An analytic approach to Van der Pol limit cycle

Alberto Strumia*

Istituto Nazionale di Alta Matematica "F. Severi" (I.N.D.A.M.) - Italy

February 3, 2018

Abstract

We propose a way to perform analytic integration of the Van der Pol unforced equation, in order to provide the Cartesian algebraic equation of the integral curve representing the Van der Pol limiting cycle. The method provides exact analytic solutions to the Van der Pol equation in the small control parameter (ϵ) approximation. A plot comparison with the results obtained with the numerical Runge-Kutta method is also examined. The present approach exhibits the advantage of dealing with the only geometry of the integral curve avoiding to involve kinematics, *i.e.,* time dependence of the co-ordinate variable during motion.

Keywords: Dynamical Systems methods Van der Pol limit cycle Location of integral curves.

1 Introduction

The search for exact analytic solutions to the Van der Pol unforced oscillator and the investigation on the related *limit cycle* solution (for an historical overview see, *i.e.* [1]) was attempted by several authors starting from the original paper of Balthasar

*Istituto Nazionale di Alta Matematica F. Severi (INDAM) - Italy – e-mail: info@albertostrumia.it – web: http://www.albertostrumia.it.

Van der Pol[6] until it was "shown that by a series of variable transformations the Van der Pol oscillator can be exactly reduced to Abel's equations of the second kind. The absence of exact analytic solutions in terms of known (tabulated) functions of the reduced equations leads to the conclusion that there are no exact solutions of the Van der Pol oscillator in terms of known (tabulated) functions."[3]

Therefore more and more attention was concentrated on numerical methods in order to obtain affordable solutions (a comparison of the efficiency of different numerical methods has been offered in[5]) and on approximated analytical ways based either on linearization (see, *e.g.*[2]) or on perturbative approaches or on Adomian decomposition.[4] Generally all those approaches are involved in studying the time dependent solutions to the original second order evolutionary Van der Pol equation, written in the equivalent form of a system of two first order o.d.e. Here we intend to propose an alternative way to attack the analytical problem concentrating ourselves on the Cartesian equation of the phase path solution to the non-evolutionary equation of integral curves arising eliminating the time parameter from the autonomous evolutionary o.d.e. Our approach will provide exact analytical solutions to the integral curve equation, when the control parameter ϵ is small respect to unity ($\epsilon \ll 1$). We will be able to integrate solutions up to the third order of approximation, employing the symbolic manipulation package *Xmaxima* (see Appendix).

A graphical comparison with the numeric method by Runge-Kutta shows a very high level of agreement even in correspondence to the upper limit value of the range $\epsilon = 1$.

2 The Van der Pol unforced oscillator

We consider the non-linear second order o.d.e. governing the unforced Van der Pol oscillator:

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0, \tag{1}$$

where:

$$x \equiv x(t), \tag{2}$$

is the unknown real function of time $t \in R$, differentiable at least twice on the time real axis and ϵ is a (generally positive) constant real control parameter.

As usual we reduce the second order eq (1) to an equivalent system of two first order equations:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \epsilon(1-x^2)y - x. \end{cases} \tag{3}$$

The system (3), being *autonomous,* it may be replaced in non-singular points ($y \neq 0$) by the *equation of integral curves*:

$$y' = \epsilon(1-x^2) - \frac{x}{y} \qquad \Longleftrightarrow \qquad yy' = \epsilon(1-x^2)y - x. \tag{4}$$

The solutions to (4):

$$y \equiv y(x), \tag{5}$$

characterize the *Cartesian equations* of the *integral curves* representing the solutions to the system (3) in the phase plane xy, prime denoting the derivative respect to x. One is unable to integrate (4) analytically for any value of ϵ. So in order to proceed analytically, we are led to involve a power series expansion approximation respect to the control parameter ϵ.

3 Power expansion respect to the parameter ϵ

In the present section we will look for analytic solutions to eq (4) which exhibit the form of a series expansion into powers of the control parameter ϵ:

$$f(x) = \sum_{k=0}^{+\infty} h_k(x)\epsilon^k. \tag{6}$$

In the assumption that ϵ is enough less than unity ($\epsilon \ll 1$) the powers ϵ^k decrease as k increases and we may drop higher order contributions according to the desired order of approximation n regardless of assumptions on the series convergence. So we consider the Taylor polynomial:

$$f_{[n]}(x) = h_0(x) + h_1(x)\epsilon + h_2(x)\epsilon^2 + \cdots + h_n(x)\epsilon^n, \tag{7}$$

which is to be introduced into the differential equation:

$$p(x,\epsilon) \equiv f(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x} - \epsilon(1-x^2)f(x) + x = 0. \tag{8}$$

Further we consider the Taylor expansion (polynomial) of $p(x,\epsilon)$ at order n, in the neighborhood of $\epsilon = 0$:

$$p_{[n]}(x,\epsilon) = p_(x,0) + \frac{\partial p}{\partial \epsilon}(x,0)\,\epsilon + \frac{1}{2}\frac{\partial^2 p}{\partial \epsilon^2}(x,0)\,\epsilon^2 + \cdots + \frac{1}{n!}\frac{\partial^n p}{\partial \epsilon^n}(x,0)\,\epsilon^n, \tag{9}$$

which will be required to be zero. We will proceed stepwise searching for analytical solutions at each stage of the expansion approximation, until we are able to proceed analytically.

3.1 Order zero $[\epsilon^0]$

At order zero we have to consider simply:

$$f_{[0]}(x) = h_0(x), \tag{10}$$

which is to be introduced into:

$$p_{[0]}(x,\epsilon) \equiv p_(x,0), \tag{11}$$

leading to the equation of the integral curves of the unforced harmonic oscillator:

$$h_{[0]}(x)\frac{\mathrm{d}h_{[0]}(x)}{\mathrm{d}x} + x = 0 \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{h_{[0]}(x)^2 + x^2}{2}\right] = 0, \tag{12}$$

the solution to which provides the Cartesian equations of the two branches:

$$h^{(\pm)}_{[0]}(x) = \pm\sqrt{A^2 - x^2} \qquad \Longrightarrow \qquad f^{(\pm)}_{[0]}(x) = \pm\sqrt{A^2 - x^2}, \tag{13}$$

of the circular integral curve of radius $|A|$, where A is constant depending on the boundary conditions (see fig. 1).

3.2 Order one (ϵ^1)

At the first order of approximation we need to consider the linear function of ϵ:

$$f_{[1]}((x) = h_{[0]}(x) + h_{[1]}(x)\,\epsilon, \tag{14}$$

where $h_{[0]}(x)$ is now provided by the previous result (13-a). Then we have to test the functions, related to two branches of the integral curve:

$$f^{(\pm)}_{[1]}((x) = \pm\sqrt{A^2 - x^2} + h^{(\pm)}_{[1]}(x)\,\epsilon, \tag{15}$$

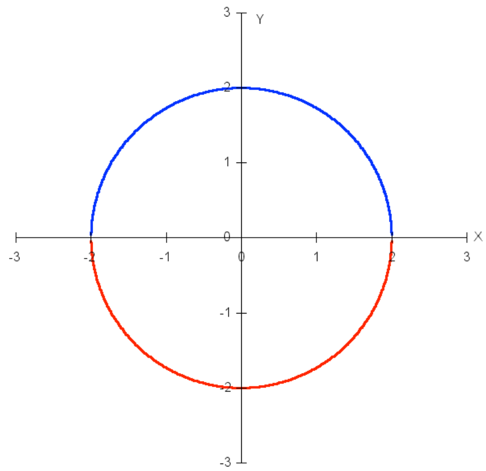

Figure 1: Analytic solution of the integral curve at order $n = 0$ when $|A| = 2$

where the $h_{[1]}^{(\pm)}(x)$ are to be determined imposing that the Taylor polynomial:

$$p_{[1]}(x, \epsilon) = p_(x, 0) + \frac{\partial p}{\partial \epsilon}(x, 0)\,\epsilon, \tag{16}$$

is equal to zero when non-linear powers of ϵ are neglected. Since at order zero the harmonic oscillator solution $h_{[0]}^{(\pm)}$, given by (13) implies that $p(x, 0)$ becomes null, it remains to impose that the coefficient of the linear term ϵ vanishes. Calculations lead to the differential equation for the unknown coefficient $h_{[1]}^{(\pm)}(x)$:

$$(A^2 - x^2)\frac{\mathrm{d}h_{[1]}^{(\pm)}(x)}{\mathrm{d}x} - x\,h_{[1]}^{(\pm)}(x) = (A^2 - x^2)(1 - x^2). \tag{17}$$

The solution to which is given by:

$$h_{[1]}^{(\pm)}(x) = \frac{1}{8}\left[\frac{A^2(4 - A^2)}{\sqrt{A^2 - x^2}}\sin^{-1}\left(\frac{x}{A}\right) + (x\,A^2 - 2x^3 + 4x)\right] + \frac{C^{(\pm)}}{\sqrt{A^2 - x^2}}, \tag{18}$$

where $C^{(\pm)}$ are integration constants. Remarkably the previous result, when $A^2 = 4$ and $C^{(\pm)} = 0$, becomes simply:

$$h_{[1]}^{(+)}(x) \equiv h_{[1]}^{(-)}(x) = x - \frac{x^3}{4}. \tag{19}$$

Then we obtain the analytic solution at the first order approximation respect to ϵ of the Van der Pol limit cycle (see fig. 2):

$$f_{[1]}^{(\pm)}(x) = \pm\sqrt{4-x^2} + \left(x - \frac{x^3}{4}\right)\epsilon. \tag{20}$$

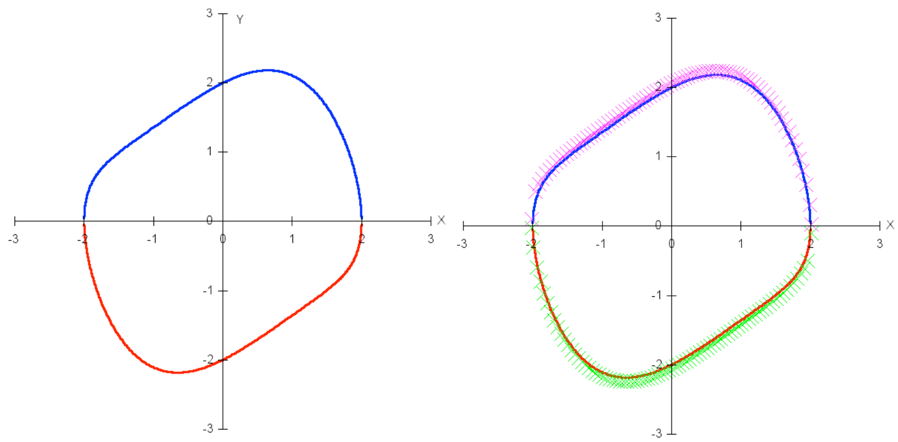

Figure 2: a) Analytic solution of the integral curve at order $n = 1$ for $|A| = 2, \epsilon = .5$
b) Comparison of the 1^{st} order analytic result with Runge-Kutta method

Moreover it happens that, at this law level of approximation, when $|A|$ is near 2 some branches of the phase trajectories appear to approach the limit cycle (see fig. 3), even if they escape elsewhere, since a more accurate order of approximations would be required for a true asymptotic behavior.

3.3 Order two (ϵ^2)

At the second order of approximation we now consider the quadratic function of ϵ:

$$f_{[2]}(x) = h_{[0]}(x) + h_{[1](x)}\epsilon + h_{[2]}(x)\epsilon^2, \tag{21}$$

where $h_{[0]}(x), h_{[1](x)}$ are known being provided respectively by (13) and (18).

It follows that:

$$f_{[2]}^{(\pm)}(x) = \pm\sqrt{A^2-x^2} + \frac{1}{8}\left[\frac{A^2(4-A^2)}{\sqrt{A^2-x^2}}\sin^{-1}\left(\frac{x}{A}\right) + \right.$$

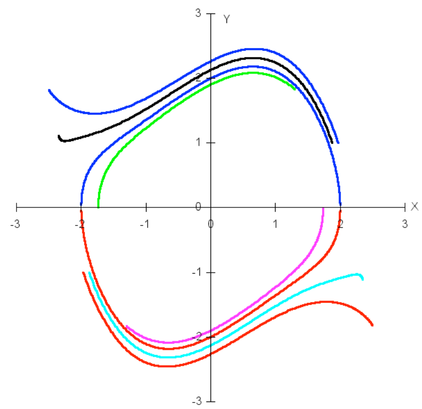

Figure 3: Analytic solutions of the integral curves at order $n = 1$ when $A = |2|, \epsilon = .5$ (limit cycle) and branches of the phase trajectories in correspondence to $|A| = 2.8, 2.4, 1.7365, C = .58, -1.3, -3$ (from top to bottom)

$$(22)$$

$$+(xA^2 - 2x^3 + 4x) + \frac{C^{(\pm)}}{\sqrt{A^2 - x^2}}\Bigg]\epsilon + h_{[2]}^{(\pm)}(x)\epsilon^2,$$

where $h_{[2](x)}$ is to be determined requiring that the Taylor quadratic polynomial:

$$p_{[2]}(x,\epsilon) = p_(x,0) + \frac{\partial p}{\partial \epsilon}(x,0)\epsilon + \frac{1}{2}\frac{\partial^2 p}{\partial \epsilon^2}(x,0)\epsilon^2, \tag{23}$$

becomes null when cubic powers of ϵ and of higher order are neglected. Substitution of (22) into (23) leads to an extremely long and complicated o.d.e., *i.e.*:

$$\begin{aligned}
&x\sin^{-1}\left(\tfrac{x}{A}\right)^2 A^8 + \sqrt{A^2-x^2}\,\Bigg\{-2x^2\sin^{-1}\left(\tfrac{x}{A}\right)A^6 + \left[4x^4\sin^{-1}\left(\tfrac{x}{A}\right) + 64\tfrac{dh_2^{(\pm)}(x)}{dx}\right]A^4 +\\
&+\left[\left(32x^2-16x^4\right)\sin^{-1}\left(\tfrac{x}{A}\right) - 128x^2\tfrac{dh_2^{(\pm)}(x)}{dx} - 64x h_2^{(\pm)}(x) + 16C^{(\pm)}x^2\right]A^2 +\\
&+64x^4\tfrac{dh_2^{(\pm)}(x)}{dx} + 64x^3 h_2^{(\pm)}(x) - 32C^{(\pm)}x^4 + 64C^{(\pm)}x^2\Bigg\} + \left[x^3 - 8x\sin^{-1}\left(\tfrac{x}{A}\right)^2\right]A^6 +\\
&+\left[16x\sin^{-1}\left(\tfrac{x}{A}\right)^2 - 16C^{(\pm)}x\sin^{-1}\left(\tfrac{x}{A}\right) - 5x^5 + 8x^3\right]A^4 + \left[64C^{(\pm)}x\sin^{-1}\left(\tfrac{x}{A}\right) +\right.\\
&\left.+8x^7 - 24x^5 + 16x^3\right]A^2 - 4x^9 + 16x^7 - 16x^5 + 64[C^{(\pm)}]^2 x = 0.
\end{aligned} \tag{24}$$

Remarkably the latter equation becomes dramatically simpler in correspondence to the parameter choices:

$$|A| = 2, \quad C^{(\pm)} = 0, \tag{25}$$

which identifies the Van der Pol limit cycle, resulting simply:

$$\frac{\mathrm{d}\, h_{[2]}^{(\pm)}(x)}{\mathrm{d}x} - \frac{x\, h_{[2]}^{(\pm)}(x)}{4 - x^2} + \frac{x^3}{16}\sqrt{4 - x^2} = 0. \tag{26}$$

Integration yields now to:

$$h_{[2]}^{(\pm)}(x) = \pm \frac{x^6 - 6x^4 + 32}{96\sqrt{4 - x^2}}. \tag{27}$$

Therefore we obtain the second order approximation for the Van der Pol limit cycle equation (see fig. 4):

$$f_{[2]}^{(\pm)}(x) = \pm\sqrt{4 - x^2} + \left(x - \frac{x^3}{4}\right)\epsilon \pm \frac{x^6 - 6x^4 + 32}{96\sqrt{4 - x^2}}\,\epsilon^2. \tag{28}$$

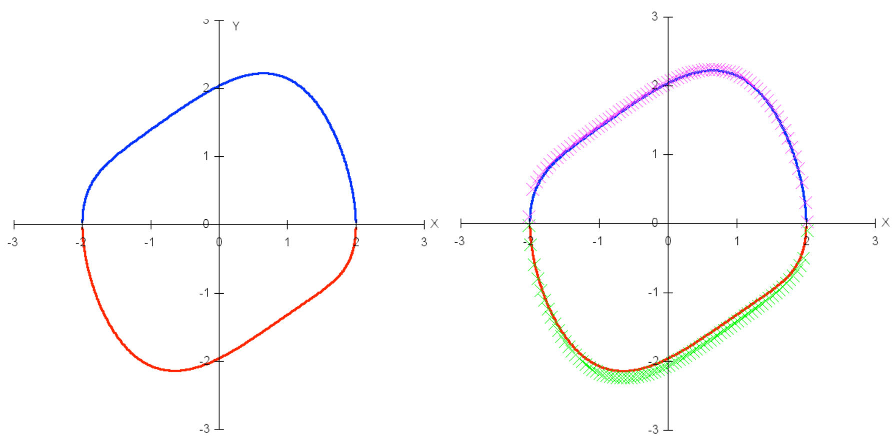

Figure 4: a) Analytic solution of the integral curve at order $n = 2$ for $|A| = 2, \epsilon = .5$
b) Comparison of the 2^{nd} order result with Runge-Kutta method

We point out that even in the upper limit value $\epsilon = 1$ there is a fine agreement between our analytic result and the Runge-Kutta numerical method (see fig. 5).

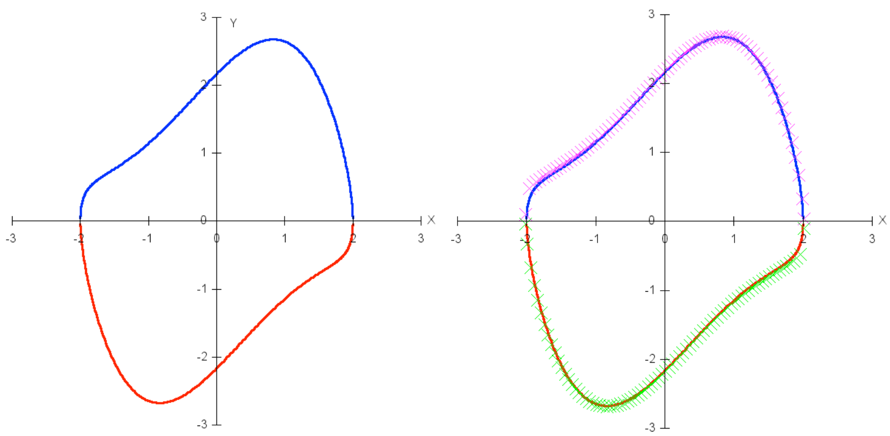

Figure 5: a) Analytic solution of the integral curve at order $n = 2$ for $|A| = 2, \epsilon = 1$
b) Comparison of the 2^{nd} order analytic result with Runge-Kutta method

3.4 Order three (ϵ^3)

Proceeding further we examine now what happens at the third order of approximation.

We have:

$$f_{[3]}(x) = h_{[0]}(x) + h_{[1](x)}\epsilon + h_{[2]}(x)\epsilon^2 + h_{[3]}(x)\epsilon^3, \tag{29}$$

where $h_{[0]}(x), h_{[1](x)}$ are known from the previous results. We limit ourselves to analyze the solution corresponding to the Van der Pol limit cycle, which is characterized by the parameter values $|A| = 2, C^{(\pm)} = 0$. According to (28) we know have:

$$f_{[3]}^{(\pm)}(x) = \pm\sqrt{4 - x^2} + \left(x - \frac{x^3}{4}\right)\epsilon \pm \frac{x^6 - 6x^4 + 32}{96\sqrt{4 - x^2}}\epsilon^2 + h_{[3]}(x)\epsilon^3. \tag{30}$$

The unknown function $h_{[3]}(x)$ will now be determined requiring that the cubic Taylor polynomial:

$$p_{[3]}(x, \epsilon) = p_(x, 0) + \frac{\partial p}{\partial \epsilon}(x, 0)\epsilon + \frac{1}{2}\frac{\partial^2 p}{\partial \epsilon^2}(x, 0)\epsilon^2 + \frac{1}{6}\frac{\partial^3 p}{\partial \epsilon^3}(x, 0)\epsilon^3, \tag{31}$$

vanishes in correspondence to the solution $f_{[3]}^{(\pm)}(x)$, if the powers of ϵ^4 and of higher order are dropped.

Substitution of (30) into (31) yields:

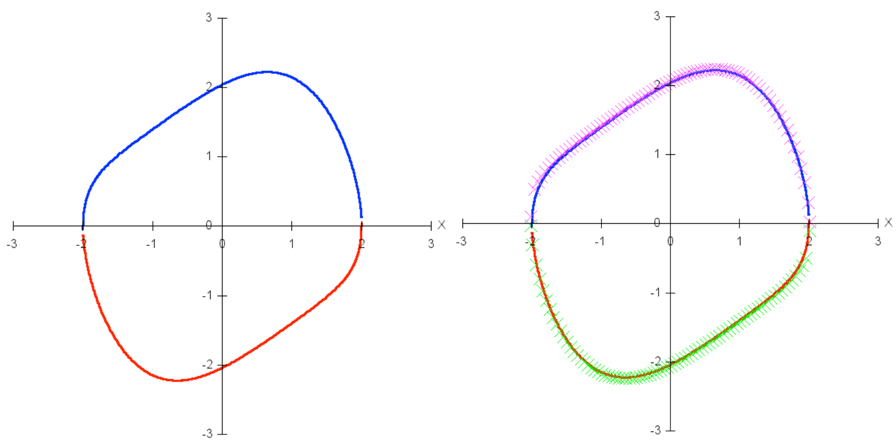

Figure 6: a) Analytic solution of the integral curve at order $n = 3$ for $|A| = 2, \epsilon = .5$
b) Comparison of the 2^{nd} order analytic result with Runge-Kutta method

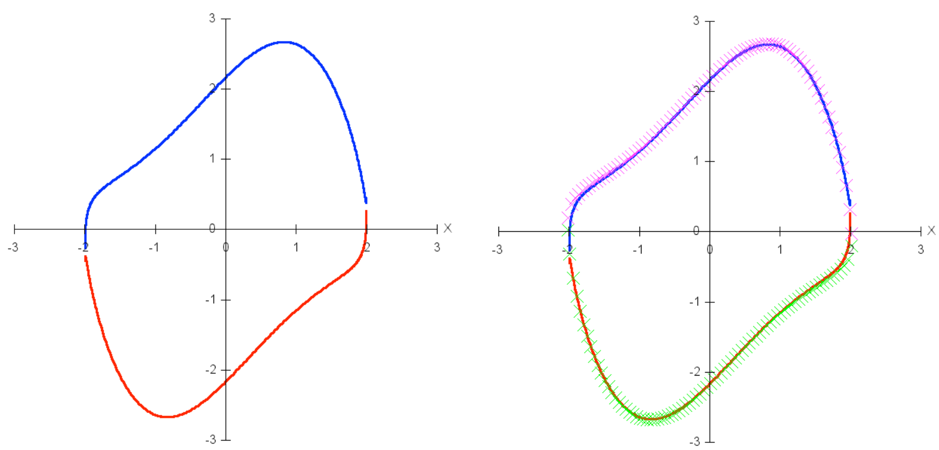

Figure 7: a) Analytic solution of the integral curve at order $n = 3$ for $|A| = 2, \epsilon = 1$
b) Comparison of the 2^{nd} order analytic result with Runge-Kutta method

$$(96x^2 - 384)\frac{\mathrm{d}\,h^{(\pm)}_{[3]}(x)}{\mathrm{d}x} + 96x\,h^{(\pm)}_{[3]}(x) + x^8 - 9x^6 + 24x^4 - 16x^2 = 0, \tag{32}$$

the solution of which results to be:

$$h^{(\pm)}_{[3]}(x) = \frac{\sin^{-1}\left(\frac{x}{2}\right)}{48\sqrt{4-x^2}} - \frac{3x^7 - 22x^5 + 34x^3 + 12x}{2304} + \frac{c^{(\pm)}}{\sqrt{4-x^2}}. \tag{33}$$

The third order approximation fo the Van der Pol cycle function is obtained setting $c^{(\pm)} = 0$ (see fig.6 and fig.7):

$$\begin{aligned} f^{(\pm)}_{[3]} &= \pm\sqrt{4-x^2} + \left(x - \frac{x^3}{4}\right)\epsilon \pm \frac{x^6 - 6x^4 + 32}{96\sqrt{4-x^2}}\epsilon^2 + \\ &+ \left(\frac{\sin^{-1}\left(\frac{x}{2}\right)}{48\sqrt{4-x^2}} - \frac{3x^7 - 22x^5 + 34x^3 + 12x}{2304}\right)\epsilon^3. \end{aligned} \tag{34}$$

At higher orders the o.d.e. become to heavy to be solved analytically even running a symbolic manipulator on a computing machine.

4 Conclusion

We presented an analytic approach to obtain solutions to the equation of integral curves in terms of polynomials of powers of the control parameter ϵ. The advantage of our method, in order to simplify the problem, is that of avoiding time dependence of the unknown functions, pointing directly to the Cartesian equation of the trajectories, being the system of the ordinary differential equations involved, autonomous. We were able to obtain analytical solutions up to the third order of approximation, which exhibit a good agreement with the numerical results provided by the Runge-Kutta method.

APPENDIX

Xmaxima List of commands

```
–>  assume(A>0)$ assume(A^2-x^2>0)$
–>  h[1](x,A,c1):=(-asin(x/A)*A^4+sqrt(A-x)*sqrt(A+x)*(x*A^2-2*x^3+4*x)+
    4*asin(x/A)*A^2+8*c1)/(8*sqrt(A-x)*sqrt(A+x))$
–>  f[2](x,A):=%epsilon*h[1](x,A,0)+%epsilon^2*h[2](x,A)$
–>  h[0](x,A):=sqrt(A^2-x^2)$ g[0](x,A):=h[0](x,A)$
–>  pp[2](x,A):=f[2](x,A)+g[0](x,A)$ pm[2](x,A):=f[2](x,A)-g[0](x,A)$
–>  qp[2](x,A):=pp[2](x,A)*diff(pp[2](x,A),x)-%epsilon*(1-x^2)*pp[2](x,A)+x$
    qm[2](x,A):=pm[2](x,A)*diff(pm[2](x,A),x)-%epsilon*(1-x^2)*pm[2](x,A)+x$
–>  Tp[2](x,A):=taylor(qp[2](x,A),%epsilon,0,2)$ Tm[2](x,A):=taylor(qm[2],%epsilon,0,2)$
–>  Cp[2](x,A):=radcan(coeff(Tp[2](x,A),%epsilon,2))$
    Cm[2](x,A):=radcan(coeff(Tm[2](x,A),%epsilon,2))$
–>  radcan(ode2(Cp[2](x,A)=0,h[2](x,A),x));
```

A=2 (limit cycle Cartesian equation)

```
–>  ratsimp(Cp[2](x,2));
–>  radcan(ode2(Cp[2](x,2)=0,h[2](x,2),x));
–>  radcan(integrate(sqrt(2-x)*sqrt(x-2)*(x^4+2*x^3),x));
–>  ratsimp(Cp[2](x,-2));
–>  h[2](x,A):=ratsimp(((%i*x^6-6*%i*x^4+32*%i)/6)/(%i*16*sqrt(4-x^2)))$
–>  h[2](x,2);
```

Plot of the limit cycle

```
–>  set_plot_option(['plot_format, 'xmaxima])$
–>  wxplot_size:[800,600];
```

%epsilon=.5

```
–>  %epsilon:.5;
–>  results:rk(%epsilon*(1-x^2)-x/y,y,0.1,[x,-2.01,2.01,.05])$
–>  wxplot2d([pp[2](x,2),pm[2](x,2)], [x,-3,3],[y,-3,3],[yx_ratio,1],[axes,solid],[box,false],[legend,false],
    [label,["X",3.1,0.0],["Y",.2,2.9]],[style,[lines,3]])$
–>  wxplot2d([[discrete,-results],[discrete,results]], [x,-3,3],[y-
    3,3],[yx_ratio,1],[axes,solid],[box,false],[legend,false], [la-
    bel,["X",3.1,0.0],["Y",.2,3.9]],[point_type,times],[style,[points,5]])$
–>  wxplot2d([pp[2](x,2),pm[2](x,2),[discrete,-results],[discrete,results]],
    [x,-3,3],[y,-3,3],[yx_ratio,1],[axes,solid],[box,false],[legend,false],
    [label,["X",3.1,0.0],["Y",.2,3.9]],[point_type,times],
    [style,[lines,3],[lines,3],[points,5],[points,5]])$
```

%epsilon=1

```
--> %epsilon:1;
--> results:rk(%epsilon*(1-x^2)-x/y,y,0.1,[x,-2.01,2.01,.05])$
--> pp[2](x,2);
--> pm[2](x,2);
--> pm[2](x,A):=-sqrt(4-x^2)-(x^6-6*x^4+32)/(96*sqrt(4-x^2))+(8*x-2*x^3)/8;
--> wxplot2d([pp[2](x,2),pm[2](x,2)], [x,-3,3],[y,-3,3],[yx_ratio,1],[axes,solid],[box,false],[legend,false],
    [label,["X",3.1,0.0],["Y",.2,2.9]],[style,[lines,3]])$
--> wxplot2d([[discrete,-results],[discrete,results]], [x,-3,3],[y,-
    3,3],[yx_ratio,1],[axes,solid],[box,false],[legend,false], [la-
    bel,["X",3.1,0.0],["Y",.2,3.9]],[point_type,times],[style,[points,5]])$
--> wxplot2d([pp[2](x,2),pm[2](x,2),[discrete,-results],[discrete,results]],
    [x,-3,3],[y,-3,3],[yx_ratio,1],[axes,solid],[box,false],[legend,false],
    [label,["X",3.1,0.0],["Y",.2,3.9]],[point_type,times],
    [style,[lines,3],[lines,3],[points,5],[points,5]])$
```

References

[1] J-M.Ginoux and Ch.Letellier, "Van der Pol and the history of relaxation oscillations: toward the emergence of a concept", Chaos **22**, 023120 (2012); doi: http://dx.doi.org/10.1063/1.3670008 (arXiv:1408.4890v1 [physics.hist-ph] 21 Aug 2014).

[2] S.S.Motsa and P.Sibanda, "A Note on the Solutions of the Van der Pol and Duffing Equations Using a Linearisation Method", Mathematical Problems in Engineering, **2012**, pp.1-10 (2012) (http://dx.doi.org/10.1155/2012/693453).

[3] D.E.Panayotounakos, N.D.Panayotounakou and A.F.Vakakis, "On the lack of analytic solutions of the Van der Pol oscillator", Z.Angew. Math. Mech. **83**, (9), pp.611-615 (2003); doi 10.1002.zamm. 200310040.

[4] P.V.Ramana and B.K.Raghu Prasad, "Modified Adomian Decomposition Method for Van der Pol equations", International Journal of Non-Linear Mechanics, **65**, pp.121-132 (2014).

[5] A.S.Soomro, G.A.Tularam and M.M.S, "A Comparison of Numerical Methods for Solving the Unforced Van Der Pol's Equation", Mathematical Theory and Modeling **3** (2), pp.66-78 (2013); doi: http://dx.doi.org/10.1063/1.3670008.

[6] B. Van der Pol, "On 'relaxation-oscillations'", The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **7** (2), pp. 978-992 (1926).