THE PRESERVATION OF NONNEGATIVITY OF SOLUTIONS OF A PARABOLIC SYSTEM WITH THE BI-LAPLACIAN

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Abstract: The paper deals with the easily verifiable necessary condition of the preservation of the nonnegativity of the solutions of a system of parabolic equations involving the bi-Laplace operator. This necessary condition is vitally important for the applied analysis community because it imposes the necessary form of the system of equations that must be studied mathematically.

AMS subject classification: 35K55, 35K57
Keywords: bi-Laplacian operator, parabolic systems, nonnegativity of solutions

1. Introduction

The solutions of many systems of convection-diffusion-reaction equations arising in biology, physics or engineering describe such quantities as population densities, pressure or concentrations of nutrients and chemicals. Thus, a natural property to require for the solutions is their nonnegativity. Models that do not guarantee the nonnegativity are not valid or break down for small values of the solution. In many cases, showing that a particular model does not preserve the nonnegativity leads to the better understanding of the model and its limitations. One of the first steps in analyzing ecological or biological or bio-medical models mathematically is to test whether solutions originating from the nonnegative initial data remain nonnegative (as long as they exist). In other words, the model under consideration ensures that the nonnegative cone is positively invariant. We recall that if the solutions (of a given evolution PDE) corresponding to the nonnegative initial data remain nonnegative as long as they exist, we say that the system satisfies the nonnegativity property.

For scalar equations with the standard Laplace operator the nonnegativity property is a direct consequence of the maximum principle (see [1] and the references therein). However, for the equations involving the bi-Laplacian the maximum principle is not valid.

In this work we aim to prove a simple and easily verifiable criterion, that is, the necessary condition for the nonnegativity of solutions of systems of nonlinear convection–diffusion–reaction equations involving the bi-Laplace operator arising in the modelling of life sciences. We believe that it could provide the modeler with a tool, which is easy to verify, to approach the question of nonnegative invariance of the model.

Presently we deal with the preservation of the nonnegativity of solutions of the following system of reaction-diffusion equations

$$\frac{\partial u}{\partial t} = -A\Delta^2 u + \sum_{i=1}^{d} \Gamma^i \frac{\partial u}{\partial x_i} - F(u), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}, \tag{1.1}$$

where $A,\ \Gamma^i,\ 1 \le i \le d$ are $N \times N$ matrices with constant coefficients. Here

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

Note that in the present article we deal with the space of an arbitrary dimension. Solvability conditions for the linearized Cahn-Hilliard equation involving the bi-Laplcian were studied in [2]. The solvability of the single equation containing the standard Laplacian with drift relevant to the fluid mechanics was treated in [3]. We assume here that (1.1) contains the square matrices with the entries constant in space and time

$$(A)_{k,j} := a_{k,j}, \quad (\Gamma^i)_{k,j} := \gamma^i_{k,j}, \quad 1 \le k, j \le N, \quad 1 \le i \le d, \quad d \in \mathbb{N}$$

and that the matrix $A + A^* > 0$ for the sake of the global well posedness of system (1.1). Here A^* stands for the adjoint of matrix A. Hence, system (1.1) can be rewritten in the form

$$\frac{\partial u_k}{\partial t} = -\sum_{j=1}^{N} a_{k,j}\Delta^2 u_j + \sum_{i=1}^{d}\sum_{j=1}^{N} \gamma^i_{k,j}\frac{\partial u_j}{\partial x_i} - F_k(u), \quad 1 \le k \le N. \tag{1.2}$$

In the present work the interaction vector function term

$$F(u) := (F_1(u), F_2(u), ..., F_N(u))^T,$$

which can be linear, nonlinear or in principle even nonlocal. We assume its smoothness in the theorem below for the sake of the well posedness of our system (1.1), although, we are not focused on the well posedness issue in the present article. Let us denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx. \tag{1.3}$$

As for the vector functions, their inner product is defined using their components as

$$(u,v)_{L^2(\mathbb{R}^d,\mathbb{R}^N)} := \sum_{k=1}^{N} (u_k, v_k)_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

Evidently, (1.4) induces the norm

$$\|u\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} = \sum_{k=1}^{N} \|u_k\|^2_{L^2(\mathbb{R}^d)}.$$

By the nonnegativity of a vector function below we mean the nonnegativity of the each of its components. Our main proposition is as follows.

Theorem 1. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$, such that $F \in \mathbb{C}^1$, the initial condition for system (1.1) is $u(x,0) = u_0(x) \geq 0$ and $u_0(x) \in L^2(\mathbb{R}^d, \mathbb{R}^N)$. We also assume that the off diagonal elements of the matrix A, are nonnegative, such that*

$$a_{k,j} \geq 0, \quad 1 \leq k, j \leq N, \quad k \neq j. \tag{1.5}$$

Then the necessary condition for system (1.1) to possess a solution $u(x,t) \geq 0$ for all $t \in [0,\infty)$ is that the matrices A and Γ^i, $1 \leq i \leq d$ are diagonal and for all $1 \leq k \leq N$

$$F_k(s_1, ..., s_{k-1}, 0, s_{k+1}, ..., s_N) \leq 0 \tag{1.6}$$

holds , where $s_l \geq 0$ and $1 \leq l \leq N$, $l \neq k$.

Remark 1. *In the case of the linear interaction term, namely when $F(u) = Lu$, where L is a matrix with elements $b_{i,j}$, $1 \leq i, j \leq N$ constant in space and time, our necessary condition leads to the condition that the matrix L must be essentially nonpositive, that is $b_{i,j} \leq 0$ for $i \neq j$, $1 \leq i, j \leq N$.*

Remark 2. *Our proof implies that, the necessary condition for preserving the nonnegative cone is carried over from the ODE (the spatially homogeneous case, as described by the ordinary differential equation $u'(t) = -F(u)$) to the case of the diffusion involving the bi-Laplacian and the convective drift term.*

Remark 3. *In the forthcoming works we intend to consider the following cases:*
a) the necessary and sufficient conditions of the present work,
b) the nonautonomous version of the present work,
c) the density-dependent diffusion matrix,
d) the effect of the delay term in the cases a), b) and c).

We proceed to the proof of our main result.

2. The preservation of the nonnegativity of the solution of the system of parabolic equations

Proof of Theorem 1. Let us note that the maximum principle actively used for the studies of solutions of single parabolic equations with the standard Laplace operator does not apply to such equations with the bi- Laplacian. Let us consider a time independent, square integrable, nonnegative vector function $v(x)$ and estimate

$$\left(\frac{\partial u}{\partial t}\bigg|_{t=0}, v\right)_{L^2(\mathbb{R}^d,\mathbb{R}^N)} = \left(\lim_{t\to 0^+}\frac{u(x,t)-u_0(x)}{t}, v(x)\right)_{L^2(\mathbb{R}^d,\mathbb{R}^N)}.$$

By virtue of the continuity of the inner product, the right side of the formula above equals to

$$\lim_{t\to 0^+}\frac{(u(x,t),v(x))_{L^2(\mathbb{R}^d,\mathbb{R}^N)}}{t} - \lim_{t\to 0^+}\frac{(u_0(x),v(x))_{L^2(\mathbb{R}^d,\mathbb{R}^N)}}{t}. \tag{2.7}$$

We choose the initial condition for our system $u_0(x) \geq 0$ and the constant in time vector function $v(x) \geq 0$ to be orthogonal to each other in $L^2(\mathbb{R}^d,\mathbb{R}^N)$. It can be achieved, for instance for

$$u_0(x) = (\tilde{u}_1(x), ..., \tilde{u}_{k-1}(x), 0, \tilde{u}_{k+1}(x), ..., \tilde{u}_N(x)), \quad v_j(x) = \tilde{v}(x)\delta_{j,k}, \tag{2.8}$$

with $1 \leq j \leq N$. Here $\delta_{j,k}$ denotes the Kronecker symbol and $1 \leq k \leq N$ is fixed. Thus, the second term in (2.7) vanishes and (2.7) is equal to

$$\lim_{t\to 0^+}\frac{\sum_{k=1}^N \int_{\mathbb{R}^d} u_k(x,t)v_k(x)dx}{t} \geq 0,$$

since all the components $u_k(x,t)$ and $v_k(x)$ involved in the formula above are nonnegative. Hence, we obtain

$$\sum_{j=1}^N \int_{\mathbb{R}^d} \frac{\partial u_j}{\partial t}\bigg|_{t=0} v_j(x)dx \geq 0.$$

By means of (2.8), only the k th component of the vector function $v(x)$ is nontrivial. This gives us

$$\int_{\mathbb{R}^d} \frac{\partial u_k}{\partial t}\bigg|_{t=0} \tilde{v}(x)dx \geq 0.$$

Therefore, using (1.2) we derive

$$\int_{\mathbb{R}^d}\Bigg[-\sum_{j=1,\ j\neq k}^N a_{k,j}\Delta^2\tilde{u}_j(x) + \sum_{i=1}^d\sum_{j=1,\ j\neq k}^N \gamma_{k,j}^i\frac{\partial \tilde{u}_j}{\partial x_i} -$$

$$-F_k(\tilde{u}_1(x), ..., \tilde{u}_{k-1}(x), 0, \tilde{u}_{k+1}(x), ..., \tilde{u}_N(x))\Bigg]\tilde{v}(x)dx \geq 0.$$

Since the nonnegative, square integrable function $\tilde{v}(x)$ can be chosen arbitrarily, we arrive at

$$-\sum_{j=1,\ j\neq k}^{N} a_{k,j}\Delta^2\tilde{u}_j(x) + \sum_{i=1}^{d}\sum_{j=1,\ j\neq k}^{N}\gamma^i_{k,j}\frac{\partial \tilde{u}_j}{\partial x_i} -$$

$$-F_k(\tilde{u}_1(x), ..., \tilde{u}_{k-1}(x), 0, \tilde{u}_{k+1}(x), ..., \tilde{u}_N(x)) \geq 0 \quad a.e. \tag{2.9}$$

For the purpose of the scaling, we replace all the $\tilde{u}_j(x)$ by $\tilde{u}_j\left(\frac{x}{\varepsilon}\right)$ in the estimate above, where $\varepsilon > 0$ is a small parameter. This gives us

$$-\sum_{j=1,\ j\neq k}^{N} \frac{a_{k,j}}{\varepsilon^4}\Delta^2\tilde{u}_j(y) + \sum_{i=1}^{d}\sum_{j=1,\ j\neq k}^{N}\frac{\gamma^i_{k,j}}{\varepsilon}\frac{\partial \tilde{u}_j(y)}{\partial y_i} -$$

$$-F_k(\tilde{u}_1(y), ..., \tilde{u}_{k-1}(y), 0, \tilde{u}_{k+1}(y), ..., \tilde{u}_N(y)) \geq 0 \quad a.e. \tag{2.10}$$

First we suppose that some of the $a_{k,j}$ involved in the sum in the left side of (2.10) are strictly positive. Evidently, the first term in the left side of (2.10) is the leading one as $\varepsilon \to 0$. Let us choose here all the $\tilde{u}_j(y),\ 1 \leq j \leq N,\ j \neq k$ to be identical, equal to e^{y^2} in a neighborhood of the origin, smooth and decaying to zero at the infinity. A straightforward computation yields

$$\Delta^2\tilde{u}_j(y)\Big|_{y=0} = 4d(d+2) > 0, \quad 1 \leq j \leq N, \quad j \neq k.$$

Therefore, $\Delta^2\tilde{u}_j(y) > 0$ in a neighborhood of the origin via the trivial continuity argument. By making the parameter ε small enough, we are able to violate the inequality in (2.10). Because the negativity of the off diagonal elements of the matrix A is ruled out via assumption (1.5), we obtain

$$a_{k,j} = 0, \quad 1 \leq k, j \leq N, \quad k \neq j.$$

Hence, from (2.10) we derive

$$\sum_{i=1}^{d}\sum_{j=1,\ j\neq k}^{N}\frac{\gamma^i_{k,j}}{\varepsilon}\frac{\partial}{\partial y_i}\tilde{u}_j(y) -$$

$$-F_k(\tilde{u}_1(y), ..., \tilde{u}_{k-1}(y), 0, \tilde{u}_{k+1}(y), ..., \tilde{u}_N(y)) \geq 0 \quad a.e. \tag{2.11}$$

In the case of $\gamma^i_{k,j} > 0,\ j \neq k$ we can choose here $\tilde{u}_j(y) = e^{-\sqrt{y^2+1}}$ in a neighborhood of the origin, smooth and decaying to zero at the infinity, such that

$$\frac{\partial}{\partial y_i}\tilde{u}_j(y) = -\frac{y_i}{\sqrt{y^2+1}}e^{-\sqrt{y^2+1}} < 0, \quad y_i > 0$$

near the origin. If $\gamma_{k,j}^i < 0, \ j \neq k$, then we can take $\tilde{u}_j(y) = e^{\sqrt{y^2+1}}$ in a neighborhood of the origin, smooth and tending to zero at the infinity, such that

$$\frac{\partial}{\partial y_i}\tilde{u}_j(y) = \frac{y_i}{\sqrt{y^2+1}} e^{\sqrt{y^2+1}} > 0, \quad y_i > 0$$

near the origin. Then the left side of (2.11) can be made as negative as possible when $\varepsilon \to 0$, which will violate bound (2.11). Let us note that the last term in the left side of (2.11) will remain bounded. Therefore, for the matrices Γ^i involved in system (1.1), the off diagonal elements should vanish, such that

$$\gamma_{k,j}^i = 0, \quad 1 \leq k, j \leq N, \quad k \neq j.$$

Thus, by means of (2.11) we arrive at

$$F_k(\tilde{u}_1(x), ..., \tilde{u}_{k-1}(x), 0, \tilde{u}_{k+1}(x), ..., \tilde{u}_N(x)) \leq 0 \quad a.e.,$$

where $\tilde{u}_j(x) \geq 0$ and $\tilde{u}_j(x) \in L^2(\mathbb{R}^d)$ with $1 \leq j \leq N, \ j \neq k$. ■

Acknowledgement. *Valuable discussions with Messoud Efendiev are gratefully acknowledged.*

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