A Derivation of Flat Galactic Rotation Curve and Baryonic Tully-Fisher Relation

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Abstract

Fundamentally for the extended disc region of a spiral galaxy, an alternative solution to Laplace equation has been presented for a potential that is radially symmetric on the disc plane. This potential, unlike newtonian one, is shown to be logarithmic in distance from the centre, which allows for the rotation velocity to be constant along the disc radius. It is also shown that this potential easily manifests into a relationship between inner mass of the galaxy and terminal rotation velocity, which has been empirically observed and known as Baryonic Tully-Fisher relations.

1 Introduction

The problem of the anomaly observed in spiral galaxy rotation curves is a longstanding one. While prevalent scientific opinion weighs towards a Dark matter hypothesis that explains the 'mass-gap' required to reconcile rotation curves with newtonian gravity, there hasn't been any direct observation of the postulated dark matter yet. On the other hand, there are theories that try to modify newton's gravity formulation to achieve the same reconciliation. But they are empirical in nature and doesn't explain how they emerge from a more fundamental understanding of the spacetime.

In this work, we've gone back to the original nonrelativistic gravity formulations of Laplace Equation. It is not necessary that newtonian gravity be the only low-energy solution of Laplace Equation. In fact, we only derive Newton's gravity as a solution only if we assume a spherical symmetry. However, consideration of the intrinsic structure of most of the spiral galaxies, should lead us more towards a cylindrically symmetric solution of Field equations.

We'll see that with the assumptions of cylindrical symmetry, one can arrive at a gravitational potential that yields a geodesic which allows flat tails of rotation curves. Also we'll see that this potential manifests into empirically observed Baryonic Tully-Fisher Relations among galactic mass and the terminal 'flat' velocity.

2 Modified gravity potential

We would start with the nonrelativistic potential function ϕ on the disc which would satisfy Laplace equation,

$$\triangle^2\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi = 0 \quad (1)$$

Here z is assumed to be the direction of central axis, as follows

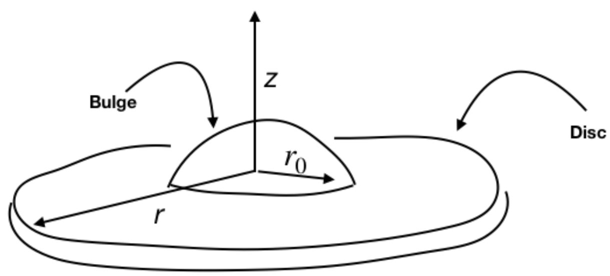

Figure 1: Disc Structure and reference frame orientation

Usually assuming ϕ to be spherically symmetric, one can derive Newtonian potential. However, because of the intrinsic planar structure of the galaxies, it would be far more reasonable to assume that $\partial_z\phi$ would vanish within the width of the disc. Beyond the disc, ϕ would have a fast tail along z-axis away from the disc, as below.

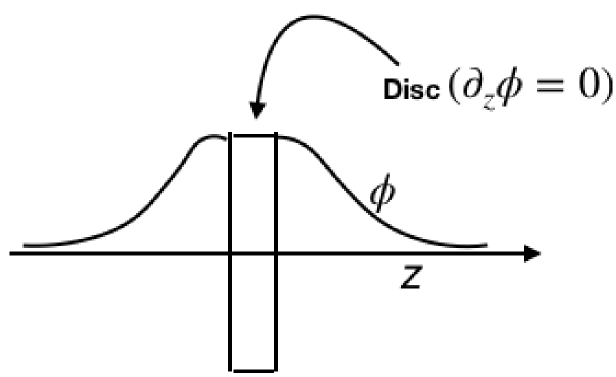

Figure 2: Distribution of potential along the z-direction

Since $\partial_z\phi = 0$ in the disc, Eq. 1 would boil down to 2-D laplacian,

$$\triangle^2\phi = \partial_x^2\phi + \partial_y^2\phi = 0 \quad (2)$$

Now if we change the coordinates to cylindrical, we know the 2-D laplacian can be written as $\triangle^2\phi = \partial_r^2\phi + \frac{1}{r}\partial_r\phi + \frac{1}{r^2}\partial_\theta^2\phi$.

But from cylindrical symmetry we know, $\partial_\theta\phi = 0$. That'll yield the differential equation to solve for ϕ,

$$\partial_r^2\phi + \frac{1}{r}\partial_r\phi = 0 \tag{3}$$

Solving the equation for ϕ we get

$$\phi = \lambda \ln \frac{r}{r_0} \tag{4}$$

where λ, is a constant and r_0 is a constant distance.

To derive the constants of motion λ, r_0, we would need boundary conditions based on smoothness of the geodesic. More specifically, if we assume that r_0 is the effective radius of the galactic baryonic mass, and within that radius the field continues to be spherically symmetric dominated by the bulge, we would require the geodesic to have a well-defined 2nd derivative at r_0 from both side.

We know that at a low energy limit, radial acceleration is

$$\frac{d^2r}{dt^2} = \partial_r\phi \tag{5}$$

At r_0, from inside $\phi = \frac{GM}{r}$ where M is the baryonic mass of the galaxy (inside r_0) and G is the gravitational constant. Hence the inside limit for radial acceleration would be $-\frac{GM}{r_0^2}$. On the other hand, same limit from the outside would be, $\frac{\lambda}{r_0}$. Hence continuity would dictate that $\lambda = -\frac{GM}{r_0}$. This would yield the final form of the disc potential to be,

$$\phi = -\frac{GM}{r_0} \ln \frac{r}{r_0} \tag{6}$$

Finally we would investigate rotation curve required by the disc potential. As usual, we would counterbalance the inward radial acceleration by outward centrifugal acceleration.

$$\frac{v^2}{r} = \frac{GM}{r_0}\frac{1}{r} \tag{7}$$

where v is the rotation velocity of a star on the disc at a distance r from the centre.

This clearly produces a flat rotation curve outside of r_0

$$v_0 = [\frac{GM}{r_0}]^{\frac{1}{2}} \tag{8}$$

Here v_0 denotes the terminal velocity at the edges of the galaxy. It would be possible to establish a relationship between terminal velocity and the baryonic mass in the galaxy if we assume ρ to be the average density of matter across the galactic disc. Then total baryonic mass could be expressed as $M = \pi\rho r_0^2$, which when plugged into the expression for v_0 yields,

$$M = \frac{v_0^4}{\pi\rho G^2} \tag{9}$$

which is exactly the empirical Tully-Fisher relationship if one assumes ρ to be consistent across galaxies.

3 Conclusion

Through a very simple proof it has been shown that there exists a potential which is differentiable everywhere (except galactic centre), that satisfy the Laplace equation. It has also been shown that such potential can produce newtonian gravity within a certain radius, and flat rotation curves outside of it. There are structural similarities between the results presented here and Modified Newtonian Dynamics (MOND). However, unlike MOND, the potential presented here doesn't attempt to alter Newton's laws of general dynamics. Rather it provides a GR based spacetime structure from which flat rotation curves emerge naturally. Additionally the emergence of the flat rotation curve is shown to be consistent with Tully-Fisher relationship which has been proven empirically.