Quantization of polynomial minimal surfaces II

Jens Hoppe

ABSTRACT. Non–commutative analogues of a class of infinitely extended 2 dimensional time–dependent surfaces that sweep out in space time 3–manifolds of vanishing mean curvature, and are described by polynomial equations, are constructed.

As found in [1]

$$
\begin{aligned}
x &= \frac{\sqrt{2}}{\tau}\sqrt{\mu^2+\varepsilon}\cos\varphi = \frac{\sqrt{2}}{\tau}\bar{x}(\mu,\varphi) = R(\tau,\mu)\cos\varphi \\
y &= \frac{\sqrt{2}}{\tau}\sqrt{\mu^2+\varepsilon}\sin\varphi = \frac{\sqrt{2}}{\tau}\bar{y}(\mu,\varphi) = R(\tau,\mu)\sin\varphi \\
\zeta &:= t-z = \frac{-\mu^2-\frac{\varepsilon}{3}}{\tau^3} = \frac{-1}{\tau^3}\bar{\zeta},\ \tau = \frac{t+z}{2},
\end{aligned} \tag{1}
$$

satisfying

$$
\begin{aligned}
&\ddot{x} = \{\{x,y\},y\},\ \ddot{y} = \{\{y,x\},x\} \\
&\ddot{\zeta} = \{\{\zeta,x\},x\} + \{\{\zeta,y\},y\}\ (=:\Delta\zeta) \\
&\ddot{\tau} = \Delta\tau(=0)
\end{aligned} \tag{2}
$$

(where $\{f,g\} := \frac{\partial f}{\partial\mu}\frac{\partial g}{\partial\varphi} - \frac{\partial g}{\partial\mu}\frac{\partial f}{\partial\varphi}$ and $\cdot = \frac{\partial}{\partial\tau}$), and resulting from a separation Ansatz for

$$\ddot{R} = R(RR')', \tag{3}$$

and solving

$$\{\{\bar{x},\bar{y}\},\bar{y}\} = \bar{x},\ \{\{\bar{y},\bar{x}\},\bar{x}\} = \bar{y} \tag{4}$$

and

$$\zeta' = \dot{R}R', \quad 2\dot{\zeta} = \dot{R}^2 + R^2R'^2, \tag{5}$$

as well as parametrizing

$$(t^2+x^2+y^2-z^2)(t+z)^2 = \frac{16}{3}\varepsilon \in \mathbb{R}, \tag{6}$$

describe 3 manifolds Σ_3 of vanishing mean curvature in $\mathbb{R}^{1,3}$ (see [5], [6] for other polynomial ones). As (4) may be written as

$$\{\bar{x},\bar{y}\} = \bar{\mu},\ \{\bar{y},\bar{\mu}\} = -\bar{x},\ \{\bar{\mu},\bar{x}\} = -\bar{y} \tag{7}$$

it is easy to see that hermitean operators X, Y, H satisfying

$$[X, Y] = iH,\ [Y, H] = -iX,\ [H, X] = -iY \tag{8}$$

(i.e. representations of $so(1,2)$; note that these do not necessarily have to give rise to group–representations, in contrast to (5.29) of [4]; so, e.g., k in (**) on p.27 of [3] need not be restricted to half–integers; any $k > 0$ would do) via $X_1 := \frac{\sqrt{2}}{\tau}X$, $X_2 := \frac{\sqrt{2}}{\tau}Y$ will then solve the 'membrane–matrix–model' [2] equations

$$\ddot{X}_i = -\big[[X_i, X_j], X_j\big],\ \sum_{i=1}^{2}[X_i, \dot{X}_i] = 0. \tag{9}$$

Just as

$$\bar{\mu}^2 - \bar{x}^2 - \bar{y}^2 = -\varepsilon, \tag{10}$$

the left hand side being a Casimir function of (7), $H^2 - X^2 - Y^2 = -Q = -C_2$ will be the standard Casimir operator, i.e. for irreducible representations of (8) (cp. [4], [3]) be proportional to the identity. Note that $\varepsilon < 0$, $\mu \geqslant \sqrt{-\varepsilon}$ will correspond to Σ_3 being time–like. Interestingly ζ, which in the classical theory is needed to (re)construct Σ_3 (once x and y are known) and usually difficult to 'quantize' (leading to the non–commutative 'membrane–matrix–model' often believed to *not* be Lorentz–invariant), in the above example does satisfy

$$\ddot{\hat{\zeta}} = -\big[[\hat{\zeta}, X], X\big] - \big[[\hat{\zeta}, Y], Y\big] =: \hat{\Delta}\zeta \tag{11}$$

for the obvious choice

$$\hat{\zeta} := \frac{-1}{\tau^3}(H^2 + \frac{\varepsilon}{3}) = \frac{-1}{\tau^3}(X^2 + Y^2 - \frac{2}{3}\varepsilon), \tag{12}$$

just as X and Y (and τ) do, so that one may think of

$$\begin{aligned} X^0 &= T = \tau + \frac{\hat{\zeta}}{2} \\ \text{and} \quad X^3 &= Z = \tau - \frac{\hat{\zeta}}{2} \end{aligned} \tag{13}$$

as the quantizations of t and z in this model (and could try to let Lorentz–transformations act on $X^\mu = (X^0, X^1, X^2, X^3)$).

Acknowledgement. I thank R.Lautenbacher for his interest, and confirming that the class of representations of (8) that one can take is not restricted to the usual list (of $SO(1,2)$ representations, derived long ago by V.Bargmann [4])

References

[1] J.Hoppe, *Some Classical Solutions of Membrane Equations in 4 Space-Time Dimensions*, Phys.Lett.B 321 1994

[2] J.Hoppe, *Quantum theory of a massless relativistic surface*, Ph.D. thesis, MIT 1982 http://dspace.mit.edu/handle/1721.1/15717,

[3] J.Hoppe, *Lectures on Integrable Systems*, Springer 1992

[4] V.Bargmann, *Irreducible Unitary Representations of the Lorentz Group*, Annals of Mathematics, Vol.48, #3, July 1947

[5] J.Hoppe, *Exact algebraic M(em)brane solutions*, arXiv:2107.00569

[6] J.Hoppe, *On the quantization of some polynomial minimal surfaces*, arXiv:2107.03319

Braunschweig University, Germany
E-mail address: jens.r.hoppe@gmail.com