Electromagnetic Foundation of Dirac Theory

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Abstract

The dynamics of classical charges subject to a particular variant of electromagnetic direct particle interaction are shown to derive from a homogeneous differential equation in a Clifford Multivector. Under appropriate conditions the multivector can be factorized to give a Dirac Equation whose bi-spinor operands are eigenvectors of the multivector, thereby giving an electromagnetic basis for the Dirac Equation.

The Clifford multivector is an ensemble of vector and bi-vector contributions from the potential and Faraday of the auxiliary ('adjunct') fields of direct particle interaction, each member generated by a unique current. The presumption of light-speed motion of the charge generates non-linear constraints on these fields that preclude their super-position in the traditional sense. Representation invariance (e.g. Fourier-space versus real space) inherent in a linear differential system survives unaffected however. These conditions are shown to be responsible for the otherwise enigmatic eigenvalue selection / 'wavefunction collapse' behavior characteristic of Dirac bi-spinors.

Though time-symmetric adjunct fields are intrinsic to the direct action paradigm, their elimination has been the main focus of works in that field - notably by Wheeler and Feynman - in an attempt to make direct particle interaction conform to Maxwell field theory. By contrast, in this work time-symmetric fields are the foundation of Dirac bi-spinors. Accidentally we also find a novel explanation of the emergence of exclusively retarded radiation from the direct action paradigm that makes no appeal to special boundary conditions.

Keywords

Quantum Theory, Dirac's Equation, Clifford Algebra, direct particle interaction, time-symmetry, Majorana spinors.

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1 Introduction

1.1 Historical Context

Direct Particle Interaction, henceforth DPI, is a version of electromagnetism distinct from the Maxwell theory first proposed by Schwarzschild [1], Tetrode [2], and Fokker [3] in which the EM fields and potentials are not independent dynamical variables, and the only electromagnetic contribution to the action comes from direct interaction between 4-currents. All electromagnetic energy and momentum is to be accounted for in the interaction between charges so that any EM energy leaving a charge must be destined for absorption by another charge. Accordingly DPI does not admit strictly vacuum degrees of freedom, strictly on-shell photons, or radiation exactly as portrayed by field theory. Since its inception a challenge for DPI has been an explanation for the observational evidence apparently in favor of exclusively retarded radiation.[1] Though Wheeler and Feynman [4,5] showed that radiation-like behavior, including radiation reaction, could arise within DPI if the future is sufficiently absorbing, the subsequent discovery of accelerating cosmological expansion rendered their explanation untenable because the universe is nearly transparent on the forward light-cone [6,7] (see also the works by Pegg [8,9]).

The books by Hoyle and Narlikar [10,11] and Davies [6] are recommended for a comprehensive review of Direct Particle Interaction. We do not discuss self-action here, which has had a complicated historical relationship with direct action. The interested reader is referred to the short review on that topic in [12].

1.2 Relation to other work

Clifford Formalism

The focus of this work is on exposing the classical foundations of Quantum Theory, employing Clifford algebra primarily as an intermediate tool, eventually departing from the Clifford formalism to obtain Dirac bi-spinors that are strictly compliant with the traditional Dirac Theory. We share with Hestenes [13,14] (see also [15]) that the Dirac equation be founded on real (versus complex) quantities, though there are differences both in how that is implemented here, and in the outcome. In particular, and as shown by Rodriguez [16,17], the Clifford object that is the operand of the Dirac-Hestenes equation

1. An outcome of this work is that the observational facts are compatible with a different interpretation.

operator is not a Dirac bi-spinor, and does not share the same Fierz Identities as those of a Dirac bi-spinor (see [18]).[2]

Pilot Wave Model

This work has in common with the pilot wave model of de Broglie [19,20] and Bohm [21-24] in both the non-relativistic (Schrödinger) domain and its relativistic extension (for example [25,26]) that the electron is a classical point charge following flow lines generated by a 'field'. The book by Holland [27] is recommended for a thorough exposition of the de Broglie Bohm theory. See [28-30] for journal-paper reviews of the Broglie Bohm theory, including its extension to quantum field theory. In common with those extensions, and of relevance to this work, the original - Schrödinger domain - model has since been re-cast by Hiley [31,32,25] in terms of Clifford algebra.

Though the psi-field and associated quantum potential of the pilot wave model are sufficient for the task of reproducing standard theory, the de Broglie Bohm model is silent on the origin of the field, which seems ad hoc and is disconnected from classical field theory.[3] Even so, that model is to be credited for its pioneering role in expanding the language employed to 'explain' QM to include a classical particle (in addition to the wavefunction) and by providing an example of a successful epistemological alternative to the Bohr / Copenhagen doctrine.

Stochastic Electrodynamics

The pilot-wave model applied to the H atom predicts that the electron is stationary relative to the nucleus, with the attractive charge opposed by the quantum potential. The position of the charge is given by a stationary classical probability distribution, an outcome that Bohm did not find satisfactory. He suggested the model be augmented by a noise source (e.g. the EM ZPF) in order that the stationarity in 3-space be replaced by statistical stationarity. There is a loose connection with stochastic electrodynamics, originating with Marshall [33,34] and extensively developed by Boyer [35-41], with contributions from Pegg [42], and many others. Whereas the de Broglie Bohm description is predictively

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2. The Fierz Identities are bi-linear relationships between the different $n \in [0,4]$ blades in the outer product $\psi\bar{\psi}$ (i.e. between the $\langle \psi\bar{\psi} \rangle_n$) as a consequence of the reduced number of degrees of freedom in a multivector restricted to this form.

3. The causal flow is one-way, from field to particle, for example.

compatible with QM, at least in the non-relativistic domain, stochastic electrodynamics is known to be compatible only in a very few special cases (see for example [43,44]).[4] Though in conventional QM the EM ZPF generates small 'radiative corrections' to the isolated (no EM coupling) Schrödinger states, of which the Lamb shift is an example, much of the stochastic electrodynamics literature is based on the possibility that the stability of the H atom states, for example, can be explained entirely within classical Maxwell theory augmented by a classical EM field that mimics the ZPF of QED. Effectively, stochastic electrodynamics posits that the EM ZPF is the source not just of the Lamb shift, but of all QM, including – if only implicitly – the potential of the de Broglie Bohm model (see for example [45], [46]). These more ambitious claims do not withstand scrutiny however. One of the (several) reasons for the failure is that the ZPF is a noise field, and cannot therefore be the foundation of processes that depend on quantum coherence – i.e. constructive and destructive interference of the wavefunction. However, though there is otherwise little overlap, this paper shares with the aspiration of stochastic electrodynamics that quantum theory is founded on classical EM.

Source Theory

In contrast to the classical ZPF paradigm of stochastic electrodynamics are works that question whether not the EM ZPF of QED is necessary or real. The ZPF-associated infinities in the EM fields can be removed by normal ordering of photon operators [47], thereby resolving the problem of the infinite Cosmological constant. The ultra-violet divergence of the mass of charged matter can be removed by renormaliztion. But spontaneous emission, Casimir forces, and the Lamb shift, all remain as apparently due to the (now residual) ZPF however.

Yet though Bohr famously proclaimed that the Casimir effect proves the reality of the ZPF, Schwinger [48-52] showed that the Casimir force could be explained by 'direct' EM interactions between the electrons without reference to the ZPF driving the motion (see also the works by Milonni [53,54] and Milton [55]). The Casimir force can be computed by superposing van der Waals forces between pairs of electron oscillators between the plates. It is amenable to Schwinger's 'source-theory' approach because the system can be analyzed as closed and in steady-state; there are no retardation effects. It is important here that if an EM-coupled system is in a steady state then any time asymmetry due to retardation of

4. Despite published claims to the contrary stochastic electrodynamics does not provide a classical explanation of the stability of the H atom for example.

EM fields will have been averaged out. Even if all EM interactions are modeled as retarded the end result will be the same as if they had been modeled as time-symmetric, or advanced. Hence there is no imperative in the Schwinger's source theory to accommodate time-symmetric electromagnetic exchanges, and no such association seems to have been made historically.

There are claims that spontaneous emission can also be accommodated within classical theory, without appeal to the ZPF [56-60], which would leave just the Lamb-shift as an irreducibly ZPF phenomenon. In [61,62] Jaynes points out that Lamb shift type corrections arise in classical coupled oscillators, without appeal to extraneous ZPF-type fields. An implication appears to be that the Lamb shift could also be accommodated within a Schwinger-type source theory. But Jaynes' suggestion requires that the ZPF allegedly responsible for the Lamb-shift be sourced by distant charged matter in such a manner that the system appears closed and in steady-state. Distinct from the Casimir case, this is not possible because our universe is transparent to radiation on the forward light cone [63,6,64] - *unless* the EM fields responsible are time-symmetric relative to their sources.[5] Jaynes' suggestion could be taken as motivation for the investigation / adoption of a theory of Direct Particle Interaction therefore, in which case DPI could be regarded as a generalization of the Schwinger source theory. Whether or not something like a 'reduced' ZPF then emerges from DPI is not pursued in this paper, which is focused instead on the acoustic rather than the optical branch of the collective modes (see Section 5). Pegg's work [42] is seemingly supportive of that outcome however.

Random Walks

There have been efforts to mimic the Schrödinger and Dirac equations with classical diffusion processes / random walks by Nagasawa [65], Nelson [66,67], Ord [68-70] and others, e.g. [71]. Such efforts face challenges similar to those of stochastic electrodynamics. Though Nelson in particular seems to have had some success in reproducing quantum behavior from diffusion processes, the rules governing the jump probabilities are exotic and lack physical motivation.[6] It is important that all of these have in common that in order to establish a 'classical' probability distribution that matches those of QM the diffusion jump probability at $\mathbf{x}$ are not Markovian, but depend on the (probabilistic) history of visits to $\mathbf{x}$.

5. Here we are ignoring the possibility that the future conformal singularity is a time-like mirror.

6. This is to be expected. The 'diffusion coefficient' in the Schrödinger equation is imaginary; the Schrödinger wavefunction is a slowly-varying envelope of a solution to a hyperbolic differential equation.

Possibly there is a connection with the ensembles of mutually exclusive possibilities that play a prominent role in this work.

Polarizable Vacuum

There have been attempts to derive the Dirac equation within classical Maxwell theory that appeal to an hypothetical omnipresent linear polarizable medium [72]. See also the remarks by Gsponer [73], and Sexl [74] on alleged Maxwell-Dirac equivalence. Though there is no (other) evidence for such a medium, nonetheless one could argue that there is some overlap with the theory presented here. Within the direct particle interaction paradigm, and subject to the constraints and qualifications described in Section 5, the collective modes of all ($\sim 10^{80}$) charges gives a local isolated charge the appearance of moving through a local polarizable medium.

Time-Symmetric Presentations of QM

Though Cramer [75-77] does not attempt to give an explicit electromagnetic foundation for the wavefunction, his 'Transactional Interpretation' of QM captures something of role of time-symmetric exchanges in this work that are crucial to the emergence of Dirac dynamics from an entirely classical EM framework. Cramer's casting - in the non-relativistic domain - of the Schrödinger wavefunction and its charge conjugate as 'offer' and 'accept' waves approximately correspond, respectively, with the retarded and advanced components of time-symmetric exchanges.

The theory of weak-value measurements due to Aharonov, Albert, and Vaidman [78] that grant equal status to the initial and final boundary conditions on the wavefunction has helped draw attention to the time-symmetry already present in traditional quantum theory, but which derives, according to this work, from the time-symmetry of the EM fields that underlie the wavefunction. Sutherland [79-82] makes a case for retro-causal influences underpinning QM, granting the final boundary condition employed to explain weak-value measurement the same status as the initial boundary condition, with the effect that the wavefunction at all intermediate times depends symmetrically on both – in all cases. With this construction he is able to give an entirely local 'ontological explanation' for entangled-state behavior such as in the Bell experiment which does not refer to a preferred frame. The claim here is not that QM is at fault predictively, or that its predictions are at odds with special relativity, but that the particle and wavefunction can be given an ontological status at all *intermediate* times consistent with special relativity.

Price, Wharton, Evans and Miller [83-86], have argued not only that the non-locality intrinsic to QM is suggestive of retro-causal influences, but have suggested (correctly, from the perspective of this work) this be taken as evidence of a direct particle interaction foundation of quantum dynamics.

Barut Zanghi Paper

We mention in passing the influential paper by Barut and Zanghi [87], which showed how to reproduce the algebraic structure of the observables of the Dirac Theory with a classical theory of a point charge augmented with spinor degrees of freedom. The goal of that work was not to reproduce the Dirac *equation*, however. Its achievement was in constructing a classical analog that was faithful to the Dirac equation so that 'canonical quantization' (the replacement of Poisson brackets with anti-commutators) reproduces the algebra of the observables of the Dirac theory. By contrast this work reproduces not only the Dirac equation '*ab initio*' from a particular variant of classical EM theory (and therefore the algebra of its observables) but also the attendant machinery of eigenvalue selection by observation, neither of which were the aim or focus of the Barut-Zanghi work.

Role of Representation

Stochastic Electrodynamics, theories of 'augmented' diffusion and the de Broglie Bohm model share a commitment to representations in 'real' 'x' space, at odds with presentations of Dirac theory as independent of the functional representation of solutions, and also with the theory presented in this paper, in which representation independence plays a crucial role in the explanation of wavefunction collapse / eigenvalue-selection.[7] Representation-independence and mutual exclusion of modes within a particular representation appear to be crucial also for the multi-particle 'second quantized' generalization of the theory presented here.

1.3 Structure of this document

The direct particle interaction paradigm is sketched Section 2, which lays the foundation for the existence of time-symmetric EM fields. Section 3 gives the motion of a classical massless light-speed charge in the presence of a given EM field, irrespective of the origin of the latter. The asymmetry typical

7. This is a reference to the representation in function-space of each component of a bi-spinor ψ. By representation independence is meant the form-invariance of expectations computed from ψ in different function-spaces - for example expressed in 'real' (x) space, versus its Fourier transform.

of the Maxwell theory between the roles of the fields and the currents is overcome by expressing the dynamics in terms of the ensembles introduced in Section 4. The outcome is a coupled differential – difference system involving the ensemble potential and Faraday (the EM multivector) of a local and all other (distant) charges, without reference to the currents. Two distinct branches for the modes of that system are identified in Section 5. The optical branch has the character of the traditional Maxwell theory. The acoustic branch, which lies outside the scope of the Maxwell theory, becomes the focus of the remainder of this document. Section 6 focusses on the dynamics of the acoustic mode multivector and its relationship to the Dirac equation. The Dirac currents are analyzed in Section 7, revisited in Appendix C. The relationship between superposition and mutual exclusion peculiar to the theory is discussed in Section 8. Section 9 summarizes the main findings.

Outer-products of Majorana bi-spinors play a prominent role in this work. Appendix B examines their properties in isolation, and Appendix C explains their role in the 4-way decomposition of both Dirac bi-spinors and the associated spin and charge currents. The notation used throughout is summarized in Appendix A.

2 Direct Particle Interaction

2.1 Action

The electromagnetic direct particle interaction is

$$I_{DPI} = -\int \mathrm{d}^4 x \int \mathrm{d}^4 x' \, \mathrm{G}\left(x - x'\right) \mathbb{j}\left(x\right) \circ \mathbb{j}\left(x'\right) \tag{1}$$

where

$$\begin{gathered} \mathrm{G}\left(x\right) = \frac{1}{4\pi} \delta\left(x^2\right) \Rightarrow \partial^2 \, \mathrm{G}\left(x\right) = \delta^4\left(x\right) \\ \mathbb{j}\left(x\right) = \left|e\right| \int \mathrm{d}\lambda \, \mathbb{v}\left(\lambda\right) \delta^4\left(x - q\left(\lambda\right)\right); \quad \mathbb{v}\left(\lambda\right) = \frac{\mathrm{d}\,\mathbb{q}\left(\lambda\right)}{\mathrm{d}\lambda} \end{gathered} . \tag{2}$$

$\mathbb{j}$ is a Lorentz vector. $\mathbb{q}\left(\lambda\right)$ is a Lorentz vector with translation-invariance excepted. $\mathbb{v}$ is a 4-vector and it is a Lorentz vector iff λ is a Lorentz scalar. A double strike font signifies the object is to be considered an element of $C\ell_{1,3}\left(\mathbb{R}\right)$ rendered in $\mathrm{M}_4\left(\mathbb{C}\right)$, where appropriate. (An exception introduced later is the EM multivector, which is in $C\ell_{1,3}\left(\mathbb{C}\right)$). x and $q\left(\lambda\right)$ are also Lorentz vectors, but so-written are considered to be represented more conventionally, i.e. in $\mathbb{R}^4$ with Minkowski norm, and $x^2 = x^\mu x_\mu$ etc. Where necessary we refer to components in 3+1 D, e.g. $x=(t,\mathbf{x})$. Hence, since they all refer to the same object, $\mathbb{q}(\lambda)\cong q(\lambda)\cong\left(q^0(\lambda),\mathbf{q}(\lambda)\right)$.

Due to the structure of (1) an anti-symmetric component of $\mathrm{G}(x)$ makes no contribution to the action. Consequently DPI effectively mandates a Green's function that is invariant under negation of any of the coordinates, and is thereby distinguished from traditional theory by its restriction to time-symmetric interactions relative to the sources.

Let the currents be broken into segments $\mathbb{q}(\lambda)\to\{\mathbb{q}_l(\lambda)\} = \{\mathbb{q}_1(\lambda),\mathbb{q}_2(\lambda),\dots,\mathbb{q}_N(\lambda)\}$ each with constant sign of $\mathrm{d}\,\mathbb{q}_l(\lambda)/\mathrm{d}\,\lambda$, so that

$$\begin{aligned}\mathbb{j}(x)&=\sum_{l=1}^{N}\mathbb{j}_l(x)\\ \mathbb{j}_l(x)&=e_l\mathbb{v}_l(t)\,\delta^3\left(\mathbf{x}-\mathbf{q}_l(t)\right)\\ \mathbb{v}_l(t)&=\left(1,\mathbf{v}_l(t)\right);\quad \mathbf{v}_l(t)=\frac{\mathrm{d}\,\mathbf{q}_l(t)}{\mathrm{d}\,t}\end{aligned}\quad. \tag{3}$$

Using (3) in (1) and denying self-action leads to

$$\begin{aligned}I_{DPI}&\to-\sum_{\substack{k,l=1\\k\neq l}}^{N}\int\mathrm{d}^4x\int\mathrm{d}^4x'\,\mathrm{G}(x-x')\,\mathbb{j}_k(x)\circ\mathbb{j}_l(x')\\ &=-\sum_{\substack{k,l=1\\k\neq l}}^{N}\frac{e_ke_l}{4\pi}\int\mathrm{d}\,t\int\mathrm{d}\,t'\,\delta\left(s_{k,l}^2(t,t')\right)\mathbb{v}_k(t)\circ\mathbb{v}_l(t');\quad s_{k,l}(t,t')=q_k(t)-q_l(t')\end{aligned}\quad. \tag{4}$$

2.2 Adjunct fields

Classical Current

With reference to the second part of Eq. (4), the subsequent introduction of x to denote pre-existing $\mathbb{R}^4$ spacetime is a mathematical abstraction. This applies to the current (2), which in DPI therefore has a derivative status relative to $q(t)$. To be consistent with the adjunct potential (see below), in this (DPI) context the subjects of (2) and (3) should properly be called *adjunct* currents, to be consistent with their use as coined by Wheeler and Feynman.

Adjunct Potential

The adjunct potential [4,5] generated by the l^{th} charge is

$$\mathbb{A}_l(x):=\int\mathrm{d}^4x'\,\mathrm{G}(x-x')\,\mathbb{j}_l(x')=e_l\int\mathrm{d}\,t'\,\mathbb{v}_l(t')\,\mathrm{G}\left(x-q_l(t')\right)=\frac{e_l}{4\pi}\int\mathrm{d}\,t'\,\mathbb{v}_l(t')\,\delta\left(\left(x-q_l(t')\right)^2\right) \tag{5}$$

a consequence of which is

$$\partial^2 \mathbb{A}_l(x) = \mathbb{j}_l(x) \quad . \tag{6}$$

The total adjunct potential from N charges is

$$\mathbb{A}(x) = \sum_{l=1}^{N} \mathbb{A}_l(x) \quad . \tag{7}$$

We will also need to refer to the potential of all but the l^{th} current:

$$\mathbb{A}_{\bar{l}}(x) := \mathbb{A}(x) - \mathbb{A}_l(x) \quad . \tag{8}$$

The technique of distinguishing between fields according to their origin is due to Leiter [88]. x in $\mathbb{A}(x)$ should not be taken to imply a pre-existing $\mathbb{R}^4$ spacetime; direct particle interaction grants the adjunct potential a physically meaningful role only on the worldlines of charges.

The Lorenz gauge is mandated by the structure of (5), in particular because the Green's function $\mathrm{G}(x,x') \to \mathrm{G}(x-x')$ depends only on the coordinate difference $x - x'$:[8]

$$\partial\!\!\!/ \circ \mathbb{A}_l(x) = \int \mathrm{d}^4 x' \, \partial\!\!\!/ \, \mathrm{G}(x-x') \circ \mathbb{j}_l(x') = -\int \mathrm{d}^4 x' \, \mathbb{j}_l(x') \circ \partial\!\!\!/' \, \mathrm{G}(x-x') = \int \mathrm{d}^4 x' \, \mathrm{G}(x-x') \, \partial\!\!\!/' \circ \mathbb{j}_l(x') = 0 \quad . \tag{9}$$

Clearly (9) implies $\partial\!\!\!/ \circ \mathbb{A}(x) = 0$. Applying (5) and (8) to Eq. (4) gives

$$I_{DPI} = -\sum_{l=1}^{N} \int \mathrm{d}^4 x \, \mathbb{j}_l(x) \circ \mathbb{A}_{\bar{l}}(x) = -\sum_{l=1}^{N} e_k \int \mathrm{d}t \, \mathbb{v}_l(t) \circ \mathbb{A}_{\bar{l}}(q_l(t)) = -\int \mathrm{d}^4 x \, \mathbb{j}(x) \circ \mathbb{A}(x) - I_{self} \tag{10}$$

where

$$I_{self} = -\sum_{l=1}^{N} \int \mathrm{d}^4 x \, \mathbb{j}_l(x) \circ \mathbb{A}_l(x) \quad . \tag{11}$$

Note that the $\mathbb{q}_l(t)$ are the only dynamical degrees of freedom - the action is not extremized by variation of the $\mathbb{A}_l(x)$.

Properties

The adjunct potential of direct particle interaction differs from a potential of traditional field theory in that the adjunct potential:

i) Is always sourced.

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8. $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ has the usual meaning. $\mathbb{a} \circ \mathbb{b} = (\mathbb{a}\mathbb{b} + \mathbb{b}\mathbb{a})/2$ is the scalar product of two Clifford vectors. Likewise $a \circ b = a^\mu b_\mu$.

ii) Necessarily satisfies the Lorenz gauge condition.

iii) Is time-symmetric relative to the source.

iv) Is physically consequential where it originates and where it is terminated.[9,10,11] (See also the footnote in [89].)

Consequent to iv) is that the solutions of $\partial^2 \mathbb{A} = \mathbb{0}$ are everywhere physically inconsequential.

Adjunct Faraday

The adjunct Faraday bi-vector is [12]

$$\mathbb{F}_l = \mathbb{F}_l\left(x\right) = \not\partial \wedge \mathbb{A}_l = \not\partial \mathbb{A}_l - \not\partial \circ \mathbb{A}_l = \not\partial \mathbb{A}_l \ . \tag{12}$$

We will need also

$$\mathbb{F}\left(x\right) = \sum_{l=1}^{N} \mathbb{F}_l\left(x\right), \quad \mathbb{F}_{\bar{l}}\left(x\right) = \mathbb{F}\left(x\right) - \mathbb{F}_l\left(x\right) \ . \tag{13}$$

Taking into account (6) (using $\not\partial^2 = \partial^2$) the 'field equations' appear to be those of the Maxwell electrodynamics in the Lorenz gauge:

$$\not\partial \mathbb{A}_l = \mathbb{F}_l, \quad \not\partial \mathbb{F}_l = \mathbb{j}_l \tag{14}$$

though from the perspective of direct particle interaction $\mathbb{A}_l$ and $\mathbb{F}_l$ are under-constrained by (14) because they admit an unphysical complementary function solution to $\partial^2 \mathbb{A}_l = \mathbb{0}$.

Eq. (4) is time-reparameterization invariant, wherein t plays the role of a 'speed parameter' for the space-time curve $q = q\left(t\right)$ in $\mathbb{R}^4$. Accordingly, the worldlines in (4) can be parameterized with any monotonic function of t. Alternatively the action can be written without any reference to t, for example as

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9. An adjunct potential may be regarded as physically consequential everywhere on the null-ray line segment(s) connecting the source to the terminations point(s) without affecting the arguments made here.

10. The Wheeler and Feynman adjunct potential satisfies i), ii) and iii) only. The termination requirement iv) is understood but not built in to the structure. Their adjunct potential is mathematically indistinguishable from a field-theory potential satisfying the same conditions (i.e. just i), ii) and iii)) because it is non-zero on all future and past oriented null rays passing through the worldline of the source. On that basis Hoyle and Narlikar have argued (incorrectly from the point of view of this work) that the electromagnetic direct action stress-energy is essentially no different from that of the Maxwell theory.

11. Feynman subsequently changed his position on the role of self-action, and so by implication on the status of the adjunct potential at its source.

12. $\mathbb{a} \wedge \mathbb{b} = \left(\mathbb{a}\mathbb{b} - \mathbb{b}\mathbb{a}\right) / 2$ is the anti-symmetric product of two vectors.

$$I_{DPI} = -\frac{e^2}{4\pi}\sum_{\substack{k,l=1\\k\neq l}}^{N}\int\int \mathrm{d}\,q_k \circ \mathrm{d}\,q_l \delta\left(\left(q_k - q_l\right)^2\right) \ . \tag{15}$$

Because there is no mention of $\mathbf{x}$ in the actions (4) and (15), the adjunct current and potential can just as well be defined as fields over Fourier $\mathbf{k}$-space (or the parameters of any generalized Fourier series) rather than $\mathbf{x}$. Representation independence of the fields will re-emerge in the subsequent description of dynamics of the currents.

2.3 Time-Symmetry

The DPI action employs a Green's function that is time-symmetric. Accordingly the adjunct potential and Faraday are time-symmetric relative to their source. The physical content of DPI, however, is restricted to the interactions at both ends of a light-like connection. These null ray line segments extend along the forward and backward light cone from a nominally local charge. Their angular distribution, and their distribution in time (forwards versus backwards) depends on the distribution of other charges in space and time. Taking into account Cosmological evolution this distribution will not generally be time-symmetric - except perhaps at the future conformal singularity. Further, potentials superpose, with the result that the total incoming response potential might in extreme cases vanish, even though it is the result of any number of other, distant charges.[13] Broadly then, though DPI is a time-symmetric theory, the manifestation of that property depends on the actual distribution of matter.

In contrast with earlier attempts to reconcile DPI with observation, in this work we allow for the possibility that the advance component of the DPI adjunct potential is not, in general, cancelled at its source by the response of other charges. An outcome is that the universal system of charges can remain tightly coupled by whatever symmetric component remains, post recombination. In Section 5 the totality of DPI modes are shown to correspond to those of an elastic lattice with optical and acoustic branches. The modes of the optical branch correspond very closely to the vacuum modes of field theory, thereby explaining the emergence of retarded radiation without appeal to a thermodynamic arrow of time. The acoustic modes are subsequently shown to underpin the Dirac wavefunction. Effectively, this work

13. This is the foundation of the Wheeler-Feynman absorber theory, wherein the presumed complete future absorption results in complete cancellation of the advanced component of the response.

resolves the difficulties previously attributed to DPI by appropriating what was once considered an undesirable side effect of the theory into the foundation of QM.

3 Light-speed charge in a given potential

3.1 Light-speed motion

We depart from classical traditional by asserting light-speed motion of the electron

$$\mathbb{v}_l^2\left(t\right) = 0 \quad \forall l \in \left[1, N\right] \ . \tag{16}$$

The justifications for the assertion are:

i) The time-symmetric interaction appears to demand that the mass be dynamically determined,[14] which (17) achieves, though not uniquely so.[15]

ii) The self-energy of a classical light-speed charge is ill-defined by traditional classical theory, which ambiguity can be removed in favor of a (definite) finite energy in that limit - without affecting the predictions of classical theory at subluminal speeds [90].

iii) The eigenvalues of the velocity operator for the Dirac electron are ± 1.

iv) The Dirac Equation is an *outcome* of this (classical) analysis.

Eq. (16) can be enforced via a semi-holonomic constraint in an action

$$I_{LSM} = -\frac{1}{2}\sum_{l=1}^{N}\int \mathrm{d}\,t\,\mu_l\left(t\right)\mathbb{v}_l^2\left(t\right) \tag{17}$$

extremized by variation of $\mu_l\left(t\right)$. For I_{LSM} to be a Lorentz scalar the $\mu_l\left(t\right)$ must transform as $\mathrm{d}\,t$. Alternatively each path can be parameterized with a monotonically increasing Lorentz scalar, including an appropriately defined frame-independent time (such as the Hubble age, for example). It turns out

14. To be submitted.

15. A property of the time-symmetric interaction is that the adjunct potential response of other nominally distant charges to the motion of the local charge is contemporaneous with the motion of the latter. The causal loop is closed with the requirement that the local electron motion in the presence of the incoming response potential is consistent with the motion that brought about that response. In the work cited here electron mass appears as an eigenvalue of the fields of that exchange.

however that the Euler equations will be such as to grant $\mu_l(t)$ the appropriate transformation property automatically.[16]

With (10) the full action is

$$I = I_{LSM} + I_{DPI} = -\sum_{l=1}^{N} \int \mathrm{d}\,t \left(\frac{1}{2} \mu_l(t)\, \mathbb{v}_l^2(t) + e_l \mathbb{v}_l(t) \circ \mathbb{A}_{\bar{l}}\left(q_l(t)\right) \right) \tag{18}$$

$\mathbb{A}_{\bar{l}}\left(q_l(t)\right)$ is the adjunct potential of all but the l^{th} charge evaluated on the path of the l^{th} charge, and is nominally the 'incoming' adjunct potential relative to the current with label l. The Euler equations are the corresponding Newton-Lorentz equations [17]

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left[\mu_l(t)\, \mathbb{v}_l(t)\right] = e_l \left\langle \mathbb{F}_{\bar{l}}\left(q_l(t)\right) \mathbb{v}_l(t) \right\rangle_1 \tag{19}$$

where, using an over-dot to identify the target of $\not{\partial}$,

$$\left\langle \mathbb{F}_{\bar{l}}\left(q_l(t)\right) \mathbb{v}_l(t) \right\rangle_1 = \left[\not{\partial} \left[\mathbb{v}_l(t) \circ \dot{\mathbb{A}}_{\bar{l}}(x) \right] \right]_{x=q_l(t)} - \frac{\mathrm{d}\,\mathbb{A}_{\bar{k}}\left(q_l(t)\right)}{\mathrm{d}\,t} \tag{20}$$

in which terms (19) can be written

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left[\mu_l(t)\, \mathbb{v}_l(t) + e_l \mathbb{A}_{\bar{l}}\left(q_l(t)\right)\right] = e_l \left[\not{\partial} \left[\mathbb{v}_l(t) \circ \dot{\mathbb{A}}_{\bar{l}}(x) \right] \right]_{x=q_l(t)} . \tag{21}$$

The left–hand side is the time rate of change of the total (mechanical plus electromagnetic) 4-momentum of the local charge. The electromagnetic part of the momentum is specific to the charge 'in' the potential $\mathbb{A}_{\bar{l}}$ at $q_l(t)$.

3.2 First integral of Newton-Lorentz equation

Null Incoming Potential

Suppose initially that the incoming potential is null. Then a particular solution of (21) is

$$\mu_l(t)\, \mathbb{v}_l(t) + e_l \mathbb{A}_{\bar{l}}\left(q_l(t)\right) = \mathbb{0} \tag{22}$$

16. This outcome is an automatic consequence of the relationship (22) established with the potential, which is a true Lorentz vector.

17. $\langle\ \rangle_1$ extracts the vector part of its Clifford operand.

and the total momentum is zero. The time-component of (22) gives that

$$\mu_l\left(t\right) = -e_l\phi_{\bar{l}}\left(q_l\left(t\right)\right) \tag{23}$$

and therefore

$$\mathbb{v}_l\left(t\right) = \mathbb{A}_{\bar{l}}\left(q_l\left(t\right)\right) / \phi_{\bar{l}}\left(q_l\left(t\right)\right) \Rightarrow \mathbf{v}_l\left(t\right) = \mathbf{A}_{\bar{l}}\left(q_l\left(t\right)\right) / \phi_{\bar{l}}\left(q_l\left(t\right)\right) \tag{24}$$

It follows that the null current follows the flow lines of an incoming null adjunct potential.

Since only derivatives of the incoming potential appear in (21) it follows that a more general solution of (21) is

$$\mu_l\left(t\right)\mathbb{v}_l\left(t\right) + e_l\mathbb{A}_{\bar{l}}\left(q_l\left(t\right)\right) = e_l\mathbb{u}_{\bar{l}} \tag{25}$$

for any constant vector $\mathbb{u}_{\bar{l}}$, though (22) will be sufficient for this work.

General Case

Any potential can be decomposed into null components.[18] It turns out to be useful to decompose not just the incoming potential, but also the incoming Faraday in an analogous way, which in combination will give rise to 4 (rather than just 2) different null potentials. Initially we presume that the charge follows just one of those null potentials, accepting the possibility of subsequent revision to account for the presence of the other potentials. It is shown in Appendix C however that it is always possible to find a decomposition in which the 4 null paths are independent of each other, provided the potentials are modes of the acoustic branch (see below). Optical branch mode potentials (Section 5.3) require separate treatment however.[19]

To implement this strategy let an arbitrary incoming potential be decomposed into r null potentials

$$\mathbb{A}_{\bar{l}}\left(x\right) = \sum_{n=1}^{r}\mathbb{A}_{\bar{l},n}\left(x\right); \quad \mathbb{A}_{\bar{l},n}^{2}\left(x\right) = 0 \ . \tag{26}$$

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18. For example, let $A = \left(\phi, \mathbf{A}\right) = A_+ + A_-$, then $A_\sigma = ½\left(1 + \sigma\phi / \left|\mathbf{A}\right|\right)\left(\sigma\left|\mathbf{A}\right|, \mathbf{A}\right)$, $\sigma = \pm$, are null. We will find subsequently however that a 4-way decomposition involving the Faraday is more appropriate.

19. As discussed in Section 6, the traditional method of coupling to EM is problematic however, not due to the $A \circ j$ form of the coupling, but due to misclassification of the null-currents encoded in the Dirac bi-spinor.

where for now the number of terms r in the decomposition is left undetermined. Each $\mathbb{A}_{\bar{l},n}\left(x\right)$ generates a set of flow-lines, the possible occupancy of each member of which by a charge will initially be considered independently, in accord with the above. Then the solution (22) can be applied to each of these:

$$\mathbb{v}_{l,n}\left(t\right)=\mathbb{A}_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)/\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)\ . \tag{27}$$

Here $q_{l,n}\left(t\right)$ is the worldline of the l^{th} charge following the flow-line of the n^{th} null potential in an r-fold decomposition of the potential of all other charges.

Connection with de Broglie Bohm Model

Eqs. (22)-(24) will generally be sufficient for the purposes of this document, and are the classical electromagnetic foundation of the de Broglie Bohm pilot-wave aspect of Dirac theory. The role of the electromagnetic 4-potential is restricted to determination of the direction of charge through the ratio in (24). That the magnitude plays no role in the dynamics of a *particular* charge is the explanation for an otherwise puzzling property of the pilot-wave of the de Broglie Bohm model. When the description is subsequently extended to cover an ensemble the time-component of the null potential (which in this context can be equated with the 'magnitude') will be 're-purposed' to carry information about the occupation probabilities of the flow lines.

3.3 Signs of mass and charge

The stipulation that the time component of $v_l\left(t\right)$ is equal to 1 forces the parameterization of all particles to be in the same direction along the time dimension. Informally this means that all charges proceed forwards in time, regardless of the sign of the charge. To align with convention we also arrange for the sign of the dynamic mass to be positive. Given that (22) implies

$$\mu_{l,n}\left(t\right)=-e_{l,n}\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right);\quad n\in\left[1,r\right]\ . \tag{28}$$

This can be achieved with the adjustments

$$e_{l,n}\rightarrow e_{l,n}\left(t\right)=-\left|e\right|\operatorname{sgn}\left(\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)\right)\Rightarrow\mu_{l,n}\left(t\right)=\left|e\right|\left|\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)\right| \tag{29}$$

with which (22) becomes

$$\mathbb{v}_{l,n}\left(t\right)=\frac{1}{\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)}\mathbb{A}_{\bar{l},n}\left(q_{l,n}\left(t\right)\right);\quad\left[\mathbb{A}_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)\right]^2=0\ . \tag{30}$$

The sign of the charge is now the negative of the sign of $\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)$. The total derivative is

$$\frac{\mathrm{d}\,\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)}{\mathrm{d}\,t}=\frac{\mathrm{d}\,\phi_{\bar{l},n}\left(t,\mathbf{q}_{l,n}\left(t\right)\right)}{\mathrm{d}\,t}=\left[\frac{\partial\phi_{\bar{l},n}\left(x\right)}{\partial t}+\mathbf{v}_{l,n}\left(t\right).\nabla\phi_{\bar{l},n}\left(x\right)\right]_{x=q_{l,n}\left(t\right)} \tag{31}$$

which, due to (30), can be written

$$\frac{\mathrm{d}\,\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)}{\mathrm{d}\,t}=\left[\frac{\partial\phi_{\bar{l},n}\left(x\right)}{\partial t}+\nabla.\mathbf{A}_{\bar{l},n}\left(x\right)\right]_{x=q_{l,n}\left(t\right)}=\left[\not{\partial}\circ\mathbb{A}_{\bar{l},n}\left(x\right)\right]_{x=q_{l,n}\left(t\right)}\ . \tag{32}$$

Consequently, if the $\mathbb{A}_{\bar{l},n}\left(x\right)$ independently satisfy the Lorenz gauge then $\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)$ is a constant (i.e., at all times on the particular worldline $q_{l,n}\left(t\right)$), and therefore

$$\begin{gathered}\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)=\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)\\ \Rightarrow\operatorname{sgn}\left\{\phi_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)\right\}=\operatorname{sgn}\left\{\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)\right\}\end{gathered} \tag{33}$$

say. With this (29) becomes

$$\begin{aligned}e_{l,n}&=-\left|e\right|\operatorname{sgn}\left\{\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)\right\}\\ \mu_{l,n}&=\left|e\right|\left|\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)\right|\end{aligned}\ . \tag{34}$$

Hence the dynamic mass of (22) is a constant on each flow-line, and one infers that $\not{\partial}\circ\mathbb{A}_{l,n}\left(x\right)=0$ is consistent with the absence of time-reversals.

Under these conditions the velocity (30) is

$$\mathbb{v}_{l,n}\left(t\right)=\frac{1}{\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)}\mathbb{A}_{\bar{l},n}\left(q_{l,n}\left(t\right)\right);\quad\left[\mathbb{A}_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)\right]^{2}=0\ . \tag{35}$$

Taking into account (35), and presuming that there is just one charge, the 4-current $\mathbb{j}_{l}\left(x\right)$ in (3) must be distinguished accordingly as one of $\mathbb{j}_{l,n}\left(x\right)$ for $n\in\left[1,r\right]$:

$$\mathbb{j}_{l,n}\left(x\right)=\frac{e_{l,n}}{\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)}\delta^{3}\left(\mathbf{x}-\mathbf{q}_{l,n}\left(t\right)\right)\mathbb{A}_{\bar{l},n}\left(q_{l,n}\left(t\right)\right) \tag{36}$$

where $q_{l,n}\left(t\right)$ is a solution of (30) (see (38) below) and $\mathbb{A}_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)$ is null. Eqs. (34) allows that this can be written

$$\mathbb{j}_{l,n}\left(x\right)=-\frac{\left|e\right|}{\left|\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)\right|}\delta^{3}\left(\mathbf{x}-\mathbf{q}_{l,n}\left(t\right)\right)\mathbb{A}_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)\ . \tag{37}$$

4 Ensembles

4.1 Initial conditions

Eq. (35) is the first order differential equation [20]

$$\frac{\mathrm{d}\, q_{l,n}\left(t\right)}{\mathrm{d}\, t} = \frac{1}{\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)} A_{\bar{l},n}\left(q_{l,n}\left(t\right)\right) \tag{38}$$

If $A_{\bar{l},n}\left(x\right)$ is given then in principle $q_{l,n}\left(t\right)$ can be found by solving (38). Dependency of the solution on the initial condition is exposed by writing the particular solution of (38) that passes through $x = q_{l,n}\left(0\right)$ (say) as $q_{l,n}\left(t; q_{l,n}\left(0\right)\right)$, where $q_{l,n}\left(0\right) = \left(0, \mathbf{q}_{l,n}\left(0\right)\right)$.[21] The dependency of the current (37) on initial conditions can be exposed by writing

$$\mathbb{j}_{l,n}\left(x; \mathbf{q}_{l,n}\left(0\right)\right) = -\frac{\left|e\right|}{\left|\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)\right|} \delta^3\left(\mathbf{x} - \mathbf{q}_{l,n}\left(t; \mathbf{q}_{l,n}\left(0\right)\right)\right) \mathbb{A}_{\bar{l},n}\left(x\right) \tag{39}$$

where n is just one of $\left\{1,2,3,4\right\}$, and $\mathbf{q}_{l,n}\left(t; \mathbf{q}_{l,n}\left(0\right)\right)$ is the particular solution of (38) that passes through $x = q_{l,n}\left(0\right)$. The null current (39) has charge $\left|e\right| \mathrm{sgn}\left\{\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)\right\}$ and passes through $x = q_{l,n}\left(0\right)$ with velocity $\mathbb{v}_{l,n}\left(0\right) = \mathbb{A}_{\bar{l},n}\left(q_{l,n}\left(0\right)\right) / \phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)$ at $t = 0$. There are r possible such null currents passing through the *same* $x = q_{l,n}\left(0\right)$ at $t = 0$, with differing velocity and / or sign of the charge. For each label n (at fixed l) there are a non-denumerably infinite number of possible currents each associated with the same $\mathbb{A}_{\bar{l},n}\left(x\right)$ passing through a *different* $\mathbf{x}$ at $t = 0$, and therefore a non-denumerably infinite number of possible r-fold null-currents. Let us write

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20. Here we revert to a component representation of the Lorentz vectors in order to avoid a discussion of functions of Clifford vectors that would be necessitated by writing (instead)

$$\mathrm{d}\, \mathbb{q}_{l,n}\left(t\right) / \mathrm{d}\, t = \mathbb{A}_{\bar{l},n}\left(\mathbb{q}_{l,n}\left(t\right)\right) / \phi_{\bar{l},n}\left(\mathbb{q}_{l,n}\left(t\right)\right)$$

21. If instead $q_{l,n}\left(t\right)$ is given then (38) can be read as a constraint on the adjunct potential on the worldline - though not elsewhere - i.e. on $A_{\bar{l},n}\left(q_{l,n}\left(t\right)\right)$. Conservation implies that the null worldline is not terminated and does not intersect another worldline, though $q_{l,n}\left(t\right)$ is otherwise unconstrained. In particular $q_{l,n}\left(t\right)$ does not incur constraints as a consequence of its associated 4-current being the source of a potential. That is: any 'conserved' null worldline is 'legal', dynamically.

$$\mathbf{q}_{l,n}\left(t;q_{l,n}\left(0\right)\right)=\mathbf{q}'_{l,n}\left(t\right)+\mathbf{q}_{l,n}\left(0\right);\quad \mathbf{q}'_{l,n}\left(0\right)=\mathbf{0} \tag{40}$$

where $\mathbf{q}'_{l,n}\left(t\right)$ is the particular solution of (38) that passes through the origin at $t=0$. Substitution of (40) into (39) gives

$$\mathbb{j}_{l,n}\left(x;\mathbf{q}_{l,n}\left(0\right)\right)=-\frac{|e|}{\left|\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)\right|}\delta^3\left(\mathbf{x}-\mathbf{q}'_{l,n}\left(t\right)-\mathbf{q}_{l,n}\left(0\right)\right)\mathbb{A}_{\bar{l},n}\left(x\right)\ . \tag{41}$$

4.2 Sum over mutually exclusive possibilities

We form a statistically-weighted ensemble, summing the currents (41) over the initial conditions (40). Let $p_{l,n}\left(\mathbf{q}_{l,n}\left(0\right)\right)$ be the weight of the n^{th} null current passing through $\mathbf{x}=\mathbf{q}_{l,n}\left(0\right)$ at $t=0$, so that the weighted ensemble of each is

$$\left\{\mathbb{j}_{l,n}\left(x\right)\right\}:=\int \mathrm{d}^3\, q_{l,n}\left(0\right)p_{l,n}\left(\mathbf{q}_{l,n}\left(0\right)\right)\mathbb{j}_{l,n}\left(x;\mathbf{q}_{l,n}\left(0\right)\right)=\int \mathrm{d}^3\, x' p_{l,n}\left(\mathbf{x}'\right)\mathbb{j}_{l,n}\left(x;\mathbf{x}'\right)\ . \tag{42}$$

$\mathbb{j}_{l,n}\left(x;\mathbf{x}'\right)$ is the delta-valued distribution over x of the n^{th} null current associated with the l^{th} charge that passes through $\mathbf{x}=\mathbf{x}'$ at $t=0$; $\mathbf{x}'$ labels the different possible null worldlines of the l^{th} charge. We use braces to signify that $\left\{\mathbb{j}_{l,n}\left(x\right)\right\}$ is an ensemble, the members of which are mutually exclusive in the event there is just one local charge. Taking into account (34), let the weights for a particular species of charge be determined from the incoming potential according to

$$p_{l,n}\left(\mathbf{q}_{l,n}\left(0\right)\right)=c_{l,n}\left|\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)\right| \tag{43}$$

for some positive real constant $c_{l,n}$, implying therefore

$$p_{l,n}\left(\mathbf{x}\right)=c_{l,n}\left|\phi_{\bar{l},n}\left(0,\mathbf{x}\right)\right|\ . \tag{44}$$

$p_{l,n}\left(\mathbf{x}\right)$ is the probability that a charge of sign $-\mathrm{sgn}\left\{\phi_{\bar{l},n}\left(0,\mathbf{x}\right)\right\}$ is present at $\mathbf{x}$ at $t=0$. Hence the probability that an electron is present at x say, is

$$p_{l,n}\left(\mathbf{x}\right)=c_{l,n}\phi_{\bar{l},n}\left(0,\mathbf{x}\right)\Theta\left(\phi_{\bar{l},n}\left(0,\mathbf{x}\right)\right)\ . \tag{45}$$

The intrinsically statistical aspect of ‘wave-mechanics’ derives in large part from (44). An implicit understanding that a probability can be engineered to have any value would normally conflict with the field status of $\phi_{\bar{l},n}\left(0,\mathbf{x}\right)$, whose value is presumably decided instead by the positions and velocities of distant charges. However, under the presumption that $\mathbb{A}_{\bar{l},n}\left(x\right)$ is null the effect of (41) is to cancel out the dependency of the potential on $\phi_{\bar{l},n}\left(0,\mathbf{x}\right)$, thereby hiding the ‘real’ $\phi_{\bar{l},n}\left(0,\mathbf{x}\right)$ from the view of the local charge. Eq. (43) can be read, therefore, as a rule that permits ‘reconstruction’ of the full incoming potential through the substitution of a probability for a missing part of the potential. We will see below that (43) closes the loop (in time) connecting the local charge to the distant charges in such a way $p_{l,n}\left(\mathbf{x}\right)$ - and therefore $\left|\phi_{\bar{l},n}\left(0,\mathbf{x}\right)\right|$ - emerge as amplitudes of a linearly-super-posable Klein-Gordon field, from which perspective (42) with (43) might then be regarded as a 'naturally-weighted' ensemble. In general other weighting schemes can be expected to give rise to different self-consistent dynamics. Whether or not the naturally-weighted ensemble can be achieved in practice is discussed in Section 7.

It will be convenient to write the coefficients $c_{l,n}$ in (43) as

$$c_{l,n} = \left|e\right|^{-1} \mu_{l,n}^2 \tag{46}$$

where $\mu_{l,n}$ has dimensions of L^{-1}. Substitution of (43), (46) and (41) into (42) and performing the integration gives

$$\left\{\mathbb{j}_{l,n}\left(x\right)\right\} = -\mu_{l,n}^2 \mathbb{A}_{\bar{l},n}\left(x\right) \ . \tag{47}$$

Notice that information particular to each path followed by a charge in $\mathbf{q}'_{l,n}\left(t\right)$ has disappeared, having been 'washed out' by the integration. Evidently the ensemble null currents are independently conserved iff each of $\mathbb{A}_{\bar{l},n}\left(x\right)$ satisfy the Lorenz gauge, and vice-versa.[22]

We now form an r-fold ensemble of null ensemble currents:[23]

$$\left\{\mathbb{j}_{l}\left(x\right)\right\} = \sum_{n=1}^{r}\left\{\mathbb{j}_{l,n}\left(x\right)\right\} \tag{48}$$

restricting the weights so that $\mu_{l,n} \rightarrow \mu_l$, i.e. so that

$$p_{l,n}\left(\mathbf{q}_{l,n}\left(0\right)\right) = \left|e\right|^{-1} \mu_l^2 \left|\phi_{\bar{l},n}\left(q_{l,n}\left(0\right)\right)\right| \ . \tag{49}$$

Summing (47) over n then gives

$$\left\{\mathbb{j}_{l}\left(x\right)\right\} = -\mu_l^2 \mathbb{A}_{\bar{l}}\left(x\right) \ . \tag{50}$$

Introducing the ensemble adjunct potential

$$\left\{\mathbb{A}_{l}\left(x\right)\right\} := \int \mathrm{d}^4 x' \, \mathrm{G}\left(x - x'\right)\left\{\mathbb{j}_{l}\left(x'\right)\right\} \tag{51}$$

the ensemble version of (14) is

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22. In Section 6 it is shown that in the absence of radiation the $\left\{\mathbb{j}_{l,n}\left(x\right)\right\}$ are independently conserved, corresponding to the Majorana currents in the Dirac theory.

23. There is no need to introduce weights for each of the r currents because such are already accommodated by $p_{l,n}\left(\mathbf{q}_{l,n}\left(0\right)\right)$.

$$\not{\partial}\left\{\mathbb{A}_l\left(x\right)\right\}=\left\{\mathbb{F}_l\left(x\right)\right\},\quad \not{\partial}\left\{\mathbb{F}_l\left(x\right)\right\}=-\mu_l^2\mathbb{A}_{\bar{l}}\left(x\right)\ . \tag{52}$$

The motivation for forming ensembles is that they simulate smooth fields satisfying differential equations. Eq.(42) bypasses the constraint that the potential exists only at the point of contact with the charge, and (48) avoids dealing with the non-linear constraint that the potential is null. The ensembles facilitate postponement of the imposition of the latter constraint, which is imposed instead on the apparently smooth and apparently linearly super-posable solutions of (52). The ensemble can also be regarded as a device to accommodate the use of singly-terminated (sourced) adjunct potentials in $\mathbb{R}^4$ in the description of the dynamics, even though the potential is physically consequential in DPI only at the point of contact with the charges. That is, upon replacing the delta-valued local current with an ensemble, the relation (50) admits, in its place, a potential of distant charges that is apparently no longer constrained to a particular worldline, but appears instead to have support in $\mathbb{R}^4$.

4.3 Post hoc enforcement of mutual exclusion

Nothing in the above enforces mutual exclusion; the $p_{l,n}\left(\mathbf{q}_{l,n}\left(0\right)\right)$ above appear to be independent. By contrast, if it is known that there is just one particle, then mutual exclusivity of flow-line occupancy requires [24]

$$p_{l,n}\left(\mathbf{q}_{l,n}\left(0\right),\mathbf{q}'_{l,n}\left(0\right)\right)=\delta^3\left(\mathbf{q}_{l,n}\left(0\right)-\mathbf{q}'_{l,n}\left(0\right)\right)p_{l,n}\left(\mathbf{q}_{l,n}\left(0\right)\right)\ , \tag{53}$$

and suitably extended to cover higher orders of correlation. An implementation, viable at least in a single particle theory, is to compute the dynamics at first ignoring mutual exclusion, treating (42) as an ordinary integral and (48) as an ordinary sum - i.e. both in the sense of a superposition - enforcing mutual exclusion only on products (of mutually exclusive possibilities). For example, squaring $\left\{\mathbb{j}_l\left(x\right)\right\}$ in (48), and supposing for simplicity that $n\in\left\{1,2\right\}$, one has

$$\left\{\mathbb{j}_l\left(x\right)\right\}^2=\left\{\mathbb{j}_{l,1}\left(x\right)\right\}^2+\left\{\mathbb{j}_{l,2}\left(x\right)\right\}^2+2\left\{\mathbb{j}_{l,1}\left(x\right)\right\}\circ\left\{\mathbb{j}_{l,2}\left(x\right)\right\} \tag{54}$$

24. If a and b are discrete and mutually exclusive then $p\left(a\mid b\right)=\delta_{a,b}$, and Bayes Theorem $p\left(a,b\right)=p\left(a\mid b\right)p\left(b\right)$ becomes $p\left(a,b\right)=\delta_{a,b}p\left(b\right)=\delta_{a,b}p\left(a\right)$.

The first two terms on the right are null, the third term vanishes because it is the product of two-mutually exclusive possibilities, and therefore $\left\{\mathbb{j}_l\left(x\right)\right\}$ is effectively null. This property extends to the $\left\{\mathbb{j}_{l,n}\left(x\right)\right\}$ given by (42): squaring (48), one has

$$\left\{\mathbb{j}_{l,n}\left(x\right)\right\}^2 = \int \mathrm{d}^3 a \int \mathrm{d}^3 b p_{l,n}\left(\mathbf{a}\right) p_{l,n}\left(\mathbf{b}\right) \mathbb{j}_{l,n}\left(x;\mathbf{a}\right) \circ \mathbb{j}_{l,n}\left(x;\mathbf{b}\right) \ . \tag{55}$$

To remove the mutually-exclusive terms one can make the replacement

$$p_{l,n}\left(\mathbf{a}\right) p_{l,n}\left(\mathbf{b}\right) \to p_{l,n}\left(\mathbf{a},\mathbf{b}\right) = p_{l,n}\left(\mathbf{a}\right) \delta^3\left(\mathbf{a}-\mathbf{b}\right) \tag{56}$$

and then invoke (53), whereupon (55) becomes

$$\left\{\mathbb{j}_{l,n}\left(x\right)\right\}^2 = \int \mathrm{d}^3 a p_{l,n}\left(\mathbf{a}\right)\left[\mathbb{j}_{l,n}\left(x;\mathbf{a}\right)\right]^2 = 0\,. \tag{57}$$

The $\mathrm{SU}\left(2\right)$ representation of a null vector can be factorized as an outer-product of Weyl spinors. It turns out that a null Faraday, which turns out also to play a prominent role in the dynamics, can be similarly factorized (see Appendix B). An advantage of expressing the dynamics of null currents in terms of Weyl spinors rather than Lorentz vectors is that nullity is then automatically preserved. When expressed in that form mutual exclusion can be enforced at the level of the Weyl spinors instead of the null current vectors, by ensuring that products of Weyl spinors that are factors of mutually exclusive null currents do not contribute to expectation of observables. The suggestive connection with the anti-commutators of QFT is not discussed in this document, which is primarily focused on the single particle theory.

The ensemble averaging process described above is independent of the ‘functional parameterization’ of solutions of the coupled system (52). That is, although the possibilities in Section 4.1 are parameterized by an initial condition $\mathbf{q}_{l,n}\left(0\right)$, one could just as well work in the Fourier domain with currents parameterized by a wave-vector $\mathbb{j}_{l,n}\left(x;\mathbf{k}\right)$. In that case preservation of the null condition requires that the currents associated with each possible different $\mathbf{k}$ are mutually exclusive. More generally, although the dynamics of the ensemble current are effectively those of a *linear* Klein-Gordon equation (see Section 5.4), once the function space is fixed the possible solutions become mutually exclusive.

5 Normal Modes

5.1 Self-consistency

Eq. (47) establishes a correspondence between an ensemble of null currents and the incoming potential of all other currents. If there is just one local charge under investigation then the members of the ensemble are mutually exclusive – at any time just one of the pair of non-denumerably infinite sets of flow-lines is occupied by a charge. When (47) is incorporated into the dynamics for the adjunct potential and Faraday of the local charge (see below) it 'closes the loop': due to the time-symmetric propagators it allows for the response of distant currents to some particular motion of the local current to act back on the local current *contemporaneously* with the particular local motion. In such a closed system one expects to be able to describe the dynamics in terms of self-consistent collective modes of the total.

5.2 Back-reaction

The above is predicated on the assumption that for the purposes of solving (38) $A_{\bar{l},n}(x)$ can be regarded as given, with all statistical aspects accommodated by $\phi_{\bar{l},n}(q_{l,n}(0))$. This implies that the larger system is undisturbed by the presence of the local charge, regardless of its path, so that the (differential) equations governing the development of $A_{\bar{l},n}(x)$ (and consequently $\{j_{l,n}(x)\}$- see below) are determined by and characteristic of the system as a whole.

But if this is not the case - if the incoming response is sensitive to the initial condition $q_{l,n}(0)$ - then one could not simply refer to $A_{\bar{l},n}(x)$ with arbitrary x in $\mathbb{R}^4$ as it appears in (41). Instead, it would be necessary to qualify the incoming potential by writing $A_{\bar{l},n}(x;q_{l,n}(0))$ in place of $A_{\bar{l},n}(x)$ in order to accommodate sensitivity of the response potential to the actual path followed by the local charge, with the possibility that $A_{\bar{l},n}(x;q_{l,n}(0)) \neq A_{\bar{l},n}(x;q_{l',n'}(0))$ unless $n = n'$ and $l = l'$.

Sensitivity of the potential to flow-line occupancy connotes a 'back-reaction' from the larger system beyond that which can be accommodated in $q_{l,n}(0)$.[25] Back-reaction will be ignored here because doing so achieves the goal of this work, which is convergence with Dirac theory in Minkowski space-time. In that

25. The connection with GR is noted but not explored further here.

case - in the absence of back-reaction - the incoming potential is no different from the ensemble of incoming potentials:

$$\mathbb{A}_{\bar{l},n}\left(x\right) = \left\{\mathbb{A}_{\bar{l},n}\left(x\right)\right\} \Rightarrow \mathbb{A}_{\bar{l}}\left(x\right) = \left\{\mathbb{A}_{\bar{l}}\left(x\right)\right\} \tag{58}$$

and (50) can be written

$$\left\{\mathbb{j}_{l}\left(x\right)\right\} = -\mu_l^2 \left\{\mathbb{A}_{\bar{l}}\left(x\right)\right\} \ . \tag{59}$$

Eqs. (52) can then be written

$$\not{\partial}\left\{\mathbb{A}_{l}\left(x\right)\right\} = \left\{\mathbb{F}_{l}\left(x\right)\right\}, \quad \not{\partial}\left\{\mathbb{F}_{l}\left(x\right)\right\} = -\mu_l^2 \left\{\mathbb{A}_{\bar{l}}\left(x\right)\right\} \ . \tag{60}$$

Note that the relationship (59) is exclusively between ensembles. There is no corresponding direct relationship between particular members of the ensemble $\mathbb{j}_{l}\left(x\right)$ and $\mathbb{A}_{\bar{l}}\left(x\right)$.[26]

5.3 Acoustic and optical branches

Eqs. (60) are equivalent to

$$\partial^2 \left\{\mathbb{A}_{l}\left(x\right)\right\} = -\mu_l^2 \left\{\mathbb{A}_{\bar{l}}\left(x\right)\right\} \tag{61}$$

subject to the constraint

$$\not{\partial} \circ \left\{\mathbb{A}_{\bar{l}}\left(x\right)\right\} = 0 \ . \tag{62}$$

Eq. (61) is a differential difference equation, with two types of solutions analogous to the 'optical' and 'acoustic' modes of an elastic lattice. We obtain these by first forming an equation 'adjoint' to (61). Suppressing arguments

$$\begin{aligned} \partial^2 \left\{\mathbb{A}_{\bar{l}}\right\} = \sum_{\substack{j=1 \\ j\neq l}}^{N} \partial^2 \left\{\mathbb{A}_{j}\right\} &= -\mu_l^2 \sum_{\substack{j=1 \\ j\neq l}}^{N} \left\{\mathbb{A}_{\bar{l}}\right\} \\ &= -\mu_l^2 \sum_{\substack{j=1 \\ j\neq lk}}^{N} \left[\left\{\mathbb{A}\right\} - \left\{\mathbb{A}_{j}\right\}\right] \\ &= -\mu_l^2 \left[\left(N-1\right)\left\{\mathbb{A}\right\} - \left\{\mathbb{A}_{\bar{l}}\right\}\right] \end{aligned} \ . \tag{63}$$

Here $\left\{\mathbb{A}\right\}$ is the total ensemble potential

26. Each member current is delta-valued on the worldline of the charge, whereas every incoming potential - every member of $\left\{\mathbb{A}\left(x\right)\right\}$ - is a smooth function of co-dimension 1 in $\mathbb{R}^4$ on the double light-cone of its source.

$$\{\mathbb{A}\} := \sum_{l=1}^{N} \{\mathbb{A}_l\} = \{\mathbb{A}_l\} + \{\mathbb{A}_{\bar{l}}\} \tag{64}$$

using which (63) can be written just in terms of $\{\mathbb{A}_l\}$ and $\{\mathbb{A}_{\bar{l}}\}$:

$$\partial^2 \{\mathbb{A}_{\bar{l}}\} = -\mu_l^2 \left[(N-1)\{\mathbb{A}_l\} + (N-2)\{\mathbb{A}_{\bar{l}}\} \right] . \tag{65}$$

Eqs. (61) and (65) form the coupled system

$$\begin{bmatrix} \partial^2 & \mu_l^2 \\ (N-1)\mu_l^2 & \partial^2 + (N-2)\mu_l^2 \end{bmatrix} \begin{bmatrix} \{\mathbb{A}_l\} \\ \{\mathbb{A}_{\bar{l}}\} \end{bmatrix} = \begin{bmatrix} \mathbb{O} \\ \mathbb{O} \end{bmatrix} . \tag{66}$$

Adding the two rows gives

$$\left[\partial^2 + \kappa_l^2\right]\{\mathbb{A}\} = \mathbb{O} \tag{67}$$

where

$$\kappa_l = \sqrt{N-1}\,\mu_l \ . \tag{68}$$

Subtracting the second row from $N-1$ times the first row gives

$$\left[\partial^2 - \mu_l^2\right]\{\tilde{\mathbb{A}}_l\} = 0 \tag{69}$$

where

$$\{\tilde{\mathbb{A}}_l\} := \{\mathbb{A}_l\} - \{\mathbb{A}_{\bar{l}}\} / (N-1) \tag{70}$$

is an anti-symmetric combination of the potential of the local charge and the potential of all other distant charges, as it acts on the local charge. The relative weights are such that the potentials of distant charges contribute coherently. The anti-symmetry is suggestive of an analogy with the optical modes of an elastic lattice. By contrast $\{\mathbb{A}\}$ describes the symmetric modes of the coupled *N*-particle system, analogous to the acoustic modes of an elastic lattice. Eq. (67) is a Klein-Gordon equation for the total adjunct ensemble potential $\{\mathbb{A}\}$ with mass-frequency κ_l. Given $N \sim 10^{80}$ say, this is of order 10^{40} times the magnitude of μ_l in (69). If κ_l corresponds to a known elementary particle then μ_l must be tiny. If for example κ_l is the Compton frequency of the electron with wavelength $2.4\times10^{-12}\,\text{m}$, then the wavelength associated with μ_l is of order of the present Hubble radius, and the frequency has a period of order of the Cosmological age. At frequencies much greater than this $\left\{\tilde{\mathbb{A}}_l\right\}$ behaves like a free (vacuum) potential:[27,28]

$$\partial^2 \left\{\tilde{\mathbb{A}}_l\right\} \approx 0\,. \tag{71}$$

Eq. (71) is a novel demonstration of the existence of *endogenous* quasi-vacuum modes in a DPI theory, without recourse to special boundary conditions for example. Examination of the connection with retarded EM radiation is outside the scope of this report, which is focused on the origin of the Dirac equation.

5.4 Acoustic branch with no radiation

If it is known that no radiation is present, i.e. $\left\{\tilde{\mathbb{A}}_l\right\} = 0$, then (70) gives that the incoming and locally-generated adjunct ensemble potentials are proportional,

$$\left\{\mathbb{A}_l\right\} = \left\{\mathbb{A}_{\bar{l}}\right\} / \left(N-1\right) \tag{72}$$

in which case

$$\left\{\mathbb{A}\right\} := \left\{\mathbb{A}_l\right\} + \left\{\mathbb{A}_{\bar{l}}\right\} = N\left\{\mathbb{A}_l\right\} \tag{73}$$

and the local potential $\left\{\mathbb{A}_l\right\}$ satisfies the Klein-Gordon equation (67). It follows from (72) that under these conditions (of no radiation), the local current is proportional to its own potential as

$$\left\{\mathbb{j}_l\right\} = -\kappa_l^2 \left\{\mathbb{A}_l\right\} \ , \tag{74}$$

and Eq. (61) now reads

$$\left[\partial^2 + \kappa_l^2\right]\left\{\mathbb{A}_l\right\} = 0 \tag{75}$$

with the subsidiary condition

$$\not{\partial} \circ \left\{\mathbb{A}_l\right\} = 0 \ \ . \tag{76}$$

27. Due to the sign of μ^2 in (69) the lowest non-negative energy mode has zero energy (no time variation) and a corresponding (Hubble radius) spatial variation. Note that, however small, μ forces agreement between field theory and direct particle interaction on the necessity that the (free) potential satisfies the Lorenz gauge.

28. Eq. (71) breaks down as the wavelength of the radiation approaches the Hubble radius. Under the prevailing assumption of light-speed motion of the source, this will occur as the acceleration approaches 10^{-10} m/s^2 from above, suggestive of a connection with anomalous dispersion of velocities in the outer arms of spiral galaxies (e.g. as characterized by MOND).

5.5 EM multivector

The Lorenz gauge constraint can be incorporated into the dynamics via use of the ensemble multivector

$$\left\{\mathbb{Q}_l\right\} = \kappa_l \left\{\mathbb{A}_l\right\} + i\left\{\mathbb{F}_l\right\} \tag{77}$$

(where $\left\{\mathbb{F}_l\right\} = \not{\partial}\left\{\mathbb{A}_l\right\}$), in which terms (75) and (76) can be combined into

$$\left[\not{\partial} + i\kappa_l\right]\left\{\mathbb{Q}_l\right\} = \mathbb{O} \tag{78}$$

and where, echoing the Feynman – Gell-Mann relation,

$$\left\{\mathbb{Q}_l\right\} = \left[i\not{\partial} + \kappa_l\right]\left\{\mathbb{A}_l\right\} \ . \tag{79}$$

Hereafter we refer to any linear combination of the potential and Faraday as an 'EM multivector' (to distinguish it from any other multivector containing other non-zero blades). We note in passing that in the Majorana representation (78) can be expressed entirely in terms of real quantities. Suppressing the particle label and re-writing as (79)

$$\left[\left[\not{\partial} + i\kappa\right]/\,i\right]\left[\left\{\mathbb{Q}\right\}/\,i\right] = \mathbb{O} \tag{80}$$

then

$$\left[\not{\partial} + i\kappa\right]/\,i = \begin{bmatrix} \frac{\partial}{\partial x} + \kappa & -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} - \frac{\partial}{\partial t} \\ -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + \kappa & -\frac{\partial}{\partial y} + \frac{\partial}{\partial t} & 0 \\ 0 & -\frac{\partial}{\partial y} - \frac{\partial}{\partial t} & \frac{\partial}{\partial x} + \kappa & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} + \frac{\partial}{\partial t} & 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + \kappa \end{bmatrix} \tag{81}$$

and

$$\left\{\mathbb{Q}\right\}/\,i = \left\{\left[\left[\begin{matrix} \kappa A_x - E_y & -\kappa A_z + B_y & E_z - B_x & -\kappa\phi + \kappa A_y + B_z + E_x \\ -\kappa A_z - B_y & -\kappa A_x - E_y & \kappa\phi - \kappa A_y + B_z + E_x & -E_z + B_x \\ E_z + B_x & -\kappa\phi - \kappa A_y - B_z + E_x & \kappa A_x + E_y & -\kappa A_z + B_y \\ \kappa\phi + \kappa A_y - B_z + E_x & -E_z - B_x & -\kappa A_z - B_y & -\kappa A_x + E_y \end{matrix}\right]\right]\right\} \tag{82}$$

wherein all components are contra-variant.

It is established in Section 6 that $\mathbb{Q}$ transforms like the outer product of a Dirac bi-spinor with its adjoint. By contrast, each of the individual columns in $\mathbb{Q}$ in (82) do not transform like a Dirac bi-spinor, even though in any frame each of those columns satisfies a Dirac-like equation. For example, although

$$\begin{bmatrix} \frac{\partial}{\partial x}+\kappa & -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y}-\frac{\partial}{\partial t} \\ -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x}+\kappa & -\frac{\partial}{\partial y}+\frac{\partial}{\partial t} & 0 \\ 0 & -\frac{\partial}{\partial y}-\frac{\partial}{\partial t} & \frac{\partial}{\partial x}+\kappa & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial y}+\frac{\partial}{\partial t} & 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial x}+\kappa \end{bmatrix} \begin{bmatrix} \kappa\left\{A_x\right\}-\left\{E_y\right\} \\ -\kappa\left\{A_z\right\}-\left\{B_y\right\} \\ \left\{E_z\right\}+\left\{B_x\right\} \\ \kappa\left\{\phi\right\}+\kappa\left\{A_y\right\}-\left\{B_z\right\}+\left\{E_x\right\} \end{bmatrix} = 0 \tag{83}$$

the current $j^\mu = \bar{V}\gamma^\mu V$, where

$$V^T = \left[\kappa\left\{A_x\right\}-\left\{E_y\right\},-\kappa\left\{A_z\right\}-\left\{B_y\right\},\left\{E_z\right\}+\left\{B_x\right\},\kappa\left\{\phi\right\}+\kappa\left\{A_y\right\}-\left\{B_z\right\}+\left\{E_x\right\}\right], \tag{84}$$

is not a true Lorentz vector, and suffers from non-conservation when viewed from a Lorentz-transformed frame.[29] By contrast, and as shown in the next section, the conserved currents can be constructed from the eigenvectors of the multivector.

5.6 Mass variation

The label l in κ_l (in (79) for example) allows for particle-specific variation in the electron mass. Dependence on the particle label will generally be well-approximated by dependence on the particle position provided the fractional rate of variation is much less than the Compton wavelength. Presuming so permits the replacement $\kappa_l \to \kappa\left(x\right)$ in (78), so that, restoring the argument, the multivector for well-separated charges satisfies

$$\left[\not{\partial}+i\kappa\left(x\right)\right]\left\{\mathbb{Q}\left(x\right)\right\} = \mathbb{0} \ . \tag{85}$$

Eq. (85) corresponds to propagation in conformal spacetime with a scale-factor that varies as $\kappa\left(x\right)$ [91]. Henceforth in this work we will restrict attention to propagation in Minkowski spacetime, and in place of (74) will write [30]

$$\left\{\mathbb{j}_l\left(x\right)\right\} = -\kappa^2\left\{\mathbb{A}_l\left(x\right)\right\} \ . \tag{86}$$

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29. By this we mean that although $\partial_\mu\left(\bar{V}\gamma^\mu V\right)=0$ in the frame in which V is as given, $\partial'_\mu\left(\bar{V}'\gamma'^\mu V'\right)\neq 0$ where the primes denote Lorentz-transformed quantities, wherein V' is constructed from the transformed elements of V , i.e. without taking V to be a bi-spinor.

30. Investigation of the connection with GR via the dependence of $\kappa\left(x\right)$ on the distribution of distant charges is outside the scope of this work.

Correspondingly, (77) and (78) become

$$\left\{\mathbb{Q}_l\left(x\right)\right\} = \boldsymbol{\kappa}\left\{\mathbb{A}_l\left(x\right)\right\} + i\left\{\mathbb{F}_l\left(x\right)\right\} \tag{87}$$

and $\mathbb{Q}_l\left(x\right)$ satisfies

$$\left[\not{\partial} + i\boldsymbol{\kappa}\right]\left\{\mathbb{Q}_l\left(x\right)\right\} = \mathbb{O} \ . \tag{88}$$

6 Dirac Equation

6.1 Multivector projections

Eqs. (87) and (88) is a coupled first order system in the components of the potential and Faraday. The focus of this work is on the ensemble current, which can found from (86) given a solution $\left\{\mathbb{Q}_l\left(x\right)\right\}$ of (88), provided the conditions described in Section 3.2 are met. If so then each flow line of each of the null components of that current is a possible - mutually exclusive - path for the local electron. Recalling the discussion at the end of Section 4.2 we show that the Dirac equation derives from (88) - re-cast so as to make those paths explicit.

One could form a Dirac equation of sorts by right multiplication of (88) with a constant 4-vector to give a 'projected' quantity in $\mathbb{C}^4$. But $\left\{\mathbb{Q}_l\left(x\right)\right\}$ times a constant 4-vector does not transform as a bi-spinor (see below). By contrast a Lorentz invariant 4-vector (bi-spinor) description of the dynamics can be obtained from a projection of the phase-space representation of the Clifford Multivector, because in that representation there is no constraint that the projection 4-vector be constant. The dimensionality of $\left\{\mathbb{Q}_l\left(x\right)\right\}$ mandates there are 4 such independent projections that generate 4 Dirac equations, each associated with a different conserved current.

6.2 Multivector eigenvectors (Dirac bi-spinors)

PHASE-SPACE REPRESENTATION

We suppress the particle index l, and distinguish between real-space, phase-space, and Fourier domain functions by their arguments. Using the transform convention

$$f\left(k\right) = \int \mathrm{d}^4\, x e^{ik\circ x} f\left(x\right) \Rightarrow f\left(x\right) = \left(2\pi\right)^{-4} \int \mathrm{d}^4\, k e^{-ik\circ x} f\left(k\right) \tag{89}$$

let

$$f\left(x,k\right)=e^{-ik\circ x}f\left(k\right) \tag{90}$$

for any function $f\left(k\right)$ so that

$$f\left(x\right)=\left(2\pi\right)^{-4}\int \mathrm{d}^4\, kf\left(x,k\right)\ . \tag{91}$$

In these terms the multivector (87) is

$$\left\{\mathbb{Q}\left(x,k\right)\right\}=\boldsymbol{\kappa}\left\{\mathbb{A}\left(x,k\right)\right\}+i\left\{\mathbb{F}\left(x,k\right)\right\}=\left[\boldsymbol{\kappa}+\Bbbk\right]\left\{\mathbb{A}\left(x,k\right)\right\} \tag{92}$$

and (88) can be written as either of [31]

$$\left[\not{\partial}+i\boldsymbol{\kappa}\right]\left\{\mathbb{Q}\left(x,k\right)\right\}=i\left[\boldsymbol{\kappa}-\Bbbk\right]\left\{\mathbb{Q}\left(x,k\right)\right\}=0\,. \tag{93}$$

The second of (93) can be written

$$\mathbb{P}_{-}\left(k\right)\left\{\mathbb{Q}\left(x,k\right)\right\}=0 \tag{94}$$

where $\mathbb{P}_{\sigma}\left(k\right)=\left(\boldsymbol{\kappa}+\sigma\Bbbk\right)/\,2\boldsymbol{\kappa},\quad \sigma=\pm1$, are a complementary pair of projection matrixes each with rank 2. Consequently, $\left\{\mathbb{Q}\left(x,k\right)\right\}$ has rank 2, and can be decomposed, therefore, as the sum of two outer-products of 4-component vectors in $\mathbb{C}^4$, though the form of that decomposition is constrained by conditions on $\left\{\mathbb{Q}\left(x,k\right)\right\}$ due to the reality of the underlying potential and Faraday, and – relatedly - the symmetries of their matrix representations.

Substitution of (92) into (93) gives the Klein-Gordon type condition $k^2=\boldsymbol{\kappa}^2$. The two roots can be accommodated by reduction of the dimensionality of the k-space integrations, replacing (91) with

$$\left\{\mathbb{Q}\left(x\right)\right\}=\left(2\pi\right)^{-3}\int \mathrm{d}^3\,k\left(\left\{\mathbb{Q}_{+}\left(\mathbf{k}\right)\right\}e^{ik_{\kappa}\circ x}+\left\{\mathbb{Q}_{-}\left(\mathbf{k}\right)\right\}e^{-ik_{\kappa}\circ x}\right) \tag{95}$$

where

$$k_{\kappa}=\left(k_{\kappa}^0,\mathbf{k}\right);\quad k_{\kappa}^0:={}_{+}\sqrt{\boldsymbol{\kappa}^2+\mathbf{k}^2}\ . \tag{96}$$

One infers from (91) that

$$\left\{\mathbb{Q}\left(x,k\right)\right\}=2\pi\left(\left\{\mathbb{Q}_{+}\left(\mathbf{k}\right)\right\}e^{ik\circ x}+\left\{\mathbb{Q}_{-}\left(\mathbf{k}\right)\right\}e^{-ik\circ x}\right)\delta\left(k^0-{}_{+}\sqrt{\boldsymbol{\kappa}^2+\mathbf{k}^2}\right) \tag{97}$$

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31. $\Bbbk$ is $\not{k}$ of the traditional Feynman slash notation.

Since $\{\mathbb{Q}(x,k)\}$ has rank 2, $\{\mathbb{Q}_+(\mathbf{k})\}$ and $\mathbb{Q}_-(\mathbf{k})$ must each have at least rank 1. (Because $\{\mathbb{Q}_+(\mathbf{k})\}e^{ik\circ x}$ and $\mathbb{Q}_-(\mathbf{k})e^{-ik\circ x}$ are functionally independent from the point of view of a Fourier decomposition of solutions of (93) it cannot be the case that one of these has rank 2, and the other rank 0.) Taking $\{\mathbb{Q}(x,k)\}$ to be the more fundamental physical quantity, we seek (possibly non-unique) rank 1 representations of $\{\mathbb{Q}_+(\mathbf{k})\}$ and $\{\mathbb{Q}_-(\mathbf{k})\}$ that have sufficient degrees of freedom to satisfy symmetry constraints on $\{\mathbb{Q}(x,k)\}$.

Relativistic Covariance

Corresponding to a Lorentz transformation

$$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu; \quad L^T L = 1 \tag{98}$$

where $L = \{L^\mu{}_\nu\}$, the transformation rule for $\partial\!\!\!/$ is

$$\partial\!\!\!/ \to \partial\!\!\!/' = \mathbb{S}\partial\!\!\!/\mathbb{S}^{-1} \tag{99}$$

for a constant matrix $\mathbb{S}(L)$. One finds

$$\partial\!\!\!/ = \gamma^\mu \frac{\partial}{\partial x^\mu} \to \gamma^\nu \frac{\partial}{\partial x'^\mu} = \gamma^\nu \frac{\partial x^\mu}{\partial x'^\nu}\frac{\partial}{\partial x^\mu} = \gamma^\nu \left(L^{-1}\right)^\mu{}_\nu \frac{\partial}{\partial x^\mu} \tag{100}$$

and therefore $\mathbb{S}(L)$ is the solution of

$$\mathbb{S}(L)\gamma^\mu\mathbb{S}^{-1}(L) = \gamma^\nu\left(L^{-1}\right)^\mu{}_\nu \tag{101}$$

which, up to an overall scalar factor, is [92]

$$\mathbb{S}(L) = e^{[\gamma_b,\gamma_a]\omega^{ab}}; \quad \omega^{ab} = \frac{1}{8}\left(g^{ab} - L^{ab}\right) \ . \tag{102}$$

Since $\partial\!\!\!/$ is a proto-typical vector it follows that the potential must transform likewise

$$\{\mathbb{A}(x)\} \to \{\mathbb{A}'(x')\} = \mathbb{S}\{\mathbb{A}(x)\}\mathbb{S}^{-1} \tag{103}$$

and therefore

$$\{\mathbb{F}(x)\} = \partial\!\!\!/\{\mathbb{A}(x)\} \to \{\mathbb{F}'(x')\} = \partial\!\!\!/'\{\mathbb{A}'(x')\} = \mathbb{S}\{\mathbb{F}(x)\}\mathbb{S}^{-1} \ . \tag{104}$$

Consequently

$$\{\mathbb{Q}(x)\} \to \{\mathbb{Q}'(x')\} = \mathbb{S}\{\mathbb{Q}(x)\}\mathbb{S}^{-1} \tag{105}$$

and (93) is invariant under Lorentz transformations:

$$\left[\not{\partial}+i\kappa\right]\left\{\mathbb{Q}\left(x\right)\right\}\to\left[\not{\partial}'+i\kappa\right]\left\{\mathbb{Q}'\left(x'\right)\right\}=\mathbb{S}\left[\not{\partial}+i\kappa\right]\left\{\mathbb{Q}\left(x\right)\right\}\mathbb{S}^{-1}=0\,. \tag{106}$$

It follows from (105) that $\left\{\mathbb{Q}\left(x\right)\right\}$ transforms as an outer-product of Dirac-theory bi-spinors $\psi\left(x\right)$, the standard transformation rule for which (see for example [92]) is

$$\psi'\left(x'\right)=\mathbb{S}\psi\left(x\right). \tag{107}$$

Since $\left\{\mathbb{Q}\left(x,k\right)\right\}$ has rank two it can be decomposed in $\mathrm{M}_4\left(\mathbb{C}\right)$ as

$$\left\{\mathbb{Q}\left(x,k\right)\right\}=r\left(x,k\right)\overline{s}\left(x,k\right)+u\left(x,k\right)\overline{v}\left(x,k\right) \tag{108}$$

where r,s transform as Dirac bi-spinors, $\overline{s},\overline{v}$ transform as adjoint bi-spinors, and the overbar has the traditional meaning for a bi-spinor ψ that $\overline{\psi}=\psi^{\dagger}\gamma^{0}$. Consistent with (108), and taking into account the discussion above, we now seek a sufficient decomposition of $\left\{\mathbb{Q}_{+}\left(\mathbf{k}\right)\right\}$ and $\left\{\mathbb{Q}_{-}\left(\mathbf{k}\right)\right\}$ in (95) as

$$\left\{\mathbb{Q}_{+}\left(\mathbf{k}\right)\right\}=r\left(\mathbf{k}\right)\overline{s}\left(\mathbf{k}\right),\quad\left\{\mathbb{Q}_{-}\left(\mathbf{k}\right)\right\}=u\left(\mathbf{k}\right)\overline{v}\left(\mathbf{k}\right) \tag{109}$$

whereupon (95) becomes

$$\left\{\mathbb{Q}\left(x\right)\right\}=\left(2\pi\right)^{-3}\int\mathrm{d}^{3}\,k\left(r\left(\mathbf{k}\right)\overline{s}\left(\mathbf{k}\right)e^{ik_{\kappa}\circ x}+u\left(\mathbf{k}\right)\overline{v}\left(\mathbf{k}\right)e^{-ik_{\kappa}\circ x}\right)\ . \tag{110}$$

We have not used the ensemble notation for the bi-spinors because $r\left(\mathbf{k}\right)$ and $\overline{s}\left(\mathbf{k}\right)$ in (109) for example are outer-product vector *factors* of an ensemble - they do not each represent an ensemble of bi-spinors.

Restriction to $C\ell_{1,3}\left(\mathbb{R}\right)$

The degrees of freedom in r,s $\overline{s},\overline{v}$ are restricted to conform with intrinsic symmetries of the gamma matrixes

$$\gamma^{\mu}=\gamma^{0}\gamma^{\mu\dagger}\gamma^{0}=\gamma^{0}\,\mathrm{C}\gamma^{\mu*}\,\mathrm{C}\gamma^{0}\,. \tag{111}$$

The first of (111) applied to a real-space potential and Faraday yield

$$\gamma^{0}\left\{\mathbb{A}^{\dagger}\left(x\right)\right\}\gamma^{0}=\left\{\mathbb{A}\left(x\right)\right\} \tag{112}$$

and

$$\gamma^{0}\left\{\mathbb{F}^{\dagger}\left(x\right)\right\}\gamma^{0}=\gamma^{0}\left[\not{\partial}\left\{\mathbb{A}\left(x\right)\right\}\right]^{\dagger}\gamma^{0}=\gamma^{0}\left\{\mathbb{A}^{\dagger}\left(x\right)\right\}\overleftarrow{\not{\partial}}^{\dagger}\gamma^{0}=\left\{\mathbb{A}\left(x\right)\right\}\overleftarrow{\not{\partial}}=-\left\{\mathbb{F}\left(x\right)\right\} \tag{113}$$

and therefore

$$\gamma^{0}\left[i\left\{\mathbb{F}\left(x\right)\right\}\right]^{\dagger}\gamma^{0}=i\left\{\mathbb{F}\left(x\right)\right\}\ . \tag{114}$$

Applied to (87) these give

$$\left\{\mathbb{Q}\left(x\right)\right\}=\gamma^{0}\left\{\mathbb{Q}^{\dagger}\left(x\right)\right\}\gamma^{0} \ . \tag{115}$$

A similar application of the second of (111) leads to

$$\left\{\mathbb{Q}\left(x\right)\right\}=\gamma^{0}\,\mathrm{C}\left\{\mathbb{Q}^{*}\left(x\right)\right\}\mathrm{C}\,\gamma^{0} \, . \tag{116}$$

Recalling (89), (90) and (91), the phase-space representations of $\left\{\mathbb{A}\right\}$, $\left\{\mathbb{F}\right\}$, and $\left\{\mathbb{Q}\right\}$ must have the same symmetries, and therefore

$$\left\{\mathbb{Q}\left(x,k\right)\right\}=\gamma^{0}\left\{\mathbb{Q}^{\dagger}\left(x,k\right)\right\}\gamma^{0}=\gamma^{0}\,\mathrm{C}\left\{\mathbb{Q}^{*}\left(x,k\right)\right\}\mathrm{C}\,\gamma^{0} \, . \tag{117}$$

The first of these implies that $\mathbb{Q}\left(x\right)\gamma^{0}$ and $\mathbb{Q}\left(x,k\right)\gamma^{0}$ are Hermitian. Applied to (110) this requires the restricted decomposition

$$\left\{\mathbb{Q}\left(x\right)\right\}=\left(2\pi\right)^{-3}\int \mathrm{d}^{3}\,k\left(r\left(\mathbf{k}\right)\overline{s}\left(\mathbf{k}\right)e^{ik_{\kappa}\circ x}+s\left(\mathbf{k}\right)\overline{r}\left(\mathbf{k}\right)e^{-ik_{\kappa}\circ x}\right). \tag{118}$$

The second of (117) connotes charge conjugation invariance of the whole matrix. Denoting the charge-conjugate of a bi-spinor ψ by

$$\psi^{c}=\gamma^{0}\,\mathrm{C}\,\psi^{*} \, , \tag{119}$$

this implies $r^{c}\left(\mathbf{k}\right)=s\left(\mathbf{k}\right)\Leftrightarrow r\left(\mathbf{k}\right)=s^{c}\left(\mathbf{k}\right)$ and therefore

$$\left\{\mathbb{Q}\left(x\right)\right\}=\left(2\pi\right)^{-3}\int \mathrm{d}^{3}\,k\left[\psi\left(½\mathbf{k}\right)\overline{\psi}^{c}\left(½\mathbf{k}\right)e^{ik_{\kappa}\circ x}+\psi^{c}\left(½\mathbf{k}\right)\overline{\psi}\left(½\mathbf{k}\right)e^{-ik_{\kappa}\circ x}\right] \tag{120}$$

for some bi-spinor $\psi\left(½\mathbf{k}\right)$. The factor of ½ in the argument of ψ affects its definition but is otherwise arbitrary. Through a change of scale of the integration (120) can be written

$$\left\{\mathbb{Q}\left(x\right)\right\}=\pi^{-3}\int \mathrm{d}^{3}\,k\left[\psi\left(x,\mathbf{k}\right)\overline{\psi}^{c}\left(x,\mathbf{k}\right)+\psi^{c}\left(x,\mathbf{k}\right)\overline{\psi}\left(x,\mathbf{k}\right)\right] \tag{121}$$

where

$$\psi\left(x,\mathbf{k}\right)=\psi\left(\mathbf{k}\right)e^{ik_{c}\circ x} \tag{122}$$

and where the wave-vector k_{c} is

$$k_{c}=\left(k_{c}^{0},\mathbf{k}\right);\quad k_{c}^{0}:={}_{+}\sqrt{\kappa_{c}^{2}+\mathbf{k}^{2}},\quad \kappa_{c}=\kappa/2 \ . \tag{123}$$

The subscript c alludes to the Compton frequency, which is half the frequency of the rest-frame adjunct potential. Returning to (95), (121) implies that $\left\{\mathbb{Q}_{-}\left(\mathbf{k}\right)\right\}=\left\{\mathbb{Q}_{+}^{c}\left(\mathbf{k}\right)\right\}$, and also that (95) could be written more efficiently as

$$\left\{\mathbb{Q}\left(x\right)\right\}=\pi^{-3}\int \mathrm{d}^{3}\,k\left(\left\{\mathbb{Q}\left(\mathbf{k}\right)\right\}e^{ik_{\kappa}\circ x}+\left\{\mathbb{Q}^{c}\left(\mathbf{k}\right)\right\}e^{-ik_{\kappa}\circ x}\right) \tag{124}$$

where

$$\left\{\mathbb{Q}^c\left(\mathbf{k}\right)\right\} := \gamma^0\,\mathrm{C}\left\{\mathbb{Q}^*\left(\mathbf{k}\right)\right\}\mathrm{C}\,\gamma^0 \tag{125}$$

and, using (120),

$$\left\{\mathbb{Q}\left(\mathbf{k}\right)\right\} = \psi\left(\tfrac{1}{2}\mathbf{k}\right)\overline{\psi}^c\left(\tfrac{1}{2}\mathbf{k}\right). \tag{126}$$

6.3 Dirac Equation

Applying (91) to (97) a Fourier phase factor form of the multivector differential equation (88) is

$$\left[\not{\partial} + i\kappa\right]\left\{\mathbb{Q}\left(x,k\right)\right\} = 0\,. \tag{127}$$

With the substitution (124) Eq. (88) can also be expressed in the form

$$\left[\not{\partial} + i\kappa\right]\left\{\mathbb{Q}\left(\mathbf{k}\right)\right\}e^{ik_\kappa\circ x} = 0 \tag{128}$$

with the component form of $\mathbb{k}_\kappa$ given in (96). This is sufficient because the charge conjugate of (128) takes care of the second term in (124). Specifically:

$$\begin{aligned}\gamma^0\,\mathrm{C}\left[\left[\not{\partial} + i\kappa\right]\left\{\mathbb{Q}\left(\mathbf{k}\right)\right\}e^{ik_\kappa\circ x}\right]^*\mathrm{C}\,\gamma^0 = 0 &\Rightarrow \gamma^0\,\mathrm{C}\left[\not{\partial}^* - i\kappa\right]\mathrm{C}\,\gamma^0\gamma^0\,\mathrm{C}\left\{\mathbb{Q}^*\left(\mathbf{k}\right)\right\}\mathrm{C}\,\gamma^0 e^{-ik_\kappa\circ x} = 0\\ &\Rightarrow \gamma^0\,\mathrm{C}\left[\not{\partial} + i\kappa\right]\left\{\mathbb{Q}^c\left(\mathbf{k}\right)\right\}e^{-ik_\kappa\circ x} = 0\end{aligned} \tag{129}$$

Expressed instead in terms of the eigenvector decomposition (126) Eq. (128) is

$$\left[\not{\partial} + i\kappa\right]\psi\left(\tfrac{1}{2}\mathbf{k}\right)\overline{\psi}^c\left(\tfrac{1}{2}\mathbf{k}\right)e^{ik_\kappa\circ x} = 0 \Rightarrow \left[\not{\partial} + i\kappa_c\right]\psi\left(\mathbf{k}\right)\overline{\psi}^c\left(\mathbf{k}\right)e^{ik_c\circ x} = 0\,. \tag{130}$$

Using (122) it follows that a sufficient condition for the satisfaction of (88) is that each phase-space component $\psi\left(x,\mathbf{k}\right),\ \forall x,\mathbf{k}$ satisfies the Dirac equation

$$\left[\not{\partial} + i\kappa_c\right]\psi\left(x,\mathbf{k}\right) = i\left[\kappa_c + \mathbb{k}_c\right]\psi\left(x,\mathbf{k}\right) = 0\,. \tag{131}$$

Suppressing arguments, that (131) implies (128) follows from

$$\left[\not{\partial} + i\kappa_c\right]\psi = 0 \Rightarrow \left[\not{\partial} + i\kappa\right]\psi\overline{\psi}^c = 0\,. \tag{132}$$

due to a shared common Fourier phase factor $\psi,\overline{\psi}^c \sim e^{-ik_\kappa\circ x/2} \Rightarrow \psi\overline{\psi}^c \sim e^{-ik_\kappa\circ x}$, and from charge-conjugation invariance of the Dirac equation, which gives $\left[\not{\partial} + i\kappa\right]\psi^c\overline{\psi} = 0$.

7 Dirac Currents

7.1 Naturally-weighted current

Solutions $\left\{\mathbb{Q}_l\left(x\right)\right\}$ of (88) can be assembled from solutions $\psi\left(x,\mathbf{k}\right)$ of (131) using (124), and (121), the bi-vector and vector parts of which are the ensemble Faraday and ensemble potential, respectively. The latter is [32]

$$\left\{\mathbb{A}\left(x\right)\right\} = \frac{1}{\pi^3\kappa}\int \mathrm{d}^3 k\left[\left\langle\psi\left(x,\mathbf{k}\right)\bar{\psi}^c\left(x,\mathbf{k}\right)\right\rangle_1 + \left\langle\psi^c\left(x,\mathbf{k}\right)\bar{\psi}\left(x,\mathbf{k}\right)\right\rangle_1\right]. \tag{133}$$

The ensemble potential is proportional to an ensemble of local currents through (74). Specifically

$$\left\{\mathbb{j}\left(x\right)\right\} = -\frac{2\kappa_c}{\pi^3}\int \mathrm{d}^3 k\left[\left\langle\psi\left(x,\mathbf{k}\right)\bar{\psi}^c\left(x,\mathbf{k}\right)\right\rangle_1 + \left\langle\psi^c\left(x,\mathbf{k}\right)\bar{\psi}\left(x,\mathbf{k}\right)\right\rangle_1\right], \tag{134}$$

where we used $\kappa = 2\kappa_c$. Let us confirm that $\left\{\mathbb{A}\left(x\right)\right\}$ and therefore $\left\{\mathbb{j}\left(x\right)\right\}$ satisfy the Lorenz gauge condition. Suppressing arguments

$$\begin{aligned}\partial\circ\left\{\mathbb{A}\right\} &= \frac{1}{\pi^3\kappa}\int \mathrm{d}^3 k\left\langle\partial\left[\left\langle\psi\bar{\psi}^c\right\rangle_1 + \left\langle\psi^c\bar{\psi}\right\rangle_1\right]\right\rangle_0 \\ &= \frac{1}{\pi^3\kappa}\int \mathrm{d}^3 k\left[\left\langle\partial\left[\psi\bar{\psi}^c\right]\right\rangle + \left\langle\partial\left[\psi^c\bar{\psi}\right]\right\rangle\right]. \\ &= \frac{1}{\pi^3\kappa}\int \mathrm{d}^3 k\left[\bar{\psi}^c\overleftrightarrow{\partial}\psi + \bar{\psi}\overleftrightarrow{\partial}\psi^c\right]\end{aligned} \tag{135}$$

This vanishes because $\partial\psi = -i\kappa_c\psi,\quad \partial\psi^c = -i\kappa_c\psi^c$, and $\bar{\psi}\overleftarrow{\partial} = i\kappa_c\bar{\psi},\quad \bar{\psi}^c\overleftarrow{\partial} = i\kappa_c\bar{\psi}^c$, and so $\left\{\mathbb{j}\left(x\right)\right\}$ is a conserved current. Recalling the discussion in Section 4.2 we will refer to the $\left\{\mathbb{j}\left(x\right)\right\}$ given by (134) as the naturally-weighted ensemble current. The overall factor κ/π^3 can be replaced to comply with a normalization condition on the charge.

To facilitate a physical interpretation of the current we express the ψ in terms of the eigenvectors of charge conjugation. This is the defining characteristic of Majorana bi-spinors, denoted here by a change of font to $\boldsymbol{\psi}$. They can be projected out of an arbitrary $\psi\left(x,\mathbf{k}\right)$ using

$$\boldsymbol{\psi}_{\sigma_e}\left(x,\mathbf{k}\right) = \hat{\mathbb{P}}_{\sigma_e}\left[\psi\left(x,\mathbf{k}\right)\right] := \frac{1}{2}\left[\psi\left(x,\mathbf{k}\right) + \sigma_e\psi^c\left(x,\mathbf{k}\right)\right]. \tag{136}$$

The inverse relations are

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32. The Minkowski components of a Clifford vector $\mathbb{V} = \left\langle\psi\bar{\psi}^c\right\rangle_1 = V^\mu\gamma_\mu$ are $V^\mu = \bar{\psi}^c\gamma_\mu\psi$. See Appendix A.

$$\psi\left(x,\mathbf{k}\right)=\psi_{+}\left(x,\mathbf{k}\right)+\psi_{-}\left(x,\mathbf{k}\right),\quad \psi^{c}\left(x,\mathbf{k}\right)=\psi_{+}\left(x,\mathbf{k}\right)-\psi_{-}\left(x,\mathbf{k}\right). \tag{137}$$

Substitution of (137) into (121) gives

$$\mathbb{Q}\left(x\right)=\frac{2}{\pi^{3}}\int \mathrm{d}^{3}k\left[\psi_{+}\left(x,\mathbf{k}\right)\bar{\psi}_{+}\left(x,\mathbf{k}\right)-\psi_{-}\left(x,\mathbf{k}\right)\bar{\psi}_{-}\left(x,\mathbf{k}\right)\right] \tag{138}$$

in which terms (134) is

$$\left\{\mathbb{j}\left(x\right)\right\}=\frac{4\kappa_{c}}{\pi^{3}}\int \mathrm{d}^{3}k\left[\left\langle\psi_{+}\left(x,\mathbf{k}\right)\bar{\psi}_{+}\left(x,\mathbf{k}\right)\right\rangle_{1}-\left\langle\psi_{-}\left(x,\mathbf{k}\right)\bar{\psi}_{-}\left(x,\mathbf{k}\right)\right\rangle_{1}\right]. \tag{139}$$

Outer-products of Majorana bi-spinors are discussed in Appendix B. Of relevance here is that the vector part is null, and therefore (137) succeeds in separating out two of the null components of the total ensemble current. The further separation of each of these into a pair of null currents, each associated with a different spin orientation, is given in Appendix C. At fixed t the ensemble $\left\{\mathbb{j}\left(x\right)\right\}$ therefore comprises 4 null currents of each charge species passing through every $\mathbf{x}$.[33]

The naturally-weighted current (139) is a signed quantity, as would be expected of a solution of the Klein-Gordon equation. Substitution of (122) into (136), the outer-products in (138) and (139) becomes

$$\begin{aligned}\psi_{\sigma_{e}}\left(x,\mathbf{k}\right)\bar{\psi}_{\sigma_{e}}\left(x,\mathbf{k}\right)&=\frac{1}{4}\left[\psi\left(\mathbf{k}\right)e^{ik_{c}\circ x}+\sigma_{e}\psi^{c}\left(\mathbf{k}\right)e^{-ik_{c}\circ x}\right]\left[\bar{\psi}\left(\mathbf{k}\right)e^{-ik_{c}\circ x}+\sigma_{e}\bar{\psi}^{c}\left(\mathbf{k}\right)e^{ik_{c}\circ x}\right]\\&=\frac{1}{4}\left[\psi\left(\mathbf{k}\right)\bar{\psi}\left(\mathbf{k}\right)+\psi^{c}\left(\mathbf{k}\right)\bar{\psi}^{c}\left(\mathbf{k}\right)\right]+\frac{\sigma_{e}}{4}\left[\psi^{c}\left(\mathbf{k}\right)\bar{\psi}\left(\mathbf{k}\right)e^{-2ik_{c}\circ x}+\psi\left(\mathbf{k}\right)\bar{\psi}^{c}\left(\mathbf{k}\right)e^{2ik_{c}\circ x}\right]\end{aligned}. \tag{140}$$

Hence at fixed $\mathbf{k}$ the two terms in (139) each comprise an oscillatory component offset by a constant mean. Moreover, the magnitude of that term is the same for both species. Consequently the static terms cancel upon substitution of (140) into (139), leaving

$$\left\{\mathbb{j}\left(x\right)\right\}=\frac{\kappa_{c}}{\pi^{3}}\int \mathrm{d}^{3}k\left[\left\langle\psi^{c}\left(\mathbf{k}\right)\bar{\psi}\left(\mathbf{k}\right)\right\rangle_{1}e^{-2ik_{c}\circ x}+\left\langle\psi\left(\mathbf{k}\right)\bar{\psi}^{c}\left(\mathbf{k}\right)\right\rangle_{1}e^{2ik_{c}\circ x}\right]. \tag{141}$$

The naturally-weighted current is purely sinusoidal therefore.

The time component of each current is proportional to

$$\bar{\psi}_{\sigma_{e}}\left(x,\mathbf{k}\right)\gamma^{0}\psi_{\sigma_{e}}\left(x,\mathbf{k}\right)=\bar{\psi}\left(\mathbf{k}\right)\gamma^{0}\psi\left(\mathbf{k}\right)+\sigma_{e}\,\mathrm{Re}\left\{\bar{\psi}\left(\mathbf{k}\right)\gamma^{0}\psi^{c}\left(\mathbf{k}\right)e^{-2ik_{c}\circ x}\right\}. \tag{142}$$

33. And therefore $r=4$ in (48).

Since

$$\bar{\psi}_{\sigma_e}(x,\mathbf{k})\gamma^0\psi_{\sigma_e}(x,\mathbf{k}) = \psi^\dagger_{\sigma_e}(x,\mathbf{k})\psi_{\sigma_e}(x,\mathbf{k}) > 0 \tag{143}$$

and

$$\bar{\psi}(\mathbf{k})\gamma^0\psi(\mathbf{k}) = \psi^\dagger(\mathbf{k})\psi(\mathbf{k}) > 0 \tag{144}$$

it follows from (142) that the sign of the charge is determined solely by the sign of the static term. The two terms in (139), $\left\langle \psi_-(x,\mathbf{k})\bar{\psi}_-(x,\mathbf{k})\right\rangle_1$ and $-\left\langle \psi_+(x,\mathbf{k})\bar{\psi}_+(x,\mathbf{k})\right\rangle_1$, are the currents of opposite signed species of equal mass. Nominally these are electrons and positrons, though which term corresponds to which species depends on the sign of the overall factor.

7.2 Naturally-weighted electron current

The naturally-weighted current is an ensemble of electron currents and positron currents. The positron component can be discarded if it is known that locally there is just one electron, say, in which case the relevant sub-set of the ensemble is

$$\left\{\mathbb{j}(x)\right\}\Big|_{\substack{\text{no local}\\ \text{positrons}}} = -\frac{4\kappa_c}{\pi^3}\int d^3k \left\langle \psi_-(x,\mathbf{k})\bar{\psi}_-(x,\mathbf{k})\right\rangle_1 \ . \tag{145}$$

Eq. (145) is (139) augmented by extra knowledge about the local state of affairs. It does not imply that $\left\langle \psi_+(x,\mathbf{k})\bar{\psi}_+(x,\mathbf{k})\right\rangle_1$ in (139) is zero. Rather, the positron current is absent from (145) because electrons are confined to a subset of flow-lines within the total.[34]

7.3 Current of quadrature bi-spinors

Nominally $\psi_+(x,\mathbf{k})$ and $\psi_-(x,\mathbf{k})$ are the positron and electron Majorana bi-spinors, respectively. Even so it is possible to modify $\psi_+(x,\mathbf{k})$ so that it becomes a generator of null electron flow lines. Let $d(\mathbf{k})$ be a 4-displacement of x in (136):

$$e^{-ik_c\circ d(\mathbf{k})}\psi_{\sigma_e}(x+d(\mathbf{k}),\mathbf{k}) = \frac{1}{2}\left[\psi(\mathbf{k}) + e^{-2ik_c\circ d(\mathbf{k})}\sigma_e\psi^c(\mathbf{k})e^{-ik_c\circ x}\right]. \tag{146}$$

Choosing $k_c \circ d(\mathbf{k}) = \pi/2$ gives a quadrature relation between the electron and positron bi-spinors:

34. The distinction here is between a current (or a bi-spinor) as a vector-valued *function* that generates flow-lines and the *occupancy* of those flowlines, which is handled in the second-quantized theory by field-operators and state-vectors, respectively.

$$\psi_{\bar{\sigma}_e}\left(x,\mathbf{k}\right) = i\psi_{\sigma_e}\left(x + d\left(\mathbf{k}\right),\mathbf{k}\right). \tag{147}$$

It will be convenient to restrict $d\left(\mathbf{k}\right)$ to a displacement in time, whereupon

$$\psi_{-}\left(x,\mathbf{k}\right) = i\psi_{+}\left(t + \tau\left(\left|\mathbf{k}\right|\right),\mathbf{x},\mathbf{k}\right); \quad \tau\left(\left|\mathbf{k}\right|\right) = \frac{\pi}{2\sqrt{\kappa_c^2 + \mathbf{k}^2}}. \tag{148}$$

Hence the electron current can be written entirely in terms of the naturally-weighted null positron current, though with quadrature bi-spinors:

$$\left\{\mathbb{j}\left(x\right)\right\}\Big|_{\substack{\text{no local}\\\text{positrons}}} = -\frac{4\kappa_c}{\pi^3}\int \mathrm{d}^3 k\left\langle\psi_{+}\left(t + \tau\left(\left|\mathbf{k}\right|\right),\mathbf{x},\mathbf{k}\right)\bar{\psi}_{+}\left(t + \tau\left(\left|\mathbf{k}\right|\right),\mathbf{x},\mathbf{k}\right)\right\rangle_1 . \tag{149}$$

Rather than 'anti-positron current' we will refer to this as a 'quadrature current', with the understanding that it is the bi-spinor, not the current, that is in quadrature (relative to its phase when acting as the generator of positron flow lines). The current is negated by this operation of course.

When both contributions (145) and (149) are equally weighted the electron current becomes

$$\left\{\mathbb{j}\left(x\right)\right\}\Big|_{\substack{\text{no local}\\\text{positrons}}} = -\frac{2\kappa_c}{\pi^3}\int \mathrm{d}^3 k\left(\left\langle\psi_{-}\left(t,\mathbf{x},\mathbf{k}\right)\bar{\psi}_{-}\left(t,\mathbf{x},\mathbf{k}\right)\right\rangle_1 + \left\langle\psi_{+}\left(t + \tau\left(\left|\mathbf{k}\right|\right),\mathbf{x},\mathbf{k}\right)\bar{\psi}_{+}\left(t + \tau\left(\left|\mathbf{k}\right|\right),\mathbf{x},\mathbf{k}\right)\right\rangle_1\right) . \tag{150}$$

Despite appearances the current (150) is the same as (145). In the Majorana representation charge conjugation is the same as complex conjugation, a consequence of which is that $\psi_{+}\left(x,\mathbf{k}\right)$ and $\psi_{-}\left(x,\mathbf{k}\right)$ are the real and imaginary parts of $\psi\left(x,\mathbf{k}\right)$ (see (136)). This is consistent with (148), which 'projects' the positron bi-spinors into the domain of the electron bi-spinors. Accordingly, a bi-spinor defined as

$$\psi\left(x,\mathbf{k}\right) = \frac{1}{\sqrt{2}}\left(i\psi_{+}\left(t + \tau\left(\left|\mathbf{k}\right|\right),\mathbf{x},\mathbf{k}\right) + \psi_{-}\left(x,\mathbf{k}\right)\right) \tag{151}$$

leaves the currents (145), (149), and (150) unchanged.

7.4 Traditional Dirac current

In contrast with (151) consider now the 'composite' bi-spinor

$$\psi\left(x,\mathbf{k}\right) = \frac{1}{\sqrt{2}}\left(\psi_{+}\left(t + \tau\left(\left|\mathbf{k}\right|\right),\mathbf{x},\mathbf{k}\right) + \psi_{-}\left(x,\mathbf{k}\right)\right). \tag{152}$$

$\psi\left(x,\mathbf{k}\right)$ is complex in the Majorana representation. Because the real and imaginary contributions to (152) are independent (each independently solves the Dirac equation), and in contrast with (151), they contribute independently to a current constructed from (152).[35] Specifically, the total electron current using (152) is

$$\left\{\mathbb{j}_{Dirac}\left(x\right)\right\} = -\frac{2\kappa_c}{\pi^3}\int \mathrm{d}^3 k \left\langle \psi\left(x,\mathbf{k}\right)\bar{\psi}\left(x,\mathbf{k}\right)\right\rangle_1 \quad . \tag{153}$$

This is the traditional Dirac current, to be compared with (134). Eq. (153) re-employs otherwise unused unpopulated positron flow-lines - that are intrinsic to the naturally-weighted self-consistent current - as electron flow lines. Note however, unlike (151), these do not come from the same naturally-weighted ensemble; Eq. (153) combines two electron bi-spinors from two independent ensembles.[36] Consequently we could just as well write

$$\psi\left(x,\mathbf{k}\right) = \frac{1}{\sqrt{2}}\left(\boldsymbol{\psi}_-^{(1)}\left(x,\mathbf{k}\right) + i\boldsymbol{\psi}_-^{(2)}\left(x,\mathbf{k}\right)\right) \tag{154}$$

or

$$\psi\left(x,\mathbf{k}\right) = \frac{1}{\sqrt{2}}\left(\boldsymbol{\psi}_-^{(1)}\left(x,\mathbf{k}\right) + \boldsymbol{\psi}_+^{(2)}\left(x,\mathbf{k}\right)\right) \tag{155}$$

where the superscripts distinguish between the two different parent ensembles, and where $\boldsymbol{\psi}_+^{(2)}\left(x,\mathbf{k}\right)$ in the second expression is unrelated to $\boldsymbol{\psi}_-^{(2)}\left(x,\mathbf{k}\right)$ in the first (they do not originate from the same parent ensemble).

7.5 External Coupling

The electron and positron Majorana bi-spinors each generate null currents. Correspondingly, a current restricted to a single charge species (145), is null. Non-null naturally-weighted currents require the presence of both species. By contrast, the utility of (155) for example is that it permits description of a non-null ensemble current of a single charge species. It should be kept in mind however that using (155) exclusively as a generator of electron flow lines hides the underlying mechanism for less than light-speed motion, which is interference of bi-spinors of different charge species. Relatedly, though (153) is an adequate estimator for a local electron current when the $\psi\left(x,\mathbf{k}\right)$ are solutions of the free Dirac equation, we observe that an interaction of the form

$$L_{int} = -\int \mathrm{d}^4 x \left\{\mathbb{j}_{\mathrm{Dirac}}\left(x\right)\right\} \circ \mathbb{A}_{\mathrm{ext}}\left(x\right) \tag{156}$$

35. They must be independent because the current is real.

36. Hence the combination (152) generates a double cover of the current vector field.

would simultaneously couple $\mathbb{A}_{\text{ext}}\left(x\right)$ to currents from two independent ensembles. In general (156) is unphysical, and is the source of problems with the traditional presentation of the single particle Dirac theory. Further discussion of interaction with external – anti-symmetric - potentials is postponed to a future work.

7.6 Dynamic independence of the currents

As shown in Appendix C the conserved charge currents can be further separated into independently conserved charged spin currents via an appropriately chosen spin projection $\mathbb{P}_s$, $\boldsymbol{\psi}_{s,\sigma_e} = \mathbb{P}_s \boldsymbol{\psi}_{\sigma_e}$, giving a total of 4 conserved null currents that can be derived from a general solution of (131). Specifically, the naturally-weighted current (139) is

$$\left\{\mathbb{j}\left(x\right)\right\} = \frac{4\kappa_c}{\pi^3} \sum_{\substack{s\in\uparrow\downarrow \\ \sigma_e=\pm 1}} \sigma_e \int \mathrm{d}^3 k \left\langle \boldsymbol{\psi}_{s,\sigma_e}\left(x,\mathbf{k}\right) \bar{\boldsymbol{\psi}}_{s,\sigma_e}\left(x,\mathbf{k}\right)\right\rangle_1 = \frac{1}{\pi^3} \sum_{\substack{s\in\uparrow\downarrow \\ \sigma_e=\pm 1}} \sigma_e \int \mathrm{d}^3 k \left\{\mathbb{j}_{s,\sigma_e}\left(x,\mathbf{k}\right)\right\} \ . \tag{157}$$

where

$$\left\{\mathbb{j}_{s,\sigma_e}\left(x,\mathbf{k}\right)\right\} := 4\kappa_c \sigma_e \left\langle \boldsymbol{\psi}_{s,\sigma_e}\left(x,\mathbf{k}\right) \bar{\boldsymbol{\psi}}_{s,\sigma_e}\left(x,\mathbf{k}\right)\right\rangle_1 \ . \tag{158}$$

The absence in the current of cross terms of the form $\boldsymbol{\psi}_{s,\sigma_e}\left(x,\mathbf{k}\right)\bar{\boldsymbol{\psi}}_{s',\sigma'_e}\left(x,\mathbf{k}\right)$ where s',σ'_e differ from s,σ_e is an outcome of the properties of the projectors. In this work these represent 4 *coinciding* ensembles of possible paths of a light-speed charge following the flow lines of a null potential, as described in Section 3. That is, at every spacetime-point there are 4 null currents that can be associated with a particular ψ, each one of which is a member of an ensemble of mutually exclusive null worldlines. Crucial to the applicability of the method of Section 3 to the general case that the incoming potential and Faraday are non-null is that these currents are dynamically independent, the demonstration of which is given Appendix C.[37]

8 Superposition and mutual exclusion

Eq. (157) is an integral superposition of an outer-product of phase-space bi-spinors, each term corresponding to a single Fourier **k**-space component of the current. The constraint that the current is

37. The construction of null currents via successive projections of ψ is in the domain of standard theory. The derivation in Appendix C is primarily to demonstrate dynamical independence.

null will be satisfied if each of the $\left\{\mathbb{j}_{s,\sigma_e}\left(x,\mathbf{k}\right)\right\}$ - i.e. for each possible $s,\sigma_e,\mathbf{k}$ at all x, - is mutually exclusive. Under these conditions each term in the superposition (the integrand) is a candidate for the role of *sole* contributor to a single instance current, and whose relative magnitude therefore corresponds to the probability of that being the case in any single, isolated, instance. Due to the co-occurrence in the integrand in (157) of $\boldsymbol{\psi}$ and $\bar{\boldsymbol{\psi}}$ with the same $\mathbf{k}$ the current can be said to be explicitly diagonal in that representation. It is implicitly diagonal also in the sub-space indexed by s,σ_e due to the automatic vanishing of cross terms $\boldsymbol{\psi}_{s,\sigma_e}\left(x,\mathbf{k}\right)\bar{\boldsymbol{\psi}}_{s',\sigma_e'}\left(x,\mathbf{k}\right)$ in the expression for the current. In the **k**-space representation the amplitude of a mode is the classical probability for the occupation of *any* flowline within that mode. In the conversion of the ensemble to a superposition nullity is preserved through mutual exclusion in the joint distribution of all such mode probabilities.

The ensemble potential can be regarded equally as the generator of flow-lines for both **x**-space and **k**-space representations - subject to the constraint that its ‘parent’ multivector satisfy (88).[38] Nullity will be preserved in the latter case (i.e. for individual flow-lines rather than **k**-space modes) by assigning mutually-exclusive classical occupation probabilities to each $\mathbf{x}$ at some fixed *t*.

Let us construct an **x**-space ensemble of legal flow-lines starting from the general solution $\varphi\left(x\right)$ of the real space Dirac equation

$$\left[\not{\partial} + i\kappa_c\right]\varphi\left(x\right) = 0 \tag{159}$$

and require current that to be equivalent to the ensemble current constructed from super-posed phase-space components (134).[39] It will be convenient initially to suppose that φ and ψ are two different bi-spinors. We let the first be defined through (159) and its relationship to the current

$$\left\{\mathrm{j}\left(x\right)\right\} = \left\langle\varphi\left(x\right)\bar{\varphi}^c\left(x\right)\right\rangle_1 + \left\langle\varphi^c\left(x\right)\bar{\varphi}\left(x\right)\right\rangle_1 \tag{160}$$

with any normalization folded into the definition of φ. We let the second be defined through the second of (131):

38. The joint distribution to enforce nullity through mutual exclusion of flow-line occupancy in the real-space representation is not simply related to the joint distribution required to enforce nullity through mutual exclusion of **k**-space mode occupancy because mutual exclusion is a non-linear constraint on the joint occupancy probability in each representation.

39. The findings of this section are unchanged if expressed in terms of Dirac currents rather than naturally-weighted currents.

$$\left[\kappa_c + \mathbb{k}_c\right]\psi\left(x,\mathbf{k}\right) = 0 \tag{161}$$

and its relation to the current (134), though with normalization similarly absorbed, so that

$$\left\{\mathbb{j}\left(x\right)\right\} = \int \mathrm{d}^3\, k\left[\left\langle\psi\left(x,\mathbf{k}\right)\bar{\psi}^c\left(x,\mathbf{k}\right)\right\rangle_1 + \left\langle\psi^c\left(x,\mathbf{k}\right)\bar{\psi}\left(x,\mathbf{k}\right)\right\rangle_1\right]. \tag{162}$$

That is, $\psi\left(x,\mathbf{k}\right)$ is a vector factor in a spectral decomposition of the phase-space representation of the current, whilst $\varphi\left(x\right)$ is a vector factor in a spectral decomposition of the real-space representation of the current. A sufficient condition for the equivalence of (160) and (162) is

$$\varphi\left(x\right)\bar{\varphi}^c\left(x\right) = \int \mathrm{d}^3\, k\psi\left(x,\mathbf{k}\right)\bar{\psi}^c\left(x,\mathbf{k}\right) \tag{163}$$

Both bi-spinors have real-space and phase-space representations. In particular

$$\pi\left(x\right) = \frac{1}{\left(2\pi\right)^3}\int \mathrm{d}^3\, k\pi\left(x,\mathbf{k}\right) = \frac{1}{\left(2\pi\right)^3}\int \mathrm{d}^3\, k\pi\left(\mathbf{k}\right)e^{-ik_c\circ x}. \tag{164}$$

the inverse relationship of which is

$$\int \mathrm{d}^3\, x\pi\left(x\right)e^{ik_c\circ x} = \frac{1}{\left(2\pi\right)^3}\int \mathrm{d}^3\, x\int \mathrm{d}^3\, k'\pi\left(\mathbf{k}'\right)e^{i\left(k_c - k'_c\right)\circ x} = \pi\left(\mathbf{k}\right) \tag{165}$$

where $k'_c = \left({}_{+}\sqrt{\kappa_c^2 + \mathbf{k}'^2}, \mathbf{k}'\right)$, and therefore

$$\pi\left(x,\mathbf{k}\right) = \int \mathrm{d}^3\, x'\pi\left(t,\mathbf{x}'\right)e^{i\mathbf{k}.\left(\mathbf{x}-\mathbf{x}'\right)} \tag{166}$$

where π is either of φ,ψ. Using the first of (164) in the left of (163) gives

$$\int \mathrm{d}^3\, k\int \mathrm{d}^3\, k'\varphi\left(x,\mathbf{k}\right)\bar{\varphi}^c\left(x,\mathbf{k}'\right) = \left(2\pi\right)^6\int \mathrm{d}^3\, k\psi\left(x,\mathbf{k}\right)\bar{\psi}^c\left(x,\mathbf{k}\right) \tag{167}$$

for which it is sufficient that

$$\varphi\left(x,\mathbf{k}\right)\bar{\varphi}^c\left(x,\mathbf{k}'\right) = \left(2\pi\right)^6\delta^3\left(\mathbf{k}-\mathbf{k}'\right)\psi\left(x,\mathbf{k}\right)\bar{\psi}^c\left(x,\mathbf{k}\right) \tag{168}$$

which implies

$$\varphi\left(\mathbf{k}\right)\bar{\varphi}^c\left(\mathbf{k}'\right) = \left(2\pi\right)^6\delta^3\left(\mathbf{k}-\mathbf{k}'\right)\psi\left(\mathbf{k}\right)\bar{\psi}^c\left(\mathbf{k}\right). \tag{169}$$

Using (166) in the right of (163) gives

$$\begin{aligned}\varphi(x)\bar{\varphi}^c(x) &= \int \mathrm{d}^3 k \int \mathrm{d}^3 x' \int \mathrm{d}^3 x'' \psi(t,\mathbf{x}') e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}')} \bar{\psi}^c(t,\mathbf{x}'') e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}'')} \\ &= (2\pi)^3 \int \mathrm{d}^3 x' \psi(t,\mathbf{x}') \bar{\psi}^c(t, 2\mathbf{x}-\mathbf{x}')\end{aligned} \tag{170}$$

for which it is sufficient that

$$\psi(t,\mathbf{x})\bar{\psi}^c(t,\mathbf{x}') = \frac{1}{\pi^3}\delta^3(\mathbf{x}-\mathbf{x}')\varphi(t,\mathbf{x})\bar{\varphi}^c(t,\mathbf{x}). \tag{171}$$

Equivalence of $\varphi(x,\mathbf{k})$ and $\psi(x,\mathbf{k})$, i.e. $\varphi(x,\mathbf{k}) = V\psi(x,\mathbf{k})$ for some scalar V requires that the bi-spinor is delta-correlated with its adjoint both in $\mathbf{k}$ space and in $\mathbf{x}$ space at a common time:

$$\begin{aligned}\psi(t,\mathbf{x})\bar{\psi}^c(t,\mathbf{x}') &= \frac{V}{\pi^3}\delta^3(\mathbf{x}-\mathbf{x}')\psi(t,\mathbf{x})\bar{\psi}^c(t,\mathbf{x}) \\ \psi(\mathbf{k})\bar{\psi}^c(\mathbf{k}') &= \frac{(2\pi)^6}{V}\delta^3(\mathbf{k}-\mathbf{k}')\psi(\mathbf{k})\bar{\psi}^c(\mathbf{k})\end{aligned}. \tag{172}$$

Even allowing for an implicit averaging process over hidden variables say, the product of ordinary functions cannot be delta-valued in $\mathbf{x}$ and $\mathbf{k}$ space simultaneously. Mutual exclusion in both representations cannot be accommodated by a bi-spinor vector of ordinary functions therefore. Eq. (172) can be achieved instead if the bi-spinors are operator-valued fields, presumed to operate on a state vector. However, since delta-correlation manifests only in the outer-product $\psi\bar{\psi}$ no approximation is entailed by ignoring it when computing the linear dynamics, such as the propagation of ψ from some initial condition. It is sufficient instead to set all off-diagonal terms to zero in the calculation of any quantity that is quadratic in ψ, and therefore of any 'expectation' $\langle \psi\bar{\psi}\,\mathrm{O}\rangle = \bar{\psi}\,\mathrm{O}\,\psi$, in whatever representation one chooses to express ψ (presumably: the representation in which the observation prediction is expressed).[40] If the states are discretely quantized due to confinement in a box, or by a Coulomb potential say, then the Dirac delta functions become Kronecker delta functions and the elimination of off-diagonal terms is straight-forward. Accommodation of continuously indexed states will require further investigation.

If the calculation is performed in a representation in which O is diagonal - i.e. in which ψ are eigenvectors of O - then there will be nothing extra to do. In that case the expectation will be a weighted sum over the eigenvalues where the coefficients are classical mutually exclusive probabilities.

40. Subject to the constraint that ψ satisfies the Dirac equation in that representation.

The technique of striking out off-diagonal terms is not restricted to the real and Fourier space representations – it applies to any functional representation of solutions of the Dirac equation.

9 Summary

The Dirac Equation is shown to derive from an equation for the Clifford multivector of the time-symmetric potential and Faraday of classical direct particle electrodynamics. The probabilistic aspect is seen to be a consequence of embedding the dynamics of a single current in an ensemble of hypothetical currents. Eigenvalue selection and representation-independence of the Dirac theory are shown to be consequences of the non-linear constraint in an otherwise linear system that the current is null.

Appendix A Notation

COMPONENT REPRESENTATIONS

x, $q_k(t)$, and $s_{k,l}(t,t') = q_k(t) - q_l(t')$ in italic font are Lorentz vectors understood to be represented traditionally, i.e. as an ordered collection of 4 components. We use the shorthand $\int \mathrm{d}^4 x = \prod_{\mu=1}^{4} \int \mathrm{d}\, x^\mu$ and $x^2 = x^\mu x_\mu$. $\mathbf{x}, \mathbf{q}(t), \mathbf{v}(t), \mathbf{A}(x)$ are Euclidian 3-vectors, in which terms $x = (t, \mathbf{x})$, $q = q(t) = (t, \mathbf{q}(t))$, $v = v(t) = (1, \mathbf{v}(t))$, $A(x) = (\phi(x), \mathbf{A}(x))$. $v(t)$ is not a true Lorentz vector. $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field 3-vectors.

CLIFFORD ELEMENTS

$\mathbb{A}(x) = A^\mu(x)\gamma_\mu$ is both an element of $C\ell_{1,3}(\mathbb{R})$ (a vector) and its representation in $\mathrm{M}_4(\mathbb{C})$ between which we make no distinction. Likewise

$$\mathbb{F}(x) = \partial\!\!\!/ \wedge \mathbb{A}(x) \equiv \tfrac{1}{2}\left(\partial\!\!\!/ \mathbb{A}(x) - \mathbb{A}(x)\overleftarrow{\partial\!\!\!/}\right) = \tfrac{1}{2} F^{\mu\nu}(x)\gamma_\mu\gamma_\nu \tag{A1}$$

is the Clifford representation of the Faraday bi-vector. We use the shorthand $\mathbb{a} \circ \mathbb{b} = \langle \mathbb{a}\mathbb{b} \rangle = \langle \mathbb{a}\mathbb{b} \rangle_0 = \tfrac{1}{2}(\mathbb{a}\mathbb{b} + \mathbb{b}\mathbb{a})$ for the scalar product and $\mathbb{a} \wedge \mathbb{b} = \langle \mathbb{a}\mathbb{b} \rangle_2 = \tfrac{1}{2}(\mathbb{a}\mathbb{b} - \mathbb{b}\mathbb{a})$ for the exterior product of arbitrary vectors $\mathbb{a}, \mathbb{b}$. $\langle \mathbb{M} \rangle = \tfrac{1}{4}\mathrm{tr}(\mathbb{M})$ extracts the scalar part of $\mathbb{M}$. $\langle \mathbb{M} \rangle_j$ $j \in [0..4]$, extracts the various 'grades' (scalar, vector, bi-vector, pseudo-vector, and pseudo-scalar) of $\mathbb{M}$:

$$\langle \mathbb{M} \rangle_0 = \langle \mathbb{M} \rangle,\ \langle \mathbb{M} \rangle_1 = \langle \mathbb{M}\gamma^\mu \rangle \gamma_\mu,\ \langle \mathbb{M} \rangle_2 = \langle \mathbb{M}\gamma^\nu\gamma^\mu \rangle \gamma_\mu\gamma_\nu,\ \langle \mathbb{M} \rangle_3 = \langle \mathbb{M}\mathbb{I}\gamma^\mu \rangle \gamma_\mu \mathbb{I},\ \langle \mathbb{M} \rangle_4 = \langle \mathbb{M}\mathbb{I} \rangle \mathbb{I}\ . \tag{A2}$$

$\mathbb{I} = i\gamma_5$ is the unit pseudo-scalar. Generally, double-strike quantities are Clifford elements, including $\mathbb{Q}$ which is a Clifford element (rather than the field of rational numbers). Exceptions are $\partial\!\!\!/ = \gamma^\mu \partial_\mu$, which retains its traditional meaning, and $\mathbb{R}$, $\mathbb{C}$, which retain their number-set meanings. The charge-conjugation matrix is written as just C. $\{\mathbb{A}\}$ denotes an ensemble of $\mathbb{A}$.

MATRIX REPRESENTATIONS

We take the chiral matrix representation of the Clifford basis corresponding to $C\ell_{1,3}(\mathbb{R}) \cong \mathrm{M}_2(\mathbb{H})$, where $\mathbb{H}$ are the real quaternions, to be

$$\gamma_\mu = \begin{bmatrix} \mathbb{O} & \sigma_\mu \\ \bar{\sigma}_\mu & \mathbb{O} \end{bmatrix}\ . \tag{A3}$$

Here $\mathbb{O}$ is a 2x2 null matrix, and the σ_μ are the 2x2 Pauli-matrixes:

$$\sigma_0 = \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{A4}$$

and

$$\bar{\sigma}_0 = \sigma_0, \quad \bar{\sigma}_j = -\sigma_j; \quad j \in \{1,2,3\} \ . \tag{A5}$$

In this representation the Clifford unit pseudo-scalar is

$$\mathrm{I} \to i\gamma_5 = i\begin{bmatrix} -\mathbb{1} & \mathbb{O} \\ \mathbb{O} & \mathbb{1} \end{bmatrix} . \tag{A6}$$

The Clifford basis in the Dirac representation is

$$\gamma_i = \begin{bmatrix} \mathbb{O} & \sigma_i \\ \bar{\sigma}_i & \mathbb{O} \end{bmatrix} = \begin{bmatrix} \mathbb{O} & \sigma_i \\ -\sigma_i & \mathbb{O} \end{bmatrix}, \quad \gamma_0 = \begin{bmatrix} \mathbb{1} & \mathbb{O} \\ \mathbb{O} & -\mathbb{1} \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} \mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{bmatrix} . \tag{A7}$$

Subscripts

Unless it refers to a Pauli matrix, σ with or without a subscript is either of ± 1, reducing to $\pm$ if used as a subscript. The overbar is its negation: $\bar{\sigma} = -\sigma$. Particles (their paths, velocities, currents) are distinguished by subscripts k and l in $[1, N]$ where $N \sim 10^{80}$. But l also identifies the 'local' quantity (a particular isolated current say) that is the focus of interest. $\bar{l}$ denotes all but the l^{th} member.

Spinors and Bi-Spinors

χ and ξ denote spinors in $\mathbb{C}^2$. $\bar{\chi}$ denotes $i\sigma_2\chi^*$, so that $\bar{\bar{\chi}} = -\chi$. With this definition if $\mathbb{A} = \chi\chi^\dagger$ then $\bar{\mathbb{A}} = \bar{\chi}\bar{\chi}^\dagger$, and likewise for higher rank expansions. Though ψ can be rendered in $\mathrm{M}_4(\mathbb{C})$, and potentially as a Clifford object, to facilitate comparison with standard Dirac theory the default interpretation of ψ in this work is as a vector (bi-spinor) in $\mathbb{C}^4$. $\bar{\psi} = \psi^\dagger\gamma_0$ also has the usual meaning. Even so, outer-products $\psi\bar{\psi}$ - which are in $\mathrm{M}_4(\mathbb{C})$ - will be equally interpreted as Clifford objects, permitting the use therefore of $\langle\psi\bar{\psi}\rangle_2$ to denote the bi-vector part, for example. We take the charge conjugate of ψ to be $\psi^c = \gamma_0 \mathrm{C} \psi^*$, where C is the charge conjugation matrix. We assume representations in which $\mathrm{C}^2 = -1$ and $\mathrm{C}^T = \mathrm{C}^\dagger = -\mathrm{C}$, which includes the Dirac, Majorana, and chiral representations. In those representations in particular $\mathrm{C} = i\gamma_0\gamma_2$, so that $\psi^c = i\gamma_2\psi^*$, and C has the property $\mathrm{C}\gamma_\mu\mathrm{C} = \gamma_\mu^T$. With this definition $(\psi^c)^c = \psi$. If $\psi^T = [\chi, \xi]$, then

$$\psi^c = \begin{bmatrix} i\sigma_2\xi^* \\ -i\sigma_2\chi^* \end{bmatrix} = \begin{bmatrix} \bar{\xi} \\ -\bar{\chi} \end{bmatrix} . \tag{A8}$$

Majorana bi-spinors are denoted by $\boldsymbol{\psi}$ and have the property $\boldsymbol{\psi}=\boldsymbol{\psi}^c$.[41] In the Dirac and chiral representations $\boldsymbol{\psi}$ has the form $\boldsymbol{\psi}^T = i\left[\chi, \bar{\chi}\right]$. In the Majorana representation disregarding an overall arbitrary phase factor $\boldsymbol{\psi}$ is entirely real.

Function Arguments

We use the symbolic shorthand that the (apparently) same function appearing with different explicit arguments denotes different but related representations of that function, rather than the same function of different arguments, although factors of 2 and ½ in the arguments have the conventional interpretation. $\mathbb{Q}\left(x\right)$ is the x-space representation of the EM multivector, and $\mathbb{Q}\left(x;\mathbf{k}\right)$ is a single Fourier component of $\mathbb{Q}\left(x\right)$ with wave-vector $\mathbf{k}$ whose time-component is $k_0 = k_0\left(\mathbf{k}\right) = {}_{+}\sqrt{\kappa^2 + \mathbf{k}^2}$. $\mathbb{Q}\left(\mathbf{k}\right)$ is the same object as $\mathbb{Q}\left(x;\mathbf{k}\right)$ but without the Fourier phase factor. Likewise for $\psi\left(x;\mathbf{k}\right)$ and $\psi\left(\mathbf{k}\right)$, though with a different phase factor. Specifically

$$\mathbb{Q}\left(x;\mathbf{k}\right) = e^{-ik\circ x}\mathbb{Q}\left(\mathbf{k}\right) = \psi\left(x,½\mathbf{k}\right)\bar{\psi}^c\left(x,½\mathbf{k}\right) \tag{A9}$$

and

$$\psi\left(x,\mathbf{k}\right) = e^{-ik_c\circ x}\psi\left(\mathbf{k}\right) \Leftrightarrow \psi\left(x,½\mathbf{k}\right) = e^{-½ik\circ x}\psi\left(½\mathbf{k}\right) \tag{A10}$$

where $k_c = \left(k_0, \mathbf{k}\right)$, and $k_0 = {}_{+}\sqrt{\kappa_c^2 + \mathbf{k}^2}$, and where $\kappa_c = \kappa / 2$. Together these imply $\psi\left(\mathbf{k}\right)\bar{\psi}^c\left(\mathbf{k}\right) = \mathbb{Q}\left(2\mathbf{k}\right)$.

41. An arbitrary overall phase-factor is here set to unity, which choice is responsible for the factor of i in $\boldsymbol{\psi}^T = i\left[\chi, \bar{\chi}\right]$.

Appendix B Maximally Null Multivectors

Clifford Representation

Let ψ be a Majorana bi-spinor. It is readily shown using the property $\psi{=}\psi^c$ that

$$\left\langle\psi\bar{\psi}\right\rangle_0 = \left\langle\psi\bar{\psi}\right\rangle_3 = \left\langle\psi\bar{\psi}\right\rangle_4 = \mathbb{O} \Rightarrow \bar{\psi}\psi{=}\bar{\psi}\gamma_5\psi{=}\bar{\psi}\gamma_\mu\gamma_5\psi{=}\,0 \qquad \forall\mu\in\left[0,3\right] \tag{B1}$$

and so $\psi\bar{\psi}$ can have vector and bi-vector parts only. Then

$$\mathbb{Q} := \psi\bar{\psi} = \kappa\mathbb{A} + i\mathbb{F} \tag{B2}$$

for some real scalar κ, vector $\mathbb{A}$, and bi-vector $\mathbb{F}$; $\kappa > 0$ without loss of generality. One sees that $\mathbb{Q}^2 = 0$ due to the first of (B1), and therefore

$$\kappa^2\mathbb{A}^2 - \mathbb{F}^2 + i\kappa\left(\mathbb{A}\mathbb{F} + \mathbb{F}\mathbb{A}\right) = \mathbb{O} \ . \tag{B3}$$

Setting the individual blades to zero:

$$\kappa^2\mathbb{A}^2 = \left\langle\mathbb{F}^2\right\rangle, \quad \mathbb{A}\mathbb{F} + \mathbb{F}\mathbb{A} = \mathbb{O}, \quad \left\langle\mathbb{F}^2\right\rangle_4 = \mathbb{O} \ . \tag{B4}$$

Similarly, from $\left\langle\psi\bar{\psi}\right\rangle_3 = \mathbb{O} \Rightarrow \left\langle\mathbb{I}\psi\bar{\psi}\right\rangle = \mathbb{O}$ it follows that $\mathbb{I}\mathbb{Q}$ is also null, and therefore

$$\begin{aligned}\left(\mathbb{I}\mathbb{Q}\right)^2 &= \mathbb{O}\\ &= \mathbb{I}\left(\kappa\mathbb{A} + i\mathbb{F}\right)\mathbb{I}\left(\kappa\mathbb{A} + i\mathbb{F}\right)\\ &= \left(\kappa\mathbb{A} - i\mathbb{F}\right)\left(\kappa\mathbb{A} + i\mathbb{F}\right)\\ &= \kappa^2\mathbb{A}^2 + \mathbb{F}^2 + i\kappa\left(\mathbb{A}\mathbb{F} - \mathbb{F}\mathbb{A}\right)\end{aligned} \ . \tag{B5}$$

Setting the individual blades to zero:

$$\kappa^2\mathbb{A}^2 + \left\langle\mathbb{F}^2\right\rangle = \mathbb{O}, \quad \mathbb{A}\mathbb{F} - \mathbb{F}\mathbb{A} = \mathbb{O} \ . \tag{B6}$$

Eqs. (B4) and (B6) together imply [42]

$$\mathbb{A}^2 = \mathbb{F}^2 = \mathbb{A}\mathbb{F} = \mathbb{F}\mathbb{A} = \mathbb{O} \ . \tag{B7}$$

That is, the potential and Faraday are both null, and the vector and pseudo-vector derived from their product are both zero. The component form of these conditions is given below.

42. This result can be inferred directly from the fact that in the chiral representation the Majorana spinor is $\psi^T = i\left[\chi, \bar{\chi}\right]$, and that a Lorentz scalar cannot be constructed from a single Weyl spinor by purely algebraic (non-operator) means.

The outer product $\psi\bar{\psi}$ is called a 'boomerang' in the Clifford literature [18] because, regarded as a multivector, it squares to itself times a Lorentz scalar: $\left[\psi\bar{\psi}\right]^2 = \lambda\psi\bar{\psi}$, where $\lambda = \bar{\psi}\psi$. But when $\psi \to \boldsymbol{\psi}$ is a Majorana bi-spinor then $\lambda = 0$, and the 'boomerang' does not return. Taking into account also the nullity of the products of its component blades in (B7) then, in conformity with the Clifford terminology, $\boldsymbol{\psi}\bar{\boldsymbol{\psi}}$ is a null boomerang.

Chiral and $\mathrm{M}_2(\mathbb{C})$ Representations

In the chiral representation the EM multivector is

$$\mathbb{Q} = \kappa\mathbb{A} + i\mathbb{F} = \begin{bmatrix} i\mathbb{F}_{2\times2} & \kappa\mathbb{A}_{2\times2} \\ \kappa\bar{\mathbb{A}}_{2\times2} & -i\mathbb{F}^{\dagger}_{2\times2} \end{bmatrix} \tag{B8}$$

where

$$\mathbb{A}_{2\times2} = A^{\mu}\boldsymbol{\sigma}_{\mu}, \quad \bar{\mathbb{A}}_{2\times2} = A^{\mu}\bar{\boldsymbol{\sigma}}_{\mu}, \quad \mathbb{F}_{2\times2} = \left(E^j + iB^j\right)\boldsymbol{\sigma}_j, \tag{B9}$$

Explicitly:

$$\mathbb{Q} = \begin{bmatrix} iE_z - B_z & iE_x + E_y + iB_y - B_x & \kappa\left(\phi + A_z\right) & \kappa\left(A_x - iA_y\right) \\ iE_x - E_y - iB_y - B_x & -iE_z + B_z & \kappa\left(A_x + iA_y\right) & \kappa\left(\phi - A_z\right) \\ \kappa\left(\phi - A_z\right) & \kappa\left(-A_x + iA_y\right) & -iE_z - B_z & -iE_x - E_y + iB_y - B_x \\ \kappa\left(-A_x - iA_y\right) & \kappa\left(\phi + A_z\right) & -iE_x + E_y - iB_y - B_x & iE_z + B_z \end{bmatrix} \tag{B10}$$

Despite the subscripts, all components are contra-variant. In the chiral representation the Majorana bi-spinors have the form

$$\boldsymbol{\psi} = i\begin{bmatrix} \chi \\ \bar{\chi} \end{bmatrix}. \tag{B11}$$

In these terms a maximally null multivector $\mathbb{Q} = \boldsymbol{\psi}\bar{\boldsymbol{\psi}}$ is

$$\mathbb{Q} = \begin{bmatrix} \chi \\ \bar{\chi} \end{bmatrix} \begin{bmatrix} \bar{\chi}^{\dagger}, \chi^{\dagger} \end{bmatrix} = \begin{bmatrix} \chi\bar{\chi}^{\dagger} & \chi\chi^{\dagger} \\ \bar{\chi}\bar{\chi}^{\dagger} & \bar{\chi}\chi^{\dagger} \end{bmatrix}. \tag{B12}$$

Comparing (B12) with (B8) one has

$$\mathbb{A}_{2\times2} = \frac{1}{\kappa}\chi\chi^{\dagger}, \quad i\mathbb{F}_{2\times2} = \chi\bar{\chi}^{\dagger}. \tag{B13}$$

The conditions (B7) are equivalent to

$$\det\left(\mathbb{A}_{2\times2}\right) = \det\left(\mathbb{F}_{2\times2}\right) = 0 \tag{B14}$$

which immediately follow from the rank deficiency in (B13).

Stereographic Projection

The 'scalar' part of the first of (B13) gives $\chi^\dagger\chi = \kappa\phi$, which can be valid only if $\phi > 0$.[43] Presuming so, then

$$\chi = \frac{Z_\chi}{2}\sqrt{\frac{\kappa\phi}{|\mathbf{A}|\left(|\mathbf{A}| - A_z\right)}}\begin{bmatrix} A_x - iA_y \\ |\mathbf{A}| - A_z \end{bmatrix} \Leftrightarrow \bar{\chi} = \frac{Z_\chi^*}{2}\sqrt{\frac{\kappa\phi}{|\mathbf{A}|\left(|\mathbf{A}| - A_z\right)}}\begin{bmatrix} |\mathbf{A}| - A_z \\ -A_x - iA_y \end{bmatrix} \tag{B15}$$

where $|\mathbf{A}| = |\phi|$ and Z_χ is phase factor that is not fixed by the first of (B13). In a spherical coordinate representation in which

$$\cos\theta = A_z / |\mathbf{A}|, \quad \tan\phi = A_y / A_x \tag{B16}$$

the first of (B15) is

$$\chi^T = Z'_\chi\sqrt{\kappa_c}\left[e^{-½i\phi}\left|\cos ½\theta\right|, e^{+½i\phi}\left|\sin ½\theta\right|\right] \tag{B17}$$

where $Z'_\chi = Z_\chi e^{-½i\phi}$ is a (new) phase factor. Eq. (B17) establishes a correspondence between χ and the stereographic projection of $\hat{\mathbf{A}} = \mathbf{A} / |\mathbf{A}|$ onto the Argand plane (see [93]).

Correspondence between the components of χ and the Faraday can be obtained by equating $\mathbb{F}_{2\times2} = i\chi\bar{\chi}^\dagger$ with the Chiral representation of $\mathbb{F}$ as it appears in (B10)

$$i\mathbb{F}_{2\times2} = \chi\bar{\chi}^\dagger = \begin{bmatrix} iE_z - B_z & iE_x + E_y + iB_y - B_x \\ iE_x - E_y - iB_y - B_x & -iE_z + B_z \end{bmatrix}. \tag{B18}$$

Note that the outcome will be consistent with (B16), (B17) and (B24). In this case the overall phase of χ, which is underdetermined in (B17), is fixed by the phase of the circular polarization, i.e. the rotational phase of the plane containing $\mathbf{E}$ and $\mathbf{B}$.

Component Forms

We start with just the constraint that $\left\langle\mathbb{FA}\right\rangle_1 = \mathbb{0}$, which in indexed notation is

43. Recall that κ was earlier taken to be real and positive, thereby transferring to ϕ responsibility for the overall sign of $\kappa\phi$.

$$F^{\mu\nu}A_\nu = 0 \tag{B19}$$

which has solutions for A only if $\det\{F^{\mu\nu}\} = 0$. By explicit calculation or otherwise one finds that this requires $\mathbf{E.B} = 0$. The space part of (B19) ($\mu \in \{1,2,3\}$) is

$$\phi\mathbf{E} + \mathbf{A}\times\mathbf{B} = \mathbf{0} \Rightarrow \mathbf{A} = \frac{\phi}{\mathbf{B}^2}\left(\mathbf{E}\times\mathbf{B} + \alpha\mathbf{B}\right) \tag{B20}$$

where α is an arbitrary scalar, and therefore

$$A = \frac{\phi}{\mathbf{B}^2}\left(\mathbf{B}^2, \mathbf{E}\times\mathbf{B} + \alpha\mathbf{B}\right) \ . \tag{B21}$$

The time component $F^{0\nu}A_\nu = 0 \Rightarrow \mathbf{E.A} = 0$ is implicit through the condition $\mathbf{E.B} = 0$. If, in addition to $\left\langle \mathbb{FA} \right\rangle_1 = \mathbb{O}$, A is null, then (B21) becomes

$$A = \frac{\phi}{\mathbf{B}^2}\left(\mathbf{B}^2, \mathbf{E}\times\mathbf{B} + \sqrt{\mathbf{B}^2 - \mathbf{E}^2}\,\mathbf{B}\right) \tag{B22}$$

which has physical solutions only if $\mathbf{E}^2 \le \mathbf{B}^2$.

A null Faraday bi-vector has the properties [44]

$$\mathbb{F}^2 = \mathbb{O} \Leftrightarrow \tilde{F}^{\mu\nu}F_{\mu\nu} = \tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = F^{\mu\nu}F_{\mu\nu} = 0 \Rightarrow \mathbf{E.B} = \mathbf{E}^2 - \mathbf{B}^2 = 0 \ . \tag{B23}$$

Hence when the Faraday is null (B22) reduces to

$$A = \phi\left(1, \hat{\mathbf{E}}\times\hat{\mathbf{B}}\right); \quad A^2 = 0 \tag{B24}$$

where the over-hats denote unit vectors, and (B19) extends to

$$\mathbb{AF} = \mathbb{FA} = \mathbb{O} \Leftrightarrow \tilde{F}^{\mu\nu}A_\nu = F^{\mu\nu}A_\nu = 0 \ . \tag{B25}$$

In summary therefore, the component forms of (B7) are

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = F^{\mu\nu}F_{\mu\nu} = \tilde{F}^{\mu\nu}A_\nu = F^{\mu\nu}A_\nu = A^\nu A_\nu = 0 \tag{B26}$$

all of which are captured by the constraints

$$A = \phi\left(1, \hat{\mathbf{E}}\times\hat{\mathbf{B}}\right), \quad \mathbf{E.B} = \mathbf{B}^2 - \mathbf{E}^2 = 0 \tag{B27}$$

which implicitly includes the condition $A^2 = 0$.

44. The condition $\mathbb{F}^2 = \mathbb{O}$ can be written in 3-vector terms as $\mathbf{f}^2 = 0$, where $\mathbf{f} = \mathbf{E} + i\mathbf{B}$ is a complex ‘sixtor’ [74].

Appendix C Conserved Currents

Introduction

The multivector subject $\mathbb{Q}(x)$ of the differential equation (88) comprises an ensemble potential and Faraday of a time-symmetric field. Likewise the subject $\mathbb{Q}(x,k)$ of (127), and $\mathbb{Q}(\mathbf{k})e^{ik_\kappa \circ x}$ of (128). The eigenvector $\psi(x,\mathbf{k})$ in the decomposition (121), and $\psi(½\mathbf{k})$ in the decomposition (126) factorize the multivector such that the potential and Faraday are the vector and bi-vector parts of the outer-product. Specifically, recalling (121)

$$\begin{aligned}\mathbb{A}(x) &= \frac{1}{\kappa}\int \mathrm{d}^3 k\left[\left\langle \psi(x,\mathbf{k})\bar{\psi}^c(x,\mathbf{k})\right\rangle_1 + \left\langle \psi^c(x,\mathbf{k})\bar{\psi}(x,\mathbf{k})\right\rangle_1\right] \\ \mathbb{F}(x) &= \frac{1}{i}\int \mathrm{d}^3 k\left[\left\langle \psi(x,\mathbf{k})\bar{\psi}^c(x,\mathbf{k})\right\rangle_2 + \left\langle \psi^c(x,\mathbf{k})\bar{\psi}(x,\mathbf{k})\right\rangle_2\right]\end{aligned}. \tag{C1}$$

Direct Particle Interaction demands that $\mathbb{A}(x)$ satisfy the Lorenz gauge. $\mathbb{A}(x)$ is a conserved current therefore, and $\int \mathrm{d}^3 x\phi(x)$ the associated conserved 'charge'. In the following we analyze $\mathbb{A}(x)$ into 'components' each of which is a vector that independently satisfies the Lorenz gauge condition. Consistent with (74) each of these is also a (conserved) current. Though an analysis of Dirac solutions into generators of conserved currents lies entirely within traditional theory, we give a derivation here tailored to DPI focused on the null current generators of the flow-lines described in Section 3. For consistency with the method of that section we will need to show that the null currents passing through each spacetime point generated by a general solution $\psi(x,\mathbf{k})$ are dynamically independent, the precise definition of which is given below.

Charge-conjugate currents

The structure of the Dirac equation is such that the vector $\left\langle \psi\bar{\psi}\right\rangle_1$ is a conserved current:

$$\not{\partial} \circ \left\langle \psi(x,\mathbf{k})\bar{\psi}(x,\mathbf{k})\right\rangle_1 = \bar{\psi}(x,\mathbf{k})\overleftrightarrow{\not{\partial}}\psi(x,\mathbf{k}) = 0\,. \tag{C2}$$

If ψ is a solution of (131) then so is ψ^c:

$$\left[\not{\partial} + i\kappa_c\right]^*\psi^* = 0 \Rightarrow \gamma^0\,\mathrm{C}\left[\not{\partial}^* - i\kappa_c\right]\gamma^0\,\mathrm{C}\gamma^0\,\mathrm{C}\psi^* = 0 = \left[\gamma^0\,\mathrm{C}\,\not{\partial}^*\gamma^0\,\mathrm{C} - i\kappa_c\right]\psi^c = -\left[\not{\partial} + i\kappa_c\right]\psi^c\,. \tag{C3}$$

It follows that $\left\langle \psi(x,\mathbf{k})\bar{\psi}(x,\mathbf{k})\right\rangle_1$ and $\left\langle \psi^c(x,\mathbf{k})\bar{\psi}^c(x,\mathbf{k})\right\rangle_1$ are independently conserved currents. It will be useful in the following to express ψ and ψ^c as linear combinations of charge conjugation eigenvectors -

which we denote by a change of font to $\boldsymbol{\psi}_{\sigma_e}(x,\mathbf{k})$. These can be projected out of an arbitrary $\psi(x,\mathbf{k})$ with the definitions

$$\boldsymbol{\psi}_{\sigma_e}(x,\mathbf{k}) = \hat{\mathbb{P}}_{\sigma_e}\left[\psi(x,\mathbf{k})\right] := \frac{1}{2}\left[\psi(x,\mathbf{k}) + \sigma_e \psi^c(x,\mathbf{k})\right]. \tag{C4}$$

The inverse relations are

$$\psi(x,\mathbf{k}) = \boldsymbol{\psi}_+(x,\mathbf{k}) + \boldsymbol{\psi}_-(x,\mathbf{k}), \quad \psi^c(x,\mathbf{k}) = \boldsymbol{\psi}_+(x,\mathbf{k}) - \boldsymbol{\psi}_-(x,\mathbf{k}). \tag{C5}$$

Charge conjugation eigenvectors are Majorana bi-spinors. In the Majorana representation Majorana bi-spinors and their charge conjugates are real times an overall phase factor. Due to the particular definition of the projector (C4) the Majorana representation of $\boldsymbol{\psi}_+(x,\mathbf{k})$ Majorana bi-spinor is entirely real, and the Majorana representation of $\boldsymbol{\psi}_-(x,\mathbf{k})$ is entirely imaginary. Irrespective of the representation, the Majorana bi-spinors $\boldsymbol{\psi}_+(x,\mathbf{k})$ and $\boldsymbol{\psi}_-(x,\mathbf{k})$ independently satisfy the Dirac equation because $\psi(x,\mathbf{k})$ and $\psi^c(x,\mathbf{k})$ independently satisfy the Dirac equation. And just as for ψ and ψ^c, the currents associated with $\boldsymbol{\psi}_+$ and $\boldsymbol{\psi}_-$ - i.e. $\left\langle \boldsymbol{\psi}_+(x,\mathbf{k})\bar{\boldsymbol{\psi}}_+(x,\mathbf{k})\right\rangle_1$ and $\left\langle \boldsymbol{\psi}_-(x,\mathbf{k})\bar{\boldsymbol{\psi}}_-(x,\mathbf{k})\right\rangle_1$ - are independently conserved.

From (C5) we have

$$\psi(x,\mathbf{k})\bar{\psi}^c(x,\mathbf{k}) + \psi^c(x,\mathbf{k})\bar{\psi}(x,\mathbf{k}) = 2\left[\boldsymbol{\psi}_+(x,\mathbf{k})\bar{\boldsymbol{\psi}}_+(x,\mathbf{k}) - \boldsymbol{\psi}_-(x,\mathbf{k})\bar{\boldsymbol{\psi}}_-(x,\mathbf{k})\right] \tag{C6}$$

and therefore, in terms of Majorana bi-spinors the multivector, (121) is

$$\mathbb{Q}(x) = 2\int \mathrm{d}^3 k\left[\boldsymbol{\psi}_+(x,\mathbf{k})\bar{\boldsymbol{\psi}}_+(x,\mathbf{k}) - \boldsymbol{\psi}_-(x,\mathbf{k})\bar{\boldsymbol{\psi}}_-(x,\mathbf{k})\right]. \tag{C7}$$

In contrast with (121) the utility of (C7) is that the vector part of the multivector is now a linear combination of two independently conserved currents. Since these contribute to the time component with opposite signs

$$\left\langle \mathbb{Q}(x)\gamma^0\right\rangle = 2\int \mathrm{d}^3 k\left|\boldsymbol{\psi}_+(x,\mathbf{k})\right|^2 - 2\int \mathrm{d}^3 k\left|\boldsymbol{\psi}_-(x,\mathbf{k})\right|^2 \tag{C8}$$

(where $\left|\psi_\sigma\left(x,\mathbf{k}\right)\right|^2 \equiv \psi_\sigma^\dagger\left(x,\mathbf{k}\right)\psi_\sigma\left(x,\mathbf{k}\right)$) it follows that $\left\langle \psi_+\left(x,\mathbf{k}\right)\bar{\psi}_+\left(x,\mathbf{k}\right)\right\rangle_1$ and $\left\langle \psi_-\left(x,\mathbf{k}\right)\bar{\psi}_-\left(x,\mathbf{k}\right)\right\rangle_1$ are currents of opposite charges, nominally of electrons and positrons, respectively.[45] The total current can be decomposed accordingly

$$\left\{\mathbb{j}\left(x\right)\right\} = \left\{\mathbb{j}_+\left(x\right)\right\} + \left\{\mathbb{j}_-\left(x\right)\right\}; \quad \left\{\mathbb{j}_{\sigma_e}\left(x\right)\right\} = \lambda\sigma_e \int \mathrm{d}^3 k \left\langle \psi_{\sigma_e}\left(x,\mathbf{k}\right)\bar{\psi}_{\sigma_e}\left(x,\mathbf{k}\right)\right\rangle_1 . \tag{C9}$$

Further, as demonstrated in Appendix B, the outer-product $\psi\bar{\psi}$ of Majorana bi-spinors is the sum of a null vector and a null bi-vector, bi-linear combinations of which are also null.[46] Consequently we have

$$\left\langle \mathbb{Q}\left(x\right)\right\rangle_1 \propto \left\{\mathbb{j}\left(x\right)\right\} = \left\{\mathbb{j}_+\left(x\right)\right\} + \left\{\mathbb{j}_-\left(x\right)\right\}; \quad \left\{\mathbb{j}_+\left(x\right)\right\}^2 = \left\{\mathbb{j}_-\left(x\right)\right\}^2 = 0, \quad \operatorname{sgn}\left(\left\{\rho_{\sigma_e}\left(x\right)\right\}\right) = \sigma_e . \tag{C10}$$

Appendix B also gives the detailed relationship between $\psi\bar{\psi}$ and the components of the time-symmetric null Faraday and null potential (or current - depending on the normalization) that it represents.

Spin currents

$\psi_{\sigma_e}\left(x,\mathbf{k}\right)$ can be further decomposed into two independent components each of which satisfies the Dirac equation, and each of which generates its own null current. It follows that though $\left\{\mathbb{j}_+\left(x\right)\right\}$ and $\left\{\mathbb{j}_-\left(x\right)\right\}$ are null, they cannot be the generators of the unique flow-lines employed in Section 3.

Bearing in mind (131), the components of any decomposition of a given solution of the Dirac equation will be independent solutions of the Dirac equation, and their associated currents will be conserved, if the eigenvectors of the associated projector commute with $\mathbb{k}_c$. A candidate projection with this property (in addition to and different from charge conjugation) is

$$\mathbb{P}_s := \frac{1}{2}\left[1 + \sigma_s \gamma_5 \mathbb{n}\right]; \quad s \in \left\{\uparrow, \downarrow\right\}, \quad \sigma_\uparrow = 1, \quad \sigma_\downarrow = -1 \tag{C11}$$

where $\mathbb{n}$ is any vector satisfying $\mathbb{n} \circ \mathbb{k}_c = 0$ and $\mathbb{n}^2 = -1$. Writing the projections as

$$\psi_s\left(x;\mathbf{k}\right) := \mathbb{P}_s \psi\left(x;\mathbf{k}\right) \tag{C12}$$

(C11) implies

45. With the particular definition (87) of $\mathbb{Q}\left(x\right)$, and taking into account (34), $\left\langle \psi_+\left(x,\mathbf{k}\right)\bar{\psi}_+\left(x,\mathbf{k}\right)\right\rangle_1$ is the electron current.

46. That the scalar, pseudo-scalar, and pseudo-vector parts of $\psi\bar{\psi}$ must vanish follows directly from (C7), i.e. given the definition (87) of the multivector $\mathbb{Q}\left(x\right)$.

$$\left[\not\partial + i\kappa_c\right]\psi_s\left(x;\mathbf{k}\right) = i\left[\kappa_c - \mathbb{k}_c\right]\psi_s\left(x;\mathbf{k}\right) = 0\,. \tag{C13}$$

$\mathbb{P}_s$ projects out the two possible spin orientations. The associated spins currents are

$$\left\{\mathbb{j}_s\left(x\right)\right\} = \lambda\int \mathrm{d}^3\, k\left\langle\psi_s\left(x,\mathbf{k}\right)\bar{\psi}_s\left(x,\mathbf{k}\right)\right\rangle_1\,. \tag{C14}$$

Conservation of $\left\{\mathbb{j}_s\left(x\right)\right\}$ follows from (C2) with $\psi\left(x,\mathbf{k}\right)$ replaced by $\psi_s\left(x,\mathbf{k}\right)$.

The charge and spin projectors commute

$$\boldsymbol{\psi}_{s,\sigma_e}\left(x;\mathbf{k}\right) := \mathbb{P}_s\hat{\mathbb{P}}_{\sigma_e}\left[\psi\left(x;\mathbf{k}\right)\right] = \hat{\mathbb{P}}_{\sigma_e}\left[\mathbb{P}_s\psi\left(x;\mathbf{k}\right)\right]. \tag{C15}$$

where $\boldsymbol{\psi}_{s,\sigma_e}\left(x;\mathbf{k}\right)$ are the Majorana bi-spinor components of a decomposition of a general $\psi\left(x,\mathbf{k}\right)$ into charge and spin components: $\psi = \boldsymbol{\psi}_{\uparrow,+} + \boldsymbol{\psi}_{\uparrow,-} + \boldsymbol{\psi}_{\downarrow,+} + \boldsymbol{\psi}_{\downarrow,-}$. The current can be decomposed likewise

$$\left\{\mathbb{j}\left(x\right)\right\} = \sum_{\substack{s\in\uparrow\downarrow\\ \sigma_e=\pm1}}\left\{\mathbb{j}_{s,\sigma_e}\left(x\right)\right\};\quad \left\{\mathbb{j}_{s,\sigma_e}\left(x\right)\right\} = \sigma_e\lambda\int \mathrm{d}^3\, k\left\langle\boldsymbol{\psi}_{s,\sigma_e}\left(x,\mathbf{k}\right)\bar{\boldsymbol{\psi}}_{s,\sigma_e}\left(x,\mathbf{k}\right)\right\rangle_1 \tag{C16}$$

for some λ, where each of the four currents is conserved and null:

$$\not\partial\circ\left\{\mathbb{j}_{s,\sigma_e}\left(x\right)\right\} = \left\{\mathbb{j}_{s,\sigma_e}\left(x\right)\right\}^2 = 0;\quad s\in\left\{\uparrow\downarrow\right\},\quad \sigma_e\in\left\{\pm1\right\}\,. \tag{C17}$$

Hence there are now four possible null-current charge paths passing through every space-time point - two for each sign of charge.[47]

Dynamic Independence

In this work $\left\{\mathbb{j}_{s,\sigma_e}\left(x\right)\right\}$; $s\in\left\{\uparrow\downarrow\right\},\sigma_e\in\left\{\pm1\right\}$ represent 4 physically coinciding but mutually exclusive ensembles of possible paths of a light-speed charge following the flow lines of a null potential, as described in Section 3. That is, at every spacetime-point there are 4 null currents that can be associated with a particular ψ, each one of which is a member of an ensemble of mutually exclusive null worldlines. It is crucial to the applicability of the method of Section 3 to the general case (i.e. that the incoming potential and Faraday are non-null) that these currents are dynamically independent. We show here that

47. Writing $\psi\left(x,\mathbf{k}\right) = \sum_{s\in\uparrow\downarrow}\sum_{\sigma_e=\pm1}\boldsymbol{\psi}_{s,\sigma_e}\left(x,\mathbf{k}\right)$, the absence in the current of cross terms $\left\langle\boldsymbol{\psi}_{s,\sigma_e}\left(x,\mathbf{k}\right)\bar{\boldsymbol{\psi}}_{s',\sigma'_e}\left(x,\mathbf{k}\right)\right\rangle_1$ - where s',σ'_e differ from s,σ_e - is easily proved from the properties of the projectors.

this constraint can be satisfied, and its satisfaction implies a particular relationship between the 4 vectors and bi-vectors generated by the 4 different ψ_{s,σ_e}.

Without commitment to a particular relationship let the two sets of labels $s \in \{\uparrow\downarrow\}, \sigma_e \in \{\pm 1\}$ and $\{1,2,3,4\}$ index the same set of 4 null currents. Employing initially the second set of labels, dynamic independence demands that current 1 does not feel a force from the Faraday of currents 2, 3, 4, etc., cyclically. Taking into account that $\{\mathbb{A}_{s,\sigma_e}\}$ and $\{\mathbb{j}_{s,\sigma_e}\}$ can be used interchangeably in this context (because a constant of proportionality is inconsequential to the determination of dynamical independence) the necessary condition for dynamic independence can be written

$$\left\langle \left\{\mathbb{A}^{(k)}\right\} \not{\partial} \left\{\mathbb{A}^{(j)}\right\} \right\rangle_1 = 0 \;\; \forall \;\; j \neq k \,. \tag{C18}$$

The case $j=k$ is excluded because, recalling (74), effective self-interaction emerges through interaction of a current with a contemporaneous time-reflected image of itself, even though 'intrinsic' self-interaction was excluded by construction in (4) and (10).

In Appendix B it is shown that the outer-product $\psi\bar{\psi}$ of Majorana bi-spinors is a multivector sum of a vector and bi-vector,[48] every bi-linear combination of which vanishes:[49]

$$\left\langle \psi_{\sigma_e} \bar{\psi}_{\sigma_e} \right\rangle_g \left\langle \psi_{\sigma_e} \bar{\psi}_{\sigma_e} \right\rangle_{g'} = 0 \quad \forall g, g' \in [0,4], \quad \sigma_e \in \{+1,-1\} \,. \tag{C19}$$

Let us identify the potential and Faraday in $\left\langle \psi_{\sigma_e} \bar{\psi}_{\sigma_e} \right\rangle$ explicitly via

$$\left\langle \psi_{\sigma_e} \bar{\psi}_{\sigma_e} \right\rangle = \kappa \left\{\mathbb{A}_{\sigma_e}\right\} + i \left\{\mathbb{F}_{\sigma_e}\right\} \tag{C20}$$

in which terms the total potential and Faraday are

$$\{\mathbb{A}\} = \{\mathbb{A}_+\} + \{\mathbb{A}_-\}, \;\; \{\mathbb{F}\} = \{\mathbb{F}_+\} + \{\mathbb{F}_-\} \tag{C21}$$

where

$$\not{\partial}\{\mathbb{A}\} = \{\mathbb{F}\}, \quad \not{\partial}\{\mathbb{F}\} = \{\mathbb{A}\} \,. \tag{C22}$$

48. $\psi\bar{\psi}^c + \psi^c\bar{\psi}$ is a vector plus a bi-vector, though $\psi\bar{\psi}$ is generally not.

49. The individual blades of the outer products $\psi\bar{\psi}$ are non-zero only for $g, g' \in [1,2]$.

Note that (C20), (C21) and (C22) do not mandate the particular relation $\left\{\mathbb{F}_{\sigma_e}\right\} = \not{\partial}\left\{\mathbb{A}_{\sigma_e}\right\}$. Due to (C19) and current conservation we have [50]

$$\not{\partial} \circ \left\{\mathbb{A}_{\sigma_e}\right\} = \left\langle\left\{\mathbb{A}_{\sigma_e}\right\}\left\{\mathbb{F}_{\sigma_e}\right\}\right\rangle_1 = 0\,. \tag{C23}$$

Comparing with (C18) we infer that

$$\left\{\mathbb{F}_+\right\} = \not{\partial}\left\{\mathbb{A}_-\right\}, \quad \left\{\mathbb{F}_-\right\} = \not{\partial}\left\{\mathbb{A}_+\right\} \tag{C24}$$

i.e. $\left\{\mathbb{F}_{\sigma_e}\right\} = \not{\partial}\left\{\mathbb{A}_{\bar{\sigma}_e}\right\}$, where, inverting (C20),

$$\left\{\mathbb{F}_{\sigma_e}\right\} := \left\langle \psi_{\sigma_e}\bar{\psi}_{\sigma_e}\right\rangle_2 / i, \quad \left\{\mathbb{A}_{\sigma_e}\right\} := \left\langle \psi_{\sigma_e}\bar{\psi}_{\sigma_e}\right\rangle_1 / \kappa\,. \tag{C25}$$

But (C23) and (C24) are not sufficient for the satisfaction of (C18). The latter demands, in addition, that $\left\langle\left\{\mathbb{A}_{s,\sigma_e}\right\}\left\{\mathbb{F}_{s',\bar{\sigma}_e}\right\}\right\rangle_1 = 0$ for one of the two possibilities that $s = s'$ or $s \neq s'$. Extracting the vector and bi-vector parts of $\psi_{\sigma_e}\bar{\psi}_{\sigma_e}$ using

$$\begin{aligned} i\left\{\mathbb{F}_{s',\bar{\sigma}_e}\right\} &= \left\langle \psi_{s',\bar{\sigma}_e}\bar{\psi}_{s',\bar{\sigma}_e}\right\rangle_2 = \frac{1}{2}\left[\psi_{s',\bar{\sigma}_e}\bar{\psi}_{s',\bar{\sigma}_e} + \gamma^5\psi_{s',\bar{\sigma}_e}\bar{\psi}_{s',\bar{\sigma}_e}\gamma^5\right] \\ \kappa\left\{\mathbb{A}_{s,\sigma_e}\right\} &= \left\langle \psi_{s,\sigma_e}\bar{\psi}_{s,\sigma_e}\right\rangle_1 = \frac{1}{2}\left[\psi_{s,\sigma_e}\bar{\psi}_{s,\sigma_e} - \gamma^5\psi_{s,\sigma_e}\bar{\psi}_{s,\sigma_e}\gamma^5\right] \end{aligned} \tag{C26}$$

we have

$$\left\langle\left\{\mathbb{A}_{s,\sigma_e}\right\}\left\{\mathbb{F}_{s',\bar{\sigma}_e}\right\}\right\rangle_1 = \frac{1}{4i\kappa}\left\langle\left[\mathbb{P}_s\psi_{\sigma_e}\bar{\psi}_{\sigma_e}\mathbb{P}_s - \gamma^5\mathbb{P}_s\psi_{\sigma_e}\bar{\psi}_{\sigma_e}\mathbb{P}_s\gamma^5\right]\left[\mathbb{P}_{s'}\psi_{\bar{\sigma}_e}\bar{\psi}_{\bar{\sigma}_e}\mathbb{P}_{s'} + \gamma^5\mathbb{P}_{s'}\psi_{\bar{\sigma}_e}\bar{\psi}_{\bar{\sigma}_e}\mathbb{P}_{s'}\gamma^5\right]\right\rangle_1. \tag{C27}$$

Multiplying out the terms and using that $\left\langle \mathbb{M}\gamma^5\right\rangle_1 = -\left\langle\gamma^5\mathbb{M}\right\rangle_1$ for any multi-vector $\mathbb{M}$ this reduces to

$$\left\langle\left\{\mathbb{A}_{s,\sigma_e}\right\}\left\{\mathbb{F}_{s',\bar{\sigma}_e}\right\}\right\rangle_1 = a\left\langle\mathbb{P}_s\psi_{\sigma_e}\bar{\psi}_{\bar{\sigma}_e}\mathbb{P}_{s'}\right\rangle_1 + b\left\langle\mathbb{P}_s\psi_{\sigma_e}\bar{\psi}_{\bar{\sigma}_e}\mathbb{P}_{s'}\gamma^5\right\rangle_1 \tag{C28}$$

where a, b are the scalars

$$a = \frac{1}{2i\kappa}\bar{\psi}_{\sigma_e}\mathbb{P}_s\mathbb{P}_{s'}\psi_{\bar{\sigma}_e}, \quad b = \frac{1}{2i\kappa}\bar{\psi}_{\sigma_e}\mathbb{P}_s\gamma^5\mathbb{P}_{s'}\psi_{\bar{\sigma}_e}\,. \tag{C29}$$

A sufficient condition that $\left\langle\left\{\mathbb{A}_{s,\sigma_e}\right\}\left\{\mathbb{F}_{s',\bar{\sigma}_e}\right\}\right\rangle_1$ vanish is that $a = b = 0$. Using that the $\psi_{\sigma_e} = \psi_{\sigma_e}\left(x, \mathbf{k}\right)$ satisfy the Dirac equation $\left[\kappa_c - \Bbbk_c\right]\psi_{\sigma_e} = 0$ we can write

$$2i\kappa b = \bar{\psi}_{\sigma_e}\Bbbk_c\mathbb{P}_s\gamma^5\mathbb{P}_{s'}\Bbbk_c\psi_{\bar{\sigma}_e}/\kappa_c^2 = -\bar{\psi}_{\sigma_e}\Bbbk_c^2\mathbb{P}_s\gamma^5\mathbb{P}_{s'}\psi_{\bar{\sigma}_e}/\kappa_c^2 = -\bar{\psi}_{\sigma_e}\mathbb{P}_s\gamma^5\mathbb{P}_{s'}\psi_{\bar{\sigma}_e} = -2i\kappa b\,. \tag{C30}$$

50. $\left\{\mathbb{F}\right\}\left\{\mathbb{A}\right\} = 0 \Rightarrow \left\langle\left\{\mathbb{F}\right\}\left\{\mathbb{A}\right\}\right\rangle_1 = \left\langle\left\{\mathbb{F}\right\}\left\{\mathbb{A}\right\}\right\rangle_3 = 0\,.$

b vanishes automatically therefore, regardless of the relative values of s and s'. Consequently it is sufficient (in order that $\left\langle \left\{ \mathbb{A}_{s,\sigma_e} \right\} \left\{ \mathbb{F}_{s',\bar{\sigma}_e} \right\} \right\rangle_1$ vanish) that $s' \neq s$, because then $\mathbb{P}_s \mathbb{P}_{s'} = 0$ and so $a = 0$.

In summary, we have shown that the 4 null currents $\left\{ \mathbb{j}_{s,\sigma_e}\left(x,\mathbf{k}\right) \right\} \propto \left\{ \mathbb{A}_{s,\sigma_e}\left(x,\mathbf{k}\right) \right\}$ (where $\sigma_e \in \left\{ \pm 1 \right\}$ and $s, s' \in \left\{ \uparrow, \downarrow \right\}$), are dynamically independent. Specifically, they satisfy

$$\left\langle \left\{ \mathbb{F}_{s',\sigma'_e}\left(x,\mathbf{k}\right) \right\} \left\{ \mathbb{A}_{s,\sigma_e}\left(x,\mathbf{k}\right) \right\} \right\rangle_1 = 0 \text{ unless } \sigma_e = \sigma'_e \text{ and } s = s' \tag{C31}$$

where

$$\left\{ \mathbb{F}_{s,\sigma_e}\left(x,\mathbf{k}\right) \right\} := \left\langle \psi_{s,\sigma_e}\left(x,\mathbf{k}\right) \bar{\psi}_{s,\sigma_e}\left(x,\mathbf{k}\right) \right\rangle_2 / i, \quad \left\{ \mathbb{A}_{s,\sigma_e}\left(x,\mathbf{k}\right) \right\} := \left\langle \psi_{s,\sigma_e}\left(x,\mathbf{k}\right) \bar{\psi}_{s,\sigma_e}\left(x,\mathbf{k}\right) \right\rangle_1 / \kappa \tag{C32}$$

and where the $\psi_{s,\sigma_e}\left(x,\mathbf{k}\right)$ are projections of a general solution of the Dirac equation (131) according to (C15). The demonstration of independence is predicated on the particular association

$$\left\{ \mathbb{F}_{s,\sigma_e}\left(x,\mathbf{k}\right) \right\} = \not{\partial} \left\{ \mathbb{A}_{s,\bar{\sigma}_e}\left(x,\mathbf{k}\right) \right\}, \tag{C33}$$

which can be accommodated within the supervening relations

$$\begin{aligned} \left\{ \mathbb{A}\left(x,\mathbf{k}\right) \right\} &= \sum_{s \in \uparrow\downarrow} \sum_{\sigma_e = \pm 1} \left\{ \mathbb{A}_{s,\sigma_e}\left(x,\mathbf{k}\right) \right\} \\ \left\{ \mathbb{F}\left(x,\mathbf{k}\right) \right\} &= \sum_{s \in \uparrow\downarrow} \sum_{\sigma_e = \pm 1} \left\{ \mathbb{F}_{s,\sigma_e}\left(x,\mathbf{k}\right) \right\} \\ \not{\partial} \left\{ \mathbb{A}\left(x,\mathbf{k}\right) \right\} &= \left\{ \mathbb{F}\left(x,\mathbf{k}\right) \right\} \end{aligned} . \tag{C34}$$

Hence the Dirac equation describes the evolution of 4 independent currents. They are independent in the sense none of them are acted on by the Faraday that derives from (is the exterior derivative of) the guiding potential of the other. It follows that the equation can be used to describe the evolution of four charges, each a member of a different (charge, spin) current. Further, because the flow lines of each current are non-intersecting as required by conservation, the multiplicity can be extended to any number of charges, up to one per flow line per current. Note however that if multiple charges are present (locally) their electromagnetic interaction must be accounted for - in addition to the influence of the time-symmetric response from distant charges that underpins the Dirac Equation.

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