}^3)} = \delta(k - k_1),\ k, k_1 \in \mathbb{R}^3.$$

Analogously, for the second operator involved in (2.2) the functions of its continuous spectrum solve

$$[-\Delta_y + U(y)]\eta_q(y) = q^2\eta_q(y), \quad q \in \mathbb{R}^3,$$

in the integral formulation

$$\eta_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta_q)(z)dz, \tag{2.10}$$

such that the orthogonality conditions $(\eta_q(y), \eta_{q_1}(y))_{L^2(\mathbb{R}^3)} = \delta(q - q_1),\ q, q_1 \in \mathbb{R}^3$ are valid. $\eta_0(y)$ corresponds to the case of $q = 0$. We denote by the double tilde sign the generalized Fourier transform with the product of these functions of the continuous spectrum

$$\tilde{\tilde{f}}(k, q) := (f(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}, \quad k, q \in \mathbb{R}^3. \tag{2.11}$$

(2.11) is a unitary transform on $L^2(\mathbb{R}^6)$. Our first main result is as follows.

Theorem 2.3. *Let Assumption 2.1 hold,* $a > 0$ *and* $f(x, y) \in L^2(\mathbb{R}^6)$*. Assume also that* $|x|f(x, y),\ |y|f(x, y) \in L^1(\mathbb{R}^6)$*. Then equation (2.1) admits a unique solution* $u(x, y) \in H^4(\mathbb{R}^6)$ *if and only if*

$$(f(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k, q) \in S^6_{\sqrt{a}} \quad a.e. \tag{2.12}$$

In the very special situation when the scalar potential functions $V(x) \equiv 0$ and $U(y) \equiv 0$ in $\mathbb{R}^3$, condition (2.12) gives us the orthogonality to the products of the corresponding standard Fourier harmonics. Let us turn our attention to the issue of the solvability in the sense of sequences for our problem. The corresponding sequence of approximate equations with $n \in \mathbb{N}$ is given by

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}^2 u_n - a^2 u_n = f_n(x, y), \quad x, y \in \mathbb{R}^3, \tag{2.13}$$

with the constant $a > 0$ and the right sides tend to the right side of (2.1) in $L^2(\mathbb{R}^6)$ as $n \to \infty$.

Theorem 2.4. *Let Assumption 2.1 hold,* $a > 0$, $n \in \mathbb{N}$ *and* $f_n(x, y) \in L^2(\mathbb{R}^6)$*, so that* $f_n(x, y) \to f(x, y)$ *in* $L^2(\mathbb{R}^6)$ *as* $n \to \infty$*. Let in addition* $|x|f_n(x, y),\ |y|f_n(x, y) \in L^1(\mathbb{R}^6),\ n \in \mathbb{N}$*, so that* $|x|f_n(x, y) \to |x|f(x, y),\ |y|f_n(x, y) \to |y|f(x, y)$ *in* $L^1(\mathbb{R}^6)$ *as* $n \to \infty$ *and the orthogonality conditions*

$$(f_n(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k, q) \in S^6_{\sqrt{a}} \quad a.e. \tag{2.14}$$

hold for all $n \in \mathbb{N}$*. Then equations (2.1) and (2.13) possess unique solutions* $u(x, y) \in H^4(\mathbb{R}^6)$ *and* $u_n(x, y) \in H^4(\mathbb{R}^6)$ *respectively, so that* $u_n(x, y) \to u(x, y)$ *in* $H^4(\mathbb{R}^6)$ *as* $n \to \infty$.

The second part of the section is devoted to the discussion of the equation

$$\{-\Delta_x - \Delta_y + U(y)\}^2 u - a^2 u = \phi(x,y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^3, \tag{2.15}$$

where $d \in \mathbb{N}$, the constant $a > 0$ and the scalar potential function involved in (2.15) is shallow and short-range under Assumption 2.1 above. The more singular case of $a = 0$ will be discussed later on in higher dimensions. The operator

$$L_U := \{-\Delta_x - \Delta_y + U(y)\}^2 : H^4(\mathbb{R}^{d+3}) \to L^2(\mathbb{R}^{d+3}). \tag{2.16}$$

Analogously to (2.2), under the stated conditions operator (2.16) considered as acting in $L^2(\mathbb{R}^{d+3})$ with domain $H^4(\mathbb{R}^{d+3})$ is self-adjoint and is unitarily equivalent to $\{-\Delta_x - \Delta_y\}^2$. Hence, operator (2.16) does not have nontrivial $L^2(\mathbb{R}^{d+3})$ eigenfunctions. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Thus, operator (2.16) is non-Fredholm. We consider another generalized Fourier transform with the standard Fourier harmonics and the perturbed plane waves

$$\tilde{\hat{\phi}}(k,q) := \left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})}, \quad k \in \mathbb{R}^d, \quad q \in \mathbb{R}^3. \tag{2.17}$$

(2.17) is a unitary transform on $L^2(\mathbb{R}^{d+3})$. We have the following result.

Theorem 2.5. *Let the potential function $U(y)$ satisfy Assumption 2.1, $a > 0$ and in addition $\phi(x,y) \in L^2(\mathbb{R}^{d+3})$, $|x|\phi(x,y)$, $|y|\phi(x,y) \in L^1(\mathbb{R}^{d+3})$, $d \in \mathbb{N}$. Then equation (2.15) admits a unique solution $u(x,y) \in H^4(\mathbb{R}^{d+3})$ if and only if*

$$\left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0, \quad (k,q) \in S_{\sqrt{a}}^{d+3} \quad a.e. \tag{2.18}$$

The final main result of the section deals with the issue of the solvability in the sense of sequences for our problem. The corresponding sequence of approximate equations with $n \in \mathbb{N}$ is given by

$$\{-\Delta_x - \Delta_y + U(y)\}^2 u_n - a^2 u_n = \phi_n(x,y), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}, \quad y \in \mathbb{R}^3, \tag{2.19}$$

where the right sides tend to the right side of (2.15) in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.

Theorem 2.6. *Let the potential function $U(y)$ satisfy Assumption 2.1, $a > 0$, $n \in \mathbb{N}$ and $\phi_n(x,y) \in L^2(\mathbb{R}^{d+3})$, $d \in \mathbb{N}$, so that $\phi_n(x,y) \to \phi(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$. Let in addition $|x|\phi_n(x,y)$, $|y|\phi_n(x,y) \in L^1(\mathbb{R}^{d+3})$, so that*

$$|x|\phi_n(x,y) \to |x|\phi(x,y), \ |y|\phi_n(x,y) \to |y|\phi(x,y)$$

in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$ and the orthogonality conditions

$$\left(\phi_n(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0, \quad (k,q) \in S_{\sqrt{a}}^{d+3} \quad a.e. \tag{2.20}$$

hold for all $n \in \mathbb{N}$. Then equations (2.15) and (2.19) have unique solutions $u(x,y) \in H^4(\mathbb{R}^{d+3})$ and $u_n(x,y) \in H^4(\mathbb{R}^{d+3})$ respectively, so that $u_n(x,y) \to u(x,y)$ in $H^4(\mathbb{R}^{d+3})$ as $n \to \infty$.

Let us emphasize that in the applications the sum of the free negative Laplacian and the Schrödinger type operator has the physical meaning of the cumulative hamiltonian of the two non-interacting quantum particles. One of these particles moves freely and another interacts with an external potential.

We conclude the section by considering problem (2.15) with $a = 0$ in the context of the solvability in the sense of sequences.

Theorem 2.7. *Let the potential function $U(y)$ satisfy Assumption 2.1, $a = 0$ and $\phi(x,y) \in L^2(\mathbb{R}^{d+3})$, $d \in \mathbb{N}$, $d \geq 4$.*

a) When $d = 4, 5$, let in addition $|x|\phi(x,y)$, $|y|\phi(x,y) \in L^1(\mathbb{R}^{d+3})$. Then equation (2.15) admits a unique solution $u(x,y) \in H^4(\mathbb{R}^{d+3})$ if and only if

$$(\phi(x,y), \eta_0(y))_{L^2(\mathbb{R}^{d+3})} = 0. \tag{2.21}$$

b) When $d \geq 6$, let in addition $\phi(x,y) \in L^1(\mathbb{R}^{d+3})$. Then problem (2.15) possesses a unique solution $u(x,y) \in H^4(\mathbb{R}^{d+3})$.

The final statement of the section is as follows.

Theorem 2.8. *Let the potential function $U(y)$ satisfy Assumption 2.1, $a = 0$, $n \in \mathbb{N}$ and $\phi_n(x,y) \in L^2(\mathbb{R}^{d+3})$, $d \in \mathbb{N}$, $d \geq 4$, so that $\phi_n(x,y) \to \phi(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.*

a) If $d = 4, 5$, let in addition $|x|\phi_n(x,y)$, $|y|\phi_n(x,y) \in L^1(\mathbb{R}^{d+3})$, so that $|x|\phi_n(x,y) \to |x|\phi(x,y)$, $|y|\phi_n(x,y) \to |y|\phi(x,y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$ and the orthogonality relations

$$(\phi_n(x,y), \eta_0(y))_{L^2(\mathbb{R}^{d+3})} = 0 \tag{2.22}$$

are valid for all $n \in \mathbb{N}$. Then problems (2.15) and (2.19) admit unique solutions $u(x,y) \in H^4(\mathbb{R}^{d+3})$ and $u_n(x,y) \in H^4(\mathbb{R}^{d+3})$ respectively, so that $u_n(x,y) \to u(x,y)$ in $H^4(\mathbb{R}^{d+3})$ as $n \to \infty$.

b) If $d \geq 6$, let in addition $\phi_n(x,y) \in L^1(\mathbb{R}^{d+3})$, so that $\phi_n(x,y) \to \phi(x,y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$. Then equations (2.15) and (2.19) have unique solutions $u(x,y) \in H^4(\mathbb{R}^{d+3})$ and $u_n(x,y) \in H^4(\mathbb{R}^{d+3})$ respectively, so that $u_n(x,y) \to u(x,y)$ in $H^4(\mathbb{R}^{d+3})$ as $n \to \infty$.

Let us note that no orthogonality relations are needed to solve problem (2.15) with $a = 0$ in $H^4(\mathbb{R}^{d+3})$ in higher dimensions $d \geq 6$. In contrast to the Fredholm case, for the proofs of Theorems 2.3-2.8 above we are using the methods of the spectral and scattering theory for

Schrödinger type operatos such as the Spectral Theorem and the studies of the Lippmann-Schwinger integral equation for the perturbed plane waves.

In the following section we will discuss preservation of the nonnegativity of the solutions of a system of parabolic equations in the case of the anomalous diffusion (see [16]).

3 Verification of biomedical processes with anomalous diffusion, transport and interaction of species

The solutions of many systems of convection-diffusion-reaction equations, which arise in biology, physics or engineering describe such quantities as population densities, pressure or concentrations of nutrients and chemicals. Thus, a natural property to require for the solutions is the nonnegativity. Models that do not guarantee the nonnegativity are not valid or break down for small values of the solution. In many situations, proving that a particular model fails to preserve the nonnegativity leads to the better understanding of the model and its limitations. One of the first steps in analyzing ecological or biological or bio-medical models mathematically is to check if the solutions which originate from the nonnegative initial data remain nonnegative (as long as they exist). In other words, the model under consideration ensures that the nonnegative cone is positively invariant. Let us recall that if the solutions (of a given evolution PDE) which corresponds to the nonnegative initial data remain nonnegative as long as they exist, we say that such system satisfies the nonnegativity property.

For the scalar problems the nonnegativity property is a direct consequence of the maximum principle (see [12] and the references therein). However, for systems of equations the maximum principle is not valid. In the particular case of monotone systems the situation resembles the case of scalar equations, sufficient conditions for preserving the nonnegative cone can be found in [12]. The existence and uniqueness of the positive solutions of certain systems of differential equations was studied in [5] and [6].

Our goal is to prove a simple and easily verifiable criterion, that is, the necessary condition for the nonnegativity of solutions of systems of nonlinear convection-anomalous diffusion-reaction equations which arise in the modelling of the life sciences. We believe that it could provide the modeler with a tool, which is easy to verify, to approach the question of positive invariance of the model.

The present section is devoted to the preservation of the nonnegativity of solutions of the following system of reaction-diffusion equations

$$\frac{\partial u}{\partial t} = -A(-\Delta_x)^s u + \sum_{l=1}^{m} \Gamma^l \frac{\partial u}{\partial x_l} - F(u), \tag{3.1}$$

where $A,\ \Gamma^l,\ 1 \le l \le m$ are $N \times N$ matrices with constant coefficients, which is relevant to the cell population dynamics in Mathematical Biology. Let us call system (3.1) as a (N, m) one.

Note that the analogous model can be used to study such branches of science as the Damage Mechanics, the temperature distribution in Thermodynamics. In this section the space variable x corresponds to the cell genotype, $u_k(x,t)$ denotes the cell density distributions for various groups of cells as functions of their genotype and time,

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The operator $(-\Delta_x)^s$ is defined by means of the spectral calculus. The probabilistic realization of the anomalous diffusion was discussed in [27]. For the simplicity of presentation we will consider the case of the one spatial dimension with $0 < s < 1/4$. Let us assume here that (3.1) contains the square matrices with the entries constant in space and time

$$(A)_{k,j} := a_{k,j}, \quad (\Gamma)_{k,j} := \gamma_{k,j}, \quad 1 \le k, j \le N$$

and that the matrix $A + A^* > 0$ for the sake of the global well posedness of system (3.1). Here A^* denotes the adjoint of matrix A. Hence, system (3.1) can be rewritten in the form

$$\frac{\partial u_k}{\partial t} = -\sum_{j=1}^{N} a_{k,j}\Big(-\frac{\partial^2}{\partial x^2}\Big)^s u_j + \sum_{j=1}^{N} \gamma_{k,j}\frac{\partial u_j}{\partial x} - F_k(u), \quad 1 \le k \le N, \tag{3.2}$$

where $0 < s < \frac{1}{4}$. In the present section of the work the interaction of species term

$$F(u) = (F_1(u), F_2(u), ..., F_N(u))^T,$$

which can be linear, nonlinear or in principle even nonlocal. Let us assume its smoothness in the theorem below for the sake of the well posedness of our problem (3.1), although, we are not focused on the well posedness issue in the present section. We choose the space dimension $d = 1$, which is related to the solvability conditions stated below for the linear Poisson type problem (3.12) involving the non-Fredholm operator in the left side. From the perspective of applications, the space dimension is not restricted to $d = 1$ since the space variable is correspondent to the cell genotype but not to the usual physical space. As for the vector functions, their inner product is defined using their components as

$$(u,v)_{L^2(\mathbb{R},\mathbb{R}^N)} := \sum_{k=1}^{N} (u_k, v_k)_{L^2(\mathbb{R})}. \tag{3.3}$$

Obviously, (3.3) induces the norm

$$\|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} = \sum_{k=1}^{N} \|u_k\|^2_{L^2(\mathbb{R})}.$$

Let us use the Sobolev spaces

$$H^{2s}(\mathbb{R}) := \Big\{ u(x) : \mathbb{R} \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}), \ \Big(-\frac{d^2}{dx^2}\Big)^s u \in L^2(\mathbb{R}) \Big\}, \quad 0 < s \le 1$$

equipped with the norm

$$\|u\|^2_{H^{2s}(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\| \left(-\frac{d^2}{dx^2} \right)^s u \right\|^2_{L^2(\mathbb{R})}. \tag{3.4}$$

By the nonnegativity of a vector function below we mean the nonnegativity of the each of its components. The main result of the section is as follows.

Theorem 3.1. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$, so that $F \in \mathbb{C}^1$, the initial condition for problem (3.1) is $u(x,0) = u_0(x) \geq 0$ and $u_0(x) \in L^2(\mathbb{R}, \mathbb{R}^N)$. Let us also assume that the off diagonal elements of the matrix A, are nonnegative, so that*

$$a_{k,l} \geq 0, \quad 1 \leq k, l \leq N, \quad k \neq l. \tag{3.5}$$

Then the necessary condition for system (3.1) to admit a solution $u(x,t) \geq 0$ for all $t \in [0, \infty)$ is that the matrices A and Γ are diagonal and for all $1 \leq k \leq N$

$$F_k(s_1, ..., s_{k-1}, 0, s_{k+1}, ..., s_N) \leq 0 \tag{3.6}$$

is valid, where $s_l \geq 0$ and $1 \leq l \leq N,\ l \neq k$.

Proof. Evidently, the maximum principle actively used for to study the solutions of single parabolic equations does not apply to systems of such equations. Let us consider a time independent, square integrable vector function $v(x)$ and estimate

$$\left(\frac{\partial u}{\partial t}\Big|_{t=0}, v \right)_{L^2(\mathbb{R},\mathbb{R}^N)} = \left(\lim_{t\to 0} \frac{u(x,t) - u_0(x)}{t}, v(x) \right)_{L^2(\mathbb{R},\mathbb{R}^N)}.$$

By virtue of the continuity of the inner product, the right side of the equality above is equal to

$$\lim_{t\to 0} \frac{(u(x,t), v(x))_{L^2(\mathbb{R},\mathbb{R}^N)}}{t} - \lim_{t\to 0} \frac{(u_0(x), v(x))_{L^2(\mathbb{R},\mathbb{R}^N)}}{t}. \tag{3.7}$$

We choose the initial condition for our problem $u_0(x) \geq 0$ and the constant in time vector function $v(x) \geq 0$ to be orthogonal to each other in $L^2(\mathbb{R}, \mathbb{R}^N)$. This can be achieved, for example for

$$u_0(x) = (\tilde{u}_1(x), ..., \tilde{u}_{k-1}(x), 0, \tilde{u}_{k+1}(x), ..., \tilde{u}_N(x)), \quad v_j(x) = \tilde{v}(x)\delta_{j,k}, \tag{3.8}$$

with $1 \leq j \leq N$, where $\delta_{j,k}$ is the Kronecker symbol and $1 \leq k \leq N$ is fixed. Hence, the second term in (3.7) vanishes and (3.7) is equal to

$$\lim_{t\to 0} \frac{\sum_{k=1}^N \int_{-\infty}^{\infty} u_k(x,t) v_k(x) dx}{t} \geq 0$$

via the nonnegativity of all the components $u_k(x,t)$ and $v_k(x)$ involved in the formula above. Therefore, we obtain

$$\sum_{j=1}^{N}\int_{-\infty}^{\infty}\frac{\partial u_j}{\partial t}\Big|_{t=0}v_j(x)dx\geq 0.$$

By means of (3.8), only the k th component of the vector function $v(x)$ is nontrivial. This gives us

$$\int_{-\infty}^{\infty}\frac{\partial u_k}{\partial t}\Big|_{t=0}\tilde{v}(x)dx\geq 0.$$

Thus, using (3.2) we derive

$$\int_{-\infty}^{\infty}\Big[-\sum_{j=1,\ j\neq k}^{N}a_{k,j}\Big(-\frac{\partial^2}{\partial x^2}\Big)^s\tilde{u}_j(x)+\sum_{j=1,\ j\neq k}^{N}\gamma_{k,j}\frac{\partial \tilde{u}_j}{\partial x}-$$

$$-F_k(\tilde{u}_1(x),...,\tilde{u}_{k-1}(x),0,\tilde{u}_{k+1}(x),...,\tilde{u}_N(x))\Big]\tilde{v}(x)dx\geq 0.$$

Because the nonnegative, square integrable function $\tilde{v}(x)$ can be chosen arbitrarily, we arrive at

$$-\sum_{j=1,\ j\neq k}^{N}a_{k,j}\Big(-\frac{\partial^2}{\partial x^2}\Big)^s\tilde{u}_j(x)+\sum_{j=1,\ j\neq k}^{N}\gamma_{k,j}\frac{\partial \tilde{u}_j}{\partial x}-$$

$$-F_k(\tilde{u}_1(x),...,\tilde{u}_{k-1}(x),0,\tilde{u}_{k+1}(x),...,\tilde{u}_N(x))\geq 0\quad a.e. \tag{3.9}$$

For the purpose of the scaling, we replace all the $\tilde{u}_j(x)$ by $\tilde{u}_j\Big(\frac{x}{\varepsilon}\Big)$ in the inequality above, where $\varepsilon>0$ is a small parameter. This implies

$$-\sum_{j=1,\ j\neq k}^{N}\frac{a_{k,j}}{\varepsilon^{2s}}\Big(-\frac{\partial^2}{\partial y^2}\Big)^s\tilde{u}_j(y)+\sum_{j=1,\ j\neq k}^{N}\frac{\gamma_{k,j}}{\varepsilon}\frac{\partial \tilde{u}_j(y)}{\partial y}-$$

$$-F_k(\tilde{u}_1(y),...,\tilde{u}_{k-1}(y),0,\tilde{u}_{k+1}(y),...,\tilde{u}_N(y))\geq 0\quad a.e. \tag{3.10}$$

Obviously, the second term in the left side of (3.10) is the leading one as $\varepsilon\to 0$. In the case of $\gamma_{k,j}>0$, let us choose here $\tilde{u}_j(y)=e^{-y}$ in a neighborhood of the origin, smooth and tending to zero at the infinities. If $\gamma_{k,j}<0$, then we can pick $\tilde{u}_j(y)=e^{y}$ around the origin and converging to zero at the infinities. Then the left side of (3.10) can be made as negative as possible which will violate inequality (3.10). Note that the last term in the left side of (3.10) will remain bounded. Thus, for the matrix Γ involved in problem (3.1), the off diagonal terms should vanish, so that

$$\gamma_{k,j}=0,\quad 1\leq k,j\leq N,\quad k\neq j.$$

Hence, from (3.10) we derive

$$-\sum_{j=1,\ j\neq k}^{N}\frac{a_{k,j}}{\varepsilon^{2s}}\Big(-\frac{\partial^2}{\partial y^2}\Big)^s\tilde{u}_j(y)-$$

$$-F_k(\tilde{u}_1(y),...,\tilde{u}_{k-1}(y),0,\tilde{u}_{k+1}(y),...,\tilde{u}_N(y))\geq 0 \quad a.e. \tag{3.11}$$

We suppose that some of the $a_{k,j}$ contained in the sum in the left side of (3.11) are strictly positive. Let us choose here all the $\tilde{u}_j(y),\ 1\leq j\leq N,\ j\neq k$ to be identical. For the equation

$$-\Big(-\frac{\partial^2}{\partial x^2}\Big)^s\tilde{u}_j(x)=\tilde{v}_j(x),\quad 0<s<\frac{1}{4}, \tag{3.12}$$

let us assume that its right side belongs to $C_c^\infty(\mathbb{R})$. Obviously, $\tilde{v}_j(x)\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ as well. Then by virtue of the part 1) of Lemma 1.6 of [41], problem (3.12) has a unique solution $\tilde{u}_j(x)\in H^{2s}(\mathbb{R})$. The orthogonality conditions here for the right side of (3.12) are not needed as distinct from the cases of $\frac{1}{4}\leq s<\frac{3}{4}$ and $\frac{3}{4}\leq s<1$ discussed in parts 2) and 3) of Lemma 1.6 of [41]. Suppose the right side of (3.12) is nonnegative on the whole real line. By means of Section 5.9 of [24] we have the explicit formula

$$\tilde{u}_j(x)=-c_s\int_{-\infty}^{\infty}|x-y|^{2s-1}\tilde{v}_j(y)dy,$$

where $c_s>0$ is a constant. Then $\tilde{u}_j(x)$ is negative on $\mathbb{R}$, which is a contradiction to our original assumption. Thus, $\tilde{v}_j(x)$ has the points of negativity on the real line. By making the parameter ε sufficiently small, we can violate the inequality in (3.11). Since the negativity of the off diagonal elements of the matrix A is ruled out due to assumption (3.5), we obtain

$$a_{k,j}=0,\quad 1\leq k,j\leq N,\quad k\neq j.$$

Hence, by virtue of (3.9) we arrive at

$$F_k(\tilde{u}_1(x),...,\tilde{u}_{k-1}(x),0,\tilde{u}_{k+1}(x),...,\tilde{u}_N(x))\leq 0\quad a.e.,$$

where $\tilde{u}_j(x)\geq 0$ and $\tilde{u}_j(x)\in L^2(\mathbb{R})$ with $1\leq j\leq N,\ j\neq k$. ■

The final section of the article deals with the solvability in the sense of sequences of the equation related to the double scale anomalous diffusion. Note that the solvability of the linear Poisson type equations is crucial for establishing the solvability of the nonlinear integro-differential equations (see e.g. [39], [41]).

4 Generalized Poisson type equation with a potential

In this section we will present the two new theorems (see Theorems 4.1 and 4.2 below) dealing with the generalized Poisson type equation with the scalar potential. Indeed, consider the equation

$$\Big\{[-\Delta + V(x)]^{s_1} + [-\Delta + V(x)]^{s_2}\Big\}u = f(x), \quad x \in \mathbb{R}^3 \tag{4.1}$$

with a square integrable right side and the powers $0 < s_1 < s_2 < 1$. The assumptions on our shallow, short-range scalar potential function $V(x)$ were stated in Section 2. The problems with the sum of the negative Laplacians without a potential raised to different fractional powers arise in the studies of the double scale anomalous diffusion (see e.g. [22]). The probabilistic realization of the anomalous diffusion was discussed in [27]. The non-Fredholm operator in the left side of (4.1)

$$L := [-\Delta + V(x)]^{s_1} + [-\Delta + V(x)]^{s_2} \tag{4.2}$$

on $L^2(\mathbb{R}^3)$ is defined via the spectral calculus. It has only the essential spectrum

$$\sigma_{ess}(L) = [0, \ +\infty)$$

and no nontrivial $L^2(\mathbb{R}^3)$ eigenfunctions. By virtue of the spectral theorem, we have

$$L\varphi_k(x) = (|k|^{2s_1} + |k|^{2s_2})\varphi_k(x)$$

with the functions of the continuous spectrum of our Schrödinger operator $\varphi_k(x)$ discussed in Section 2 above. The function $\varphi_0(x)$ in the theorem below will correspond to the case of $k = 0$. In the argument below we will use

$$\tilde{f}(k) = (f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}, \quad k \in \mathbb{R}^3. \tag{4.3}$$

(4.3) is a unitary transform on $L^2(\mathbb{R}^3)$. Corollary 2.2 of [35] under the conditions stated below gives us the estimate

$$|\tilde{f}(k)| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f(x)\|_{L^1(\mathbb{R}^3)}, \tag{4.4}$$

where $I(V) < 1$ is the left side of inequality (2.4). The first result of the section is as follows.

Theorem 4.1. *Let $V(x)$ satisfy (2.3), (2.4) and (2.6) of Assumption 2.1 above, the powers $0 < s_1 < s_2 < 1$ and $f(x) \in L^2(\mathbb{R}^3)$.*

1) If $0 < s_1 < \frac{3}{4}$, let in addition $f(x) \in L^1(\mathbb{R}^3)$. Then equation (4.1) admits a unique solution $u(x) \in L^2(\mathbb{R}^3)$.

2) If $\frac{3}{4} \le s_1 < 1$, let in addition $xf(x) \in L^1(\mathbb{R}^3)$. Then problem (4.1) possesses a unique solution $u(x) \in L^2(\mathbb{R}^3)$ if and only if the orthogonality condition

$$(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0 \tag{4.5}$$

is valid.

Proof. Let us suppose that equation (4.1) admits two solutions $u_1(x), u_2(x) \in L^2(\mathbb{R}^3)$. Then their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3)$ is a solution of the homogeneous problem

$$Lw = 0.$$

Since the operator L on $L^2(\mathbb{R}^3)$ does not have any nontrivial zero modes, $w(x)$ vanishes in $\mathbb{R}^3$.

We apply the generalized Fourier transform (4.3) to both sides of our equation (4.1) and arrive at

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \le 1\}} + \frac{\tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| > 1\}}. \tag{4.6}$$

Here and below χ_A will stand for the characteristic function of a set $A \subseteq \mathbb{R}^3$. Clearly, the second term in the right side of (4.6) can be estimated from above in the absolutely value as

$$\left| \frac{\tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| > 1\}} \right| \le \frac{|\tilde{f}(k)|}{2} \in L^2(\mathbb{R}^3)$$

via the one of our assumptions. Let us first consider the situation when $0 < s_1 < \frac{3}{4}$. Then the first term in the right side of (4.6) can be bounded from above in the absolutely value using inequality (4.4) as

$$\left| \frac{\tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \le 1\}} \right| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f(x)\|_{L^1(\mathbb{R}^3)} \frac{\chi_{\{|k| \le 1\}}}{|k|^{2s_1}} \in L^2(\mathbb{R}^3),$$

which completes the proof of part 1) of our theorem. Let us conclude the argument by treating the situation when the power $\frac{3}{4} \le s_1 < 1$. We will use the representation formula

$$\tilde{f}(k) = \tilde{f}(0) + \int_0^{|k|} \frac{\partial \tilde{f}(q, \sigma)}{\partial q} dq.$$

Here and below σ denotes the angle variables on the sphere and

$$\tilde{f}(0) = (f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)}.$$

This enables us to express the first term in the right side of (4.6) as

$$\frac{\tilde{f}(0)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \le 1\}} + \frac{\int_0^{|k|} \frac{\partial \tilde{f}(q,\sigma)}{\partial q} dq}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \le 1\}}.$$

Evidently, the second term in the sum above can be estimated from above in the absolute value as

$$\left| \frac{\int_0^{|k|} \frac{\partial \tilde{f}(q,\sigma)}{\partial q} dq}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \le 1\}} \right| \le \|\nabla_q \tilde{f}(q)\|_{L^\infty(\mathbb{R}^3)} |k|^{1-2s_1} \chi_{\{|k| \le 1\}} \in L^2(\mathbb{R}^3).$$

Note that under the given conditions $\nabla_q \tilde{f}(q) \in L^\infty(\mathbb{R}^3)$ by means of Lemma 2.4 of [35]. Therefore, it remains to analyze the term

$$\frac{\tilde{f}(0)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \le 1\}}. \tag{4.7}$$

It can be easily verified that (4.7) belongs to $L^2(\mathbb{R}^3)$ if and only if orthogonality condition(4.5) holds. ■

Note that in the first case of our theorem we do not need an orthogonality condition to solve equation (4.1) in $L^2(\mathbb{R}^3)$. Let us introduce the approximate equations

$$\Big\{[-\Delta + V(x)]^{s_1} + [-\Delta + V(x)]^{s_2}\Big\} u_n = f_n(x), \quad x \in \mathbb{R}^3 \tag{4.8}$$

with $n \in \mathbb{N}$ and $0 < s_1 < s_2 < 1$ and establish the solvability in the sense of sequences for our problem (4.1). The final result of the article is as follows.

Theorem 4.2. *Let $V(x)$ satisfy (2.3), (2.4) and (2.6) of Assumption 2.1 above, $n \in \mathbb{N}$, the powers $0 < s_1 < s_2 < 1$ and $f_n(x) \in L^2(\mathbb{R}^3)$, so that $f_n(x) \to f(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$.*

1) If $0 < s_1 < \frac{3}{4}$, let in addition $f_n(x) \in L^1(\mathbb{R}^3)$, $n \in \mathbb{N}$, so that $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$. Then equations (4.1) and (4.8) possess unique solutions $u(x) \in L^2(\mathbb{R}^3)$ and $u_n(x) \in L^2(\mathbb{R}^3)$ respectively, so that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$.

2) If $\frac{3}{4} \le s_1 < 1$, let in addition $xf_n(x) \in L^1(\mathbb{R}^3)$, $n \in \mathbb{N}$, so that $xf_n(x) \to xf(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$ and

$$(f_n(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0 \tag{4.9}$$

is valid for all $n \in \mathbb{N}$. Then problems (4.1) and (4.8) admit unique solutions $u(x) \in L^2(\mathbb{R}^3)$ and $u_n(x) \in L^2(\mathbb{R}^3)$ respectively, so that $u_n(x) \to u(x)$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$.

Proof. Clearly, each equation (4.8) has a unique solution $u_n(x)$ in $L^2(\mathbb{R}^3)$, $n \in \mathbb{N}$ due to the result of Theorem 4.1 above. It can be easily verified that in the case 2) of our theorem the limiting orthogonality relation

$$(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0 \tag{4.10}$$

will hold. Indeed, by means of (4.9) along with estimate (4.4)

$$|(f(x), \varphi_0(x))_{L^2(\mathbb{R}^3)}| = |(f(x) - f_n(x), \varphi_0(x))_{L^2(\mathbb{R}^3)}| \le$$

$$\le \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)} \to 0, \quad n \to \infty.$$

Note that via the assumptions of the second part of the theorem we have $f_n(x) \to f(x)$ in $L^1(\mathbb{R}^3)$ as $n \to \infty$ by virtue of Lemma 3.3 of [34]. Therefore, in both cases of the theorem, limiting equation (4.1) admits a unique solution $u(x) \in L^2(\mathbb{R}^3)$ according to Theorem 4.1. Let us apply the generalized Fourier transform (4.3) to both sides of equation (4.8). This yields

$$\tilde{u}_n(k) = \frac{\tilde{f}_n(k)}{|k|^{2s_1} + |k|^{2s_2}}. \tag{4.11}$$

Formulas (4.11) and (4.6) give us

$$\tilde{u}_n(k) - \tilde{u}(k) = \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \le 1\}} + \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| > 1\}}. \tag{4.12}$$

Evidently, the second term in the right side of (4.12) can be estimated from above in the absolute value as

$$\left| \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \right| \chi_{\{|k| > 1\}} \le \frac{|\tilde{f}_n(k) - \tilde{f}(k)|}{2}.$$

Hence,

$$\left\| \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| > 1\}} \right\|_{L^2(\mathbb{R}^3)} \le \frac{1}{2} \|f_n(x) - f(x)\|_{L^2(\mathbb{R}^3)} \to 0, \quad n \to \infty$$

as assumed. Let us first discuss the case when $0 < s_1 < \frac{3}{4}$. By means of (4.4), we have

$$|\tilde{f}_n(k) - \tilde{f}(k)| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)}.$$

Then the first term in the right side of (4.12) can be bounded from above in the absolute value as

$$\left| \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \le 1\}} \right| \le \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{1 - I(V)} \|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)} \frac{1}{|k|^{2s_1}} \chi_{\{|k| \le 1\}},$$

so that

$$\left\| \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k| \le 1\}} \right\|_{L^2(\mathbb{R}^3)} \le \frac{1}{\sqrt{2}\pi} \frac{1}{1 - I(V)} \|f_n(x) - f(x)\|_{L^1(\mathbb{R}^3)} \frac{1}{\sqrt{3 - 4s_1}} \to 0$$

as $n \to \infty$ as assumed, which completes the proof of the first part of the theorem.

Finally, we proceed to treating the case when $\frac{3}{4} \le s_1 < 1$. Orthogonality relations (4.10) and (4.9) imply that

$$\tilde{f}(0) = 0, \quad \tilde{f}_n(0) = 0, \quad n \in \mathbb{N}.$$

Therefore,

$$\tilde{f}(k) = \int_0^{|k|} \frac{\partial \tilde{f}(q, \sigma)}{\partial q} dq, \quad \tilde{f}_n(k) = \int_0^{|k|} \frac{\partial \tilde{f}_n(q, \sigma)}{\partial q} dq, \quad n \in \mathbb{N}.$$

This enables us to write the first term in the right side of (4.12) as

$$\frac{\int_0^{|k|}\left[\frac{\partial \tilde{f}_n(q,\sigma)}{\partial q} - \frac{\partial \tilde{f}(q,\sigma)}{\partial q}\right]dq}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k|\leq 1\}},$$

which can be easily estimated from above in the absolute value by

$$\|\nabla_q[\tilde{f}_n(q) - \tilde{f}(q)]\|_{L^\infty(\mathbb{R}^3)} |k|^{1-2s_1} \chi_{\{|k|\leq 1\}}.$$

Thus,

$$\left\| \frac{\tilde{f}_n(k) - \tilde{f}(k)}{|k|^{2s_1} + |k|^{2s_2}} \chi_{\{|k|\leq 1\}} \right\|_{L^2(\mathbb{R}^3)} \leq \|\nabla_q[\tilde{f}_n(q) - \tilde{f}(q)]\|_{L^\infty(\mathbb{R}^3)} \frac{2\sqrt{\pi}}{\sqrt{5-4s_1}}.$$

By means of the result of Lemma 3.4 of [34] under the given conditions we have

$$\|\nabla_q[\tilde{f}_n(q) - \tilde{f}(q)]\|_{L^\infty(\mathbb{R}^3)} \to 0, \quad n \to \infty,$$

which completes the proof of the theorem. ■

References

[1] M.S. Agranovich. *Elliptic boundary problems,* Encyclopaedia Math. Sci., vol. 79, Partial Differential Equations, IX, Springer, Berlin (1997), pp. 1–144.

[2] N. Apreutesei, N. Bessonov, V. Volpert, V. Vougalter. *Spatial Structures and Generalized Travelling Waves for an Integro- Differential Equation,* Discrete Contin. Dyn. Syst. Ser. B, **13** (2010), no. 3, 537–557.

[3] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik. *The non-local Fisher-KPP equation: traveling waves and steady states,* Nonlinearity, **22** (2009), no. 12, 2813–2844.

[4] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon. *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Springer-Verlag, Berlin (1987), 319 pp.

[5] E.N. Dancer. *On positive solutions of some pairs of differential equations,* Trans. Amer. Math. Soc., **284** (1984), no. 2, 729–743.

[6] E.N. Dancer. *On positive solutions of some pairs of differential equations. II,* J. Differential Equations, **60** (1985), no. 2, 236–258.

[7] E.N. Dancer, Y. Du, M.A. Efendiev. *Quasilinear elliptic equations on half -and quarter-spaces,* Adv. Nonlinear Stud., **13** (2013), no. 1, 115–136.

[8] Y. Du, M.A. Efendiev. *Existence and exact multiplicity for quasilinear elliptic equations in quarter-spaces,* Patterns of dynamics, Springer Proc. Math. Stat., **205**, Springer, Cham (2017), 128–137.

[9] A. Ducrot, M. Marion and V. Volpert. *Reaction-diffusion problems with non Fredholm operators,* Advances Diff. Equations, **13** (2008), no. 11-12, 1151–1192.

[10] M.A. Efendiev. *Fredholm structures, topological invariants and applications,* American Institute of Mathematical Sciences (2009), 205 pp.

[11] M.A. Efendiev. *Finite and infinite dimensional attractors for evolution equations of mathematical physics.* GAKUTO International Series. Mathematical Sciences and Applications, **33**. *Gakkōtosho Co., Ltd., Tokyo*, (2010). 239 pp.

[12] M.A. Efendiev. *Evolution equations arising in the modelling of life sciences.* International Series of Numerical Mathematics, **163**. Birkhäuser/Springer Basel AG, Basel, (2013). 217 pp.

[13] M.A. Efendiev, *Symmetrization and stabilization of solutions of nonlinear elliptic equations,* Fields Institute Monographs, **36**. Fields Institute for Research in Mathematical Sciences, Toronto, ON; Springer, Cham (2018), 258 pp.

[14] M.A. Efendiev, L.A. Peletier. *On the large time behavior of solutions of fourth order parabolic equations and ε- entropy of their attractors,* C. R. Math. Acad. Sci. Paris, **344** (2007), no. 2, 93–96.

[15] M.A. Efendiev, V. Vougalter. *Solvability of some integro-differential equations with drift,* Osaka J. Math., **57** (2020), no. 2, 247–265.

[16] M.A. Efendiev, V. Vougalter. *Verification of Biomedical Processes with Anomalous Diffusion, Transport and Interaction of Species,* DOI: 10.1007/978-981-15-9034-4_5, Book: Nonlinear Dynamics, Chaos, and Complexity, Springer, (2021).

[17] M.A. Efendiev, V. Vougalter. *Solvability in the sense of sequences for some fourth order non-Fredholm operators,* J. Differential Equations, **271** (2021), 280–300.

[18] M.A. Efendiev, V. Vougalter. *Existence of solutions for some non-Fredholm integro-differential equations with mixed diffusion,* J. Differential Equations, **284** (2021), 83–101.

[19] M.A. Efendiev, S.V. Zelik. *The attractor for a nonlinear reaction-diffusion system in an unbounded domain,* Comm. Pure Appl. Math., **54** (2001), no. 6, 625–688.

[20] H.G. Gebran, C.A. Stuart. *Fredholm and properness properties of quasilinear elliptic systems of second order,* Proc. Edinb. Math. Soc. (2), **48** (2005), no. 1, 91–124.

[21] H.G. Gebran, C.A. Stuart. *Exponential decay and Fredholm properties in second-order quasilinear elliptic systems,* J. Differential Equations, **249** (2010), no. 1, 94–117.

[22] N. Guerngar, E. Nane, R. Tinaztepe, S. Ulusoy, H.W. Van Wyk. *Simultaneous inversion for the fractional exponents in the space-time fractional diffusion equation* $\partial_t^\beta u = -(-\Delta)^{\frac{\alpha}{2}} u - (-\Delta)^{\frac{\gamma}{2}} u$, Fract. Calc. Appl. Anal., **24** (2021), no. 3, 818–847.

[23] T. Kato. *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann., **162** (1965/1966), 258–279.

[24] E.H. Lieb, M. Loss. *Analysis. Graduate Studies in Mathematics*, **14**, American Mathematical Society, Providence (1997), 278 pp.

[25] J.L. Lions, E. Magenes. *Problèmes aux limites non homogènes et applications.* Volume 1. Dunod, Paris, 1968.

[26] V.G. Maz'ja, M. Otelbaev. *Imbedding theorems and the spectrum of a certain pseudodifferential operator*, (Russian) Sibirsk. Mat. Z., **18** (1977), no. 5, 1073–1087, 1206.

[27] R. Metzler, J. Klafter. *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep., **339** (2000), 1–77.

[28] P.J. Rabier, C.A. Stuart. *Fredholm and properness properties of quasilinear elliptic operators on* $\mathbb{R}^N$, Math. Nachr., **231** (2001), 129–168.

[29] M. Reed, B. Simon. *Methods of modern mathematical physics. III. Scattering theory*, Academic Press (1979), 463 pp.

[30] I. Rodnianski, W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math., **155** (2004), No. 3, 451–513.

[31] L.R. Volevich. *Solubility of boundary problems for general elliptic systems,* Mat. Sbor., **68** (1965), 373–416; English translation: Amer. Math. Soc. Transl., **67** (1968), Ser. 2, 182–225.

[32] V. Volpert. *Elliptic partial differential equations.* Volume I. Fredholm theory of elliptic problems in unbounded domains. Birkhäuser, Basel (2011).

[33] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski. *Solvability conditions for elliptic problems with non-Fredholm operators,* Appl. Math., **29** (2002), no. 2, 219–238.

[34] V. Volpert, V. Vougalter. *Solvability in the sense of sequences to some non-Fredholm operators,* Electron. J. Differential Equations, **160** (2013), 16 pp.

[35] V. Vougalter, V. Volpert. *Solvability conditions for some non Fredholm operators,* Proc. Edinb. Math. Soc. (2), **54** (2011), no. 1, 249–271.

[36] V. Vougalter, V. Volpert. *On the existence of stationary solutions for some non-Fredholm integro-differential equations,* Doc. Math., **16** (2011), 561–580.

[37] V. Vougalter, V. Volpert. *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2** (2012), no. 4, 473–496.

[38] V. Vougalter, V. Volpert. *On the solvability in the sense of sequences for some non-Fredholm operators,* Dyn. Partial Differ. Equ., **11** (2014), no. 2, 109–124.

[39] V. Vougalter, V. Volpert. *Existence of stationary solutions for some integro-differential equations with anomalous diffusion,* J. Pseudo-Differ. Oper. Appl., **6** (2015), no. 4, 487–501.

[40] V. Vougalter, V. Volpert. *Existence of stationary solutions for some non-Fredholm integro-differential equations with superdiffusion,* J. Pseudo-Differ. Oper. Appl., **9** (2018), no. 1, 1–24.

[41] V. Vougalter, V. Volpert. *Solvability of some integro-differential equations with anomalous diffusion.* Regularity and stochasticity of nonlinear dynamical systems, Nonlinear Syst. Complex., **21**, Springer, Cham (2018), 1–17.

[42] V. Vougalter, V. Volpert. *Existence in the sense of sequences of stationary solutions for some non-Fredholm integro- differential equations,* J. Math. Sci. (N.Y.) **228**, no. 6, Problems in mathematical analysis, **90** (Russian) (2018), 601–632.