ON THE SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON-FREDHOLM OPERATORS IN HIGHER DIMENSIONS

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Abstract: We study solvability of some linear nonhomogeneous elliptic equations and establish that under reasonable technical conditions the convergence in $L^2(\mathbb{R}^d)$ of their right sides yields the existence and the convergence in $L^2(\mathbb{R}^d)$ of the solutions. The problems contain the fractional powers of the sums of second order non-Fredholm differential operators and we use the methods of the spectral and scattering theory for Schrödinger type operators analogously to our preceding article [27].
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1. Introduction

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.1}$$

with $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ tends to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of equation (1.1) contains the origin. As a consequentce, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimensions of its kernel and the codimension of its image are not finite. The present article is devoted to the studies of some properties of the operators of this kind. We recall that elliptic problems containing non Fredholm operators were treated extensively in recent years (see [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], also [6]) along with their potential applications to the theory of reaction-diffusion equations (see [8], [9]). Non-Fredholm operators are also crucial when studying wave systems with an infinite number of localized traveling waves (see [1]). Particularly, when $a = 0$ the operator A satisfies the Fredholm property in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is significantly different and the method developed in these articles cannot be used.

One of the important issues about equations with non-Fredholm operators concerns their solvability. Let us address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Since the operator A fails to satisfy the Fredholm property, the sequence u_n may not be convergent. We call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of equation $Au = f$ (see [17]). If such sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this problem. Solution in the sense of sequences is equivalent in this case to the usual solution. However, in the case of the non-Fredholm operators, this convergence may not hold or it can occur in some weaker sense. In this case, solution in the sense of sequences may not imply the existence of the usual solution. In the present article we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent. Solvability in the sense of sequences for the sums of Schrödinger type operators without Fredholm property was studied in [27]. In the first part of the article we consider such operators raised to fractional powers, namely

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}^s u - au = f(x, y), \quad x, y \in \mathbb{R}^3, \quad 0 < s < 1, \tag{1.2}$$

where the constant $a \geq 0$. The operator

$$H_{U,\,V} := \{-\Delta_x + V(x) - \Delta_y + U(y)\}^s \tag{1.3}$$

here is defined by means of the spectral calculus. Here and further down the Laplace operators Δ_x and Δ_y are with respect to the x and y variables respectively. The fractional powers of second order differential operators are actively used, for example in the studies of the anomalous diffusion problems (see e.g. [28], [29] and the references therein). The form boundedness criterion for the relativistic Schrödinger operator was established in [14]. The work [13] deals with proving the imbedding theorems and the studies of the spectrum of a certain pseudodifferential operator.

The scalar potential functions involved in operator (1.3) are assumed to be shallow and short-range, satisfying the assumptions analogous to the ones of [20] and [22].

Assumption 1. *The potential functions $V(x), U(y) : \mathbb{R}^3 \to \mathbb{R}$ satisfy the estimates*

$$|V(x)| \leq \frac{C}{1 + |x|^{3.5+\varepsilon}}, \quad |U(y)| \leq \frac{C}{1 + |y|^{3.5+\varepsilon}}$$

with some $\varepsilon > 0$ and $x, y \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}} \frac{9}{8} (4\pi)^{-\frac{2}{3}} \|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}} \|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1, \tag{1.4}$$

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|U\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|U\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \tag{1.5}$$

and

$$\sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi, \quad \sqrt{c_{HLS}}\|U\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here and below C stands for a finite positive constant and c_{HLS} given on p.98 of [12] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\right| \leq c_{HLS}\|f_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^2, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

The norm of a function $f_1 \in L^p(\mathbb{R}^d),\ 1 \leq p \leq \infty,\ d \in \mathbb{N}$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^d)}$. We designate the inner product of two functions as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \tag{1.6}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x) \in L^\infty(\mathbb{R}^d)$, like for instance the functions of the continuos spectrum of the Schrödinger operators discussed below, then the integral in the right side of (1.6) is well defined. By virtue of Lemma 2.3 of [22], under Assumption 1 above on the scalar potentials, operator (1.3) considered as acting in $L^2(\mathbb{R}^6)$ with domain $H^2(\mathbb{R}^6)$ is self-adjoint and is unitarily equivalent to $\{-\Delta_x - \Delta_y\}^s$ on $L^2(\mathbb{R}^6)$ via the product of the wave operators (see [11], [16])

$$\Omega_V^\pm := s - \lim_{t\to\mp\infty} e^{it(-\Delta_x+V(x))}e^{it\Delta_x}, \quad \Omega_U^\pm := s - \lim_{t\to\mp\infty} e^{it(-\Delta_y+U(y))}e^{it\Delta_y},$$

with the limits here understood in the strong L^2 sense (see e.g. [15] p.34, [7] p.90). Thus, operator (1.3) has no nontrivial $L^2(\mathbb{R}^6)$ eigenfunctions. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Therefore, operator (1.3) fails to satisfy the Fredholm property. The functions of the continuos spectrum of the first operator involved in (1.3) are the solutions the Schrödinger equation

$$[-\Delta_x + V(x)]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3,$$

in the integral form the Lippmann-Schwinger equation

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{1.7}$$

and the orthogonality conditions $(\varphi_k(x), \varphi_{k_1}(x))_{L^2(\mathbb{R}^3)} = \delta(k - k_1),\ k, k_1 \in \mathbb{R}^3$. The integral operator involved in (1.7)

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi(x) \in L^\infty(\mathbb{R}^3).$$

Let us consider $Q : L^{\infty}(\mathbb{R}^3) \to L^{\infty}(\mathbb{R}^3)$ and its norm $\|Q\|_{\infty} < 1$ under Assumption 1 via Lemma 2.1 of [22]. In fact, this norm is bounded above by the k-independent quantity $I(V)$, which is the left side of inequality (1.4). Similarly, for the second operator involved in (1.3) the functions of its continuous spectrum solve

$$[-\Delta_y + U(y)]\eta_q(y) = q^2\eta_q(y), \quad q \in \mathbb{R}^3,$$

in the integral formulation

$$\eta_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta_q)(z)dz, \tag{1.8}$$

such that the the orthogonality conditions $(\eta_q(y), \eta_{q_1}(y))_{L^2(\mathbb{R}^3)} = \delta(q - q_1),\ q, q_1 \in \mathbb{R}^3$ hold. The integral operator involved in (1.8) is

$$(P\eta)(y) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta)(z)dz, \quad \eta(y) \in L^{\infty}(\mathbb{R}^3).$$

For $P : L^{\infty}(\mathbb{R}^3) \to L^{\infty}(\mathbb{R}^3)$ its norm $\|P\|_{\infty} < 1$ under Assumption 1 by virtue of Lemma 2.1 of [22]. As before, this norm can be bounded from above by the q-independent quantity $I(U)$, which is the left side of inequality (1.5). We denote by the double tilde sign the generalized Fourier transform with the product of these functions of the continuous spectrum

$$\tilde{\tilde{f}}(k, q) := (f(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}, \quad k, q \in \mathbb{R}^3. \tag{1.9}$$

(1.9) is a unitary transform on $L^2(\mathbb{R}^6)$. Our first main statement is as follows.

Theorem 2. *Let Assumption 1 hold and* $f(x, y) \in L^2(\mathbb{R}^6)$.

a) When $a = 0$*, let in addition* $f(x, y) \in L^1(\mathbb{R}^6)$*. Then problem (1.2) possesses a unique solution* $u(x, y) \in L^2(\mathbb{R}^6)$*.*

b) When $a > 0$*, let in addition* $xf(x, y),\ yf(x, y) \in L^1(\mathbb{R}^6)$*. Then equation (1.2) admits a unique solution* $u(x, y) \in L^2(\mathbb{R}^6)$ *if and only if*

$$(f(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k, q) \in S^6_{a^{\frac{1}{2s}}}. \tag{1.10}$$

Here and further down S^d_a denotes the sphere in $\mathbb{R}^d$ of radius a centered at the origin. Such unit sphere will be denoted as S^d and its Lebesgue measure as $|S^d|$. Note that in the case when $a = 0$ of the theorem above no orthogonality relations are needed to solve problem (1.2) in $L^2(\mathbb{R}^6)$.

Let us turn our attention to the issue of the solvability in the sense of sequences for our problem. The corresponding sequence of equations with $n \in \mathbb{N}$ is given by

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}^s u_n - au_n = f_n(x, y), \quad x, y \in \mathbb{R}^3, \tag{1.11}$$

where $0 < s < 1$, the constant $a \geq 0$ and the right sides converge to the right side of (1.2) in $L^2(\mathbb{R}^6)$ as $n \to \infty$.

Theorem 3. *Let Assumption 1 hold, $n \in \mathbb{N}$ and $f_n(x,y) \in L^2(\mathbb{R}^6)$, such that $f_n(x,y) \to f(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$.*

a) When $a = 0$, let in addition $f_n(x,y) \in L^1(\mathbb{R}^6),\ n \in \mathbb{N}$, such that $f_n(x,y) \to f(x,y)$ in $L^1(\mathbb{R}^6)$ as $n \to \infty$. Then problems (1.2) and (1.11) admit unique solutions $u(x,y) \in L^2(\mathbb{R}^6)$ and $u_n(x,y) \in L^2(\mathbb{R}^6)$ respectively, such that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$.

b) When $a > 0$, let in addition $xf_n(x,y),\ yf_n(x,y) \in L^1(\mathbb{R}^6),\ n \in \mathbb{N}$, such that $xf_n(x,y) \to xf(x,y),\ yf_n(x,y) \to yf(x,y)$ in $L^1(\mathbb{R}^6)$ as $n \to \infty$ and the orthogonality conditions

$$(f_n(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k,q) \in S^6_{a^{\frac{1}{2s}}}. \tag{1.12}$$

hold for all $n \in \mathbb{N}$. Then equations (1.2) and (1.11) possess unique solutions $u(x,y) \in L^2(\mathbb{R}^6)$ and $u_n(x,y) \in L^2(\mathbb{R}^6)$ respectively, such that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$.

In the second part of the work we study the equation

$$\{-\Delta_x - \Delta_y + U(y)\}^s u - au = \phi(x,y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^3, \tag{1.13}$$

where $d \in \mathbb{N},\ 0 < s < 1$ and the scalar potential function involved in (1.13) is shallow and short-range under Assumption 1 above. The operator

$$L_U := \{-\Delta_x - \Delta_y + U(y)\}^s \tag{1.14}$$

here is defined by virtue of the spectral calculus. Analogously to (1.3), under the given assumptions operator (1.14) considered as acting in $L^2(\mathbb{R}^{d+3})$ with domain $H^2(\mathbb{R}^{d+3})$ is self-adjoint and is unitarily equivalent to $\{-\Delta_x - \Delta_y\}^s$. Hence, operator (1.14) does not have nontrivial $L^2(\mathbb{R}^{d+3})$ eigenfunctions. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$ and such that operator (1.14) is non Fredholm. We consider another generalized Fourier transform with the standard Fourier harmonics and the perturbed plane waves

$$\tilde{\hat{\phi}}(k,q) := \left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})}, \quad k \in \mathbb{R}^d, \quad q \in \mathbb{R}^3. \tag{1.15}$$

(1.15) is a unitary transform on $L^2(\mathbb{R}^{d+3})$. We have the following proposition.

Theorem 4. *Let the potential function $U(y)$ satisfy Assumption 1 and $\phi(x,y) \in L^2(\mathbb{R}^{d+3}),\ d \in \mathbb{N}$.*

a) When $a = 0$, let in addition $\phi(x,y) \in L^1(\mathbb{R}^{d+3})$. Then problem (1.13) possesses a unique solution $u(x,y) \in L^2(\mathbb{R}^{d+3})$.

b) When $a > 0$, let in addition $x\phi(x,y),\ y\phi(x,y) \in L^1(\mathbb{R}^{d+3})$. Then equation (1.13) admits a unique solution $u(x,y) \in L^2(\mathbb{R}^{d+3})$ if and only if

$$\left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0, \quad (k,q) \in S^{d+3}_{a^{\frac{1}{2s}}}. \tag{1.16}$$

Note that in the case when $a = 0$ of this theorem no orthogonality conditions are required to solve equation (1.13) in $L^2(\mathbb{R}^{d+3})$.

Our final main statement deals with the issue of the solvability in the sense of sequences for our problem. The corresponding sequence of approximate equations with $n \in \mathbb{N}$ is given by

$$\{-\Delta_x - \Delta_y + U(y)\}^s u_n - a u_n = \phi_n(x,y), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}, \quad y \in \mathbb{R}^3, \tag{1.17}$$

where $0 < s < 1$ and the right sides converge to the right side of (1.13) in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.

Theorem 5. *Let the potential function $U(y)$ satisfy Assumption 1, $n \in \mathbb{N}$ and $\phi_n(x,y) \in L^2(\mathbb{R}^{d+3})$, $d \in \mathbb{N}$, such that $\phi_n(x,y) \to \phi(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.*

a) When $a = 0$, let in addition $\phi_n(x,y) \in L^1(\mathbb{R}^{d+3})$, $n \in \mathbb{N}$, such that $\phi_n(x,y) \to \phi(x,y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$. Then problems (1.13) and (1.17) admit unique solutions $u(x,y) \in L^2(\mathbb{R}^{d+3})$ and $u_n(x,y) \in L^2(\mathbb{R}^{d+3})$ respectively, such that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.

b) When $a > 0$, let in addition $x\phi_n(x,y),\ y\phi_n(x,y) \in L^1(\mathbb{R}^{d+3})$, such that $x\phi_n(x,y) \to x\phi(x,y),\ y\phi_n(x,y) \to y\phi(x,y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$ and the orthogonality conditions

$$\left(\phi_n(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0, \quad (k,q) \in S^{d+3}_{a^{\frac{1}{2s}}} \tag{1.18}$$

hold for all $n \in \mathbb{N}$. Then equations (1.13) and (1.17) have unique solutions $u(x,y) \in L^2(\mathbb{R}^{d+3})$ and $u_n(x,y) \in L^2(\mathbb{R}^{d+3})$ respectively, such that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.

Note that (1.10), (1.12), (1.16), (1.18) are the orthogonality relations containing the functions of the continuous spectrum of our Schrödinger operators, as distinct from the Limiting Absorption Principle in which one orthogonalizes to the standard Fourier harmonics (see e.g. Lemma 2.3 and Proposition 2.4 of [10]). Let us proceed to the proof of our propositions.

2. Solvability in the sense of sequences with two potentials

Proof of Theorem 2. To prove the uniqueness of solutions for our problem, we suppose that equation (1.2) has two square integrable solutions $u_1(x,y)$ and $u_2(x,y)$. Then their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in L^2(\mathbb{R}^6)$ satisfies the equation

$$H_{U,\,V}w = aw.$$

Since operator (1.3) has no nontrivial square integrable eigenfunctions in the whole space as discussed above, we have $w(x,y) = 0$ a.e. in $\mathbb{R}^6$.

Let us first treat the case of our theorem when $a = 0$. We apply the generalized Fourier transform (1.9) to both sides of equation (1.2). This gives us

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}} + \frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}> 1\right\}}, \tag{2.19}$$

where $k, q \in \mathbb{R}^3$. Here and below χ_A will stand for the characteristic function of a set $A \subseteq \mathbb{R}^d$. Clearly, the second term in the right side of (2.19) can be bounded from above in the absolute value by $|\tilde{\tilde{f}}(k,q)| \in L^2(\mathbb{R}^6)$ via the one of our assumptions. By virtue of Corollary 2.2 of [22] (see also [20]) under the given assumptions for $k, q \in \mathbb{R}^3$ we have $\varphi_k(x),\ \eta_q(y) \in L^\infty(\mathbb{R}^3)$ and

$$\|\varphi_k(x)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{1-I(V)}\frac{1}{(2\pi)^{\frac{3}{2}}}, \quad \|\eta_q(y)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{1-I(U)}\frac{1}{(2\pi)^{\frac{3}{2}}}. \tag{2.20}$$

This allows us to estimate the first term in the right side of (2.19) from above in the absolute value by by

$$\frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f\|_{L^1(\mathbb{R}^6)}\frac{\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}}{\{k^2+q^2\}^s}.$$

Thus,

$$\left\|\frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}\right\|_{L^2(\mathbb{R}^6)} \le \frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f\|_{L^1(\mathbb{R}^6)}\sqrt{\frac{|S^6|}{6-4s}},$$

which is finite due to the assumptions of the theorem. Therefore, for the unique solution of problem (1.2) in the case a) of the theorem we have $u(x,y) \in L^2(\mathbb{R}^6)$.

Let us conclude the proof by considering the case b) of our theorem. We apply the generalized Fourier transform (1.9) to both sides of equation (1.2) and obtain

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s - a}.$$

For the technical purposes we introduce the set

$$A_\delta := \{(k,q) \in \mathbb{R}^6 \mid a^{\frac{1}{2s}} - \delta \le \sqrt{k^2+q^2} \le a^{\frac{1}{2s}} + \delta\}, \quad 0 < \delta < a^{\frac{1}{2s}}, \tag{2.21}$$

such that

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s - a}\chi_{A_\delta} + \frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s - a}\chi_{A_\delta^c}. \tag{2.22}$$

Here and below for a set $A \subseteq \mathbb{R}^d$ we designate its complement as A^c. Clearly, the second term in the right side of (2.22) can be estimated from above in the absolute value by $\frac{\left|\tilde{\tilde{f}}(k,q)\right|}{(a^{\frac{1}{2s}}+\delta)^{2s} - a}$ for $\sqrt{k^2+q^2} > a^{\frac{1}{2s}} + \delta$ and by $\frac{\left|\tilde{\tilde{f}}(k,q)\right|}{a - (a^{\frac{1}{2s}}-\delta)^{2s}}$ when $0 \le \sqrt{k^2+q^2} < a^{\frac{1}{2s}} - \delta$, such that

$$\left|\frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s - a}\chi_{A_\delta^c}\right| \le C\left|\tilde{\tilde{f}}(k,q)\right| \in L^2(\mathbb{R}^6)$$

due to the one of our assumptions. Obviously, we can express

$$\tilde{\tilde{f}}(k,q) = \tilde{\tilde{f}}(a^{\frac{1}{2s}}, \sigma) + \int_{a^{\frac{1}{2s}}}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}(\xi,\sigma)}{\partial \xi} d\xi.$$

Here and further down σ will stand for the angle variables on the sphere. This allows us to write the first term in the right side of (2.22) as

$$\frac{\tilde{\tilde{f}}(a^{\frac{1}{2s}},\sigma)}{\{k^2+q^2\}^s - a}\chi_{A_\delta} + \frac{\int_{a^{\frac{1}{2s}}}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}(\xi,\sigma)}{\partial \xi} d\xi}{\{k^2+q^2\}^s - a}\chi_{A_\delta}. \tag{2.23}$$

Apparently, we can bound the second term in (2.23) from above in the absolute value by

$$\|(\nabla_k + \nabla_q)\tilde{\tilde{f}}(k,q)\|_{L^\infty(\mathbb{R}^6)} \left|\frac{\sqrt{k^2+q^2} - a^{\frac{1}{2s}}}{\{k^2+q^2\}^s - a}\right| \chi_{A_\delta} \le$$

$$\le C\|(\nabla_k + \nabla_q)\tilde{\tilde{f}}(k,q)\|_{L^\infty(\mathbb{R}^6)}\chi_{A_\delta} \in L^2(\mathbb{R}^6).$$

Here and further down ∇_k and ∇_q will stand for the gradients acting on variables k and q respectively. Clearly, under the given assumptions $(\nabla_k + \nabla_q)\tilde{\tilde{f}}(k,q) \in L^\infty(\mathbb{R}^6)$ by virtue of Lemma 11 of [20]. A straightforward computation shows that the first term in (2.23) belongs to $L^2(\mathbb{R}^6)$ if and only if $\tilde{\tilde{f}}(a^{\frac{1}{2s}}, \sigma)$ vanishes. This is equivalent to orthogonality relation (1.10). ■

Then we turn our attention to the solvability in the sense of sequences for our problem in the case of two scalar potentials.

Proof of Theorem 3. In the case when the constant $a = 0$, equations (1.2) and (1.11) admit unique solutions $u(x, y), u_n(x, y) \in L^2(\mathbb{R}^6)$ respectively with $n \in \mathbb{N}$ due to the part a) of Theorem 2 above. We apply the generalized Fourier transform (1.9) to both sides of problems (1.2) and (1.11). This gives us

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s}, \quad \tilde{\tilde{u}}_n(k,q) = \frac{\tilde{\tilde{f}}_n(k,q)}{\{k^2+q^2\}^s}, \quad 0 < s < 1, \quad n \in \mathbb{N}.$$

Hence $\tilde{\tilde{u}}_n(k,q) - \tilde{\tilde{u}}(k,q)$ can be expressed as

$$\frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}} + \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}> 1\right\}}. \tag{2.24}$$

Obviously, the second term in (2.24) can be easily estimated from above in the absolute value by $|\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)|$. Therefore,

$$\left\| \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}> 1\right\}} \right\|_{L^2(\mathbb{R}^6)} \leq \|f_n(x,y) - f(x,y)\|_{L^2(\mathbb{R}^6)} \to 0$$

as $n \to \infty$ via the one of our assumptions. The first term in (2.24) can be bounded from above in the absolute value by virtue of inequalities (2.20) by

$$\frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f_n(x,y) - f(x,y)\|_{L^1(\mathbb{R}^6)}\frac{\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}}{\{k^2+q^2\}^s}.$$

Thus,

$$\left\| \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}} \right\|_{L^2(\mathbb{R}^6)} \leq$$

$$\leq \frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f_n(x,y) - f(x,y)\|_{L^1(\mathbb{R}^6)}\sqrt{\frac{|S^6|}{6-4s}} \to 0, \quad n \to \infty$$

due to the one of our assumptions. Hence, $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$ in the case a) of our theorem.

Let us complete the proof of the theorem by considering the part b). For each $n \in \mathbb{N}$ problem (1.11) has a unique solution $u_n(x,y) \in L^2(\mathbb{R}^6)$ by virtue of the result of the part b) of Theorem 2 above. By means of (1.12) along with inequalities (2.20), we obtain for $(k,q) \in S^6_{a^{\frac{1}{2s}}}$

$$|(f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}| = |(f(x,y) - f_n(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}| \leq$$

$$\leq \frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f_n(x,y) - f(x,y)\|_{L^1(\mathbb{R}^6)} \to 0, \quad n \to \infty.$$

Note that under the given conditions $f_n(x,y) \to f(x,y)$ in $L^1(\mathbb{R}^6)$ by means of the simple argument on p.114 of [27]. Thus, we arrive at

$$(f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k,q) \in S^6_{a^{\frac{1}{2s}}}. \tag{2.25}$$

Hence, problem (1.2) possesses a unique solution $u(x,y) \in L^2(\mathbb{R}^6)$ via the result of the part b) of Theorem 2 above. Let us apply the generalized Fourier transform (1.9) to both sides of equations (1.2) and (1.11). This yields

$$\tilde{\tilde{u}}_n(k,q) - \tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s - a}\chi_{A_\delta} + \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s - a}\chi_{A_\delta^c}, \tag{2.26}$$

where the set A_δ is defined in (2.21). Evidently, the second term in the right side of (2.26) can be estimated from above in the absolute value by $\dfrac{|\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)|}{(a^{\frac{1}{2s}}+\delta)^{2s} - a}$ for $\sqrt{k^2+q^2} > a^{\frac{1}{2s}} + \delta$ and by $\dfrac{|\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)|}{a - (a^{\frac{1}{2s}}-\delta)^{2s}}$ for $0 \leq \sqrt{k^2+q^2} < a^{\frac{1}{2s}} - \delta$. Hence

$$\left\| \frac{\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^s - a}\chi_{A_\delta^c} \right\|_{L^2(\mathbb{R}^6)} \leq C\|f_n(x,y) - f(x,y)\|_{L^2(\mathbb{R}^6)} \to 0, \quad n \to \infty$$

via the one of our assumptions. Orthogonality relations (1.12) and (2.25) give us

$$\tilde{\tilde{f}}(a^{\frac{1}{2s}}, \sigma) = 0, \quad \tilde{\tilde{f}}_n(a^{\frac{1}{2s}}, \sigma) = 0, \quad n \in \mathbb{N}.$$

Therefore,

$$\tilde{\tilde{f}}(k,q) = \int_{a^{\frac{1}{2s}}}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}(\xi,\sigma)}{\partial \xi} d\xi, \quad \tilde{\tilde{f}}_n(k,q) = \int_{a^{\frac{1}{2s}}}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}_n(\xi,\sigma)}{\partial \xi} d\xi, \quad n \in \mathbb{N}.$$

This allows us to express the first term in the right side of (2.26) as

$$\frac{\int_{a^{\frac{1}{2s}}}^{\sqrt{k^2+q^2}} \left[\frac{\partial \tilde{\tilde{f}}_n(\xi,\sigma)}{\partial \xi} - \frac{\partial \tilde{\tilde{f}}(\xi,\sigma)}{\partial \xi}\right] d\xi}{\{k^2+q^2\}^s - a}\chi_{A_\delta}. \tag{2.27}$$

Clearly, (2.27) can be estimated from above in the absolute value by

$$\|(\nabla_k + \nabla_q)(\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q))\|_{L^\infty(\mathbb{R}^6)} \left| \frac{\sqrt{k^2+q^2} - a^{\frac{1}{2s}}}{\{k^2+q^2\}^s - a} \right| \chi_{A_\delta} \leq$$

$$\leq C\|(\nabla_k + \nabla_q)(\tilde{\tilde{f}}_n(k,q) - \tilde{\tilde{f}}(k,q))\|_{L^\infty(\mathbb{R}^6)}\chi_{A_\delta}.$$

This enables us to obtain the upper bound for the $L^2(\mathbb{R}^6)$ norm of (2.27) by

$$C\|(\nabla_k+\nabla_q)(\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q))\|_{L^\infty(\mathbb{R}^6)}\to 0, \quad n\to\infty$$

via the part a) of Lemma 5 of [27] under the given conditions. Hence, $u_n(x,y)\to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n\to\infty$. ■

In the final section of the work we consider the case when the free Laplacian is added to the three dimensional Schrödinger operator.

3. Solvability in the sense of sequences with Laplacian and a single potential

Proof of Theorem 4. To show the uniqueness of solutions for our problem, let us suppose that (1.13) has two square integrable solutions $u_1(x,y)$ and $u_2(x,y)$. Then their difference $w(x,y):=u_1(x,y)-u_2(x,y)\in L^2(\mathbb{R}^{d+3})$ solves the equation

$$L_U w = aw.$$

Because operator (1.14) does not possess nontrivial square integrable eigenfunctions in the whole space as discussed above, we have $w(x,y)=0$ a.e. in $\mathbb{R}^{d+3}$.

First we consider the case a) of our theorem when the parameter a vanishes. Let us apply the generalized Fourier transform (1.15) to both sides of problem (1.13). This gives us

$$\tilde{\tilde{u}}(k,q)=\frac{\tilde{\tilde{\phi}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}+\frac{\tilde{\tilde{\phi}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}}, \tag{3.28}$$

where $k\in\mathbb{R}^d,\ q\in\mathbb{R}^3$. Evidently, the second term in the right side of (3.28) can be estimated from above in the abosolute value by $|\tilde{\tilde{\phi}}(k,q)|\in L^2(\mathbb{R}^{d+3})$ via the one of our assumptions. By means of (2.20), we easily obtain

$$|\tilde{\tilde{\phi}}(k,q)|\le\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}.$$

Hence, the first term in the right side of (3.28) can be bounded from above in the abosolute value by

$$\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\frac{\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}}{\{k^2+q^2\}^s}.$$

Therefore,

$$\left\|\frac{\tilde{\tilde{\phi}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}\right\|_{L^2(\mathbb{R}^{d+3})}\le$$

$$\leq \frac{1}{(2\pi)^{\frac{d+3}{2}}} \frac{1}{1-I(U)} \|\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})} \sqrt{\frac{|S^{d+3}|}{d+3-4s}} < \infty$$

via our assumptions. Hence, $u(x,y) \in L^2(\mathbb{R}^{d+3})$ in the case a) of our theorem.

We conclude the proof by treating the case b) of our theorem. Let us apply the generalized Fourier transform (1.15) to both sides of equation (1.13) and arrive at

$$\tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{\phi}}(k,q)}{\{k^2+q^2\}^s - a}.$$

For the technical purposes we will use the set

$$B_\delta := \{(k,q) \in \mathbb{R}^{d+3} \mid a^{\frac{1}{2s}} - \delta \leq \sqrt{k^2+q^2} \leq a^{\frac{1}{2s}} + \delta\}, \quad 0 < \delta < a^{\frac{1}{2s}}. \quad (3.29)$$

Thus

$$\tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{\phi}}(k,q)}{\{k^2+q^2\}^s - a} \chi_{B_\delta} + \frac{\tilde{\hat{\phi}}(k,q)}{\{k^2+q^2\}^s - a} \chi_{B_\delta^c}. \quad (3.30)$$

The second term in the right side of (3.30) can be bounded from above in the absolute value by $\frac{|\tilde{\hat{\phi}}(k,q)|}{(a^{\frac{1}{2s}}+\delta)^{2s} - a}$ for $\sqrt{k^2+q^2} > a^{\frac{1}{2s}} + \delta$ and by $\frac{|\tilde{\hat{\phi}}(k,q)|}{a - (a^{\frac{1}{2s}}-\delta)^{2s}}$ for $0 \leq \sqrt{k^2+q^2} < a^{\frac{1}{2s}} - \delta$, such that

$$\left| \frac{\tilde{\hat{\phi}}(k,q)}{\{k^2+q^2\}^s - a} \chi_{B_\delta^c} \right| \leq C |\tilde{\hat{\phi}}(k,q)| \in L^2(\mathbb{R}^{d+3})$$

via the one of our assumptions. We express

$$\tilde{\hat{\phi}}(k,q) = \tilde{\hat{\phi}}(a^{\frac{1}{2s}}, \sigma) + \int_{a^{\frac{1}{2s}}}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(\xi,\sigma)}{\partial \xi} d\xi.$$

This enables us to write the first term in the right side of (3.30) as

$$\frac{\tilde{\hat{\phi}}(a^{\frac{1}{2s}}, \sigma)}{\{k^2+q^2\}^s - a} \chi_{B_\delta} + \frac{\int_{a^{\frac{1}{2s}}}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(\xi,\sigma)}{\partial \xi} d\xi}{\{k^2+q^2\}^s - a} \chi_{B_\delta}. \quad (3.31)$$

Clearly, we have the estimate from above for the second term in (3.31) in the absolute value by

$$\|(\nabla_k + \nabla_q) \tilde{\hat{\phi}}(k,q)\|_{L^\infty(\mathbb{R}^{d+3})} \left| \frac{\sqrt{k^2+q^2} - a^{\frac{1}{2s}}}{\{k^2+q^2\}^s - a} \right| \chi_{B_\delta} \leq$$

$$\leq C \|(\nabla_k + \nabla_q) \tilde{\hat{\phi}}(k,q)\|_{L^\infty(\mathbb{R}^{d+3})} \chi_{B_\delta} \in L^2(\mathbb{R}^{d+3}).$$

Evidently, under the stated assumptions $(\nabla_k + \nabla_q)\hat{\tilde{\phi}}(k,q) \in L^{\infty}(\mathbb{R}^{d+3})$ due to Lemma 12 of [20]. It can be easily shown that, the first term in (3.31) belongs to $L^2(\mathbb{R}^{d+3})$ if and only if $\hat{\tilde{\phi}}(a^{\frac{1}{2s}}, \sigma)$ vanishes. This is equivalent to orthogonality condition (1.16). ∎

Let us finish the article with showing the solvability in the sense of sequences for our equation when the free Laplacian is added to a three dimensional Schrödinger operator.

Proof of Theorem 5. In the case when $a = 0$, problems (1.13) and (1.17) possess unique solutions $u(x,y), u_n(x,y) \in L^2(\mathbb{R}^{d+3})$ respectively for $n \in \mathbb{N}$ due to the part a) of Theorem 4 above. Let us apply the generalized Fourier transform (1.15) to both sides of equations (1.13) and (1.17). This yields

$$\hat{\tilde{u}}(k,q) = \frac{\hat{\tilde{\phi}}(k,q)}{\{k^2+q^2\}^s}, \quad \hat{\tilde{u}}_n(k,q) = \frac{\hat{\tilde{\phi}}_n(k,q)}{\{k^2+q^2\}^s}, \quad 0 < s < 1, \quad n \in \mathbb{N}.$$

This allows us to write $\hat{\tilde{u}}_n(k,q) - \hat{\tilde{u}}(k,q)$ as

$$\frac{\hat{\tilde{\phi}}_n(k,q) - \hat{\tilde{\phi}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}} + \frac{\hat{\tilde{\phi}}_n(k,q) - \hat{\tilde{\phi}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}> 1\right\}}. \tag{3.32}$$

Evidently, the second term in (3.32) can be easily bounded from above in the absolute value by $|\hat{\tilde{\phi}}_n(k,q) - \hat{\tilde{\phi}}(k,q)|$. Hence

$$\left\| \frac{\hat{\tilde{\phi}}_n(k,q) - \hat{\tilde{\phi}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}> 1\right\}} \right\|_{L^2(\mathbb{R}^{d+3})} \le \|\phi_n(x,y) - \phi(x,y)\|_{L^2(\mathbb{R}^{d+3})} \to 0$$

as $n \to \infty$ due to the one of our assumptions. We derive the estimate from above in the the absolute value for the first term in (3.32) via (2.20) by

$$\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi_n(x,y) - \phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\frac{\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}}}{\{k^2+q^2\}^s}.$$

Hence

$$\left\| \frac{\hat{\tilde{\phi}}_n(k,q) - \hat{\tilde{\phi}}(k,q)}{\{k^2+q^2\}^s}\chi_{\left\{\sqrt{k^2+q^2}\le 1\right\}} \right\|_{L^2(\mathbb{R}^{d+3})} \le$$

$$\le \frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi_n(x,y) - \phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\sqrt{\frac{|S^{d+3}|}{d+3-4s}} \to 0, \quad n \to \infty$$

due to the one of our assumptions. This implies that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$ in the case a) of our theorem.

We finish the work by proceeding to the proof of the part b) of the theorem. For each $n \in \mathbb{N}$ equation (1.17) admits a unique solution $u_n(x,y) \in L^2(\mathbb{R}^{d+3})$ due to the result of the part b) of Theorem 4 above. By virtue of (1.18) along with the second inequality in (2.20), we derive for $(k,q) \in S^{d+3}_{a^{\frac{1}{2s}}}$

$$\left|\left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})}\right| = \left|\left(\phi(x,y)-\phi_n(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})}\right| \leq$$

$$\leq \frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi_n(x,y)-\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})} \to 0, \quad n \to \infty.$$

Note that under the given conditions $\phi_n(x,y) \to \phi(x,y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$ via the trivial argument on p.116 of [27]. Hence, we obtain

$$\left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})} = 0, \quad (k,q) \in S^{d+3}_{a^{\frac{1}{2s}}}. \tag{3.33}$$

Thus, equation (1.13) admits a unique solution $u(x,y) \in L^2(\mathbb{R}^{d+3})$ by means of the result of the part b) of Theorem 4 above. We apply the generalized Fourier transform (1.15) to both sides of problems (1.13) and (1.17). This gives us

$$\tilde{\hat{u}}_n(k,q) - \tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\{k^2+q^2\}^s - a}\chi_{B_\delta} + \frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\{k^2+q^2\}^s - a}\chi_{B_\delta^c}, \tag{3.34}$$

where the set B_δ is defined in (3.29). Obviously, the second term in the right side of (3.34) can be bounded from above in the absolute value by $\dfrac{|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)|}{(a^{\frac{1}{2s}}+\delta)^{2s} - a}$ for $\sqrt{k^2+q^2} > a^{\frac{1}{2s}} + \delta$ and by $\dfrac{|\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)|}{a - (a^{\frac{1}{2s}}-\delta)^{2s}}$ for $0 \leq \sqrt{k^2+q^2} < a^{\frac{1}{2s}} - \delta$. Therefore,

$$\left\|\frac{\tilde{\hat{\phi}}_n(k,q) - \tilde{\hat{\phi}}(k,q)}{\{k^2+q^2\}^s - a}\chi_{B_\delta^c}\right\|_{L^2(\mathbb{R}^{d+3})} \leq C\|\phi_n(x,y)-\phi(x,y)\|_{L^2(\mathbb{R}^{d+3})} \to 0, \quad n \to \infty$$

via the one of our assumptions. Orthogonality conditions (1.18) and (3.33) give us

$$\tilde{\hat{\phi}}(a^{\frac{1}{2s}}, \sigma) = 0, \quad \tilde{\hat{\phi}}_n(a^{\frac{1}{2s}}, \sigma) = 0, \quad n \in \mathbb{N}.$$

Therefore,

$$\tilde{\hat{\phi}}(k,q) = \int_{a^{\frac{1}{2s}}}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}(\xi,\sigma)}{\partial \xi} d\xi, \quad \tilde{\hat{\phi}}_n(k,q) = \int_{a^{\frac{1}{2s}}}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\hat{\phi}}_n(\xi,\sigma)}{\partial \xi} d\xi, \quad n \in \mathbb{N}.$$

This enables us to write the first term in the right side of (3.34) as

$$\frac{\int_{a^{\frac{1}{2s}}}^{\sqrt{k^2+q^2}} \left[\frac{\partial \tilde{\tilde{\phi}}_n(\xi,\sigma)}{\partial \xi} - \frac{\partial \tilde{\tilde{\phi}}(\xi,\sigma)}{\partial \xi}\right] d\xi}{\{k^2+q^2\}^s - a} \chi_{B_\delta}. \tag{3.35}$$

Evidently, (3.35) can be estimated from above in the absolute value by

$$\|(\nabla_k + \nabla_q)(\tilde{\tilde{\phi}}_n(k,q) - \tilde{\tilde{\phi}}(k,q))\|_{L^\infty(\mathbb{R}^{d+3})} \left| \frac{\sqrt{k^2+q^2} - a^{\frac{1}{2s}}}{(k^2+q^2)^s - a} \right| \chi_{B_\delta} \leq$$

$$\leq C\|(\nabla_k + \nabla_q)(\tilde{\tilde{\phi}}_n(k,q) - \tilde{\tilde{\phi}}(k,q))\|_{L^\infty(\mathbb{R}^{d+3})} \chi_{B_\delta}.$$

This allows us to obtain the upper bound for the $L^2(\mathbb{R}^{d+3})$ norm of (3.35) as

$$C\|(\nabla_k + \nabla_q)(\tilde{\tilde{\phi}}_n(k,q) - \tilde{\tilde{\phi}}(k,q))\|_{L^\infty(\mathbb{R}^{d+3})} \to 0, \quad n \to \infty$$

by means of the result of the part b) of Lemma 5 of [27] under the stated conditions. Therefore, $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$. ■

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