ELEMENTARY PROOF OF THE CARTAN MAGIC FORMULA

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Abstract. In this short note we present very simple proof of the famous Cartan homotopy formula

$$L_v\omega = di_v\omega + i_v d\omega.$$

The Proof

Let M be a smooth manifold, $\dim M = m$, and let v, ω be a smooth vector field and a smooth differential k-form respectively.

Our aim is to prove the following Cartan formula

$$L_v\omega = di_v\omega + i_v d\omega.$$

Here L_v is the Lie derivative and i_v is the interior product.

First we check the formula for each point $x \in M$ such that $v(x) \neq 0$. It is well known [1] that if $v(\tilde{x}) \neq 0$ then in some neighbourhood of the point $\tilde{x}$ there are local coordinates $x = (x^1, \dots, x^m)$ such that in these coordinates the vector field v is presented as follows $v = \partial_1$.

The corresponding flow has the form

$$g^t(x) = (x^1 + t, x^2, \dots, x^m). \tag{0.1}$$

By linearity of L_v, d, i_v it is sufficient to check the Cartan formula for the monomials of the following two sorts:

1) $\omega = a(x)dx^1 \wedge dx^{j_1} \wedge \dots \wedge dx^{j_{k-1}}, \quad 1 < j_1 < \dots < j_{k-1} \le m$;
and

2) $\gamma = b(x)dx^{l_1} \wedge \dots \wedge dx^{l_k}, \quad 1 < l_1 < \dots < l_k \le m$.

Consider the case 1); the case two 2) is carried out similarly.

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By direct calculation we obtain

$$i_v\omega = a dx^{j_1}\wedge\ldots\wedge dx^{j_{k-1}},\quad di_v\omega=\sum_{s=1}^m\frac{\partial a}{\partial x^s}dx^s\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_{k-1}};$$

and

$$d\omega=\sum_{r=2}^m\frac{\partial a}{\partial x^r}dx^r\wedge dx^1\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_{k-1}};$$

$$i_v d\omega=-\sum_{r=2}^m\frac{\partial a}{\partial x^r}dx^r\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_{k-1}}.$$

On the other hand by formula (0.1) it follows that

$$\begin{aligned}L_v\omega &= \frac{d}{dt}\Big|_{t=0}a(x^1+t,x^2,\ldots,x^m)d(x^1+t)\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_{k-1}}\\ &=\frac{\partial a}{\partial x^1}dx^1\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_{k-1}}.\end{aligned}$$

This proves the Cartan formula at each point of the set

$$F=\{x\in M\mid v(x)\ne 0\}.$$

This set is open. By continuity, the Cartan formula remains valid in the closure $\overline{F}$.

The set $N=M\backslash\overline{F}$ is open and $v\mid_N=0$. This implies that in any local coordinates all partial derivatives of v vanish at each point of N. Consequently, on the set N the Cartan formula takes the trivial form:

$$0=0.$$

The Cartan formula is proved.

References

[1] Taylor M.E. (2011) Basic Theory of ODE and Vector Fields. In: Partial Differential Equations I. Applied Mathematical Sciences, vol 115. Springer, New York, NY