# Non-deterministic Quantum Mechanics and Particle Spin[1]

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The preceding paper [1] proposed an alternative interpretation of quantum mechanics which is distinct from the Copenhagen Interpretation and the pilot wave theory. This alternative interpretation and its associated non-deterministic velocity component are tested here with reference to particle spins. A mechanism for generating particle spins is rigorously identified for the case of a free particle and the case of the hydrogen electron. The mechanism can be generalised for electrons in other atoms and other particles. The incomplete nature of the Schrödinger equation with regard to its inability to incorporate particle spin is demonstrated. The adoption of Born's rule by the physicist community for interpreting the probability density is seen as implicitly acknowledging the mechanism for generating particle spin but this mechanism was far from being made explicit by that rule. The crucial question of the need for sensitive non-deterministic information input to maintain particle spin and to satisfy Born's rule is raised. This question probes the limit of our human ability to penetrate into non-deterministic processes and events. The paper concludes with an assessment of the proposed alternative interpretation of quantum mechanics with respect to the criteria of conceptual coherence and correspondence to experiments.

## 1.0    Introduction

The previous paper [1] takes the position that a particle has an unambiguous position and an unambiguous momentum at any instant in time even though the total velocity of the particle is non-determinate. It identifies a deterministic component and a non-deterministic component of the velocity of a particle which renders the total velocity non-determinate. The non-deterministic velocity component is tangential to the *S* (phase) surface and the particle can be visualised as 'surfing' on the *S* surface as it is carried forward by the deterministic velocity component. This paper explores further the possible non-deterministic motion of the particle on the *S* surface which can be seen as a mechanism for generating the spin of a particle.

As hinted in the previous paper [1], in a disorderly and violent universe, the non-deterministic free motion of the particle on the *S* surface has little constraint. However, in a reasonable and orderly universe like ours, even though the free velocity on the *S* surface is not determinate, it may still be subject to some kind of constraint. We now investigate the constraint on its energy, its possible direction and its relation to particle spin.

## 2.0  The Energy of a Particle

The Schrödinger equation

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$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi + U\psi$$

can be written as the following two equations:

$$\hbar \frac{\partial S}{\partial t} + \frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + U = 0 \tag{1}$$

$$\frac{\partial R^2}{\partial t} + Div\left(\frac{\hbar}{m}R^2\nabla S\right) = 0 \ . \tag{2}$$

where the wave function has been written as

$$\psi = Re^{iS} \ ,$$

$m$ is the mass of the particle, $\hbar$ is the reduced Planck constant, $U$ is the potential 'experienced' by the particle. Equation (2) is called the pseudo continuity equation in [1] and is the focus of that paper. In this paper, the focus is on equation (1) which is called the energy equation. As in the previous paper, the total velocity of a particle is written as

$$\underset{\sim}{v} = \underset{\sim}{v}_1 + \lambda_2 \underset{\sim}{v}_2 + \lambda_3 \underset{\sim}{v}_3$$

where $\lambda_2$ and $\lambda_3$ are constant over space but can be functions of time, and with $\rho \equiv R^2$ the velocity components can be written as

$$\underset{\sim}{v}_1 \equiv \frac{\hbar}{m}\nabla S \ , \ \underset{\sim}{v}_2 \equiv \frac{\hbar}{2m}\frac{\nabla \rho}{\rho}\wedge \vec{s} \ , \ \underset{\sim}{v}_3 \equiv \frac{\hbar}{m}\frac{\nabla S}{\rho}\wedge \vec{a}$$

where $\vec{s}$ at the particle's position is a unit vector perpendicular to $\nabla\rho$ and lies on the plane formed by $\nabla S$ and $\nabla\rho$, $\vec{a}$ is the unit vector in the direction of $\underset{\sim}{v}_2$ at the point where the particle is.[3] $\underset{\sim}{v}_1$ is the deterministic velocity component and the non-deterministic velocity on a $S$ surface is given by $\underset{\sim}{v}_s \equiv \lambda_2 \underset{\sim}{v}_2 + \lambda_3 \underset{\sim}{v}_3$ and the particle is said to surf on the $S$ surface with this velocity. Also, at the position where the particle is, $\underset{\sim}{v}_1$ , $\underset{\sim}{v}_2$ , $\underset{\sim}{v}_3$ form a set of orthogonal velocities. The energy equation can be written as

$$\hbar \frac{\partial S}{\partial t} + \frac{1}{2}m|\underset{\sim}{v}_1|^2 + \frac{1}{2}m|\underset{\sim}{v}_s|^2 - \left[\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + \frac{1}{2}m|\underset{\sim}{v}_s|^2\right] + U = 0 \tag{3}$$

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[3] See [1] for details.

2

where $\frac{1}{2}m|v_s|^2$ is called the Quantum Kinetic Energy (*QKE*) of the particle. $\frac{1}{2}m|v_1|^2 + \frac{1}{2}m|v_s|^2$ is the total kinetic energy of the particle. The fourth term (enclosed by the brackets) with its minus sign is called the Quantum Potential Energy (*QPE*) of the particle such that

$$QKE + QPE = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$$

which in this paper is called the Total Quantum Energy (*TQE*). Usually, it is called the quantum potential but this term is not used in this paper in order not to confuse it with *QPE*. *U* is the classical potential and we can treat the sum of the classical potential energy and the Quantum Potential Energy (*QPE*) as the total potential energy of the particle. Hence, the sum of the second and third terms represents the total kinetic energy of the particle and the sum of the fourth and fifth terms represents the total potential energy of the particle. The reason for having these different terms in the energy equation will become clear later. At this point, it is sufficient to note that the sum of the second, third, fourth and fifth terms on the left hand side represents the total energy of the particle, *E*, and

$$\hbar\frac{\partial S}{\partial t} = -E \ . \tag{4}$$

### 3.0 A Particle in Steady Motion State

Taking the gradient of the energy equation (1), we have

$$\hbar\frac{\partial \nabla S}{\partial t} + \nabla\left[\frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + U\right] = 0$$

If $\nabla S$ is constant with respect to time but varies with space, i.e., if $v_1$ is constant in time (but can vary in space), then the total energy

$$E \equiv \frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + U$$

is constant over space. We called this Steady Motion State (*SMS*). The only steady part of the motion is the deterministic velocity component, $v_1$ , while the non-deterministic component can vary in time and hence can be unsteady. A Steady Motion State is not to be confused with the usual steady state motion where all components of the motion are constant in time. The total energy, *E*, of the particle, consists of (i) the classical terms for kinetic energy and potential energy (first and third term) and (ii) the Total Quantum Energy (second term, which can be divided into Quantum Kinetic Energy and Quantum Potential Energy as explained above). *E* is uniform over space for *SMS*, i.e., whichever position the particle happens to have at a certain instant, its total energy is the same. It is natural and highly reasonable to suggest that the total energy, *E*, of a particle is conserved in time

for *SMS*. If this were not the case, since *E* is constant over space for *SMS*, varying *E* in time would mean varying *E* over the whole space in time and this demands some enormous source or sink of energy over the whole space, the cause of which we have no idea. This paper chooses the simplest option, i.e., for *SMS* the total energy, *E*, of a particle is conserved in time and is uniform over space, i.e., whatever time and wherever the particle happens to be at, it has the same total energy. Since *E* is constant, equation (4) can be integrated in time to yield

$$S = -\frac{E}{\hbar}t + b(\underset{\sim}{r}) \tag{5}$$

where $\underset{\sim}{r}$ is a position vector and *b* is a scalar function independent of time.

## 4.0 A Particle in Steady Motion State With Uniform Translational Velocity and Uniform *U*

Let $\underset{\sim}{v}_1$ be named as the translational component of the velocity. We now consider the case where this translational component of the velocity (and hence $\nabla S$) is uniform over space and the classical potential, *U*, is uniform over space and time. These give rise to the following energy equation:

$$\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = \frac{\hbar^2}{2m}(\nabla S)^2 + U - E \tag{6}$$

where each term on the right hand side is a constant so that the right hand side can be treated as a single constant. Let this constant be written as $-\frac{\hbar^2}{2m}a^2$ for a suitable *a*. The above equation becomes

$$\nabla^2 R + a^2 R = 0 \ .$$

This is in the form of the Helmholtz Equation. Note that *a* is chosen to be real so that the right hand side of equation 6 is negative; this can be accommodated or ensured by taking a sufficiently negative value of the classical potential energy, *U*, if necessary, without disturbing the dynamics of the system. If *a* is imaginary, we will show rigorously (in Appendix A) that it will lead to unphysical scenarios. Also, the solution for *R* is a function of space only (since *a* does not vary with time), so that the first term of equation (2), $\frac{\partial R^2}{\partial t}$ , is zero. Hence, $\rho$ is constant in time but varying in space while $\underset{\sim}{v}_1$, $\nabla S$ and *U* are constant in time and in space. Furthermore, from equation (2) we have

$$Div(\rho\underset{\sim}{v}_1) = 0 \implies \underset{\sim}{v}_1 \cdot \nabla\rho = 0 \ . \tag{7}$$

Now, the above Helmholtz Equation for constant $\underset{\sim}{v}_1$ and constant *U* can be solved in cylindrical co-ordinates, *r, θ, z*. But first we have to choose the orientation of the cylindrical co-ordinates

appropriately. Let $z$ be in the direction of $\nabla S$, and hence in the direction of $v_1$. This means that the particle is travelling with constant translational velocity ($v_1$) in the $z$ direction and cutting through the $S$ surface perpendicularly. A $S$ surface is then identical to the $(r, \theta)$ plane (the $z$ plane). The scenario constructed so far corresponds to a particle travelling in free space (with uniform $U$) with uniform translational velocity ($v_1$) in the $z$ direction. In this case, using (7) above, $\nabla\rho$ is perpendicular to the $z$ axis which implies that

$$\frac{\partial \rho}{\partial z} = 0 \quad \Longrightarrow \quad \frac{\partial R}{\partial z} = 0 \; .$$

Also, we can appeal to isotropy with respect to $\theta$ and take $\dfrac{\partial}{\partial \theta} = 0$ so that the Helmholtz Equation becomes

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + a^2 R = 0 \tag{8}$$

which is the radial equation with one independent variable, $r$. This equation is in a form of the Bessel equation and has the solution

$$R = AJ_0(ra) + BY_0(ra)$$

where $J_0, Y_0$ are Bessel functions of the first and second kind respectively. Since $Y_0(0) = \infty$, if $R$ is to have the usual meaning, i.e., $\rho \equiv R^2$ is the probability density, then it seems that $B = 0$; however, the situation is somewhat more complicated than that but we will later show rigorously that indeed $Y_0$ indeed is not a suitable solution when we consider the 'spin energy' of the particle. First, we consider a mechanism for generating the spin of the particle.

**5.0 Particle Spin in Steady Motion State With Uniform Translational Velocity in Free Space – Stage 1 of Exploration**

**5.1 Bessel Function of the First Kind**

Here we consider this solution:

$$R = AJ_0(ra) \; , \quad \rho = A^2 J_0^2(ra) \; .$$

This means that the contours of constant $\rho$ on the $(r, \theta)$ plane, i.e., on the $S$ surface, are concentric circles.
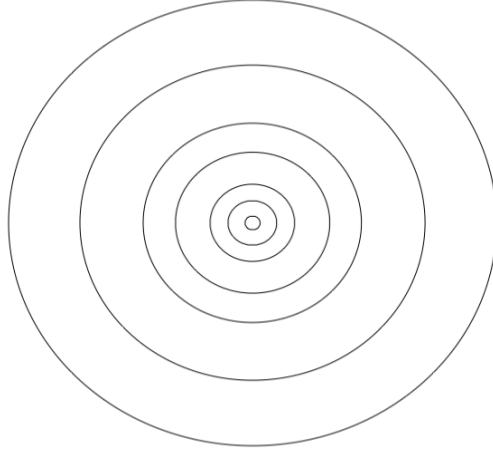
Figure 1: Contours of constant $\rho$ on the $(r, \theta)$ plane (or $S$ surface); the $z$ direction and hence the direction of $v_1$ is perpendicular to the page

To look at the structure of the $\rho$ contours, a plot of $\rho$ with respect to $r$ is necessary. At this point, it is sufficient to plot the square of the Bessel function of the first kind, i.e., without the constant $A$.
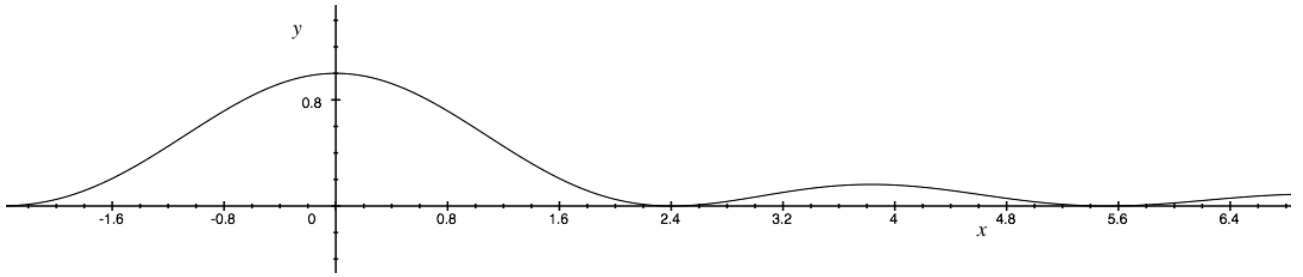


Figure 2: $y = J_0^2(x)$

Now, $\rho$ has a maximum at $r = 0$ and it is zero at $ra = 2.4$, i.e., at $r = 2.4/a$. If $\rho$ is to have the usual meaning in quantum mechanics, we expect to find the particle between $r = 0$ and $r = 2.4/a$. What happens beyond this latter point, i.e., $r > 2.4/a$? To answer this question, we need to explore the possible nature of the spin of the particle.

In the previous paper, it was suggested that the motion of the particle on the $S$ surface is non-determinate. This implies that, for the case being considered, the particle's motion on the $(r, \theta)$ plane is non-determinate. How then is it possible to explore the motion of the particle on this plane/ surface and possibly relate it to its spin? Remember that in the previous paper, it was concluded that there are two degrees of freedom for the particle's motion on the $S$ surface, corresponding to the two unspecified magnitudes of $\lambda_2 v_2$ and $\lambda_3 v_3$ . Here, it is suggested that one can specify the magnitude of one component of the particle's velocity on the $S$ surface while leaving the other component, which is perpendicular to it, non-determinate in magnitude so that the overall motion on the $S$ surface is still non-determinate.  Now, we specify $\lambda_2$ to be a specific constant over time in

addition to its constancy in space, thus specifying one component of the velocity, $\lambda_2 v_2$ , on the $S$ surface and leaving $\lambda_3 v_3$ to be non-determinate in its magnitude even though its direction, being perpendicular to $\lambda_2 v_2$ on the $S$ surface, is determined. This non-determinate nature of its magnitude will be crucially utilised later. At the position where the particle is, $\lambda_2 v_2$ lies on the $(r, \theta)$ plane (which is a $S$ surface) and it is tangential to the circular $\rho$ contour there, while the direction of $\lambda_3 v_3$ goes through the origin or the centre.

There is the critical point at $r = 2.4/a$ where $\rho = 0$ and it needs to be considered carefully. For convenience, we define $L$ to be $2.4/a$. In general,

$$\frac{|\nabla \rho|}{\rho} = 2a \left| \frac{J_{-1}(ra)}{J_0(ra)} \right| \quad \left(\texttt{using } \frac{d}{dr} J_0(ra) = a J_{-1}(ra) \right).$$

At the critical point,

$$\lim_{r \to L} \rho = 0 \ , \ \lim_{r \to L} |\nabla \rho| = 0 \ ,$$

$$\lim_{r \to L} \frac{|\nabla \rho|}{\rho} = 2a \lim_{r \to L} \frac{J_{-1}(ra)}{J_0(ra)} = 2a \lim_{x \to 2.4} \frac{J_{-1}(x)}{J_0(x)} = \infty \ .$$

This means that the velocity $\lambda_2 v_2$ , now called the *spin velocity*, is tangential to the $\rho$ contour and tends to infinity at the critical point of $r = L$. In order to avoid such a non-physical scenario, we can limit the motion of the particle to be within the boundary of the critical point by insisting that $r < L$, which is reasonable since $\rho$ vanishes at the critical point (reasonable if $\rho$ retains the usual meaning in quantum mechanics).

$\lambda_2 v_2$ will be very large as $r$ approaches the critical value but this is alleviated by the fact that $\rho$ is very small in the neighbourhood of the critical point which will therefore be infrequently visited. Note also that $\lambda_2 v_2$ is zero at $r = 0$ and the magnitude of $\lambda_2 v_2$ steadily increases (exponentially like) as $r$ increases towards the critical point.

Now, we come to the point of postulating a possible mechanism for the spin of a particle. Hestenes [2] [3] suggested that 'the *zitterbewegung* [ZBW] is a local circulatory motion of the electron presumed to be the basis of the electron spin and magnetic moment'. Others have followed his idea, e.g., [4-6], and some used the term 'helical' to describe the motion. However, Hestenes concentrated on the case where the trajectory of the particle has constant curvature. This is so because he concentrated on the time-averaged angular momentum of the particle (with respect to a reference frame) which thus became a determinate case. However, this paper explicitly considers the non-determinate nature of the motion of the particle whose trajectory does not have a constant curvature. Nevertheless, this paper does suggest, as Hestenes did, that the spin of the particle is

related to the circulatory motion of the particle. In the case being considered here, the circulatory motion is given by $\lambda_2 v_2$ tangential to the $\rho$ contours on the $S$ surface. We now explore the spin of the particle in two stages.

The first stage, for the sake of instructive illustration, will consider the particle to be simultaneously present at the whole region of $r < L$. This has the flavour of the Copenhagen interpretation but the particle is simultaneously present only on a particular $S$ surface and within radius $L$. That particular $S$ surface is determined by the $z$ co-ordinate of the particle which is in turn determined by the constant deterministic velocity in the $z$ direction, $v_1$ . This can be seen in the following derivation of the possible positions of the particle. The rate of change of $S$ following a particle is

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \underset{\sim}{v} \cdot \nabla S = \frac{\partial S}{\partial t} + (v_1 + v_s) \cdot \nabla S = \frac{\partial S}{\partial t} + v_1 \cdot \nabla S = \frac{\partial S}{\partial t} + \frac{\hbar}{m}(\nabla S)^2 = K_s$$

since $v_s \cdot \nabla S = 0$ , $\frac{\partial S}{\partial t} = -E/\hbar$ , and $K_s$ is a constant. Hence, the rate of change of $S$ following a particle is constant no matter where the particle is. A particle beginning with a certain initial position with an initial $S_i$ at an initial time of $t_i$ has the freedom to surf on the $S$ surface while moving forward with constant $v_1$ . It can therefore have different trajectories. However, at any instant after the initial time, $t_i + \triangle t$, no matter what trajectory it has taken, it will have the same $S$ value. Beginning from a single position, the area covering the possible positions of the particle at a future time of $t_i + \triangle t$ expands (since $v_s$ is non-zero) with every position in the area having the same $S$ value. Given enough time (i.e., when $\triangle t$ is large enough), the area covering the possible positions of the particle expands to fill the whole area of $r < L$ on the $S$ surface. Equivalently, at $t_i + \triangle t$, the particle can be located at any neighbourhood within $r < L$ on the $S$ surface (which has the value of $S_i + K_s \triangle t$ ); and one can add that each neighbourhood has its corresponding probability of finding the particle there. This interpretation of the particle's position in this case is slightly different from the Copenhagen Interpretation where the particle can be located anywhere in the physical space. Here, the position of the particle is still non-determinate, and in that sense it is similar to the Copenhagen Interpretation, but its range of non-determinacy is confined to the $S$ surface with $r < L$. This will facilitate the calculation of the spin of the particle in this first stage of exploration, as will be seen later.

The second stage will consider the motion of the particle, which occupies an unambiguous position at any one instant, over a short period which will then essentially lead to the same result as from the first stage. This will be considered in section 6.

For this first stage, the particle is assumed to be simultaneously present at every point in the region $r < L$ with probability $\rho$ (cf. the Copenhagen interpretation), but for

$$r \geq L \, , \, \rho = 0 \, .$$

To account for this fact, we use the normalisation procedure, i.e.,

$$\int_0^L 2\pi r \rho \, dr = 1 \, .$$

The magnitude of the integrated angular momentum of the particle with respect to the $z$ axis at the centre ($r = 0$) on the $S$ surface, or the $(r, \theta)$ plane, at an instant is:

$$\int_0^L mr\lambda_2|\underline{v}_2| \, (2\pi r \rho) \, dr = -\int_0^L mr\frac{\lambda_2\hbar}{2m\rho}\frac{d\rho}{dr}(2\pi r\rho) \, dr = -\int_0^L \lambda_2\hbar\pi r^2\frac{d\rho}{dr} \, dr$$

where $\lambda_2\underline{v}_2$ can be in one of the two directions which are opposite to one another, and the contribution from $\lambda_3\underline{v}_3$ to the angular momentum is zero since it goes through the centre ($r = 0$). Now, the last integral can be evaluated by integration by parts so that it becomes

$$-\lambda_2\hbar\pi\left[\left[\rho r^2\right]_0^L - 2\int_0^L r\rho \, dr\right] \, . \tag{9}$$

With the carefully chosen $L$ such that $\rho = 0$ at $r = L$ (see figure 2), the first term is zero and the integrated angular momentum of the particle is

$$\lambda_2\hbar\int_0^L 2\pi r\rho \, dr = \lambda_2\hbar$$

where the integral on the LHS after $\lambda_2\hbar$ is the normalisation integral (see above) and is therefore 1.

One can readily see the physical significance of $\lambda_2$, i.e., it corresponds to the kind of particle being considered and is the spin number of the particle, e.g., for electron, protons and neutron, its value is 1/2, for photon it is 1 and so on.[4] Note that the explicit form of $\rho$ is not needed in the last few lines of integration; what is required is its value at the boundary points. This crucial feature will be utilised further later to produce the same spin for a particle under different situations.

We can also calculate the integrated spin energy of the particle by looking at the energy in the velocity component, $\lambda_2\underline{v}_2$, while leaving out the $\lambda_3\underline{v}_3$ component:

$$\int_0^L \frac{m}{2}\lambda_2^2|\underline{v}_2| \, (2\pi r\rho) \, dr = \frac{\pi(\lambda_2\hbar)^2}{4m}\int_0^L \frac{r}{\rho}\left(\frac{d\rho}{dr}\right)^2 \, dr \, . \tag{10}$$

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[4] Since the integrated angular momentum does not depend on the mass of the particle, the case of zero mass photon can be seen as the limiting case as mass tends to zero.

Using $\rho = A^2 J_0^2(ra)$ and $\dfrac{d}{dr} J_0(ra) = a J_{-1}(ra)$ ), the integrated spin energy of the particle becomes

$$\frac{\pi(\lambda_2 \hbar)^2}{4m} \int_0^L 4a^2 r J_{-1}^2(ra)\, dr = \frac{\pi(\lambda_2 \hbar)^2}{m} \int_0^{2.4} x J_{-1}^2(x)\, dx = 1.5586 \frac{\pi(\lambda_2 \hbar)^2}{m} .$$

Note that even though the radial limit of the spinning motion, $r = L = 2.4/a$, depends on $a$, the integrated angular momentum (spin) and the integrated spin energy of the particle are both independent of $a$. According to the definition of $a$ above, this means that the spin and the integrated spin energy do not depend on the magnitude of the translational velocity, the magnitude of the potential $U$ (as long as it does not make $a$ imaginary), or the total energy $E$. The controlling variable in both of these quantities is $\lambda_2$ which characterises the particular kind of particle in question. This may give us some important insight into why the spin of a particle is invariably constant.

### 5.2 Bessel Function of the Second Kind

We now return to the possible solution of $R = B Y_0(ra)$ , where $Y_0$ is the Bessel function of the second kind.
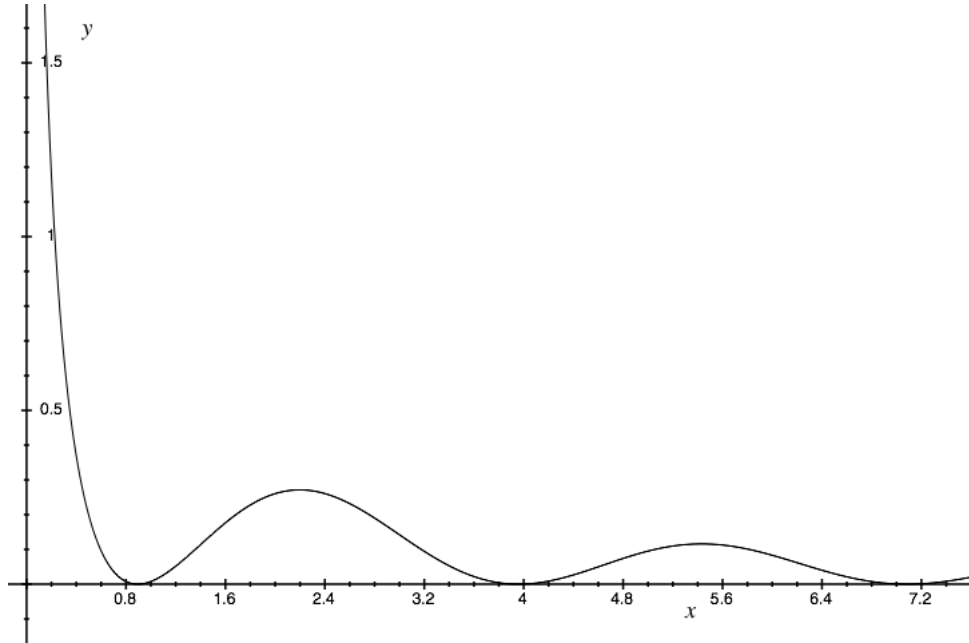


Figure 3: $y = Y_0^2(x)$

Similar to $J_0$, $Y_0$ has a root; $Y_0(0.8936) = 0$ . Similarly, we can confine the motion of the particle within the radius of $L = 0.8936/a$. Furthermore, by choosing the right constant $B$, the normalisation condition can also be satisfied:

$$\int_0^L 2\pi r \rho\, dr = \int_0^L 2\pi r B^2 Y_0^2(ra)\, dr = 1$$

despite the fact that $Y_0(0) = \infty$. This is so because the large value of $Y_0$ near $r = 0$ is multiplied by the small $r$ there; in fact $\lim_{r \to 0} rY_0^2(ra) = 0$. We can repeat the above procedure for evaluating the integrated momentum of the particle and come to the same expression,

$$-\lambda_2 \hbar \pi \left[ \left[ \rho r^2 \right]_0^L - 2 \int_0^L r\rho \, dr \right] .$$

Similarly as before, with $Y_0(La) = Y_0(0.8936) = 0$, the first term is zero and the integrated angular momentum of the particle is

$$\lambda_2 \hbar \int_0^L 2\pi r\rho \, dr = \lambda_2 \hbar$$

which is identical to the case for the solution with the Bessel function of the first kind, $J_0$. Does that mean that given the same $a$, it is possible to have two distinct dynamical regimes for the particle with identical integrated angular momentum (spin)? It turns out that the second regime corresponding to the Bessel function of the second kind, $Y_0$, is non-physical when we consider its integrated spin energy. The crucial factor for this energy is (see equation 10 above)

$$\int_0^L \frac{r}{\rho} \left( \frac{d\rho}{dr} \right)^2 dr$$

whose integrand approaches infinity as $r$ approaches zero. This integral is therefore undefined so that there is no meaningful integrated spin energy in this case which consequently is unphysical.

Appendix A considers the Helmholtz Equation of the other form:

$$\nabla^2 R - a^2 R = 0 .$$

There, using a similar argument, it is also shown that the solutions, given by the modified Bessel functions of the first and second kind, $I_0$, $K_0$, are also unphysical. We are therefore left with Bessel function of the first kind, $J_0$, as the solution to the Helmholtz Equation with real $a$.

## 6.0 Particle Spin in Steady Motion State With Uniform Translational Velocity in Free Space – Stage 2 of Exploration

The above wave-like interpretation of the particle requires the particle to be present simultaneously over all points within the circular region of motion on the $S$ surface with some probability density distribution given by $\rho$. This wave-like model gives the correct calculation for the spin of the

particle. However, we may have a conceptual difficulty in accepting that the particle is simultaneously present in the circular region on the *S* surface. Hence, in this section we consider a particle interpretation where the particle occupies an unambiguous position at any one instant within the circular region of motion, and we endeavour to show that the time-averaged angular momentum of the particle over a suitably short period is no different in form and in substance to the one derived from the wave-like interpretation given above.

Let us now consider the radial component of the particle's velocity on the *S* surface, $\lambda_3 \underset{\sim}{v}_3$ . This velocity component is perpendicular to $\lambda_2 \underset{\sim}{v}_2$ which is tangential to the $\rho$ contour. As we have seen, this radial component does not generate any angular momentum with respect to the *z*-axis going through the origin or centre. As $\lambda_2 \underset{\sim}{v}_2$ carries the particle along a $\rho$ contour, $\lambda_3 \underset{\sim}{v}_3$ carries the particle radially across the circular $\rho$ contour towards the centre or away from the centre. Let us consider the time spent by the particle in the area between two circular $\rho$ contours which have a radial distance of small $\triangle r$ between them; let this time be $\triangle t$. It is possible that the particle visits this area between these two $\rho$ contours, i.e., the circular strip, a number of times within a period of *T* and $\triangle t$ is the total time spent in the circular strip over the given number of visits. Thus, the particle spends a fraction of the period *T* in the strip, and the fraction is $\triangle t$ / *T* which is the non-dimensional time. $\rho \equiv R^2$ is usually interpreted as the density of the probability of finding a particle within the small neighbourhood of a certain point *at an instant*, or equivalently as the probability per unit volume *at an instant*. Now, so far in this paper $\rho$ has been interpreted, according to this sense with a slight variation, as the probability per unit area on the *S* surface *at an instant* (see 5.1 above). However, the Schrödinger equation in itself does not force such an interpretation; the usual meaning of $\rho$ is only an interpretation fashioned by Born [7]. We now re-interpret the meaning of $\rho$ in the following way. The 'non-dimensional time' is a fraction of the period *T* , e.g., $\triangle t$ / *T* ; it is a non-dimensional measure of the time duration with respect to the period *T*. The 'non-dimensional time' density, defined for the circular strip with area $2\pi r \triangle r$ (where *r* is the radius of the inner or outer circle of the circular strip)*,* is the *expected* non-dimensional time $\triangle t$ / *T* to be spent by the particle over the period of *T* within that strip divided by the area of that strip. We interpret $\rho$ to be this 'non-dimensional time' density for the circular strip:

$$\rho = \frac{\triangle t / T}{2\pi r \triangle r} \, .$$

(11)

Note that $\rho$ is defined with respect to expected events *over the period T* and is thus not defined with respect to an *instant* in time as in the usual interpretation of $\rho$. For a given *r*, $\rho$ tends to a limit as $\triangle r$ and $\triangle t$ tend to zero so that $\rho$ is a function of *r*. And since $\rho$ is derived from *R*, it has to be consistent with the reduced form of the Helmholtz Equation in *R*, i.e., the radial equation (8) where the only

independent variable is $r$. Thus, $\rho$ as a function of $r$ (expressed via the Bessel function of the first kind as $A^2 J_0^2(ra)$) determines the non-dimensional time to be spent by the particle in a circular strip, which is given by *$2\pi r \triangle r \rho$. This puts a clear constraint on the time to be spent in the strip.* Now, the angular momentum of the particle for the duration of $\triangle t$ spent within a circular strip of radius $r$ is $mr\lambda_2|\underset{\sim}{v_2}|$. If we multiply this with the time duration $\triangle t$, we have $mr\lambda_2|\underset{\sim}{v_2}|\triangle t$, which can be called the 'time angular momentum' of the particle for the duration of $\triangle t$.

By using the above expression (11) for $\rho$, this 'time angular momentum' can be written as $mr\lambda_2|\underset{\sim}{v_2}|(2\pi r\rho)T\triangle r$, which is a function of $r$ since $\underset{\sim}{v_2}$ and $\rho$ are functions of $r$. We then sum up all such contributions of 'time angular momentum' from every circular strip with $r$ ranging from 0 to $L = 2.4/a$ over the period of $T$. Before taking the step of this summing up, we need to consider the period $T$. We propose that over a suitable period of $T$, the particle does spend the expected amount of time at the positions between $r = 0$ and $r = L$ such that the radial equation (in the Bessel form) is satisfied, i.e., the trajectory of the particle during the period $T$ is such that it does satisfy the clear constraint on the time to be spent in each strip whose $r$ ranges between 0 and $L$. We shall later see how this constraint can indeed be satisfied. Now with the above considerations duly given, it is time to sum up the contributions of 'time angular momentum' from every circular strip with radius between 0 and $L$ for the period $T$, and do it in such a way as to take the limit of the summation as $\triangle r$ tends to zero:

$$\lim_{\triangle r \to 0} \sum_{i=1}^{N} Tmr\lambda_2|\underset{\sim}{v_2}|(2\pi r_i\rho_i)\triangle r = T\int_0^L mr\lambda_2|\underset{\sim}{v_2}|(2\pi r\rho)\,dr$$

where the disc of motion, $r < L$, is divided into $N$ circular strips of $\triangle r$, $\underset{\sim}{v_2}$ in the discrete sum (on the LHS) is a function of $r_i$, and $N$ tends to infinity as $\triangle r$ tends to 0. We can follow the same mathematical procedure as given in section 5.1 above (by using integration by parts) and express this integrated sum of 'time angular momentum' for the period $T$ as

$$-T\lambda_2\hbar\pi\left[\left[\rho r^2\right]_0^L - 2\int_0^L r\rho\,dr\right]$$

where the first term is 0 and the second term can be expressed as:

$$T\lambda_2\hbar\int_0^L 2\pi r\rho\,dr = T\lambda_2\hbar\lim_{\triangle r \to 0}\sum_{i=1}^{N}2\pi r_i\rho_i\triangle r = T\lambda_2\hbar\lim_{\triangle r \to 0}\sum_{i=1}^{N}\triangle t_i/T = T\lambda_2\hbar$$

where $\triangle t_i$ is the time spent by the particle in the *i*th circular strip and the expression in (11) has been used. If we take the time average of this integrated sum of 'time angular momentum' for the period *T* by dividing it by *T*, this is the average angular momentum of the particle over the period *T* and has the value of $\lambda_2 \hbar$ which is the same as the angular momentum derived in section 5.1 for the case where the particle is present simultaneously at one instant at all points of the disc of motion on the *S* surface. Likewise, the time averaged spin kinetic energy of the particle is $1.5586\pi(\lambda_2\hbar)^2/m$, as given in section 5.1.

Now, with the given interpretation of $\rho$ as the 'non-dimensional time' density, one can connect this interpretation with the concept of probability density. Suppose a particle visits the same circular strip with area $2\pi r \triangle r$ at three different intervals (or over three stints) within the period *T* and the intervals $\triangle t_1$ , $\triangle t_2$ , $\triangle t_3$ add up to $\triangle t$. The probability of finding the particle within this strip is either 1 at any instant of the three intervals or 0 at any instant outside the three intervals.
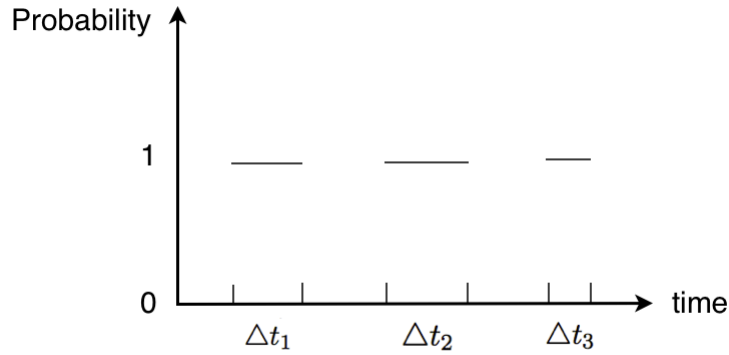


Figure 4: Probability of finding the particle in the strip

The time integrated probability of finding the particle within the strip, over the period *T*, is $\triangle t_1 + \triangle t_2 + \triangle t_3 = \triangle t$. Hence, the *time averaged probability* of finding the particle within the strip is $\triangle t / T$. This averaged probability of finding the particle there is highly reasonable since it spends a fraction, $\triangle t / T$, of the period *T* in the strip. If one checks if the particle is in the strip at an instant randomly picked within *T*, the probability of finding the particle there will be $\triangle t / T$. Also, if one does a high frequency sampling at a small regular time interval within *T*, the probability of finding the particle there will also be $\triangle t / T$. Now this $\triangle t / T$, interpreted as the time averaged probability (over the period *T*) of finding the particle in the strip can also be written as follows (using expression (11) above):

$$\frac{\triangle t}{T} = 2\pi r \rho \triangle r.$$

For a given $r$, $\rho$ tends to a limit as $\triangle r$ and $\triangle t$ tend to zero. From this expression and by interpreting $\triangle t / T$ as the time averaged probability over the period $T$, we can see that $\rho$ can also be interpreted as the time averaged probability density over the period $T$ for the circular strip. Therefore, we have two equivalent interpretations of $\rho$:

1.  the 'non-dimensional time' density for the period $T$ and
2.  the time averaged probability density for the period $T$.


If we compare the second interpretation, the time averaged probability density *over a period*, with the usual interpretation, i.e., probability density *at an instant* (as in section 5.1), then we can see the similarity between the two in their probability density, and the difference between the two in relation to their time duration: the latter relates to an instant while the former relates to a period. We can understand the treatment and derivation of the spin of a particle at one single instant in section 5.1 as the result of compressing the history of the particle over a period of $T$ into a single instant in time. That is, if we suppose that the treatment and derivation in the current section involving the history of the particle over a period of $T$ is reality (or something close to reality with the particle having unambiguous position at any instant), then the treatment in section 5.1 is a convenient shorthand summary of this history into an instant, by using the time averaged probability density over the period $T$ as the probability density for a hypothetical compressed instant. In other words, $\rho$ is taken as the probability density for that 'compressed moment' where the particle is present at different positions at the same time. Even though this shorthand treatment in section 5.1, as this paper suggests, may not be the reality, it nevertheless captures the essence of the dynamics of the particle in its history over the period $T$. And if the period $T$ is very short, the dynamics of the 'compressed moment' is a good representation of the history of the particle over the short period $T$. In fact, as $T$ tends to the limiting case of zero, the dynamics of the system tends to the the dynamics of the 'compressed moment' which is thus the limiting case.

In the treatment in section 5.1, the particle is required to have different radial distances, different $r$'s, at the same time, but this is not necessary in the present treatment as the particle has an unambiguous radial distance at any one point in time. Likewise, in section 5.1 as the particle is simultaneously present over the whole disc of motion, in one instant it covers the full range of the value for the azimuthal co-ordinate, $\theta$. Here, the particle has one value for $\theta$ at any one instant. For the motion within a circular strip, the range of $\theta$ covered by the particle within the time of $2\pi r\rho\triangle r$ depends on the magnitude of $\lambda_2 v_2$. If the magnitude is large, the full range of 360 degrees (or more) can be covered within the radial distance of $\triangle r$ as the particle moves in the radial direction.

The discussion so far touches on the fundamental ontology of particles: are they waves or simply particles? The question of the duality of wave and particle in Quantum Mechanics is a perplexing one. The discussion in this paper suggests that a particle goes through cycles of motion (both in the radial and azimuthal directions) according to some periodic time and in that sense it

behaves *functionally* like a wave. On the other hand, the proposition that the particle occupies a definite unambiguous position at any one moment in time suggests that as an entity, *ontologically* it is a particle. Hence, this paper suggests that the *functional* description of the *motion* of a particle can be captured by some kind of *continuous* wave while its *ontology* is given by the *discrete* property of a particle.

## 7.0    How Can the Radial Equation be Satisfied by the Motion of the Discrete Particle?

For our purpose of our analysis here, the most useful interpretation of $\rho$ is the 'non-dimensional time' density, as a function of $r$, for the period $T$. The time spent by a particle within a circular strip of radius $r$ is given by $2\pi r \triangle r \rho T$, and this time can be the summation of the time intervals for the many visits made by the particle to that strip. It has been assumed in section 6 that over a suitable period of $T$, the particle does spend the expected amount of time at the positions between $r = 0$ and $r = L$ such that the radial equation (in the Bessel form) is satisfied. We have established that $\rho$ has to be expressed via the Bessel function of the first kind as $A^2 J_0^2(ra)$. Before we demonstrate how this expression can be satisfied (or approximately satisfied), it is necessary to have a more concrete grasp of the value of $A$, and $a$ which is the parameter in the Helmholtz Equation. We recall that the time-averaged angular momentum (spin) and the time-averaged spin energy of the particle are both independent of $a$. It seems that $a$ is a free parameter which has to be determined from experiment. Suppose $a$ has some specific value, it then specifies $J_0^2(ra)$ in the expression for $\rho$ and $L=2.4/a$ but the constant $A$ in $\rho$ is yet to be specified. However, the normalisation constraint will yield the value of $A$:

$$\int_0^L 2\pi r A^2 J_0^2(ra)\, dr = 1 \,.$$

We can see that as $a$ increases, $L$ becomes smaller, i.e., the area of the spin disc becomes smaller or more compact while at the same time $A$ will have to increase to maintain the normalisation constraint. Conversely, as $a$ decreases, the area of the spin disc spreads out and $A$ has to decrease. The curve for $r\rho$ is plotted as a function of $r$ below in Figure 4 with $a=1$, $L=2.4$, $A=0.4519$. This case for $a=1$ will be used in this section for illustrative purpose. Despite its specificity, this illustration can be very general in nature because $L/2.4=1/a$ is a measure of length, and if we define the unit length to be the length of $L/2.4$, then with that length convention $a=1$. That is, if we adjust the definition of the unit length according to what $L$ is in reality, then we can guarantee that 1 is correct value for $a$.
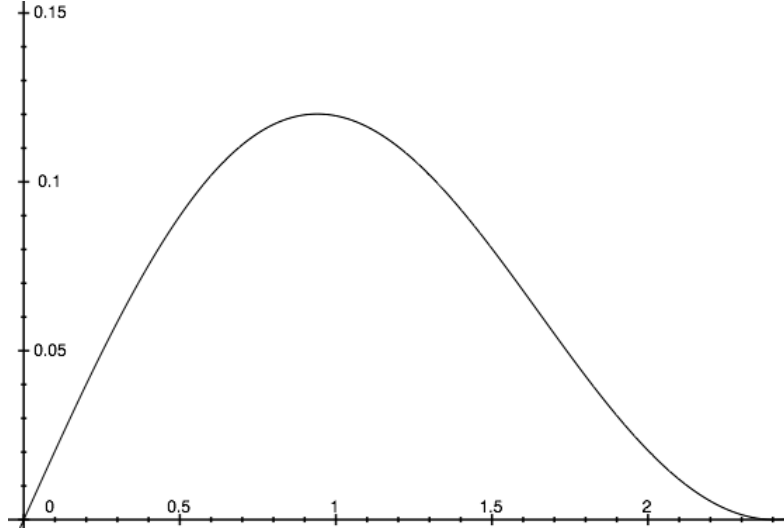
Figure 4: y(r) = $r\rho$ with $a$=1, $L$=2.4, $A$=0.4519

We will now make the first (but not final) attempt to satisfy the radial equation by using (11) to write:

$$\frac{\triangle r}{\triangle t} = \frac{1}{2\pi T r \rho}.$$ (12)

In this first attempt, we suppose that the particle visits each strip only *once* within the given period *T*. In that case, we can interpret $\triangle r/\triangle t$ to be the average speed of the particle in the radial direction, and this average radial speed is inversely proportional to $r\rho$. And if we take the limit of $\triangle r/\triangle t$ as $\triangle$r and $\triangle$t tends to 0, then since $\rho$ is a function of *r* we have a unique radial speed profile for the particle as a function of *r;* and one cycle involves the particle travelling from *r*=0 to *r*=*L* over the period *T*. That is, given the constants *T* and *a*, it seems that we have managed to fix the radial speed of the particle from *r*=0 to *r*=*L*. Remember we have already fixed the magnitude and direction of the velocity component, $\lambda_2 v_2$ (the spin velocity), on the *S* surface whose magnitude is a function of $\rho$ and thus of *r*. Now, if we also fix the radial speed as a function of *r* as suggested above, we have effectively fixed $\lambda_3 v_3$ (to be called the radial velocity) on the *S* surface whose direction is perpendicular to the spin velocity and whose magnitude is a function of *r*. This means that the velocity of the particle on the *S* surface, $\lambda_2 v_2 + \lambda_3 v_3$ , is effectively determined over the whole disc of spinning motion. Now, we have already fixed $v_1$ so that we have effectively determined $v = v_1 + \lambda_2 v_2 + \lambda_3 v_3$. That implies that the total velocity of the particle is deterministic. Have we then reduced quantum mechanics to deterministic classical mechanics? But is it the case?

Apart from the prescribed constant *T* in the system which can conceivably be made to vary over time and become a source of indeterminacy, we have the following grave boundary conditions at *r*=*L* and *r*=0, where *r$\rho$*=0 either due to *r*=0 or $\rho$=0 at *r*=*L*, which are the two singularity points and thus violate the rather simplistic derivation of the deterministic $\lambda_3 v_3$ and the deterministic total

velocity in the last paragraph. According to the expression in (12), the radial speed has to be infinite at the two boundary points of $r=L$ and $r=0$; hence they are also called singularity points. Furthermore, if we envisage the motion of the particle over two cycles with the first cycle beginning when the particle is at $r=0$, not only will the particle possess infinite speed at that point, the particle will slow down away from $r=0$ (according to the increasing $r\rho$), then accelerate towards $r=L$ (according to the decreasing $r\rho$) reaching infinite speed at $r=L$ *and instantaneously* begin the second cycle by reversing the direction of its radial velocity but maintaining its infinite magnitude! Surely, this scenario of the particle having infinite radial speed and *instantaneously* reversing its direction but maintaining infinite radial speed at $r=L$ is not a scenario that we wish to accept as physical.[5] Because of this, we have reasons to believe that non-deterministic quantum mechanics cannot be reduced to deterministic classical mechanics by adopting the above deterministic expression of the radial speed of a particle. But how can the radial equation be satisfied or approximated? In particular, how can the two singularity points be dealt with?

By writing

$$\frac{\triangle t}{\triangle r} = 2\pi T r \rho \,, \tag{13}$$

intuitively one can see from this expression and Figure 4 that the particle needs to spend more time (cumulatively) at a strip with higher $r\rho$ and less time at a strip with lower $r\rho$, and the time spent at a strip can be built up over a number of visits. For example, where $r\rho$ is at the maximum (at $r=0.9364$), the particle will spend the largest amount of time at a strip around that point but this can be built up through multiple visits (or possibly built up through the maximum number of visits amongst the strips) within the period $T$. Conversely, where $r\rho$ is small, the particle will spend less time there and the time can be built up with fewer number of visits within the period $T$. Regarding the neighbourhoods of the extreme boundary points, we envisage that the particle only visits them once within the period $T$ of one cycle. One can visualise that, within the period $T$, the particles moves in a series of oscillations around the point of maximum $r\rho$ with variable amplitude. In this way, it is conceivable that the $r\rho$ curve in Figure 4 can be approximated. The following diagram illustrates one possible trajectory within one cycle of period $T$ although there can be many variations from this trajectory, as we shall see.

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[5] The situation at $r=0$ is also grave with infinite speed but at least the infinite radial velocity there does not have to reverse its direction as it goes through $r=0$ as is necessary at $r=L$.
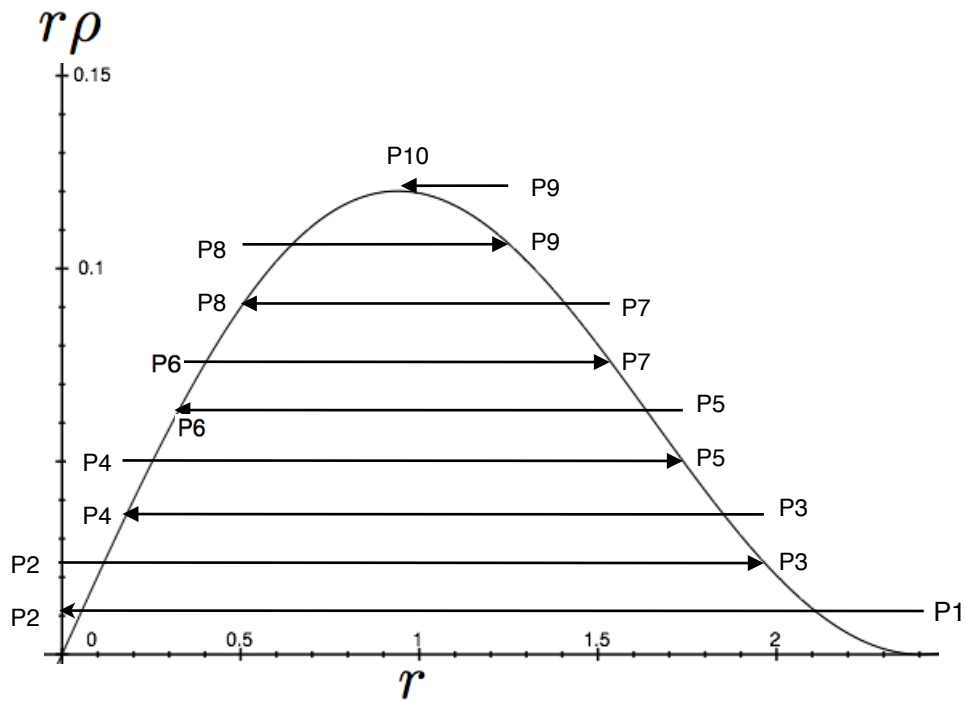
Figure 5: Possible radial trajectory, from P1 to P10, of a particle within the period $T$

The particle begins the motion for period $T$ from position P1(one of the two boundary points) with zero radial speed, having just completed the motion in the previous cycle, at the end of which the particle moved radially outward towards P1 and stopped there. At the beginning of the new cycle, it moves radially inward towards P2 (another boundary point) and stops at P2 before moving outward to P3, although strictly speaking it does not have to stop radially at P2 as it can continue through P2 to reach a negative $r$ value which is allowed by the motion (and the radial trajectory in Figure 5 will also apply in an equivalent manner). In either case, from P1 the particle with its spin velocity and inward radial velocity spirals towards the centre; then it spirals out from the centre. As the particle reaches P3, it stops radially and begins to move inward towards P4 and stop there radially before moving towards P5, and so on, until it reaches P10 at the end of the cycle. From P10, a new cycle begins (trajectory not drawn) but it will eventually reach P1 at the end of that cycle (or possibly at an earlier point of the cycle as long as P1 is covered at some stage).

Now, we seek to find an alternative trajectory which can equally be as valid as the previous one described. For example, it can follow the same radial trajectory as the previous one until a point – as it moves from P4 to P5 it stops at a point between these two points and that point is P7. From P7, the alternative trajectory deviates from the previous one and the particle moves radially inward to P6, stops there, moves radially outward, and instead of stopping at P7 it carries on beyond P7 and stops at P5 which was missed in its last radially outward movement from P4. From P5, it moves to P8, P9 and then to P10. This alternative trajectory covers the same distance and positions as the previous one described, albeit in a different order. One can see that *many other alternative trajectories* can be constructed in many different ways. But can these trajectories have the

appropriate speed profile that will satisfy the requirement in (13) which is necessary for constant time-averaged angular momentum over period $T$? Because the magnitude of the radial velocity, $\lambda_3 v_3$, strictly speaking is non-determinate, the radial speed of the particle within one cycle of period $T$ can be adjusted accordingly to yield the desirable $r\rho$ profile except in the neighbourhoods of the two boundary singularity points which pose a greater challenge. We now investigate further what happens at the two boundary singularity points.

Since at $r=L$ the particle has to stop in order to reverse the direction of its radial velocity, the time spent in the neighbourhood of $r=L$ at a strip with $\triangle r$ cannot be represented by $(2\pi T\triangle r)r\rho$; had it been represented by this expression involving the factor of $r\rho$, we will have the singularity point at $r=L$ (since $\rho(L)=0$ ) with reversing infinite radial velocity there. Let the actual time spent around $r=L$ at a strip with $\triangle r$ be represented by $(2\pi T\triangle r)r\rho'$ where $\rho'(r)$, the actual non-dimensional time density, is different from $\rho(r)$, the idealised non-dimensional time density derived from the radial equation. We envisage the following variation of $\rho'$ from $\rho$ near $r=L$. We call the neighbourhood near $r=L$ the outer tail end. (A similar treatment can be given for the case of the neighbourhood near $r=0$, the inner tail end).

In Figure 5, the positions between P1 and P3 are only visited once by the particle in one cycle. This corresponds to the least amount of time to be spent at the outer tail end. For brevity, P3 is called $B$. $B$ is used to reference both this point and its distance from $r=0$, and this naming convention, already used for $L$, will be used for other points as well in the following. The outer tail end is defined as the neighbourhood, $B < r < L$. We envisage that $B$ is close to $L$ and that before $B$, $\rho' = \rho$; and for $r > B$, $\rho' \neq \rho$. This means that away from the outer tail end (and away from the inner tail end at $r=0$), $r\rho = r\rho'$. Within the outer tail end where $r > B$, $r\rho \neq r\rho'$ but we maintain that the spin velocity is still a function of $\rho$, not $\rho'$. We also introduce $L'$ which is just slightly less than (or before) $L$ for the following reason. Remember that at $r=L$ the spin velocity also goes to infinity with $\rho=0$. If $\rho'(L) \neq 0$, then we will incur a very large, possibly infinite, angular momentum contribution from the neighbourhood of $r=L$. To prevent this happening, we have to insist that $\rho'(L) = 0$, i.e., the particle does not actually visit the point $L$ at all as it stops at the point $L'$ before $L$. $L'$ is thus the furtherest point from $r=0$ which the particle does visit so that $\rho'(r>L')=0$. $L'$ is envisaged to be close to $L$. The points configuration at the outer tail end is given in Figure 6 below.
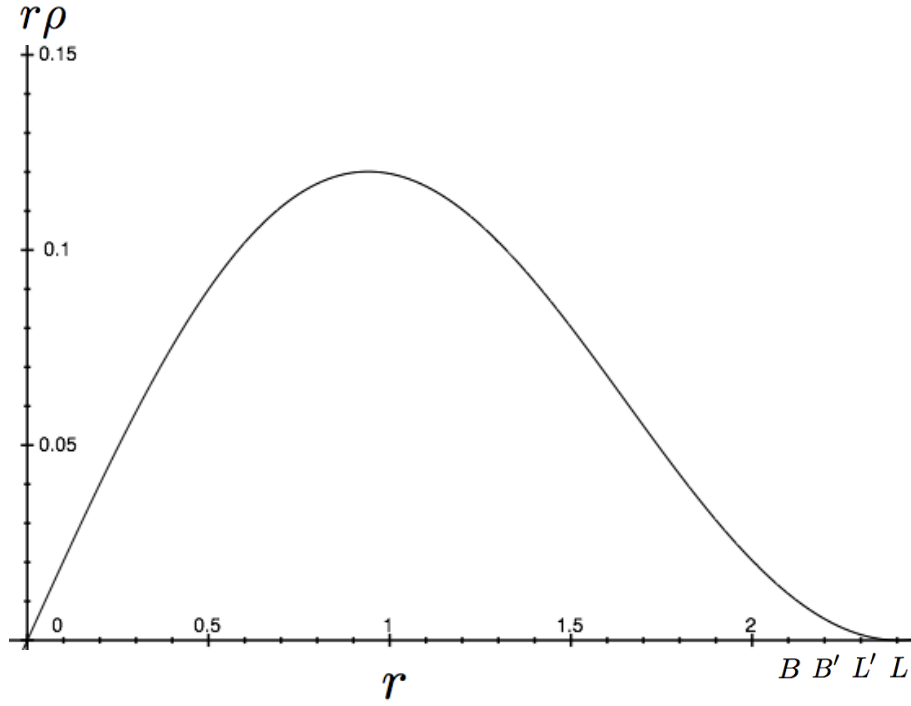
Figure 6: The Outer Tail End ($r>B$) with Positions of $B, B', L', L$

We now configure the new $\rho'$ for $B < r < L'$ such that $\rho'$ at the outer tail end satisfies the following three constraints:

1. its continuity with $\rho$ at $B$,
2. the normalisation constraint,
3. the angular momentum constraint.

Since the particle only visits the outer tail end once in a cycle, the $\triangle t$ in $\dfrac{\triangle r}{\triangle t} = \dfrac{1}{2\pi T r \rho'}$ is accumulated through only one visit. Taking the limit of $\triangle r/\triangle t$ as $\triangle r$ and $\triangle t$ tends to 0, we have a radial speed profile for the particle at the outer tail end:

$$\frac{dr}{dt} = \frac{1}{2\pi T r \rho'} \tag{14}$$

which is a function of $r$ since $\rho'$ is a function of $r$. We now need to find a suitable radial speed profile at the outer tail end to yield $\rho'$ such that $\rho'$ satisfies the three constraints given above. Since the Schrödinger equation puts no constraint on the radial speed, in principle we have great freedom (in fact unlimited degrees of freedom) in choosing a suitable radial speed profile at the outer tail end and our attempt to find such a profile satisfying all the three constraints should be successful. Here, we illustrate this possibility and success by using one example which is one amongst many possibilities. Hence, this example should not be taken as definitive.

In this example, the particle begins with zero radial speed at $L$, as in other acceptable possibilities. We model the radial speed profile within the outer tail end by using two parameters

21

while in other more elaborate possibilities many more parameters can be used. The radial speed is given by the equation of a simple harmonic motion where the acceleration is given by

$$\frac{d^2r}{dt^2} = -\omega^2(r - B') .$$

The two parameters are therefore $\omega$ and $B'$ where $B'$ is between $B$ and $L'$. Remember that this equation applies only in the outer tail end; after the particle reaches B from $L'$, it will moves towards the centre with speed not determined by this equation (hence there is no complete cycle of the simple harmonic motion in the usual sense). After we apply the two boundary conditions at $t = 0$, i.e., $r=L'$ and $dr/dt = 0$, we have

$$r = (L' - B')\cos \omega t + B' \tag{15}$$
$$\frac{dr}{dt} = -\omega(L' - B')\sin \omega t .$$

(The radial speed begins with zero at $r=L'$ and gains its maximum at $r=B'$.) Using these two relations and the expression for $dr/dt$ in (14) above, $\rho'$ can be expressed as a function of $r$, with parameters $B'$, $L'$ and $\omega$:

$$\rho' = \frac{1}{2\pi rT\omega(L' - B')\sin \omega t} = \frac{\left(1 - \left(\frac{r-B'}{L'-B'}\right)^2\right)^{-1/2}}{2\pi rT\omega(L' - B')} .$$

Or, one can write $\rho'= \rho'(r, B', L', \omega)$. We now consider the three constraints.

Firstly, given a value of $B$, since $\rho(B)$ is known (as we have set $a=1$), the constraint of the continuity between with $\rho$ and $\rho'$ at $B$ gives

$$\rho(B) = \rho'(B, B', L', \omega).$$

This gives a relationship between $B$, $B'$, $L'$ and $\omega$. Strictly speaking, this constraint is not indispensable as an actual trajectory is possible with $\rho(B) \neq \rho'(B, B', L', \omega)$ as long as the normalisation constraint and the angular momentum constraint are maintained.

Secondly, regarding the normalisation constraint, this requires

$$\int_B^L 2\pi r\rho \, dr = \int_B^{L'} 2\pi r\rho' \, dr .$$

Since $\rho$ is known, the integral on the LHS is known and is a function of $B$. Let its value be $T_t/T$ where $T_t$ (subscript $t$ for tail) is the time to be spent by the particle between $B$ and $L$ in the idealised case. $T_t$ is a function of $B$. The RHS corresponds to the same length of time but that time is spent between $L'$ and $B$ in an actual case. This can be expressed, evaluating (15) at $r=B$, as

$$\frac{B - B'}{L' - B'} = \cos \omega T_t \, .$$

Since $T_t$ is a function of $B$, this is the second relationship between $B$, $B'$, $L'$ and $\omega$.

Thirdly, regarding the angular momentum constraint, we recall that the 'time angular momentum' for the period $T$ is

$$-T\lambda_2 \hbar \pi \left[ \left[ \rho r^2 \right]_0^L - 2 \int_0^L r\rho \, dr \right]$$

for the idealised case. Since away from the two tail ends $\rho' = \rho$, for the outer tail end in the present consideration, the same angular momentum will be contributed in the actual case as in the idealised case if

$$-T\lambda_2 \hbar \pi \left[ \left[ \rho r^2 \right]_B^L - 2 \int_B^L r\rho \, dr \right] = -T \int_B^{L'} r \frac{\lambda_2 \hbar}{2\rho} \frac{d\rho}{dr} (2\pi r \rho') \, dr$$

where $(2\pi r \rho')dr$ represents the infinitesimal non-dimensional time actually spent in the infinitesimal strip represented by $dr$. Since the LHS and the RHS depend on $B$, and on the RHS $\rho' = \rho'(r, B', L', \omega)$, this angular momentum constraint yields the third relationship between $B$, $B'$, $L'$ and $\omega$.

Collecting the three constraints together, we have three relationships between the four parameters, $B$, $B'$, $L'$ and $\omega$. We are therefore free to choose one parameter, e.g., $B$, such that all the three constraints can be satisfied simultaneously. If we do away with the first constraint which is not strictly necessary, then we have two free parameters and we may choose, for example, $B$ and $L'$, such that they are very close to $L$, thus making the actual $r\rho'$ almost identical to the idealised $r\rho$.[6] In this case, the probability of finding the actual particle at a certain strip is virtually given by the idealised $2\pi r\rho \triangle r$. If we use more parameters for the radial speed at the outer tail end than the present two parameters of $B'$ and $\omega$, more elaborate radial speed profiles can be constructed while satisfying the constraints for normalisation and angular momentum.

The procedure for the outer tail end as described above can be similarly carried out for the inner tail end, with the difference that the radial speed at $r=0$ does not need to be zero.

## 8.0    Determinism or Non-determinism?

If one insists on determinism, one could argue that parameters for the outer tail end such as $B$, $B'$, $L'$ and $\omega$ can be fixed (and similarly those parameters for the inner tail end), and that the trajectory

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[6] In the limiting case, $B$ and $L'$ will coincide with $L$. This will be the case where the particle reaches $L$ with infinite speed and instantaneously reverses with infinite speed, which is not a physically acceptable scenario.

such as given in Figure 5 can also be fixed for each cycle (including fixed P1 to P10) so that the motion of the particle in each cycle is fixed and therefore deterministic. In fact, for the region away from the two tail ends, one can simplify the trajectory by having the particle travelling from one tail end directly to the other tail end without any interim stop and reversal of the direction of radial motion so that the cycle is covered by one sweep of radial motion, as opposed to multiple sweeps as given in Figure 5. In that case, the radial speed away from the tail ends are simply, explicitly and deterministically given by

$$\frac{dr}{dt} = \frac{1}{2\pi T r \rho} \, .$$

It seems that if one is prepared to sacrifice everything for determinism, it is possible conceptually. Suarez [8], when considering nonlocal 'realistic' Leggett models wrote:

> The before-before experiment demonstrates that Nature works out the quantum correlations in a nonlocal and non-deterministic way. This means that the measurement outcomes (for instance A = +1 and B = −1, in the experiment ...) imply a true choice on the part of Nature, and are not determined by pre-existing properties the particles carry independent of the act of measurement. It is important to stress that to draw [...] these consequences one tacitly assumes the freedom of the experimenter. If one rejects this freedom one can explain the nonlocal correlations in a fully deterministic and local relativistic way by pictures like "Super-determinism" or "Many Worlds" ... . One can speak with Anton Zeilinger's about "two freedoms": The freedom of the experimenter implies the freedom on the part of Nature.

Super-determinism is always an option if one wishes to insist on determinism but that will involve the denial of one's free will. This is echoed by John Bell [9] but also in a disapproving manner.

If there is no other option than the kind of determinism suggested above, then one has to accept it, however grudgingly. But, as we have seen above, there are unlimited number of options for the trajectory so that non-determinism is possible. The points, P3 to P10, can be chosen in an unlimited number of ways to create an infinite number of possible trajectories. These infinite number of possible trajectories can be accommodated because of the non-determinate nature of the radial speed. Also, *it is not strictly necessary to assume that parameters for the tail ends such as B, B′, L′ and ω (or even T) are fixed for every cycle*; hence, these parameters can also be a source of non-determinacy. Since *there is great capacity for non-determinacy of the particle's motion even if one insists that the time-averaged angular momentum over period T has to be determinate with a certain value (* $\lambda_2 \hbar$ *)*, one is able to state that non-determinism is genuinely possible and one can believe that non-determinism is the most sensible option compared to super-determinism.

If one believes in non-determinism, then one wonders whether it is possible to use some kind of stochastic process to simulate the particle's trajectory such that the radial equation is satisfied for a certain period, *T*. However, if there is one stochastic process which can produce the correct simulation, one wonders if there may be other stochastic processes which can also produce the desirable simulation. If there are, then we do not know which particular stochastic process is closest to reality. And even if we can identify the process with the best simulation performance, one does

not really know if the process represents what actually happens with the particle in reality. Here, we may be very close to the limit of what we can know, if we are not already at the limit.

### 9.0   Quantum Kinetic Energy, Quantum Potential Energy and Dark Energy

In section 2, the energy equation is written as

$$\hbar\frac{\partial S}{\partial t} + \frac{1}{2}m|v_1|^2 + \frac{1}{2}m|v_s|^2 - \left[\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + \frac{1}{2}m|v_s|^2\right] + U = 0$$

and the Total Quantum Energy (*TQE*) is defined as

$$QKE + QPE = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$$

where Quantum Kinetic Energy (*QKE*) is $\frac{1}{2}m|v_s|^2$, with $v_s \equiv \lambda_2 v_2 + \lambda_3 v_3$ which is the velocity on the *S* surface consisting of the spin velocity and the radial velocity. Quantum Potential Energy (*QPE*) is defined by the above relation. In the case being studied here, i.e., particle travelling in free space with constant translational velocity, the RHS of the last equation is equal to a positive constant (see eqn. (8)) so that *TQE* is positive and constant over space and time. Since *QKE* can vary over a great range, e.g., with $\lambda_2 v_2$ very large near *r=L*, it is conceivable that in some positions *QPE* could be very small and even negative in order to maintain the above budget for the quantum kinetic and quantum potential energies.

Some justification is necessary for splitting the Total Quantum Energy into Quantum Kinetic Energy and Quantum Potential Energy in the above manner. We recall that if the particle is in Steady Motion State (*SMS*), i.e., if $\nabla S$ (and therefore $v_1$) is constant respect to time (but is allowed to vary over space if that is the case), then the total energy of the particle,

$$E \equiv \frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + U \tag{16}$$

is invariant over space and time, i.e., whatever time and wherever the particle happens to be at, it has the same total energy. Since $\nabla S$ at any point does not change in time and is therefore determinate at any time, if $U$ at any point is also determinate at any time, then $-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$ at any point is also determinate at any time.[7] But since the motion on the *S* surface is non-determinate, the kinetic energy contribution from that motion on the *S* surface is also non-determinate. How can this non-determinate kinetic energy be represented in the above expression for the total energy of the

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[7] In the case being studied here, i.e., free particle with constant translational velocity ($v_1$) and constant classical potential *U*, these determinate conditions are satisfied.

particle, given the fact that each of the three three terms is determinate? The non-determinate kinetic energy can be linked to one determinate term such that the determinate term is seen as a sum of the non-determinate kinetic energy and a corresponding (and complementary) non-determinate term. But which of the three terms should be linked to the non-determinate kinetic energy? Since $\hbar$ is associated with non-determinate quantum mechanics, i.e., setting it to zero reduces the non-determinate kinetic energy on the $S$ surface to zero (see (16)) and reduces the system to the classical one (with the total energy involving $\nabla S$ and $U$ only), it is highly reasonable to link the non-determinate quantum kinetic energy to the second term (with the factor $\hbar$ ) which also becomes zero when $\hbar$ is set to zero. Hence, the second term, the Total Quantum Energy which is determinate, is expressed as a sum of the the non-determinate Quantum Kinetic Energy (contribution from the motion on the $S$ surface) and non-determinate Quantum Potential Energy.

The question of dark energy and the associated cosmological constant is perplexing; however, it has been connected with the 'quantum potential' which is called the Total Quantum Energy (*TQE*) in this paper [10-12]. From the perspective of this paper, one can see that the motion of the particle on the $S$ surface gives rise to its spin which involves much Quantum Kinetic Energy (*QKE*). The sum of *QKE* and *QPE*, understood as the Total Quantum Energy (*TQE*) as explained above, gives us a better physical grasp of the total energy of the particle and could lead to some useful insight into the nature of dark energy. Without considering the *TQE* of a particle, this contribution to the total energy of the particle is overlooked or becomes hidden. If we can take into consideration the spin related *QKE* and the *QPE* of all particles in the universe, it may help to account for the mysterious dark energy. Further studies involving relativistic extension of the present study could be useful for understanding more about this possibility. However, it is highly probable that the main features of the spin motion and the energy therein for the relativistic case have already been captured by the non-relativistic case studied in this paper.


## 10.0 The Electron in a Hydrogen Atom

The case studied above is for a free particle travelling with constant translational velocity while its velocity on the $S$ surface, being perpendicular to the translational velocity, is non-determinate and hence unpredictable. By prescribing the spin velocity ($\lambda_2 v_2$ ) on the $S$ surface, the radial velocity, though non-determinate, is subject to an overall constraint – i.e., over a period $T$ it has to satisfy the radial equation – if $\rho$ is to have the meaning of time averaged probability density over the period and if the time averaged angular momentum is to be maintained constant over each cycle. Based on the understanding of the dynamics of this relatively uncomplicated case, we now consider the possible trajectories of the electron in a hydrogen atom where the translational velocity is no longer constant but varies in space. Using spherical co-ordinates and setting the potential $U$ to be inversely

proportional to the distance to the centre, the solutions for the wave function of a hydrogen electron can be found in text books and a few of these solutions will be considered here to illustrate the possible trajectories. Before we examine these solutions and their possible trajectories, we briefly consider how one can use the energy equation (1) to arrive at the same solutions as those produced by the usual approach (as found in text books) which uses the time-independent Schrödinger equation as the energy equation.

## 10.1 The Energy of the Hydrogen Electron

In the above analysis of a particle travelling in free space, we have already used the energy equation to solve for $R$; the case of the electron in the hydrogen atom is more complicated because of the presence of varying potential in space.

Again, taking the gradient of the energy equation (1), treating $\nabla S$ (and hence $v_1$) to be constant with respect to time only (hence Steady Motion State), the following equation in spherical coordinates, $(r, \theta, \phi)$, expresses the constant total energy of the electron in space and time:

$$\frac{\hbar^2}{2m_e}(\nabla S)^2 - \frac{\hbar^2}{2m_e}\frac{\nabla^2 R}{R} - \frac{e^2}{r} = E \tag{17}$$

where the second term is the Total Quantum Energy and the third term is the potential, $E$ is the total energy of the electron, and $m_e$ is the mass of the electron.

As before (see (5)), $S = -\frac{E}{\hbar}t + b(r)$, where $r$ is a position vector. If we set $b(r) = m\phi$ where $m$ is an integer, then $\frac{m_e}{\hbar}|v_1| = |\nabla S| = \frac{m}{r\sin\theta}$ which means that the orbital angular momentum of the electron with respect to the $\theta=0$ axis (normally vertical) is constant and is $m\hbar$ (see more about this later in section 10.2).[8] With

$$\psi = R\,e^{im\phi}\,e^{-\frac{iE}{\hbar}t}$$

we express $R = F(r)f(\theta)$ and substitute $R$ into the energy equation (17). Using the above expression for $|\nabla S|$ and rearranging the reduced Planck constant and mass, the energy equation becomes

$$\frac{\nabla^2 R}{R} - \frac{m^2}{r^2\sin^2\theta} + \frac{2m_e}{\hbar^2}\frac{e^2}{r} = -\frac{2m_e}{\hbar^2}E\,. \tag{18}$$

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[8] Earlier in the paper, *m* is used as the mass of a particle. Here, *m* is the magnetic quantum number, not the mass of the electron.

Note that there is the explicit inclusion of the Total Quantum Energy which is hidden in the usual approach. Furthermore, the second term involves the modulus square of $\nabla S$ which is interpreted in this paper as $\frac{m_e}{\hbar} v_1$, i.e., in this energy equation we have explicitly included the kinetic energy of the translational component of the velocity of the electron, but this translational component is also hidden in the usual text book approach where the time-independent Schrödinger equation is used as the energy equation. The other two terms are the potential 'experienced' by the electron and the total energy of the particle which are also found in the usual text book approach. Substituting $R = F(r)f(\theta)$ into the above energy equation (where $F$ and $f$ are yet to be solved) separates the variables and yields two equations, one in $\theta$ and one in $r$ (the radial equation). $f(\theta)\,e^{im\phi}$ is the spherical harmonics with integer parameters of $m$ and $l$, and writing $G = rF(r)$ the radial equation can be expressed as

$$\frac{\partial^2 G}{\partial r^2} + \left[\frac{2m_e}{\hbar^2}\left(\frac{e^2}{r} + E\right) - \frac{l(l+1)}{r^2}\right]G = 0 \,. \tag{19}$$

Let us briefly compare the present approach with the usual text book approach.

In the usual text book approach, one begins with the Schrödinger equation, separates the spatial and time dependence by writing $\psi = \psi_p\, e^{-\frac{iE_p}{\hbar}t}$, where $\psi_p$ depends on the spatial position and $E_p$ is the constant given by the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m_e}\nabla^2\psi_p - \frac{e^2}{r}\psi_p = E_p\psi_p \,. \tag{20}$$

$E_p$ in this equation is the total energy of the electron. Whether this $E_p$ is identical to $E$ in (18) remains to be seen. One separates the variables in this equation by writing $\psi_p = F_p(r)\, f_p(\theta)\, e^{im\phi}$ and substitutes this expression into the time-independent Schrödinger equation. We know that $f_p(\theta)\, e^{im\phi}$ is the spherical harmonics with integer parameters of $m$ and $l$, and writing $G_p = rF_p(r)$ the radial equation can be expressed as

$$\frac{\partial^2 G_p}{\partial r^2} + \left[\frac{2m_e}{\hbar^2}\left(\frac{e^2}{r} + E_p\right) - \frac{l(l+1)}{r^2}\right]G_p = 0 \,. \tag{21}$$

The spherical harmonics of the two approaches, $f(\theta)\, e^{im\phi}$ and $f_p(\theta)\, e^{im\phi}$, can easily be shown to be identical. Also, the two radial equations of the two approaches, (19) and (21), are exactly of the same form so that the allowable quantised energies for the usual approach, $E_p$, are identical to those for the approach in this paper, $E$. Hence, the wave function solutions for these two approaches are identical, and the expression for $E$ in (18) expresses faithfully the total energy of the electron and

gives a more detailed breakdown of the constituents of the total energy. For example, the translational component of the velocity of the electron, $v_1$, which is determinate and unchanging in time while varying in space, appears explicitly in this energy equation while the usual text book approach (often following the Copenhagen interpretation of quantum mechanics) does not produce such a certainty for any velocity. Also, the Total Quantum Energy is given explicitly in (18) while it is implicit in the usual approach (see (20)). For these reasons, the present approach gives us a closer and more detailed feel to the dynamics of the electron.

Just in case the reader thinks the approach taken here is no different from the pilot wave approach, we must point out the fundamental difference between the two. While both approaches have the same time-independent determinate translational velocity, $v_1$ (which is proportional to $\nabla S$ and therefore in a direction perpendicular to the $S$ surface which is also a $\phi$ surface in this case), in the present approach the particle has the additional non-determinate surfing motion on the $\phi$ surface while the pilot wave approach leaves out this vital non-determinate component of the total velocity. The pilot wave approach moves one step beyond classical mechanics by allowing the Total Quantum Energy (usually called quantum potential) to influence the velocity component, $v_1$; however, it does not include the non-determinate surfing motion on the $\phi$ (or $S$) surface represented by the Quantum Kinetic Energy which is accommodated in the Total Quantum Energy. *In the interpretation of quantum mechanics presented in this paper (and the previous paper), it is the availability of the Total Quantum Energy for*

1. *influencing the determinate $v_1$ (since it is sensitive to TQE) and*
2. *accommodating the non-determinate surfing motion on the S surface (by providing the QKE)*

*that make quantum mechanics truly distinct from classical mechanics.* The pilot wave theory has element (1) but not element (2). In sum, *we have distinguished the present approach for the interpretation of quantum mechanics from the pilot wave approach and the text book approach* which often follows the trend of the orthodox or Copenhagen interpretation.

## 10.2 Trajectories and Momenta of the Hydrogen Electron

We now consider the possible trajectories and momenta of the electron for the second excited state corresponding to $n=3$. For $\psi_{321}$, its modulus can be expressed as

$$R_{321} \equiv |\psi_{321}| = Cr^2 e^{-r/3r_0} \sin\theta \, \cos\theta$$

where $C$ is a normalisation constant and $r_0$ is the Bohr radius. If we use $r_0$ as the length unit and $r$ is measured with reference to this unit, the contour of constant $R_{321}$ is given by
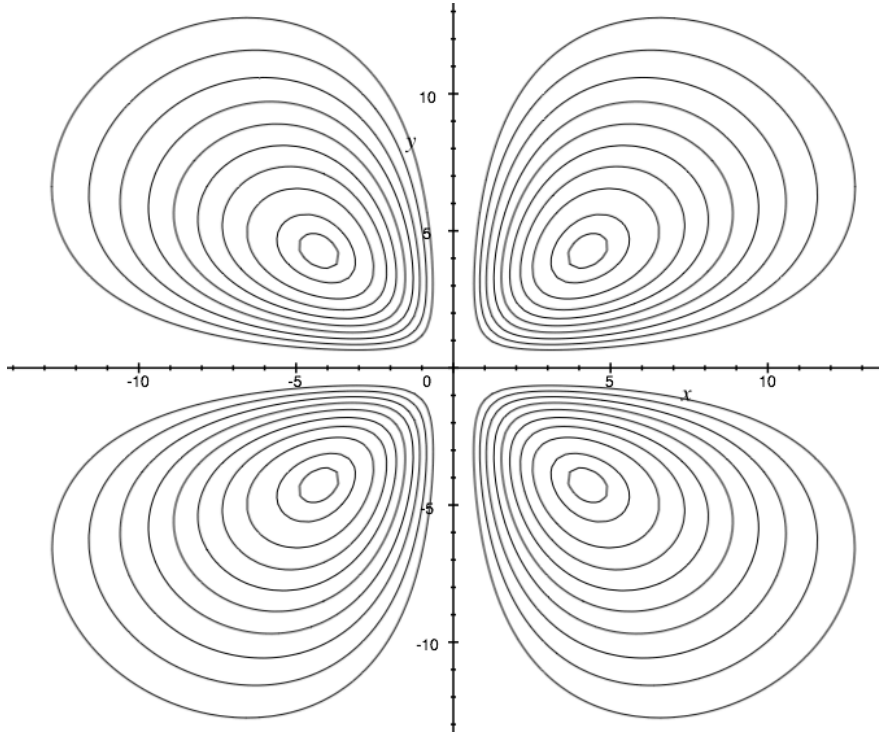
$$r^2 e^{-r/3} \sin\theta \, \cos\theta = constant \, .$$



Figure 7: Contours of $R_{321}$ on a $(r, \theta)$ plane (or a $\phi$ surface) given by $R_{321}/C = 0.7$ to 2.3 (interval 0.2) and 2.4 (innermost contour shown), and their negative counterparts (The upper vertical axis corresponds to $\theta = 0$)

A $\rho$ contour coincides with the corresponding $R$ contour but with a different constant. The right half of the above diagram corresponds to $\phi = 0$ and the left half corresponds to $\phi = \pi$. We have already stated that a $\phi$ surface is also a $S$ surface, $\nabla S$ is perpendicular to the $S$ or $\phi$ surface, $\underset{\sim}{v}_1 = \dfrac{\hbar}{m}\nabla S$ is the translational velocity; hence, the electron's translational velocity is in a direction perpendicular to the $\phi$ surface (into or out of the page). Crucially, in addition there is the additional non-determinate surfing velocity on the $\phi$ surface which accounts for the spin of the electron (as indicated in the free particle case, also see below). Before the non-determinate surfing motion is considered, the angular momentum of the translational velocity deserves close attention. Considering this angular momentum for the general case of $m$ ($m$=1 for $\psi_{321}$),

$$m_e \underset{\sim}{v}_1 \wedge \underset{\sim}{r} = \hbar \nabla S \wedge \underset{\sim}{r} = \pm \, \hbar |\nabla S| \, r\vec{\theta} = -\frac{m\hbar}{\sin\theta}\vec{\theta}$$

where $\vec{\theta}$ is the unit vector in the direction of increasing $\theta$ perpendicular to $\underset{\sim}{r}$, and the choice of '+' or '-' depends on the sign of $m$. Note that the magnitude and direction of this angular momentum varies with the position vector $\underset{\sim}{r}$. And its projection on to the vertical axis is $m\hbar$ which is constant for a given $m$. In quantum mechanics text books, $m\hbar$ is the component of the orbital angular

momentum in the $z$-direction, $L_z$, while the the orbital angular momentum vector has the fixed magnitude, $\hbar\sqrt{l(l+1)}$ (where $l$ is the orbital angular momentum quantum number), but whose direction cannot be ascertained uniquely and therefore is said to lie on a cone. There are some significant distinctions between the present approach and the text book approach to angular momentum even though both affirm that the projection of the angular momentum on to the vertical axis is $m\hbar$. While the direction of the angular momentum cannot be uniquely determined in the text book approach, in the present approach the direction of the angular momentum for the translational velocity can be uniquely determined and varies with the motion of the particle. While the magnitude of the angular momentum is fixed in the text book approach, in the present approach the magnitude of the angular momentum also varies with the motion of the particle. It is remarkable that in terms of the component of the angular momentum in the vertical direction, both approaches produce the same value, $m\hbar$. However, it has to be said that the angular momentum considered in the present approach uses the normal definition of angular momentum which involves a definite velocity and a definite position of the particle in relation to a reference point; hence it deserves serious consideration. This approach produces the horizontal component of the angular momentum vector: $L_h = -m\hbar\dfrac{cos\theta}{sin\theta}$. And it can be shown that, for a given $\theta$, the probability density of $L_h$ is

$$\lim_{\triangle L_h \to 0} \frac{P(\triangle L_h)}{\triangle L_h} = \frac{sin^2\theta}{m\hbar} \int_0^\infty r\rho \, dr$$

which is a function of $\theta$.[9] For small $\theta$, $v_1$ is very large, and $L_h$ is very large in magnitude but the probability density for $L_h$ with small $\theta$ is very small. This sense of the horizontal component of the angular momentum vector and its probability density are not dealt with in the usual text book approach.

Another distinction between the text book approach and the present approach is that in the latter, apart from the contribution of the translational velocity to the angular momentum of the particle, the additional non-determinate surfing motion on the $\phi$ surface is included explicitly to account for the spin angular momentum of the particle (as indicated by the free particle case, see also below). Since the Schrödinger equation has nothing to say about the direction of the particle's motion on the $\phi$ surface (see [1]) and it has very little to say regarding the kinetic energy of the surfing motion on the surface, *on its own it has nothing to say about the spin of the particle*. This important fact and the analysis above regarding orbital angular momentum strongly suggests that *the Schrödinger equation can only account for the (orbital) angular momentum contribution from the translational velocity* which is a velocity identified in the pilot wave theory and in this paper. That is why in the text book approach, a spin term which is alien to the Schrödinger equation has to

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[9] It is assumed that the particle does not cross from the upper quadrants to the lower quadrants through the horizontal nodal line (see below), hence the same probability contribution from a lower quadrant is not included here.

be added to the orbital angular momentum as derived from the Schrödinger equation to express the total angular momentum but the mechanism for the generation of the spin represented by this added term is not explained. The present approach overcomes the limitation of the Schrödinger equation by considering the spin-producing surfing motion on the $\phi$ surface as part of the total motion, and its contribution to the total angular momentum will be included explicitly with dynamical details (see below). This alerts us to the need to look at other cases where, in the usual approach, the contributions from the spin-producing surfing motion on the $\phi$ surface towards other calculations of some important quantities also are not captured by the Schrödinger equation. In these cases, it is very probable that additional terms need to be added to account for the contribution from the spin-producing surfing motion, if this has not already been done.

A brief comparison with the pilot wave theory is also in order. In that theory, the non-determinate surfing motion on the $\phi$ surface is also omitted so that the electron in that theory moves according to a fixed determined trajectory which has a constant $r$ and a constant $\theta$. This means that the electron returns to the same position in a determined fashion after completing one orbit in $\phi$. For the electron to cover a different trajectory, it must have a different initial position. However, in the present approach, the electron beginning with a certain initial position can vary its $r$ and $\theta$ with a significant degree of freedom through its surfing motion on the $\phi$ surface while orbiting around the vertical axis. This degree of freedom is synonymous with its non-determinacy.

Regarding the non-determinate surfing motion on the $\phi$ surface, based on the analysis so far in this paper, we may also suggest that on that surface the electron has (i) a velocity component parallel to the $\rho$ contour with the speed $\frac{\lambda_2 \hbar}{2m_e} \frac{|\nabla \rho|}{\rho}$ (where $\lambda_2 = 1/2$ is the spin number associated with the electron) and at the same time (ii) a velocity component on the same surface perpendicular to the $\rho$ contour with an indeterminate speed. One may also suggest that this indeterminate speed satisfies some overall constraint within a time period of $T$, and the constraint is that the 'non-dimensional time' density distribution for the period $T$ – evaluated at suitably small neighbourhoods for the plotting of this distribution[10] – which is equivalent to the time averaged probability density for the period $T$, conforms to the $\rho$ pattern. That is, within the period $T$, in the neighbourhood of highest $\rho$ (at $r = 6$ and $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$) we expect to see the electron for the longest accumulated time (e.g., through a slower speed or/and many visits), and in the neighbourhood of lowest $\rho$, we expect to see the electron for the shortest accumulated time (e.g., through a fast speed or/and minimum number of visit). But what happens in a neighbourhood around $\rho = 0$, which is called a nodal neighbourhood. For example, what happens near the vertical and horizontal axes where $\rho = 0$? This question is of a similar nature to the one considered in section 7.0 where the challenge of singularities at the two tail ends of the free particle case is discussed and a rigorous

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[10] We cannot invoke the small circular strip to evaluate the 'non-dimensional' time density as previously in section 7 since the area between two $\rho$ contours here does not form such a neat circular strip.

attempt has been made to deal with it. A similar answer, but with philosophical underpinning, will be given later. In the meantime, we deal with the question of the electron passing from one quadrant to another. For the electron to move from the right upper (or lower) quadrant to the corresponding left upper (or lower) quadrant is not a problem as the electron circulates around the vertical axis with constant orbital angular momentum to reach the left quadrant. However, does the electron move from the upper two quadrants to the lower two quadrants (or vice versa) by crossing the horizontal axis of $\rho = 0$?

This paper suggests that the electron does not cross the horizontal axis of $\rho = 0$. That is, if the electron is in the upper quadrants, it will remain in the upper quadrants unless it receives energy or loses energy to move to another orbital. There are two reasons for this suggestion. Firstly, the speed of the electron along the $\rho = 0$ contour will reach infinity (as in the case of the free particle in section 7). This can be seen in the expression of the magnitude of that speed which is proportional to $\dfrac{|\nabla \rho|}{\rho}$ where

$$\frac{\nabla \rho}{\rho} = 2\frac{\nabla R}{R} = 2\left(\frac{2}{r} - \frac{1}{3}\right)\vec{r} + \frac{2}{r}\left(\frac{cos^2\theta - sin^2\theta}{sin\theta cos\theta}\right)\vec{\theta}$$

where $r_0$, the Bohr radius, is used as the unit length and $\vec{r}$ is the unit vector in the direction of the position vector and $\vec{\theta}$ is the unit vector in the direction of increasing $\theta$. At the horizontal axis where $\theta = \pi/2$, and $\theta = 3\pi/2$, the $\vec{\theta}$ component will go to infinity and so will the speed of the electron along the axis. This can be avoided if the electron does not cross the horizontal axis. Secondly, the electron while moving in the top right quadrant, for example, will have a certain sense of rotation (i.e., spin) with respect to the axis which goes through the maximum $\rho$ at $r = 6$ and $\theta = \pi/4$ (the axis is perpendicular to the $\phi$ surface). Say, it has an anti-clockwise sense of rotation. If the electron is to move to the lower right quadrant, it should maintain that sense of rotation. But this means that the electron does not only incur infinite speed at the horizontal axis, but it incurs infinite speed with opposite directions at the same time! Hence, this cannot be a physical scenario and should be rejected. But in this case how can the $\rho$ contours in the lower quadrants be accounted for?

These contours in the lower quadrants can be accounted for if they are interpreted as corresponding to another occasion where the electron happens to be there, and while the electron is in the upper quadrants, $\rho=0$ in the lower quadrants. Whether the electron will land in the upper or lower quadrants will depend on the specific situation in which it enters this orbit after a change in its energy. But this remaining in the upper or lower quadrants raises the question of the meaning of $\rho$, the integration of which over all space should be equal to one according to the normalisation constraint. It seems that setting $\rho=0$ in the lower quadrants will only yield a half as the integrated value of $\rho$. However, this question can be dissolved if we retain the meaning of $\rho$ as adopted in the present paper, which is the time averaged probability density over a period $T$. If the electron spends

the whole period $T$ in the upper quadrants, then the time averaged probability density over the period $T$ integrated throughout the upper quadrants (which are understood in terms of three dimensional space) will be one. Or equivalently and evidently, the non-dimensional time density over the period $T$ integrated over the whole upper three dimensional space will be one. Practically, this amounts scaling up the original $\rho$ by a factor of two in the quadrants where the electron is found.

If the electron does not cross the horizontal axis and remains in the upper quadrants (or equivalently the lower quadrants), it still incurs very high speed along the contours close to the vertical and horizontal axes as the term $\left(\frac{cos^2\theta - sin^2\theta}{sin\theta cos\theta}\right)$ tends to infinity as the electron approaches the axes. As in section 7, we may suggest that the electron does not in fact reach the axes and its velocity component, $\lambda_3 v_3$ which is perpendicular to the $\rho$ contour but whose magnitude is non-determinate,[11] reaches zero at some point as the electron approaches any of the axes and pulls away from the axis with non-zero $\lambda_3 v_3$. The treatment is therefore analogous to the one given in section 7 for the case of a particle in free space which also deals with the question of the satisfaction of the normalisation constraint and the maintenance of angular momentum in the critical neighbourhoods of the tail ends. Such similar questions in the present case of the electron in the hydrogen atom can be dealt with in a similar manner and it is not necessary to go through the details here since at the end of the day we do not know what parameters are actually active in those critical neighbourhoods, as discussed in section 7. In addition to these critical neighbourhoods near the axes, there is the region of small or vanishing $\rho$ when $r$ is large. Surely, the electrons cannot visit all the neighbourhoods with large $r$ within a finite period of $T$ so that the only alternative is that there has to be a critical cut-off point at a certain low value of $\rho$ where $\lambda_3 v_3$ will reach zero and the electron will pull back towards the point of maximum $\rho$ near to the centre of the atom, rendering $\rho' = 0$ beyond the critical cut-off $\rho$ contour. The above suggested motion of the electron in all these critical neighbourhoods with small $\rho$ implies that the actual time averaged probability density in these neighbourhoods, $\rho'$ (or equivalently the non-dimensional time density), is different from the idealised $\rho$ calculated from the Schrödinger equation. This implication requires some philosophical discussion and explanation.

### 10.2.1 Knowability of the Electron's Trajectories

It has been pointed out in the previous paper that the Schrödinger equation does not actually put any constraint on the motion of the particle (electron in this case) on the $S$ surface (the $\phi$ surface in this case). The traditional meaning of $\rho$ as the density of the probability of finding the particle in an infinitesimal neighbourhood is an interpretation which is not strictly required by the Schrödinger

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[11] This is equivalent to the radial speed in the case of the particle in free space.

equation. In an alternative and possible universe where the Schrödinger equation is still valid, the surfing motion of a particle on the $S$ surface may pay no respect to that interpretation of $\rho$ so that the actual time averaged probability density for the motion over any period $T$ bears no resemblance to the $\rho$ pattern at all even though that $\rho$ pattern is derived from the Schrödinger equation. In that universe, the spin value of a particle will not be constant; indeed it could be wildly unpredictable giving rise to a chaotic universe where the chemical elements will lose their spin-dependent properties as we know them in this universe. However, we are not suggesting that this present universe is that kind of chaotic universe which is not life sustaining and therefore uninhabitable. Nevertheless, the point is that the idealised $\rho$ does not strictly govern the motion of the electron on the $\phi$ surface. And if in a certain very brief period of time in period $T$ the electron ventures into a neighbourhood of $\rho = 0$ (a nodal neighbourhood) and produces a $\rho'$ (the actual time averaged probability density) that deviates from the $\rho$ pattern, while for the rest of the time in period $T$ the electron's motion away from those nodal neighbourhoods produces a $\rho'$ that closely follows the $\rho$ pattern, the Schrödinger equation would have voiced no complaint about such motion producing such $\rho'$ in the nodal neighbourhoods. Here, we are running into the philosophical distinction between idealism and realism. In our mind with our mathematical ideas of perfect mathematical objects in perfect form, we may like the motion of the electron to follow strictly a pattern which will reproduce the idealised $\rho$ pattern in our mind, in which case the electron will incur infinite speed at $\rho = 0$ along the two axes. If we think that our mathematical ideas of perfect mathematical objects in perfect form represent reality in the real world, then we are taking the philosophical position of 'realism' (absolute objectivity). However, if we say our mathematical ideas of perfect mathematical objects in perfect form are merely objects in our mental world which do not correspond to the reality in the world, then we are adopting the position of 'epistemological idealism' (strong subjectivity [13]). The extreme form of idealism (absolute subjectivity) will be very pessimistic about the capacity of our perfect mathematical objects to correspond to reality. However, the philosophical position of critical realism, taking the middle way, allows the correspondence between our perfect mathematical objects and the reality to a good but not perfect extent. With 'critical realism', one can say this: using our mathematical ideas and objects, to a good extent and for most of the time we can have a reasonable knowledge of the real behaviour of the electron, but we cannot describe or deduce the total real behaviour of the electron with certainty all of the time, i.e., we cannot attain absolute knowledge of reality. When the term 'a reasonable knowledge of the real behaviour of the electron' is used in the last sentence, it refers to the knowledge of the behaviour of the electron away from the nodal areas and even this knowledge has to be qualified as reasonable, as opposed to total knowledge, because we do not actually know the non-determinate motion on the $\phi$ surface at any particular instant (even away from the nodal neighbourhoods) since we only know the electron's time averaged motion on that surface (which is the motion along the $\rho$ contours since $\lambda_3 \upsilon_3$ averages to zero in the period $T$ or over multiple $T$'s).

With this modest position of critical realism, we can make sense of the unknowability of different aspects of the electron's motion while affirming what we can know about its average behaviour in regions of our interest.

## 10.3 Spin of the Hydrogen Electron

In the case of $\psi_{321}$ , it is tempting to calculate the time averaged angular momentum (spin) produced by the electron's circulating motion over the period $T$ around the axis perpendicular to the $\phi$ surface and going through the point of maximum $\rho$. Apart from the mathematical complexity involved in the calculation since the $\rho$ contours are not concentric circles, there is another reason for not doing this calculation – as the electron orbits around the vertical axis in period $T$, whatever time averaged spin angular momentum is generated within a narrow range of a certain $\phi$ , the time averaged spin angular momentum over the whole period $T$ integrated over the range of $2\pi$ of $\phi$ is zero. This is so because whatever time averaged spin is generated over $T$ in the neighourhood of a certain $\phi$, it is cancelled by the time averaged spin generated at the neighourhood of $\pi + \phi$. Hence, an orbital with non-zero $m$, i.e., non-zero orbital angular momentum around the vertical axis, will not generate any net spin over the period $T$. But how about the case when $m = 0$?

Of particular interest is the case of $\psi_{100}$ (the ground state):

$$\psi_{100} = \frac{1}{\sqrt{\pi}} e^{-r} \ , \ \rho = \frac{1}{\pi} e^{-2r}$$

where the Bohr radius is used as the unit length. The $\rho$ contours are circular and centred at the origin. Since $v_1 = \frac{\hbar}{m} \nabla S = 0$ , the translational velocity of the electron is zero, the electron is not *orbiting* around any axis but its *circulating* motion along the $\rho$ contours creates a time averaged spin angular momentum over a period $T$ around an axis which runs through the origin and has no preferred direction while the non-determinate velocity component perpendicular to the $\rho$ contours (and hence going through the origin) moves the electron to visit different $\rho$ contours but contributes nothing to the angular momentum. The direction of the spin axis, around which the circulating motion along the $\rho$ contours happens, depends on the situation in which the electron entered the orbit and any of its interaction with a magnetic field while in that orbit (see the Stern-Gerlach experiment later).
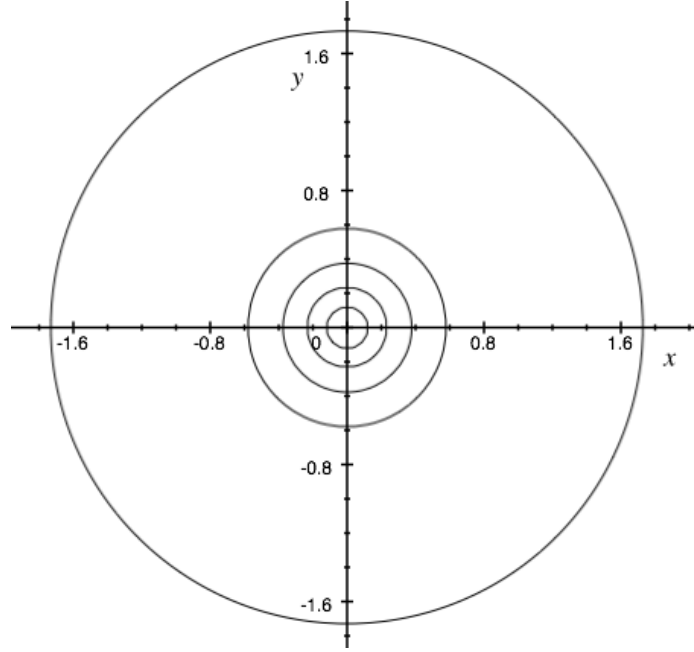
Figure 8: $\psi_{100}$, $\rho = \dfrac{1}{\pi} e^{-2r}$, $\rho = 0.25$ to $0.1$ (interval: $-0.05$), and $\rho = 0.01$.

To calculate the time averaged spin angular momentum over a period $T$, we use the idealised $\rho$ pattern, acknowledging that the electron will not visit the outer $\rho$ contours with very small $\rho$ so that the idealised angular momentum based on the idealised $\rho$ pattern is a very good approximation to the actual time averaged spin angular momentum.

$$\int_0^\infty m_e r \lambda_2 |v_2| \, (2\pi r \rho) \, dr = -\int_0^\infty m_e r \frac{\lambda_2 \hbar}{2 m_e \rho} \frac{d\rho}{dr} (2\pi r \rho) \, dr = -\int_0^\infty \lambda_2 \hbar \pi r^2 \frac{d\rho}{dr} \, dr$$

The last integral can be integrated by parts to yield

$$-\lambda_2 \hbar \pi \left[ \left[ \rho r^2 \right]_0^\infty - 2 \int_0^\infty r\rho \, dr \right].$$

The last two lines of integration are quite general in that *they do not require explicit details of $\rho$*. However, to evaluate the integral, we need the explicit values of $\rho$ at infinite $r$ and at $r=0$. In this case, $\rho=0$ at infinite $r$ and $\rho$ is finite at $r=0$ yield the integral value to be

$$\lambda_2 \hbar \int_0^\infty 2\pi r \rho \, dr = \lambda_2 \hbar$$

where the normalisation constraint has been used. Since the $\lambda_2$ value for the electron is 1/2 (or -1/2), the spin angular momentum has the expected value for an electron.

Since $\psi_{100}$ has no dependence on $\phi$ or $\theta$, $\psi_{100} = R_{100}$, one can substitute $R_{100}$ for $\psi_{100}$ in the time independent Schrödinger equation (20) and divide by the common factor of $R_{100}$,

$$-\frac{\hbar^2}{2m_e}\frac{\nabla^2 R}{R} - \frac{e^2}{r} = E_p .$$

The first term is the Total Quantum Energy and the second term is the potential energy. There is no kinetic energy in the translational velocity since this velocity component in this case is zero (as $m=0$). Here, we see that the TQE is a real entity whose magnitude increases as $r$ tends to zero to balance the very large negative potential energy of the electron. Similar statements can be made for other $\psi_{n00}$ orbits. For orbits with non-zero $l$ and $m$, the more general energy equation (18) also involving the TQE term will be satisfied.

It is worth pointing out that the spin velocity in the $\psi_{100}$ case has constant magnitude:

$$\lambda_2 |v_2| = \frac{1}{2}\frac{\hbar}{m_e r_0} .$$

For $\psi_{300}$, again using the Bohr radius as the unit length,

$$\psi_{300} = R_{300} = Ae^{-r/3}\left(1 - \frac{2r}{3} + \frac{2}{27}r^2\right)$$

where $A$ is the normalisation constant. At $r=1.9$ and 7.1, there are two nodal contours ($\rho=0$). At these nodal contours, the magnitude of the spin velocity

$$|v_2| \propto \left|\frac{\nabla \rho}{\rho}\right| = 2\left|\frac{\nabla R}{R}\right| = \left|-\frac{2}{3} + \left(-\frac{4}{3} + \frac{8}{27}r\right)\Big/\left(1 - \frac{2r}{3} + \frac{2}{27}r^2\right)\right|$$

will reach infinity. It is therefore suggested that there are three possible circular sections of motion for the electron which are divided by the two nodal contours such that the electron cannot cross a nodal contour to reach a neighbouring section.
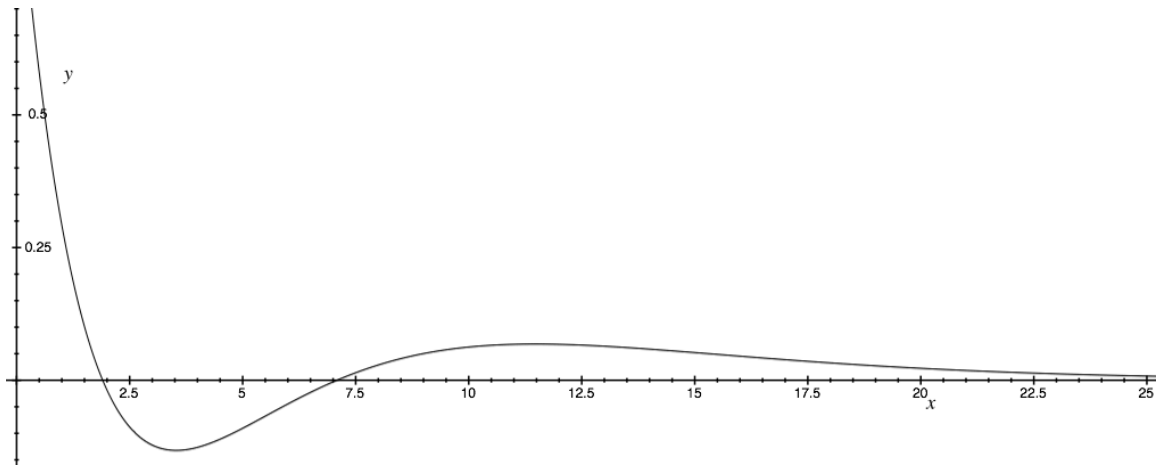


Figure 9: $\psi_{300}$ as a function of $r$

Which section the electron is in depends on the specific situation in which it enters this orbit as a result of energy change; different scenarios will lead to the electron entering different sections of the orbit. *If* there is an equal chance for the electron to land in any of the three sections, then the higher values of $\rho$ in the inner section do not necessarily means that the electron will land in that section more frequently than the other two. If we take $\rho$ as the time averaged probability density over the period $T$ during which the electron stays within the same section of the orbit (see analogously the case of $\psi_{321}$ above where the electron is confined to the upper or lower quadrants only), then the $\rho$ value in each section has to be scaled up by a factor to $\rho_s$ to satisfy the normalisation constraint which is applied independently on each section. Since $\left|\dfrac{\nabla\rho}{\rho}\right|$ is not affected by this scaling, the magnitude of the spin velocity in each section is not affected.

Even if the electron's motion is confined to only one particular section, in each case its spin angular momentum will be the same ($\lambda_2\hbar$ which is $\hbar/2$ for an electron) as the integral for this momentum

$$\int m_e r \lambda_2 |v_2| \left(2\pi r \rho_s\right) dr = -\int \lambda_2 \hbar \pi r^2 \frac{d\rho_s}{dr} \, dr = -\lambda_2 \hbar \pi \left[\left[\rho_s r^2\right] - 2\int r\rho_s \, dr\right]$$

will yield the same value when evaluated with these three pairs of limits, (0, 1.9), (1.9 to 7.1), (7.1 to $\infty$ ), since the first term on the RHS will vanish and the second term is $\lambda_2\hbar$ due to the normalisation constraint. This result for $\psi_{300}$ can be generalised to the $\psi_{n00}$ cases where there will be $n$ circular sections of motion and the electron's spin for each section will have the same correct value. Again, as seen in the expression for the wave function, there is no preferred direction for the spin axis which may depend on the situation in which the electron entered the orbit and any interaction with a magnetic field.

## 11.0 The Electron's Spin in the Stern-Gerlach Experiment

The above result is not only significant for hydrogen but it is also illuminating in understanding the Stern-Gerlach experiment where the silver atom was used. Silver has 47 electrons and they are spread over different orbitals of different energies. We need to analyse its orbital angular momentum and its spin angular momentum both of which can have effect in the Stern-Gerlach experiment. Firstly, we analyse their orbital angular momentum. The electrons in the orbitals 1s, 2s, 3s, 4s and 5s have zero orbital angular momentum since $m=0$. The electrons in the orbits belonging to 2p, 3p and 4p have $m = -1, 0, 1$ where $m=0$ again produces zero orbital angular momentum, and $m = -1$ and 1 with their opposite orbital angular momenta cancel out one another so that these orbits also produce zero net orbital angular momentum. Similarly, the electrons in the orbits belonging to

3d and 4d with $m$ = -2, -1, 0, 1, 2 also produce zero net orbital angular momentum. Hence, all 47 electrons produces zero orbital angular momentum so that the only angular momentum active in the Stern Gerlach experiment is the spin angular momentum of the electrons which warrants the following analysis. 46 of the 47 electrons (i.e., not counting the one in 5s) are in orbitals each of which has a pair of electrons of opposite spin signs so that even for those orbitals with $m=0$, the electrons there cannot yield any net spin. However, for the 5s electron, $m=0$ and the electron is unpaired so that it can generate spin angular momentum of $\hbar/2$ in a manner similar to the electron in the $\psi_{300}$ orbit of the hydrogen atom as indicated above, with five sections of possible motion. Even though $\rho_s$ for the electron in the silver atom has a somewhat different form to the one for hydrogen, the calculation of the spin angular momentum does not depend on the explicit form of $\rho_s$, as it only depends on the values of $\rho r^2$ at the limits in evaluating the angular momentum integral, and those values are the same at the nodal limits and the centre in both cases of hydrogen and silver – zero. This result also applies to other cases and explains the ubiquitous nature of the constancy of spin of electrons (and indeed other particles), giving some support to the credibility of the interpretation of quantum mechanics proposed in this paper and the previous paper.

Hence, we can conclude that the magnitude of the net angular momentum of the 47 electrons of the silver atom is still the same as that of the single electron of the hydrogen $\psi_{n00}$ case, which is $\hbar/2$. (Incidentally, an electron travelling in free space also has the same spin value, as given in section 4.) Again, there is no preferred direction for the spin axis which may depend on the situation in which the electron entered the orbit. As the silver atom enters the magnetic field area, the spin axis is open to adjustment and is therefore aligned to the direction of the magnetic field so that the up or down deflection can take place with greater efficiency. To sum up, it is the unpaired 5s electron in the silver atom which effectively generates the spin angular momentum which is captured in the Stern-Gerlach experiment.

Incidentally, another atom with an unpaired electron in an orbit where $m=0$ will also serve well in the Stern-Gerlach experiment, e.g., copper has 29 electrons and 28 of them together produces zero orbital angular momentum and zero spin angular momentum while the other electron, in the 4s orbital, has $m=0$ and is unpaired so that it can yield spin angular momentum and can be active in the Stern-Gerlach experiment. Similarly, the unpaired electron in the 4s orbit of potassium can also be active in that experiment.

In this paper the interpretation of the mechanism for the generation of non-zero electron spin requires that the electron is unpaired in an orbit with $m=0$. It may be useful to experiment with an unpaired electron and then a pair of electrons in the same non-zero $m$ orbit in the following manner to verify that such electrons cannot produce non-zero net spin over a period of $T$. For the unpaired electron in a non-zero $m$ orbit, the average total angular momentum vector over the period $T$ can be written as $J_1 = L + S$ where the first term is the time averaged orbital angular momentum vector

and the second term is the time averaged spin angular momentum vector. *Suppose* the net or time averaged spin angular momentum vector is non-zero over a period of $T$. When the second electron is added to fill the orbit, the average total angular momentum of the two electrons over the period $T$ is

$$\underset{\sim}{J}_2 = (\underset{\sim}{L} + \underset{\sim}{S}) + (\underset{\sim}{L} - \underset{\sim}{S}) = 2\underset{\sim}{L} \ .$$

If the time averaged spin angular momentum vector is indeed non-zero, then $2\underset{\sim}{J}_1 \neq \underset{\sim}{J}_2$. However, if experiments show that $2\underset{\sim}{J}_1 = \underset{\sim}{J}_2$, then the supposition is wrong and the time averaged spin angular momentum vector is indeed zero, as suggested in this paper.

It will also be interesting to see if any of the electrons responsible for interacting with the magnetic field in the Stern-Gerlach experiments performed so far lies in an orbit with *m* not equal to zero. The paper suggests that $T$ is so short that a typical experimental timescale is greater than $T$ so that electrons in orbit with non-zero *m* will not produce net spin.

The cases of $\psi_{320}$ and $\psi_{322}$ for the hydrogen electron will be briefly presented in Appendix B.


## 12.0 Conclusion


Philosophically, a theory cannot be proved to be correct in an absolute sense. However, a theory's truthfulness can be assessed by its internal conceptual coherence and its correspondence to experimental reality. This paper and the previous paper suggest a distinctive theoretical interpretation of quantum mechanics. It is also true that this theoretical interpretation cannot be proved to be correct in an absolute sense. However, we ought to assess it by the two criteria given above: internal coherence and correspondence to experimental reality. Firstly, we assess its internal conceptual coherence.

Compared to the Copenhagen Interpretation, it does not assume that a particle does not have an unambiguous position and momentum; rather, the whole interpretation is based on the assertion that a particle has a definite position and a definite momentum at any instant even when such properties are not measured. It does not require a particle to be in a superposition of states at a given instant. Rather, it sees such a hypothetical superposition of states (e.g., positions) with their corresponding probability densities as a compression of the history of the particle's states over period $T$ into a single hypothetical instant by means of averaging the time varying probability densities of the states over the period $T$. This paper suggests that *the particle possesses all the states it traverses in time within the period T, not in a single instant*. Its introduction of time-averaged probability density, and the equivalent 'non-dimensional time' density, thus makes sense of the meaning of $\rho$ in a slightly different manner to the one given by Born's rule. It is true that given any random instant within $T$, the probability of finding the particle in a certain neighbourhood is $\rho$, and

in this sense $\rho$ in the present interpretation still has an almost identical meaning to the one in Born's rule. However, we insist that it is precisely because $\rho$ has the meaning as given in the present interpretation – i.e., in relation to a period $T$ – that it can have the usual meaning as found in Born's rule, i.e., for a single instant. That is, the meaning of $\rho$ as given in the present interpretation is more fundamental than the one given in Born's rule. And this meaning of $\rho$ has served well the purpose of understanding spin in the free particle case and the cases of the hydrogen electron and silver electron (see later in relation to verification by experiment). In terms of comprehending the physical reality that we have in this universe, the assertion of unambiguous position and momentum makes more conceptual sense (or common sense) than the Copenhagen Interpretation. It avoids such paradox as the Schrödinger's cat and does not introduce any new paradox, as far as one can see.

Compared to the pilot wave theory, the present theoretical interpretation does not require the universe to be deterministic. The pilot wave theory only has one velocity component for the particle (called the translational velocity in this paper), the improved versions of this theory by Salesi, Recami and Esposito [4-6] have two velocity components. However, the present theoretical interpretation employs the full three dimensions of the particle's velocity and the three velocity components form a tidy orthogonal set. While the translational velocity is deterministic, the motion produced by the other two components lying on a $S$ surface is non-deterministic. These two velocity components can be structured in vastly different ways in different universes where the Schrödinger equation still holds, with some of these universes being extremely chaotic and uninhabitable, e.g., the spin of the particle could be unpredictable. The present theoretical interpretation suggests that in our universe, one of the two components of the velocity on the $S$ surface is also deterministic – the spin velocity parallel to a $\rho$ contour whose magnitude is proportional to the spin number corresponding to the kind of particle in question. Given that the direction (and magnitude) of the spin velocity is deterministic, it follows that the direction of the third velocity component is also deterministic, i.e., it has to lie on the $S$ surface and is perpendicular to the spin velocity. However, because of the magnitude of this third velocity component is indeterminate, it produces the fundamental non-deterministic nature of our universe. The magnitude is indeterminate in the sense that even if we know all the laws governing particle motions in this universe and even if we know all the history of the trajectories and momenta of all particles in the universe from the beginning of time until the present moment, there is not sufficient information for us to know any future motion of any particle. Conceptually, is this a problem? Einstein would think that it is a problem because he liked to see deterministic laws governing the whole of the universe ('God does not play dice'). However, conceptually the non-deterministic nature of our universe, as proposed by the present theoretical interpretation, makes sense since it asserts that the universe has a high degree of regularity through the two deterministic velocity components but it also has a significant degree of freedom through the non-determinate magnitude of the third velocity component. Therefore, conceptually it has an elegant balance between regularity on one hand and spontaneous freedom on the other hand such that the universe is not like a clock, nor is it so free that it becomes chaotic and uninhabitable.

In light of the above comparisons with the Copenhagen Interpretation and pilot wave theory, it can be argued that the present theoretical interpretation reaps the best of both interpretations: (i) the non-determinism of the Copenhagen Interpretation and (ii) the reality of the particles with their definite positions and momenta at all times of the pilot wave theory. However, the present theoretical interpretation avoids the shortcomings of both interpretations which can be expressed as the converse of the last sentence: (ii) the lack of the reality of the particles with their definite positions and momenta in the Copenhagen Interpretation and (ii) the lack of spontaneous freedom in the pilot wave theory. Therefore, according to these comparisons from the conceptual perspective, the present theoretical interpretation deserves serious consideration.

Secondly, we assess the present theoretical interpretation with respect to experimental reality. Experiments have shown the ubiquitous constancy of a particle's spin but there is no serious attempt to explain the origin of this spin, except Hestenes [2-3] and those who follow his idea (but see the differences between his approach and the present approach later). However, this paper has shown that

(i) in the case of free particle,

(ii) in the case of the hydrogen electron in $\psi_{n00}$ orbits,

(iii) in the case of the 5s silver electron in the Stern-Gerlach experiment and

(iv) in the cases of other electrons of other elements in $\psi_{n00}$ orbits which can be similarly tested as the silver electron in the Stern-Gerlach experiment,

the non-determinate surfing motion on the $S$ surface as described in this paper – i.e., with the prescribed spin velocity and with the time spent at different portions of its trajectory mimicking the $\rho$ pattern according to Born's rule (as interpreted in the sense given in this paper) – will produce the observed spin angular momentum (the cases of $\psi_{nl0}$ with non-zero $l$ are yet to be studied). The form of the spin velocity is crucial in obtaining this remarkable result where the spin number is seen as a constant for the particle, and where the explicit expression for $\rho$ is not necessary except for evaluating $\rho r^2$ at the centre or the nodal $\rho$ boundary contours which invariably and conveniently

gives the value of zero such that the normalisation constraint ensures that the correct spin value is produced. Hence, the theoretical interpretation gives a rigorous and arguably elegant explanation for the ubiquitous constancy of a particle's spin. However, it cautions that for non-zero $m$, i.e., with non-zero orbital angular momentum about the vertical axis, because the spin vector is in the direction of the particle's translational velocity, the time averaged spin over the period $T$ (involving the completion of at least one orbit or many orbits) will be zero while its instantaneous value will be non-zero.

Both the pilot wave approach and the text book approach cater for spin not by including the surfing motion on the $S$ surface in addition to their use of the Schrödinger equation; rather, they cater for spin by making the wave function a complex-valued vector and by adding a spin term to the Schrödinger equation. The spin term does not give us any understanding of the origin or generation of the spin; it merely represents its effect in interaction with magnetic field. However,

the present approach gives details of the origin of particle spin by studying intently the surfing motion on the *S* surface. Hestenes [1-2] and those who followed his idea, e.g., Salesi, Recami and Esposito [4-6], had suggested the link between a circulatory motion of a particle and its spin. However, in dealing with a free electron in the relativistic case, Hestenes limited his study by looking at the mean motion in the following way:

> we define the zitter mean as an average over the free particle zitter period that keeps the *zitter center velocity v* and the spin vector ... fixed. ... This approximation ignores variations in zitter radius and mass over a zitter period. ([2], p. 12)

He ignored the radial motion on the *S* surface by taking a certain time average which yields a fixed radius for the circulatory motion. This amounts to studying a deterministic system where the trajectory of the particle has a constant curvature. This does not do justice to the inherently non-deterministic nature of the particle's surfing motion on the *S* surface so that it does not give a detailed account of the non-deterministic mechanism for the generation of spin. Those who followed his suggestion took an explicitly deterministic approach and therefore did not succeed in accounting for the generation of particle spin even though they helpfully suggested an expression for the spin velocity which is generalised in this paper.

Lastly, this paper suggests, albeit in a tentative manner, a possible explanation for some of the dark energy in the universe still to be accounted for. The surfing motion on the *S* surface with its Quantum Kinetic Energy and its associated Quantum Potential Energy, which together constitute the Total Quantum Energy (usually called the quantum potential), could be a possible candidate for such an account.

In assessing the present theoretical interpretation, we have looked at its conceptual coherence and its correspondence to experimental results. In terms of conceptual coherence, the present interpretation incorporates the strength of the Copenhagen Interpretation – non-determinacy – and the strength of the pilot wave theory – unambiguous position and momentum of a particle at any time. Conversely, it also avoids their weaknesses: ambiguous position and momentum of the Copenhagen Interpretation; determinism of the pilot wave theory. In terms of explanatory value for observed results from experiments in relation to spin, the present interpretation provides an explicit explanation for the origin and generation of spin of different particles while the Copenhagen Interpretation and the pilot wave theory manage to represent the effect of spin by adding a spin term to the Schrödinger equation but without giving a detailed account of the origin and generation of spins of particles. The present interpretation also suggests its potential for accounting for some of the dark energy in the universe. In sum, in view of its conceptual coherence and its robustness in explaining the origin of particle spins, it can be argued that the present theoretical interpretation has good potential for corresponding to reality (in the sense of critical realism) and deserves further attention and study.

We now look at the nature of the Schrödinger equation and the underlying assumptions which may have been implicitly operating behind the awareness of quantum physicists. This paper

suggests that the Schrödinger equation is a budget equation giving an overall bulk constraint over a period of a particle's motion. Hence, it cannot describe the instantaneous properties of particles, e.g., position and momentum; it can only give the probability of a particle having a certain position and momentum. However, *it can only do so reliably if the surfing motion on the S surface co-operates to yield those probabilities*. If the surfing motion on the S surface is chaotic, $\rho$ will lose its usual meaning as the probability density of finding a particle in a certain neighbourhood, and the ability of the Schrödinger equation to yield probabilities of position and momentum of a particle will also be lost. That is, for the Schrödinger equation to serve its normal expected functions amongst the physicist community, it has to be assumed that the surfing motion produces a time averaged probability density over a period of $T$ which matches or closely mimics the $\rho$ distribution on the S surface (which itself is derived from the Schrödinger equation). This necessary assumption was not spelled out in the thinking of the physicist community because it was not aware of this assumption but this paper argues that this assumption is a necessary and correct one, and it needs to be made explicit. For the surfing motion to match or mimic the $\rho$ distribution on the S surface, we have suggested that two other conditions, apart from the constraint given by the Schrödinger equation, need to be in operation. The first condition is that the spin velocity, $\lambda_2 v_2$, has to have a certain form as given in this and the previous paper. This means that the Schrödinger equation, at least in the description of this non-chaotic universe, is really incomplete (as Einstein suggested [14]): even though the continuity equation derived from the Schrödinger equation gives a clear expression of the translational velocity, it gives us absolutely no clue as to the form of the spin velocity (see previous paper [1]) while this form is absolutely crucial in matching the $\rho$ distribution and the generation of spin. However, adding this form of the spin velocity to the Schrödinger equation to form a larger system is not enough to match the $\rho$ distribution – another step is necessary, i.e., there has to be a non-determinate velocity, whose direction with the directions of the spin velocity and translational velocity form an elegant orthogonal set, while its magnitude is non-determinate. This non-determinate velocity is also crucial in matching the $\rho$ distribution. It can do the matching not in a random uncontrolled manner (as the $\rho$ distribution will then not be matched) but in a flexible and yet controlled manner (as we have suggested in this paper). The Schrödinger equation, apart from having nothing to say about the spin velocity, also has nothing to say about this non-determinate velocity in regard to its direction and magnitude. It is in these two senses that the Schrödinger equation is incomplete and in these senses Einstein's intuition was right (but only partly, see below). And the Schrödinger equation has served the physicist community well because, through adopting Born's rule for the interpretation of $\rho$, these two additional conditions have been assumed unknowingly and made implicit in operating with the Schrödinger equation; yet the details of their operation were not made explicit as it has been done here. It is safe to say that Born's rule effectively governs the surfing motion on the S surface even though this rule does not spell out how this rule is satisfied. However, this paper has suggested the details of the explanation why this rule

is valid in this universe: the suitable form of the spin velocity and the indeterminate and yet appropriate magnitude of the third velocity component ensure that the rule is valid in the sense as given in this paper. Put in another way, the appropriate surfing motion on the $S$ surface creates Born's rule which would have been gravely invalid if the surfing motion was not well organised to produce or create the probabilities expected in that rule. In another universe where the Schrödinger equation is still valid, the surfing motion there may not realise this rule and a very different and probably chaotic universe will emerge.

It is noteworthy that as Born's rule is being matched by the surfing motion, the characteristic spins of the particles are also generated. The crucial form of the spin velocity is such that it will help to yield those characteristic spins as long as the boundary conditions are satisfied, i.e., $\rho r^2$ vanishing at the centre or at certain $\rho$ contours, creating discrete regions or sections of space for the particle's motion. This explains the ubiquitous nature of the constancy of particles' spins. Without knowing such details about the mechanism for the generation of spin, the physicist community has nevertheless managed to represent the effect of spin by adding an extra term to the Schrödinger equation. This addition of the spin term again witnesses to the fact that the Schrödinger equation on its own is incomplete in describing the full motion of a particle, and in that sense the physicist community has agreed with Einstein regarding the incomplete nature of the Schrödinger equation.

Despite Einstein's insight into the incomplete nature of the Schrödinger equation, the author has to disagree with him on the appropriateness of finding all the necessary constraints or equations to completely determine the system. Although it is possible for someone passionate about determinism to adopt a deterministic interpretation of quantum mechanics as outlined in section 8 which will include determining the magnitude of $\lambda_3 v_3$, that is not the only option possible. If there is no other option available, then one has to take the deterministic position, whether unhappily or happily. However, since there are infinite number of possible trajectories for the particle other than the deterministic ones, one can freely and justifiably choose to take the non-deterministic option and leave the non-determinate magnitude of $\lambda_3 v_3$ as it is. Super-determinism will afford us with only illusory or apparent human freedom which is not freedom in the real and desirable sense.

Even though the motion of a particle on the $S$ surface is non-determinate due to the non-deterministic magnitude of $\lambda_3 v_3$, it does not mean that the motion is random. One must be careful to make the distinction between randomness and non-determinacy. If the magnitude of $\lambda_3 v_3$ is random, then the surfing motion on the $S$ surface will not match the $\rho$ distribution on that surface. The non-determinate motion on that surface has to be well co-ordinated to match or mimic the desirable $\rho$ distribution. One may ask: how does the particle know how to behave in order to match or mimic the desirable $\rho$ distribution and in so doing produce the characteristic spin? It requires some kind of knowledge or information of the recent history of the trajectory of the particle, at least within the current period of $T$. Yet, even with this information available, this information along with all other information and laws operating in the universe do not prescribe a deterministic velocity or

trajectory at any succeeding moment in time for the particle. All one can say is that over the period of $T$, *somehow* the particle traces out a trajectory with suitable non-random and non-deterministic velocity to mimic the $\rho$ distribution on the $S$ surface and thus produces the characteristic spin. This requires new information on top of the information of the particle's recent history (for the current cycle covered so far). The baffling question is: what causes such delicately balanced trajectories and velocities, including those for the sensitive regions of the tail ends? And what is the source of the new information? This information might be simulated by some stochastic process but such simulating processes do not inform us any more about the causes of the particle's highly organised non-determinate motion on the $S$ surface since (i) these stochastic processes are simulations in the first place and (ii) they can only simulate the effect but not the cause of such motion. Here, we may be reaching the limit of what we can know. Deterministic processes are amenable to our knowledge; the causes of non-deterministic but highly organised processes cannot be so easily pinned down.

What is the nature of the period $T$ which has been invoked so many times? Since experiments show that the spin values of particles are observed to be constant (within experimental error) each according to its own type, it follows that the period $T$ (which may be a constant for all particles or it may vary with different particles) is short compared to the typical timescale operating in those experiments, e.g., the time for the silver atoms to pass through the magnetic field of the Stern-Gerlach experiment. Had $T$ been much longer than those experimental timescales, since the spin angular momentum of the particle varies significantly within the period $T$, those experiments would have captured different sections of the period $T$ and would have reported rather different magnitudes of the particle spin in different experimental runs. However, this is not reported even though inevitably there is a limited extent of spread in the extent of deflection in the Stern-Gerlach experiment. It is also possible that the fast varying angular momentum of the particle within the short period $T$ may contribute to those spread to a small degree. If the timescale in the Stern-Gerlach experiment can be reduced sufficiently but it still long enough to deflect the particles up or down, this could produce a greater spread in the extent of deflection in the up and down direction.

It was shown in section 10.1 that the Total Quantum Energy (TQE) is a genuine constituent term of the total energy of a particle. This term does not appear in classical mechanics. However, this term in quantum mechanics provides the kinetic energy (QKE) for the motion of a particle on a $S$ surface which is critical for generating spin. TQE can serve to account for the tunneling phenomenon which classical mechanics cannot account for. Physicists in the pilot wave camp have already suggested this possibility, e.g., [15], even though they did not see it as a source of kinetic energy for the surfing motion on the $S$ surface.

One can raise questions about the treatment of singularities at the tail ends in section 7. Singularities are not uncommon in theoretical studies of the physical world. There are two options open to us in the present case of quantum mechanics. One option is to stand on the Copenhagen position and say that the particle is spread out as a wave over the physical space – which in the case of the free electron is the whole disc with $r < L$, and in the case of the hydrogen electron is the

whole orbital space – so that the particle is generating spin simultaneously over the whole physical space to yield the characteristic spin of the particle (see section 5.1). The option taken by this paper is that the particle has unambiguous position and momentum at any one instant which means that the singularities at the tail ends (for the particle in free space, and similarly for the electron in the hydrogen case) need to be contended with. Section 7 makes a serious rigorous attempt to demonstrate that the singularity problem is not insurmountable. The $\rho'$ at the tail ends will be somewhat different from the idealised $\rho$ derived from the Schrödinger equation but the normalisation constraint and the constraint of the characteristic spin of the particle can still be satisfied due to the flexible nature of the magnitude of the third velocity. Given the choice between (i) no definite position or momentum of a particle but with the actual $\rho'$ matching perfectly with the idealised or theoretical $\rho$ (including the tail ends) as derived from the Schrödinger equation (the Copenhagen Interpretation) and (ii) definite position and definite momentum of a particle but with the actual $\rho'$ somewhat different from the idealised or theoretical $\rho$ *only at the tail ends*, the author is willing to sacrifice perfect matching between $\rho'$ and $\rho$ at the tail ends to adopt position (ii) as that sacrifice is easier to make compared to the sacrifice of definite position and momentum of a particle.

Finally, as is already evident in the last paragraph, this paper takes seriously the particle interpretation of quantum mechanics, i.e., *ontologically* the particle is an entity which is at one definite place at any given instant; however, equally it takes seriously the wave-like behaviour of the particle, i.e., *functionally* the evolution of the particle's trajectory and velocity is partly governed by the Schrödinger wave equation while the particle has freedom to surf on the $S$ surface.

## Appendix A: Solution of the Radial Equation with Modified Bessel Functions of the First and Second Kind

Now we consider the Helmholtz Equation of the other form with $-a^2$ as the parameter ($a$ is real):

$$\nabla^2 R - a^2 R = 0 .$$

The radial equation is

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - a^2 R = 0$$

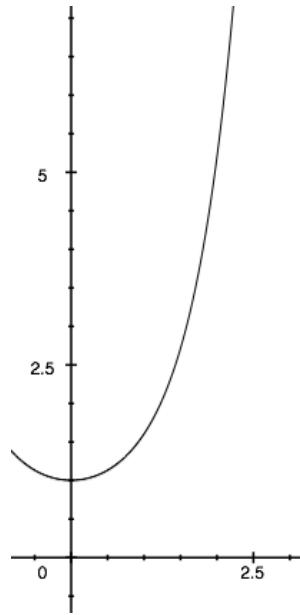with $R$ given by the modified Bessel functions of the first and second kind, $I_0, K_0$.

Figure 10: Square of the Modified Bessel functions of the first kind, $I_0^2$, as a function of $r$

For $I_0$, the normalisation integral does not converge so that it is not a physically meaningful solution.

For $K_0$, the normalisation integral converges, and the integral for the integrated angular momentum converges to the correct value. However, the integrated spin energy of the particle

$$\frac{\pi(\lambda_2\hbar)^2}{4m} \int_0^L \frac{r}{\rho}\left(\frac{d\rho}{dr}\right)^2 dr$$

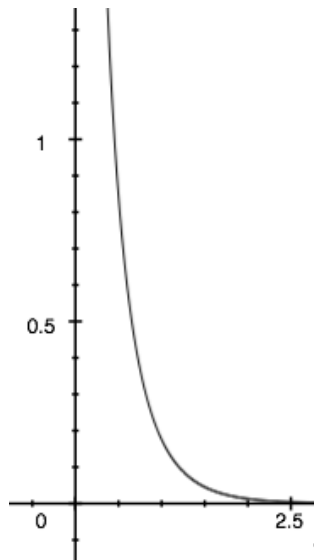does not converge. See Figure 11 for the function of $K_0^2$. Hence, this solution is also not physical.



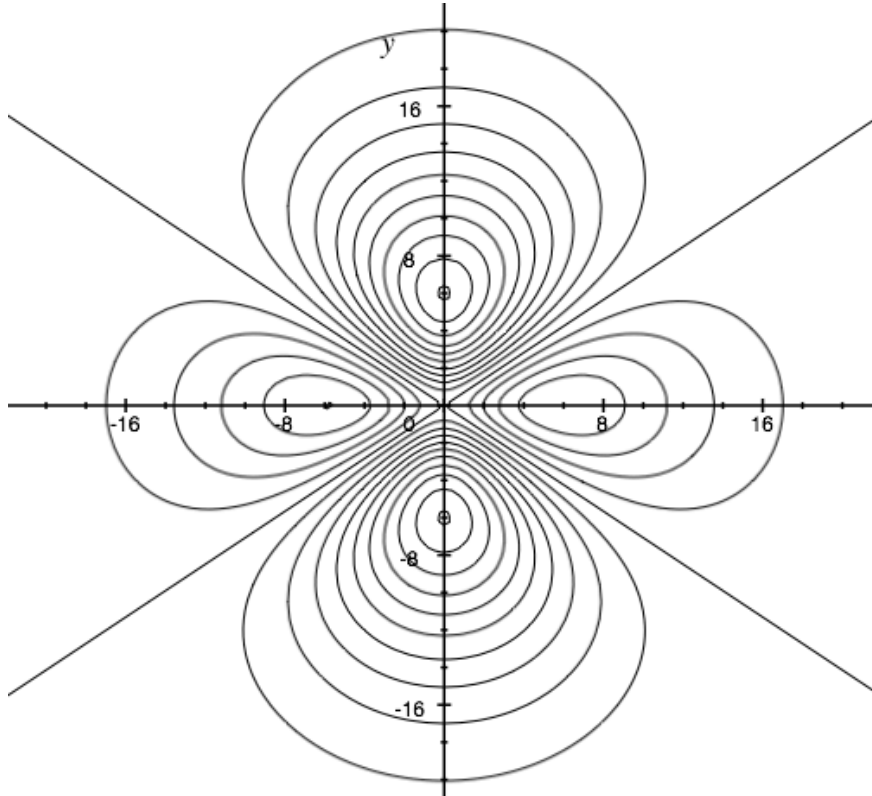Figure 11: Square of the Modified Bessel functions of the second kind, $K_0^2$, as a function of $r$

Figure 12: Contours of $R$ for $\psi_{320}$ given by $r^2 e^{-r/3}(3cos^2\theta - 1) = constant$, the values of the constants are 1 to 9 (interval 1), 9.7, 0 to -4 (interval -1), and -4.85

Since $m=0$, the angular momentum about the vertical axis is zero. The electron can cross the vertical and horizontal axes without incurring infinite speed. There are four nodal contours (joining to form two diagonal axes) giving four regions of possible motion for the electrons. In each region, the electron circulates along the $\rho$ contours while being moved by the third non-deterministic velocity on the $\phi$ surface. To find an analytic expression for the spin of the electron around a point of maximum $\rho$ in each region, one could assume that the net contribution to the spin angular momentum from the non-deterministic velocity over the period $T$, or multiple periods of $T$, will be zero. However, it is still not easy to find the analytic expression as the contours are not circular. A numerical approach may produce an interesting result.
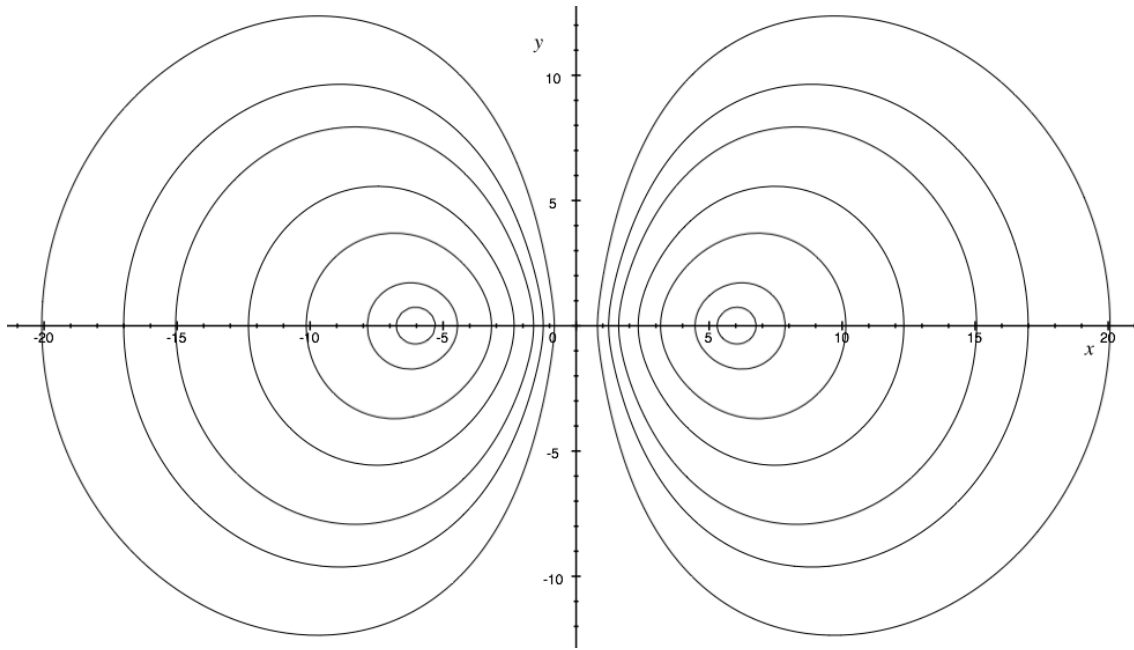
Figure 13: Contours of *R* for $\psi_{322}$ given by $r^2 e^{-r/3} sin^2\theta = constant$ , the values of the constants are 0.5 to 4.5 (interval 0.5) and 4.8

In the case of $\psi_{322}$ , the orbital angular momentum around the vertical axis is non-zero. The electron can cross the horizontal axis without incurring infinite speed but cannot cross the vertical axis as it will incur infinite speed. The electron circulates along the $\rho$ contours while being moved by the third non-deterministic velocity on the $\phi$ surface and at the same time orbiting around the vertical axis. There is a fully three dimensional motion for the electron in this case, as in $\psi_{321}$, since both has non-zero *m*. However, the net spin angular momentum over a period *T* is predicted to be zero because of the non-zero *m* even though its instantaneous value is not zero.

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