SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON FREDHOLM OPERATORS WITH DRIFT AND SUPERDIFFUSION

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Abstract: We study the solvability of certain linear nonhomogeneous elliptic problems and prove that under some technical assumptions the convergence in L^2 of their right sides implies the existence and the convergence in H^1 of the solutions. The equations contain first order differential operators with or without Fredholm property, in particular the square root of the one dimensional negative Laplacian, on the whole real line or on a finite interval with periodic boundary conditions. We establish that the drift term involved in these problems provides the regularization of solutions.

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1. Introduction

Let us consider the equation

$$\sqrt{-\Delta + V(x)}u - au = f, \tag{1.1}$$

where $u \in E = H^1(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant, and the function $V(x)$ tends to 0 at infinity. The operator $\sqrt{-\Delta + V(x)}$ can be defined via the spectral calculus under the appropriate technical conditions on the scalar potential $V(x)$ (see Assumption 3 of [21]). If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$, which corresponds to the left side of problem (1.1) contains the origin. As a consequence, this operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. In the present article we will study some properties of such operators. Note that elliptic problems containing non-Fredholm operators were treated extensively in recent years (see [13], [16], [17], [18], [19], also [3]) along with their potential applications to the theory of reaction-diffusion equations (see [7], [8]). In the particular case when a vanishes, the operator A satisfies the Fredholm property in some properly chosen weighted spaces (see [1], [2], [3], [5], [6]). However, the case where $a \neq 0$ is considerably different and the method developed in these works cannot be used.

One of the important issues about problems with non-Fredholm operators concerns their solvability. We will study it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^1(\mathbb{R}^d)$ such that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Since the operator A does not satisfy the Fredholm property, the sequence u_n may not be convergent. Let us call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of problem $Au = f$ (see [12]). If this sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this equation. Solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of non-Fredholm operators this convergence may not hold or it can occur in some weaker sense. In this case, the solution in the sense of sequences may not imply the existence of the usual solution. In this article we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent.

In the first part of the work we consider the equation with the drift term

$$\sqrt{-\frac{d^2}{dx^2}}u - b\frac{du}{dx} - au = f(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $a \geq 0$ and $b \in \mathbb{R}, \ b \neq 0$ are constants and the right side is square integrable. The operator $\sqrt{-\frac{d^2}{dx^2}}$ can be defined via the spectral calculus and is extensively used, for example in the studies of the superdiffusion and related equations (see [20] and the references therein). Superdiffusion can be described as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of normal diffusion, but this is not the case for superdiffusion. Asymptotic behavior at infinity of the probability density function determines the value of the power of the Laplace operator (see [11]). The problem with drift in the context of the Darcy's law describing the fluid motion in the porous medium was considered in [18]. The drift term is significant when studying the emergence and propagation of patterns arising in the theory of speciation (see [14]). Nonlinear propagation phenomena for the reaction-diffusion type problems involving the drift term was investigated in [4]. Weak solutions of the Dirichlet and Neumann problems with drift were considered in [10]. Evidently, the operator involved in the left side of (1.2)

$$L_{a,\ b} := \sqrt{-\frac{d^2}{dx^2}} - b\frac{d}{dx} - a : \quad H^1(\mathbb{R}) \to L^2(\mathbb{R}) \tag{1.3}$$

is non-selfadjoint. By virtue of the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ipx} dx, \quad p \in \mathbb{R} \tag{1.4}$$

it can be easily obtained that the essential spectrum of the operator $L_{a,\ b}$ is given by

$$\lambda_{a,\ b}(p) := |p| - a - ibp, \quad p \in \mathbb{R}.$$

Clearly, when $a > 0$ the operator $L_{a,\ b}$ is Fredholm, since its essential spectrum does not contain the origin. But when $a = 0$ the operator $L_{0,\ b}$ fails to satisfy the Fredholm property because the origin belongs to its essential spectrum.

Note that in the absence of the drift term we are dealing with the self-adjoint operator

$$\sqrt{-\frac{d^2}{dx^2}} - a : \quad H^1(\mathbb{R}) \to L^2(\mathbb{R}), \quad a > 0,$$

which is non Fredholm (see [21]). We write down the corresponding sequence of approximate equations with $m \in \mathbb{N}$ as

$$\sqrt{-\frac{d^2}{dx^2}} u_m - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in \mathbb{R}, \tag{1.5}$$

with the right sides convergent to the right side of (1.2) in $L^2(\mathbb{R})$ as $m \to \infty$. The inner product of two functions

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x) dx, \tag{1.6}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R})$ and $g(x)$ is bounded, then obviously the integral considered above makes sense, like for instance in the case of functions involved in the orthogonality relations (1.8) and (1.9) of Theorems 1.1 and 1.2 below. For our equations on the finite interval $I := [0, 2\pi]$ with periodic boundary conditions, we will use the inner product analogous to (1.6), replacing the real line with I. In the first part of the present work we will consider the space $H^1(\mathbb{R})$ equipped with the norm

$$\|u\|^2_{H^1(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\|\frac{du}{dx}\right\|^2_{L^2(\mathbb{R})}. \tag{1.7}$$

When dealing with the norm $H^1(I)$ later on, we will replace $\mathbb{R}$ with I in formula (1.7). Our first main statement is as follows.

Theorem 1.1. *Let $f(x):\mathbb{R}\to\mathbb{R}$ and $f(x)\in L^2(\mathbb{R})$.*

a) If $a>0$, then problem (1.2) has a unique solution $u(x)\in H^1(\mathbb{R})$.

b) Suppose that $a=0$ and $xf(x)\in L^1(\mathbb{R})$. Then equation (1.2) admits a unique solution $u(x)\in H^1(\mathbb{R})$ if and only if the orthogonality relation

$$(f(x),1)_{L^2(\mathbb{R})}=0 \tag{1.8}$$

holds.

Obviously, the expression in the left side of (1.8) makes sense by means of the trivial argument analogous to the proof of Fact 1 of [16]. Let us turn our attention to establishing the solvability in the sense of sequences for our problem on the real line.

Theorem 1.2. *Let $m\in\mathbb{N},\ f_m(x):\mathbb{R}\to\mathbb{R}$ and $f_m(x)\in L^2(\mathbb{R})$. Furthermore, $f_m(x)\to f(x)$ in $L^2(\mathbb{R})$ as $m\to\infty$.*

a) If $a>0$, then problems (1.2) and (1.5) admit unique solutions $u(x)\in H^1(\mathbb{R})$ and $u_m(x)\in H^1(\mathbb{R})$ respectively, such that $u_m(x)\to u(x)$ in $H^1(\mathbb{R})$ as $m\to\infty$.

b) Suppose that $a=0$, $xf_m(x)\in L^1(\mathbb{R})$, and $xf_m(x)\to xf(x)$ in $L^1(\mathbb{R})$ as $m\to\infty$. Moreover,

$$(f_m(x),1)_{L^2(\mathbb{R})}=0, \quad m\in\mathbb{N} \tag{1.9}$$

holds. Then equations (1.2) and (1.5) possess unique solutions $u(x)\in H^1(\mathbb{R})$ and $u_m(x)\in H^1(\mathbb{R})$ respectively, such that $u_m(x)\to u(x)$ in $H^1(\mathbb{R})$ as $m\to\infty$.

Note that in the parts a) of Theorems 1.1 and 1.2 above the orthogonality relations are not needed, as distinct from the case without a drift term treated in the parts c) of Theorem 1 and 2 of [21]. Therefore, the introduction of the drift term provides the regularization for the solutions of our equations. In the parts b) of Theorems 1.1 and 1.2 of the present work only a single orthogonality condition is required, analogously to the cases a) of Theorems 1 and 2 of [21].

In the second part of the article we consider our problem on the finite interval with periodic boundary conditions, i.e. $I:=[0,\ 2\pi]$, namely

$$\sqrt{-\frac{d^2}{dx^2}}u-b\frac{du}{dx}-au=f(x), \quad x\in I, \tag{1.10}$$

where $a\geq 0$ and $b\in\mathbb{R},\ b\neq 0$ are constants and the right side of (1.10) is bounded and periodic. Evidently,

$$\|f\|_{L^1(I)}\leq 2\pi\|f\|_{L^\infty(I)}<\infty, \quad \|f\|_{L^2(I)}\leq\sqrt{2\pi}\|f\|_{L^\infty(I)}<\infty. \tag{1.11}$$

Hence $f(x) \in L^1(I) \cap L^2(I)$ as well. Let us use the Fourier transform

$$f_n := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}, \tag{1.12}$$

such that

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{inx}}{\sqrt{2\pi}}.$$

Apparently, the non-selfadjoint operator involved in the left side of (1.10)

$$\mathcal{L}_{a,\ b} := \sqrt{-\frac{d^2}{dx^2}} - b\frac{d}{dx} - a : \quad H^1(I) \to L^2(I) \tag{1.13}$$

is Fredholm. By virtue of (1.12), it can be easily verified that the spectrum of $\mathcal{L}_{a,\ b}$ is given by

$$\lambda_{a,\ b}(n) := |n| - a - ibn, \quad n \in \mathbb{Z}$$

and the corresponding eigenfunctions are the Fourier harmonics $\dfrac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$. The eigenvalues of the operator $\mathcal{L}_{a,\ b}$ are simple, as distinct from the case without the drift term, when the eigenvalues corresponding to $n \neq 0$ are double-degenerate. The appropriate function space here $H^1(I)$ is

$$\{u(x) : I \to \mathbb{R} \mid u(x), u'(x) \in L^2(I), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)\}.$$

For the technical purposes, let us use the following auxiliary constrained subspace

$$H_0^1(I) = \{u(x) \in H^1(I) \mid (u(x), 1)_{L^2(I)} = 0\}, \tag{1.14}$$

which is a Hilbert spaces as well (see e.g. Chapter 2.1 of [9]). Evidently, for $a > 0$, the kernel of the operator $\mathcal{L}_{a,\ b}$ is trivial. When a vanishes, let us consider

$$\mathcal{L}_{0,\ b} : \quad H_0^1(I) \to L^2(I).$$

Obviously, this operator has the trivial kernel as well. Let us write down the corresponding sequence of the approximate equations with $m \in \mathbb{N}$, namely

$$\sqrt{-\frac{d^2}{dx^2}} u_m - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in I, \tag{1.15}$$

where the right sides are bounded, periodic and tend to the right side of (1.10) in $L^\infty(I)$ as $m \to \infty$. The goal of Theorems 1.3 and 1.4 below is to demonstrate the formal similarity of the results on the finite interval with periodic boundary conditions to the ones obtained for the whole real line situation in Theorems 1.1 and 1.2 above.

Theorem 1.3. *Let $f(x) : I \to \mathbb{R}$, such that $f(0) = f(2\pi)$ and $f(x) \in L^\infty(I)$.*

a) If $a > 0$, then problem (1.10) possesses a unique solution $u(x) \in H^1(I)$.

b) If $a = 0$, then equation (1.10) has a unique solution $u(x) \in H_0^1(I)$ if and only if the orthogonality relation

$$(f(x), 1)_{L^2(I)} = 0 \tag{1.16}$$

holds.

Our final main statement deals with the solvability in the sense of sequences for our problem on the finite interval I.

Theorem 1.4. *Let $m \in \mathbb{N},\ f_m(x) : I \to \mathbb{R}$, such that $f_m(0) = f_m(2\pi)$. Moreover, $f_m(x) \in L^\infty(I)$ and $f_m(x) \to f(x)$ in $L^\infty(I)$ as $m \to \infty$.*

a) If $a > 0$, then problems (1.10) and (1.15) admit unique solutions $u(x) \in H^1(I)$ and $u_m(x) \in H^1(I)$ respectively, such that $u_m(x) \to u(x)$ in $H^1(I)$ as $m \to \infty$.

b) Suppose that $a = 0$ and

$$(f_m(x), 1)_{L^2(I)} = 0, \quad m \in \mathbb{N}. \tag{1.17}$$

Then equations (1.10) and (1.15) have unique solutions $u(x) \in H_0^1(I)$ and $u_m(x) \in H_0^1(I)$ respectively, such that $u_m(x) \to u(x)$ in $H_0^1(I)$ as $m \to \infty$.

Note that in the cases a) of Theorems 1.3 and 1.4 above the orthogonality conditions are not needed. When there is no drift term in our equations, the situation is more singular (see formulas (3.1) and (3.7) below with $a = n_0,\ n_0 \in \mathbb{N}$).

2. The whole real line case

Proof of Theorem 1.1. First of all, let us establish that it would be sufficient to solve our problem in $L^2(\mathbb{R})$. Indeed, if $u(x)$ is a square integrable solution of (1.2), directly from this equation under the given conditions we arrive at

$$\sqrt{-\frac{d^2}{dx^2}}u - b\frac{du}{dx} \in L^2(\mathbb{R})$$

as well. Using the standard Fourier transform (1.4), we obtain $(|p| - ibp)\widehat{u}(p) \in L^2(\mathbb{R})$. Thus, $\int_{-\infty}^{\infty} p^2|\widehat{u}(p)|^2 dp < \infty$, such that $\frac{du}{dx} \in L^2(\mathbb{R})$. Therefore, $u(x) \in H^1(\mathbb{R})$ as well.

To prove the uniqueness of solutions of (1.2), we suppose that $u_1(x),\ u_2(x) \in H^1(\mathbb{R})$ solve (1.2). Then their difference $w(x) := u_1(x) - u_2(x) \in H^1(\mathbb{R})$ satisfies the homogeneous equation

$$\sqrt{-\frac{d^2}{dx^2}}w - b\frac{dw}{dx} - aw = 0.$$

Because the operator $L_{a,\,b}$ defined in (1.3) does not have any nontrivial zero modes in $H^1(\mathbb{R})$, the function $w(x) \equiv 0$ on $\mathbb{R}$.

Let us apply the standard Fourier transform (1.4) to both sides of problem (1.2). This implies

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p| - a - ibp}. \tag{2.1}$$

Hence,

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{|\widehat{f}(p)|^2}{(|p|-a)^2 + b^2p^2} dp. \tag{2.2}$$

First we treat the case a) of our theorem. (2.2) yields

$$\|u\|^2_{L^2(\mathbb{R})} \leq \frac{1}{C}\|f\|^2_{L^2(\mathbb{R})} < \infty$$

via the one of our assumptions. Here and throughout the article C will stand for a finite, positive constant.

Let us turn our attention to the case of $a = 0$. From (2.1), we easily write

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p| - ibp}\chi_{\{|p|\leq 1\}} + \frac{\widehat{f}(p)}{|p| - ibp}\chi_{\{|p|>1\}}. \tag{2.3}$$

Here and below χ_A will stand for the characteristic function of a set $A \subseteq \mathbb{R}$. Clearly, the second term in the right side of (2.3) can be bounded from above in the absolute value by $\frac{|\widehat{f}(p)|}{\sqrt{1+b^2}} \in L^2(\mathbb{R})$ since $f(x)$ is square integrable due to our assumption. We express

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^p \frac{d\widehat{f}(s)}{ds} ds.$$

Thus, the first term in the right side of (2.3) can be written as

$$\frac{\widehat{f}(0)}{|p| - ibp}\chi_{\{|p|\leq 1\}} + \frac{\int_0^p \frac{d\widehat{f}(s)}{ds} ds}{|p| - ibp}\chi_{\{|p|\leq 1\}}. \tag{2.4}$$

Using definition (1.4) of the standard Fourier transform, we easily derive

$$\left| \frac{d\widehat{f}(p)}{dp} \right| \leq \frac{1}{\sqrt{2\pi}}\|xf(x)\|_{L^1(\mathbb{R})}.$$

Hence, the second term in (2.4) can be estimated from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}} \frac{\|xf(x)\|_{L^1(\mathbb{R})}}{\sqrt{1+b^2}}\chi_{\{|p|\leq 1\}} \in L^2(\mathbb{R}).$$

Evidently, the first term in (2.4) belongs to $L^2(\mathbb{R})$ if and only if $\widehat{f}(0) = 0$, which is equivalent to orthogonality condition (1.8). ■

We proceed to proving the solvability in the sense of sequences for our equation on the real line.

Proof of Theorem 1.2. Let us first suppose that problems (1.2) and (1.5) have unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R}),\ m \in \mathbb{N}$ respectively, such that $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. This will yield that $u_m(x)$ also tends to $u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$. Indeed, from (1.2) and (1.5) we easily derive

$$\left\| \sqrt{-\frac{d^2}{dx^2}}(u_m - u) - b\frac{d(u_m - u)}{dx} \right\|_{L^2(\mathbb{R})} \leq \|f_m - f\|_{L^2(\mathbb{R})} + a\|u_m - u\|_{L^2(\mathbb{R})}.$$

The right side of the inequality above converges to zero as $m \to \infty$ via our assumptions. Using the standard Fourier transform (1.4), we easily obtain

$$\int_{-\infty}^{\infty} p^2 |\widehat{u}_m(p) - \widehat{u}(p)|^2 dp \to 0, \quad m \to \infty.$$

Hence, $\dfrac{du_m}{dx} \to \dfrac{du}{dx}$ in $L^2(\mathbb{R})$ as $m \to \infty$, such that $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ as well.

Let us apply the standard Fourier transform (1.4) to both sides of (1.5), which yields

$$\widehat{u}_m(p) = \frac{\widehat{f}_m(p)}{|p| - a - ibp}, \quad m \in \mathbb{N}. \tag{2.5}$$

First we consider the case a) of our theorem. By virtue of the part a) of Theorem 1.1, problems (1.2) and (1.5) possess unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R}),\ m \in \mathbb{N}$ respectively. By means of (2.5) along with (2.1), we arrive at

$$\|u_m - u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{|\widehat{f}_m(p) - \widehat{f}(p)|^2}{(|p| - a)^2 + b^2p^2} dp.$$

Thus,

$$\|u_m - u\|_{L^2(\mathbb{R})} \leq \frac{1}{C}\|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

due to the one of our assumptions. This yields that in the case when $a > 0$ we have $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ via the argument above.

We conclude the proof of our theorem by considering the situation when the parameter a vanishes. By virtue of the result of the part a) of Lemma 3.3 of [15], under the given conditions

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{2.6}$$

holds. Then by means of the part b) of Theorem 1.1, equations (1.2) and (1.5) admit unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R}),\ m \in \mathbb{N}$ respectively when $a = 0$. Formulas (2.5) and (2.1) yield

$$\widehat{u}_m(p) - \widehat{u}(p) = \frac{\widehat{f}_m(p) - \widehat{f}(p)}{|p| - ibp}\chi_{\{|p|\le 1\}} + \frac{\widehat{f}_m(p) - \widehat{f}(p)}{|p| - ibp}\chi_{\{|p|>1\}}. \tag{2.7}$$

Evidently, the second term in the right side of (2.7) can be estimated from above in the $L^2(\mathbb{R})$ norm by

$$\frac{1}{\sqrt{1+b^2}}\|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

due to our assumption. By means of orthogonality relations (2.6) and (1.9), we have

$$\widehat{f}(0) = 0, \quad \widehat{f}_m(0) = 0, \quad m \in \mathbb{N}.$$

Hence

$$\widehat{f}(p) = \int_0^p \frac{d\widehat{f}(s)}{ds}ds, \quad \widehat{f}_m(p) = \int_0^p \frac{d\widehat{f}_m(s)}{ds}ds, \quad m \in \mathbb{N}. \tag{2.8}$$

Therefore, it remains to estimate the norm of the term

$$\frac{\int_0^p [\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}]ds}{|p| - ibp}\chi_{\{|p|\le 1\}}.$$

By virtue of the definition of the standard Fourier transform (1.4), we easily arrive at

$$\Big|\frac{d\widehat{f}_m(p)}{dp} - \frac{d\widehat{f}(p)}{dp}\Big| \le \frac{1}{\sqrt{2\pi}}\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})},$$

such that

$$\left\|\frac{\int_0^p [\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}]ds}{|p| - ibp}\chi_{\{|p|\le 1\}}\right\|_{L^2(\mathbb{R})} \le \frac{\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}}{\sqrt{\pi(1+b^2)}} \to 0$$

as $m \to \infty$ via the one of our assumptions. Thus, $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. By means of the argument above we obtain that $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ in the case b) of our theorem as well. ∎

3. The problem on the finite interval

Proof of Theorem 1.3. First of all, let us show that it would be sufficient to solve our equation in $L^2(I)$. Indeed, if $u(x)$ is a square integrable solution of (1.10), periodic on I along with its first derivative, directly from our problem under the stated assumptions we obtain

$$\sqrt{-\frac{d^2}{dx^2}}u - b\frac{du}{dx} \in L^2(I).$$

(1.12) yields $(|n| - ibn)u_n \in l^2$. Hence, $\sum_{n=-\infty}^{\infty} n^2|u_n|^2 < \infty$, such that $\frac{du}{dx} \in L^2(I)$. Therefore, $u(x) \in H^1(I)$ as well.

To demonstrate the uniqueness of solutions of (1.10), let us consider the case of $a > 0$. When $a = 0$, we are able to exploit the similar ideas in the constrained subspace $H_0^1(I)$. We suppose that $u_1(x),\ u_2(x) \in H^1(I)$ satisfy (1.10). Then their difference $w(x) := u_1(x) - u_2(x) \in H^1(I)$ solves the homogeneous problem

$$\sqrt{-\frac{d^2}{dx^2}}w - b\frac{dw}{dx} - aw = 0.$$

Because the operator $\mathcal{L}_{a,\ b}$ defined in (1.13) does not possess any nontrivial $H^1(I)$ zero modes, the function $w(x)$ vanishes on I.

Let us apply the Fourier transform (1.12) to both sides of equation (1.10). Thus

$$u_n = \frac{f_n}{|n| - a - ibn}, \quad n \in \mathbb{Z}, \tag{3.1}$$

such that

$$\|u\|^2_{L^2(I)} = \sum_{n=-\infty}^{\infty} \frac{|f_n|^2}{(|n| - a)^2 + b^2n^2}. \tag{3.2}$$

Let us first treat the case a) of the theorem. By means of (3.2), we obtain

$$\|u\|^2_{L^2(I)} \leq \frac{1}{C}\|f\|^2_{L^2(I)} < \infty$$

due to the one of our assumptions (see (1.11)). By virtue of the argument above, $u(x) \in H^1(I)$ as well.

We conclude the proof of our theorem by considering the situation when a vanishes, such that from (3.1)

$$u_n = \frac{f_n}{|n| - ibn}, \quad n \in \mathbb{Z}. \tag{3.3}$$

Apparently, the right side of (3.3) belongs to l^2 if and only if

$$f_0 = 0, \tag{3.4}$$

such that

$$\|u\|^2_{L^2(I)} = \frac{1}{1+b^2} \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{|f_n|^2}{n^2} \leq \frac{1}{1+b^2}\|f\|^2_{L^2(I)} < \infty,$$

via the one of our assumptions and (1.11). The argument above yields that $u(x) \in H^1_0(I)$ as well. Clearly, (3.4) is equivalent to orthogonality relation (1.16). ■

We turn our attention establishing the solvability in the sense of sequences for our equation on the interval I with periodic boundary conditions.

Proof of Theorem 1.4. Using the given conditions, we derive

$$|f(0) - f(2\pi)| \leq |f(0) - f_m(0)| + |f_m(2\pi) - f(2\pi)| \leq 2\|f_m - f\|_{L^\infty(I)} \to 0$$

as $m \to \infty$. Thus, $f(0) = f(2\pi)$. By virtue of (1.11) for $f_m(x),\ f(x)$ bounded on our interval I, we have $f_m(x),\ f(x) \in L^1(I) \cap L^2(I),\ m \in \mathbb{N}$. (1.11) also yields

$$\|f_m(x) - f(x)\|_{L^1(I)} \leq 2\pi\|f_m(x) - f(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty. \tag{3.5}$$

Hence, $f_m(x) \to f(x)$ in $L^1(I)$ as $m \to \infty$. Similarly, (1.11) gives us

$$\|f_m(x) - f(x)\|_{L^2(I)} \leq \sqrt{2\pi}\|f_m(x) - f(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty. \tag{3.6}$$

Thus, $f_m(x) \to f(x)$ in $L^2(I)$ as $m \to \infty$ as well. Let us apply the Fourier transform (1.12) to both sides of (1.15). We arrive at

$$u_{m,n} = \frac{f_{m,n}}{|n| - a - ibn}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}. \tag{3.7}$$

Let us first treat the case a) of the theorem. By means of the part a) of Theorem 1.3, equations (1.10) and (1.15) admit unique solutions $u(x) \in H^1(I)$ and $u_m(x) \in H^1(I),\ m \in \mathbb{N}$ respectively. (3.7) along with (3.1) and (3.6) imply that

$$\|u_m - u\|^2_{L^2(I)} = \sum_{n=-\infty}^{\infty} \frac{|f_{m,n} - f_n|^2}{(|n| - a)^2 + b^2 n^2} \leq \frac{1}{C}\|f_m - f\|^2_{L^2(I)} \to 0, \quad m \to \infty.$$

Hence, $u_m(x) \to u(x)$ in $L^2(I)$ as $m \to \infty$. We will prove that $u_m(x)$ tends to $u(x)$ in $H^1(I)$ as $m \to \infty$. Indeed, by virtue of (1.10) and (1.15)

$$\left\| \sqrt{-\frac{d^2}{dx^2}}(u_m - u) - b\frac{d(u_m - u)}{dx} \right\|_{L^2(I)} \leq \|f_m - f\|_{L^2(I)} + a\|u_m - u\|_{L^2(I)}.$$

The right side of this estimate tends to zero as $m \to \infty$ via (3.6). Using the Fourier transform (1.12), we obtain

$$\sum_{n=-\infty}^{\infty} n^2 |u_{m,n} - u_n|^2 \to 0, \quad m \to \infty.$$

Thus, $\dfrac{du_m}{dx} \to \dfrac{du}{dx}$ in $L^2(I)$ as $m \to \infty$, such that $u_m(x) \to u(x)$ in $H^1(I)$ as $m \to \infty$ as well.

Finally, we turn our attention to the case when the parameter $a = 0$. By virtue of (1.17) along with (3.5), we derive

$$|(f(x), 1)_{L^2(I)}| = |(f(x) - f_m(x), 1)_{L^2(I)}| \le \|f_m - f\|_{L^1(I)} \to 0, \quad m \to \infty,$$

such that the limiting orthogonality relation

$$(f(x), 1)_{L^2(I)} = 0 \tag{3.8}$$

holds. By means of the part b) of Theorem 1.3 above problems (1.10) and (1.15) have unique solutions $u(x) \in H_0^1(I)$ and $u_m(x) \in H_0^1(I),\ m \in \mathbb{N}$ respectively when a vanishes. Formulas (3.1) and (3.7) give us

$$u_{m,n} - u_n = \frac{f_{m,n} - f_n}{|n| - ibn}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}. \tag{3.9}$$

Orthogonality conditions (3.8) and (1.17) yield

$$f_0 = 0, \quad f_{m,0} = 0, \quad m \in \mathbb{N}.$$

We estimate the norm

$$\|u_m - u\|_{L^2(I)} = \sqrt{\sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{|f_{m,n} - f_n|^2}{(1+b^2)n^2}} \le \frac{\|f_m - f\|_{L^2(I)}}{\sqrt{1+b^2}} \to 0, \quad m \to \infty$$

due to (3.6). Thus, $u_m(x) \to u(x)$ in $L^2(I)$ as $m \to \infty$. Therefore, $u_m(x) \to u(x)$ in $H_0^1(I)$ as $m \to \infty$ as well via the argument analogous to the one above in the proof of the case a) of our theorem. ■

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