ON THE RESULT OF KILLIP, MOLCHANOV, AND SAFRONOV

OLEG SAFRONOV

Abstract. We study the properties of operators $(-\Delta)^l \pm V$. We discuss a recent result of Killip, Molchanov, and Safronov [4] which states that if the negative spectra of these operators are discrete, then their positive spectra do not have gaps. Similar statements are also proved for more general operators of the form $\alpha(i\nabla) \pm V$ and operators on the lattice $\mathbb{Z}^d$. Here, we give a more detailed description of the matter.

Notations. For an open domain $\Omega \subset \mathbb{R}^d$, the symbol $\mathcal{H}^l(\Omega)$ denotes the Sobolev space of functions $u : \Omega \mapsto \mathbb{C}$ satisfying the condition

$$||u||^2_{\mathcal{H}^l} = \int_\Omega \sum_{n=0}^{l} \sum_{j_1+\dots+j_d=n} \Big| \frac{\partial^n u}{(\partial x_1)^{j_1} \dots (\partial x_d)^{j_d}} \Big|^2 dx < \infty.$$

The class of smooth functions $u : \mathbb{R}^d \mapsto \mathbb{C}$, such that

$$\sup_{x\in\mathbb{R}^d} \Big((1+|x|)^m |(-\Delta)^l u(x)| \Big) < \infty, \qquad \forall l, m \in \{0, 1, 2, \dots\},$$

is denoted by $\mathcal{S}(\mathbb{R}^d)$. By $\mathfrak{B}$ we denote the class of bounded operators on a Hilbert space $\mathfrak{H}$. We use the notation $\mathfrak{S}_\infty$ for the class of compact operators. For a linear densely defined operator T, the symbols $\mathcal{D}(T)$, $\sigma(T)$ denote the domain and the spectrum of this operator. If T is self-adjoint, then the symbol $E_T(\cdot)$ stands for the (operator-valued) spectral measure of T.

1. Introduction and main results

In this paper, we discuss spectral properties of the self-adjoint operator

$$(-\Delta)^l + V(x), \qquad x \in \mathbb{R}^d, \qquad l \in \mathbb{N}_+ = \{1, 2, 3, \dots\},$$

acting in the Hilbert space $L^2(\mathbb{R}^d)$. It is easy to show that if $V = 0$, then the spectrum of this operator coincides with the set $[0, \infty)$ and is absolutely continuous.

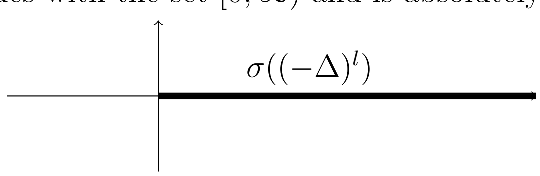

Fig. 1.The spectrum of $(-\Delta)^l$

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However, numerous examples of such operators in quantum mechanics show that the spectrum might look different if $V \neq 0$. In particular, it might have negative eigenvalues. Typically, if V decays at infinity, then the spectrum of this operator looks like the set displayed in the picture below.

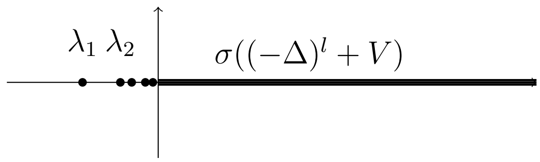

Fig. 2. The spectrum of $(-\Delta)^l + V$

It turns out, that there is a relation between the left and the right parts of this picture, i.e. a relation between the continuous spectra and the sets of negative eigenvalues of such operators. However, in order to describe this relation, one needs to consider two operators

$$H_\pm = (-\Delta)^l \pm V.$$

We begin with the following result which is one of the theorems proved in [8].

Theorem 1.1. [cf.[3]**,** [8]**]** *Let $l = 1$. Let V be a real-valued bounded function on $\mathbb{R}^d$. Suppose that the spectra of $H_+ = -\Delta + V$ and $H_- = -\Delta - V$ in $(-\infty, 0)$ consist of eigenvalues λ_j^+ and λ_j^-, satisfying*

$$\sum_j |\lambda_j^+|^{1/2} < \infty, \qquad \sum_j |\lambda_j^-|^{1/2} < \infty. \tag{1.1}$$

Then the absolutely continuous spectrum of each operator H_+, H_- is essentially supported on the set $[0, \infty)$.

Remark. Although this theorem can be proved in any dimension d, the paper [8] proves it only for $d = 3$. The arguments in $d \neq 3$ require a small modification. Still, the proof in [8] is more detailed compared to the less preferable versions from [5] and [7]. The case $d = 1$ was considered by Damanik and Remling in [3]. For a similar result handling the case of finitely many negative eigenvalues see the articles [2] and [6].

In order to formulate the result of Killip, Molchanov and Safronov, we need to recall the relation between self-adjoint operators and quadratic forms.

Let $a[\cdot, \cdot]$ be the sesquilinear non-negative form defined by

$$a[u, u] = \int_{\mathbb{R}^d} |\xi|^{2l} |\hat{u}(\xi)|^2 d\xi, \qquad \hat{u} = \Phi u, \qquad l \in \mathbb{N}_+ = \{1, 2, 3, \dots\}, \tag{1.2}$$

where Φ is the Fourier transform operator. If we define the domain $d[a]$ as the Sobolev space $\mathcal{H}^l(\mathbb{R}^d)$, this form will be closed in $L^2(\mathbb{R}^d)$. The latter means that the domain is a complete Hilbert space with the inner product

$$a_1[u, v] = a[u, v] + (u, v), \qquad u, v \in d[a]. \tag{1.3}$$

The form $a[\cdot, \cdot]$ generates a unique operator $(-\Delta)^l$ self-adjoint in $L^2(\mathbb{R}^d)$.

Let V be a real-valued measurable function and let $v[\cdot,\cdot]$ be the Hermitian form

$$v[u,u] = \int_{\mathbb{R}^d} V(x)|u(x)|^2 dx, \qquad u \in d[a].$$

We will assume that

$$\sup_{n\in\mathbb{Z}^d} \int_{[0,1)^d+n} |V|^p dx < \infty, \tag{1.4}$$

where

$$\begin{cases} p > d/2l, & \text{if} \quad d \geq 2l; \\ p = 1, & \text{if} \quad d < 2l. \end{cases} \tag{1.5}$$

It follows from Sobolev's embedding theorems that, under these conditions, the form v satisfies

$$|v[u,u]| \leq \varepsilon a_1[u,u] + C(\varepsilon)||u||^2, \qquad \forall u \in d[u], \forall\varepsilon > 0. \tag{1.6}$$

Consider now the sesquilinear form $h_\pm = a \pm v$ on $d[a]$. It follows from (1.6) that $h_\pm$ is semi-bounded and closed on $d[a]$. Let $H_\pm$ be the self-adjoint operator corresponding to the form $h_\pm$. Then, the domain $\mathcal{D}(H_\pm)$ is a subset of $d[a]$. The two conditions

$$u \in \mathcal{D}(H_\pm), \text{ and } H_\pm u = w$$

are equivalent to the fact that

$$h_\pm[u,v] = (w,v), \qquad \forall v \in d[a].$$

By selecting an appropriate $\gamma > 0$, one can achieve that $h_\pm + \gamma \geq 1$. Then the norm corresponding to the inner product (1.3) will be equivalent to the norm corresponding to the inner product

$$h_\pm[u,v] + \gamma(u,v), \qquad u, v \in d[a].$$

The latter is equivalent to the relations

$$\begin{aligned} &((-\Delta)^l + I)^{1/2}(H_\pm + \gamma I)^{-1/2} \in \mathfrak{B}, \\ &(H_\pm + \gamma I)^{1/2}((-\Delta)^l + I)^{-1/2} \in \mathfrak{B}, \end{aligned} \tag{1.7}$$

where $\mathfrak{B}$ denotes the class of bounded operators.

When reading Theorem 1.1, one gets the impression that the rate of accumulation of the eigenvalues $\lambda_j^\pm$ to zero determines the properties of the positive spectra of the operators H_+, H_-. In the theorem of Killip, Molchanov and Safronov [4], one considers the case where $\lambda_j^\pm \to 0$ in an arbitrary way, when no information about the rate of accumulation of the eigenvalues to 0 is given. We formulate it in a very general setting for an arbitrary integer $l > 0$.

Theorem 1.2. (see [4]**)** *Let V be a real-valued measurable function on $\mathbb{R}^d$ satisfying* (1.4) *with p described in* (1.5). *Let H_+ and H_- be the operators corresponding to the forms $a+v$ and $a-v$. Suppose that the spectra of H_+ and H_- in $(-\infty,0)$ are discrete. Then the spectrum of each operator H_+, H_- contains the interval* $[0,\infty)$.

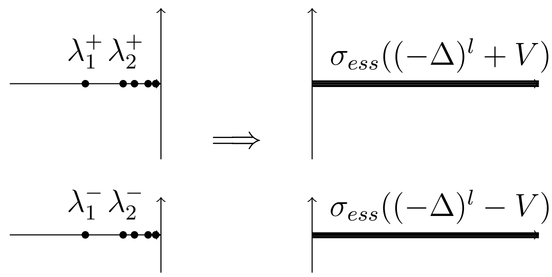

Fig. 3. The left parts of the pictures imply the right parts

Remark. Theorem 1.2 does not hold for arbitrary bounded perturbations V, which do not have to be multiplication operators. A counterexample is the case $V = E_{(-\Delta)^l}\big([1,2]\big) \cdot (-\Delta)^l$.

One of the advantages of Theorem 1.2 is that one does not impose any restriction on the dimension d and the integer parameter l. The case $d = l = 1$ should be mentioned separately, since the relation between the negative and positive spectra of one-dimensional Schrödinger operators was found a long time ago (in 2004) by Damanik and Remling. However, the corresponding paper [3] containing one-dimensional versions of Theorems 1.1 and 1.2 was published a little bit later in 2007. A similar observation for a one(and two)-dimensional discrete Schrödinger operator was made even earlier in 2003 by Damanik, Hundertmark, Killip and Simon [1].

The case $d = l = 1$ seems to be different from the case $d > 2l$. If $d = l = 1$, the statement of Theorem 1.2 follows from the fact that the operator

$$\Big(-\frac{d^2}{dx^2} - z\Big)^{-1} - (H_\pm - z)^{-1}, \qquad z \in \mathbb{C} \setminus \mathbb{R}$$

is compact. On the other hand, the conditions of Theorem 1.2 **do not** imply that

$$((-\Delta)^l - z)^{-1} - (H_\pm - z)^{-1}, \qquad z \in \mathbb{C} \setminus \mathbb{R}$$

is compact if $d > 2l$. Still, our proof does not feel any difference between the cases corresponding to different values of the parameters d and l.

Our arguments rely on a classical result established by H. Weyl. We start with the definition.

A sequence u_n is called singular for a self-adjoint operator A and $\lambda \in \mathbb{R}$, if

1) $u_n \in \mathcal{D}(A), \quad \inf_n ||u_n|| > 0$;
2) u_n converges to zero weakly;
3) $(A - \lambda)u_n$ converges to zero strongly (in the norm topology).

Theorem 1.3. [H. Weyl] *Let A be a self-adjoint operator in a separable Hilbert space. The condition that $\lambda \in \mathbb{R}$ is a point of the essential spectrum of A is equivalent to existence of a singular sequence for A and λ.*

2. Proof of Theorem 1.2

Most of the arguments are borrowed from [4]. The first statement of this section allows one to reduce the study of spectra of the unbounded operators H_+ and H_- to the study of the properties of continuous operators.

Proposition 2.1. *Let $-\gamma < \inf \sigma(H_\pm)$. A non-negative number $\lambda \geq 0$ is a point of the essential spectrum of $H_\pm$ if and only if 0 is a point of the essential spectrum of the operator*

$$Q_\pm = (H_\pm - \lambda)(H_\pm + 2\gamma I)^{-1}. \tag{2.1}$$

Proof. Indeed, denote the spectral measure of a self-adjoint operator $T = T^*$ by $E_T(\cdot)$. A real number s is a point of the essential spectrum of T if and only if $\mathrm{rank}\big[E_T(s-\varepsilon, s+\varepsilon)\big] = \infty$ for all $\varepsilon > 0$. Choose now $0 < \varepsilon < 1/2$. Then $E_{Q_\pm}(-\varepsilon, \varepsilon) = E_{H_\pm}(\Omega)$, where $\Omega = \big(\frac{\lambda - 2\gamma\varepsilon}{1+\varepsilon}, \frac{\lambda + 2\gamma\varepsilon}{1-\varepsilon}\big)$. Consequently,

$$\mathrm{rank}\big[E_{Q_\pm}(-\varepsilon, \varepsilon)\big] = \infty,\ \forall \varepsilon > 0 \qquad \Longleftrightarrow \qquad \mathrm{rank}\big[E_{H_\pm}(\lambda - \varepsilon, \lambda + \varepsilon)\big] = \infty,\ \forall \varepsilon > 0.$$

The proof is completed. $\square$

The proof of Theorem 1.2 is based on an application of Theorem 1.3. In order to construct singular sequences for the operators $H_\pm$, we first define functions u_n on $\mathbb{R}^d$ as follows. Let $\varphi \in C^\infty(\mathbb{R}^d)$ be a function with the compactly supported Fourier transform $\hat{\varphi} \in C_0^\infty(\mathbb{R}^d)$ and the property

$$\int_{\mathbb{R}^d} |\varphi(x)|^2 dx = 1.$$

Set

$$u_n(x) = n^{-d/2} \varphi(x/n), \qquad \forall n \in \mathbb{N}_+ = \{1, 2, 3, \dots\} \tag{2.2}$$

Let us now use the information about the negative spectra of the operators H_+ and H_- to prove the following statement.

Proposition 2.2. *Assume that conditions of Theorem 1.2 are fulfilled. Then*

$$|H_\pm|^{1/2} u_n \to 0, \ \text{as } n \to \infty. \tag{2.3}$$

Proof. Note that the sequence u_n converges to 0 weakly. Consequently, any compact operator T maps u_n onto a sequence Tu_n strongly convergent to 0. In particular,

$$\Big(E_{H_\pm}(-\infty, 0)[(H_\pm + \gamma I)^{1/2} - \gamma^{1/2} I]\Big) u_n \to 0, \quad \text{as} \quad n \to \infty \tag{2.4}$$

and

$$E_{H_\pm}(-\infty, 0)|H_\pm|^{1/2} u_n \to 0, \quad \text{as} \quad n \to \infty. \tag{2.5}$$

On the other hand,

$$h_+[u_n, u_n] + h_-[u_n, u_n] = 2a[u_n, u_n] = 2n^d \int_{\mathbb{R}^d} |\xi|^{2l} |\hat{\varphi}(n\xi)|^2 d\xi \to 0 \tag{2.6}$$

as $n \to \infty$. Combining (2.4) with (2.6) we obtain that

$$||E_{H_\pm}[0,\infty)(H_\pm+\gamma I)^{1/2}u_n||^2-\gamma||E_{H_\pm}[0,\infty)u_n||^2\to 0, \quad \text{as } n\to\infty.$$

The latter relation simply means that

$$\lim_{n\to\infty}||E_{H_\pm}[0,\infty)|H_\pm|^{1/2}u_n||=0. \tag{2.7}$$

Thus (2.3) follows from (2.5) and (2.7). $\square$

Let $k \in \mathbb{R}^d$. Our construction of a singular sequence for $H_\pm$ and $\lambda=|k|^{2l}$ involves the unitary operator U defined by

$$(Uu)(x)=\exp(ikx)u(x), \qquad u\in L^2(\mathbb{R}^d). \tag{2.8}$$

Proposition 2.3. *Let U be the operator defined in (2.8). Let $u \in \mathcal{S}(\mathbb{R}^d)$ and let $v \in d[a]$. Assume that V satisfies the condition (1.4) with p described in (1.5). Then*

$$h_\pm[Uu,v]=h_\pm[u,U^*v]+(\big[(-\Delta)^l,\,U\big]u,v), \tag{2.9}$$

where $[(-\Delta)^l,\,U]=(-\Delta)^lU-U(-\Delta)^l$ defined on $\mathcal{D}([(-\Delta)^l,\,U])=\mathcal{S}(\mathbb{R}^d)$.

Proof. According to the definition of Sobolev's derivative,

$$a[Uu,v]=a[u,U^*v]+(\big[(-\Delta)^l,\,U\big]u,v)$$

for all $u\in C_0^\infty(\mathbb{R}^d)$. By a density argument, this relation also holds for any $u\in\mathcal{S}(\mathbb{R}^d)$. The latter statement implies (2.9). $\square$

Corollary 2.4. *Let $\lambda=|k|^{2l}$ where $k\in\mathbb{R}^d$. Let U be the operator (2.8). Assume that V satisfies the condition (1.4) with p as in (1.5). Then, for any $u\in\mathcal{S}(\mathbb{R}^d)$,*

$$\begin{aligned}(H_\pm-\lambda)(H_\pm+2\gamma)^{-1}Uu=(H_\pm+2\gamma)^{-1}([(-\Delta)^l,\,U]-\lambda U)u+\\ \big(|H_\pm|^{1/2}U^*(H_\pm+2\gamma)^{-1}\big)^*S_\pm|H_\pm|^{1/2}u,\end{aligned} \tag{2.10}$$

where $S_\pm=E_{H_\pm}[0,\infty)-E_{H_\pm}(-\infty,0)$.

Proof. It is sufficient to show that

$$\begin{aligned}H_\pm(H_\pm+2\gamma)^{-1}Uu=(H_\pm+2\gamma)^{-1}[(-\Delta)^l,\,U]u+\\ \big(|H_\pm|^{1/2}U^*(H_\pm+2\gamma)^{-1}\big)^*S_\pm|H_\pm|^{1/2}u.\end{aligned}$$

For that purpose, consider the inner product

$$\big(H_\pm(H_\pm+2\gamma)^{-1}Uu,w\big)=h_\pm\big[Uu,(H_\pm+2\gamma)^{-1}w\big], \qquad \forall w\in L^2(\mathbb{R}^d).$$

According to (2.9),

$$\begin{aligned}\big(H_\pm(H_\pm+2\gamma)^{-1}Uu,w\big)=h_\pm\big[u,U^*(H_\pm+2\gamma)^{-1}w\big]+(\big[(-\Delta)^l,\,U\big]u,(H_\pm+2\gamma)^{-1}w)=\\ \big(S_\pm|H_\pm|^{1/2}u,|H_\pm|^{1/2}U^*(H_\pm+2\gamma)^{-1}w\big)+((H_\pm+2\gamma)^{-1}\big[(-\Delta)^l,\,U\big]u,w), \qquad \forall w\in L^2(\mathbb{R}^d).\end{aligned}$$

The last line implies (2.10). $\square$

Set now

$$\psi_n = U u_n, \qquad n \in \mathbb{N}_+, \tag{2.11}$$

where u_n is the sequence defined in (2.2).

Proposition 2.5. *Let $\lambda > 0$. Assume that conditions of Theorem 1.2 are fulfilled. Then there exists a singular sequence for the operator $H_\pm$ and the point λ.*

Proof. Let us first find $k \in \mathbb{R}^d$ such that $\lambda = |k|^{2l}$. According to Proposition 2.1, it is enough to show that the sequence ψ_n defined in (2.11) is singular for the operator $Q_\pm$ and the point 0. Since the operator $Q_\pm$ is bounded, $\psi_n \in \mathcal{D}(Q_\pm)$ for all n. It is also obvious that $||\psi_n|| = 1$ and that the sequence ψ_n converges to 0 weakly. It remains to show that

$$||Q_\pm \psi_n|| \to 0 \text{ as } \ n \to \infty. \tag{2.12}$$

For that purpose, we note that the operator

$$|H_\pm|^{1/2} U^* (H_\pm + 2\gamma)^{-1} =$$

$$\Big(|H_\pm|^{1/2}(H_\pm+\gamma)^{-1/2}\Big) \cdot \Big((H_\pm+\gamma)^{1/2}((-\Delta)^l + I)^{-1/2}\Big) \cdot$$

$$\Big(((-\Delta)^l + I)^{1/2} U^* ((-\Delta)^l + I)^{-1/2}\Big) \cdot \Big(((-\Delta)^l + I)^{1/2}(H_\pm + 2\gamma)^{-1}\Big)$$

is bounded. Therefore, (2.12) follows from (2.3) and (2.10) combined with the obvious fact that

$$\lim_{n\to\infty} ||\big(\big[(-\Delta)^l,\, U\big] - \lambda U\big) u_n|| = 0.$$

The proof is completed. $\square$

Now Theorem 1.2 follows from Theorem 1.3 and Proposition 2.5.

3. Additional remarks

In this section, we state and prove three different results from [4]. One of them is formulated for operators in an arbitrary Hilbert space, another theorem holds for operators on $\mathbb{R}^d$ under more general conditions than restrictions of Theorem 1.2. The third result pertains to the theory of operators on the lattice $\mathbb{Z}^d$.

1. Let $a[\cdot,\cdot]$ be a sesquilinear non-negative form in a separable Hilbert space $\mathfrak{H}$ defined on a linear subset $d[a]$ dense in $\mathfrak{H}$. Assume that this form is closed in $\mathfrak{H}$. The latter means that the domain is a complete Hilbert space with the inner product

$$a_1[u,v] = a[u,v] + (u,v), \qquad u, v \in d[a].$$

The form $a[\cdot,\cdot]$ generates a unique operator A self-adjoint in $\mathfrak{H}$.

Let $v[\cdot,\cdot]$ be a Hermitian form satisfying

$$|v[u,u]| \le \varepsilon a_1[u,u] + C(\varepsilon)||u||^2, \qquad \forall u \in d[u], \forall \varepsilon > 0. \tag{3.1}$$

Consider now the sesquilinear form $h_\pm = a \pm v$ on $d[a]$. It follows from (3.1) that $h_\pm$ is semibounded and closed on $d[a]$. Therefore, $h_\pm$ generates a self-adjoint operator $H_\pm$.

Theorem 3.1. *Let $a[\cdot,\cdot]$ be a sesquilinear non-negative closed form in a Hilbert space $\mathfrak{H}$. Assume that the essential spectrum of the self-adjoint operator A corresponding to the form $a[\cdot,\cdot]$ contains* 0. *Let H_+ and H_- be the operators corresponding to the forms $a+v$ and $a-v$, where v satisfies* (3.1). *Suppose that the spectra of H_+ and H_- in $(-\infty,0)$ are discrete. Then 0 is a point of the essential spectrum of each operator H_+, H_-.*

Proof. Indeed, let $u_n \in d[a]$ be a sequence of vectors such that $||u_n|| = 1$ and $||A^{1/2}u_n|| \to 0$. We will also assume that u_n converges to zero weakly. Such a sequence exists, because 0 is a point of the essential spectrum of the operator A. Since

$$h_+[u_n,u_n] + h_-[u_n,u_n] = 2||A^{1/2}u_n||^2,$$

we conclude that $||\,|H_\pm|^{1/2}u_n||$ tends to zero as $n\to\infty$. Consequently, 0 is a point of the essential spectrum of $H_\pm$. $\square$

2. Let $\mathfrak{H} = L^2(\mathbb{R}^d)$ again. Instead of defining the form $a[\cdot,\cdot]$ by (1.2), we set

$$a[u,u] = \int_{\mathbb{R}^d} \alpha(\xi)|\hat{u}(\xi)|^2 d\xi, \qquad \hat{u} = \Phi u, \tag{3.2}$$

where Φ is the Fourier transform operator and α is a non-negative continuous function on $\mathbb{R}^d$ such that

$$C_1(1+|\xi|^{2l}) \le 1+\alpha(\xi) \le C_2(1+|\xi|^{2l}), \qquad l \in \mathbb{N}_+ = \{1,2,3,\dots\}, \tag{3.3}$$

with some positive constants C_1 and C_2. If we define the domain $d[a]$ as the the Sobolev space $\mathcal{H}^l(\mathbb{R}^d)$, this form will be closed in $L^2(\mathbb{R}^d)$. The form $a[\cdot,\cdot]$ generates a unique self-adjoint operator $A = \alpha(i\nabla)$.

Let now V be a real-valued measurable function obeying the condition (1.4) with p as in (1.5). Then the form

$$v[u,u] = \int_{\mathbb{R}^d} V(x)|u(x)|^2 dx, \qquad u \in d[a],$$

satisfies (1.6) with a defined by (3.2). The theorem below is an analogue of Theorem 1.2 formulated for the operators $H_\pm$ corresponding to the forms $h_\pm = a \pm v$ with a described by (3.2).

Theorem 3.2. (see [4]**)** *Assume that $\alpha \in C(\mathbb{R}^d)$ appearing in the definition* (3.2) *of the form $a[\cdot,\cdot]$ is a continuous function obeying* (3.3) *and such that*

$$\min_{\xi\in\mathbb{R}^d} \alpha(\xi) = 0.$$

Let V be a real-valued measurable function on $\mathbb{R}^d$ satisfying (1.4) *with p described in* (1.5). *Let H_+ and H_- be the operators corresponding to the forms $a+v$ and $a-v$. Suppose that the spectra of H_+ and H_- are discrete in the interval $(-\infty,0)$. Then the spectrum of each operator H_+, H_- contains the half-line $[0,\infty)$.*

Proof. Let $\lambda \geq 0$. We are going to construct a singular sequnce for the operator $H_\pm$ and the point λ. According to the assumptions of the theorem, there exist two vectors $\xi_0 \in \mathbb{R}^d$ and $\xi_* \in \mathbb{R}^d$ in the space $\mathbb{R}^d$ with the properties

$$\alpha(\xi_0) = 0, \qquad \alpha(\xi_*) = \lambda. \tag{3.4}$$

Set

$$u_n(x) = n^{-d/2} e^{i\xi_0 x} \varphi(x/n), \qquad \forall n \in \mathbb{N}_+ = \{1, 2, 3, \dots\},$$

where φ is a function whose Fourier transform $\hat{\varphi} \in C_0^\infty(\mathbb{R}^d)$ is compactly supported and such that

$$\int_{\mathbb{R}^d} |\varphi(x)|^2 dx = 1.$$

Our arguments follow closely the arguments of the proof of Theorem 1.2. We also define ψ_n by (2.11). However, in the formula (2.8) defining the operator U, we set

$$k = \xi_* - \xi_0. \tag{3.5}$$

In this case, $\Phi U \Phi^*$ is a shift operator that turns the function $\hat{u}(\xi)$ into the function $\hat{u}(\xi - k)$. Therefore, to prove the relation

$$\lim_{n\to\infty} ||\big([A,\, U] - \lambda U\big) u_n|| = 0,$$

we can use the representation

$$||\big([A,\, U] - \lambda U\big) u_n||^2 = \int_{\mathbb{R}^d} \Big| \Big(\big(\alpha(\xi) - \alpha(\xi - k)\big) - \lambda \Big) \hat{u}_n(\xi - k) \Big|^2 d\xi$$

and the fact that $|\hat{u}_n(\xi)|^2$ converges to the Dirac delta-function $\delta(\xi - \xi_0)$ in the sense of distributions. Consequently,

$$||\big([A,\, U] - \lambda U\big) u_n||^2 \to \Big| \big(\alpha(\xi_0 + k) - \alpha(\xi_0)\big) - \lambda \Big|^2 \qquad \text{as } n \to \infty.$$

It remains to note that $\big|(\alpha(\xi_0 + k) - \alpha(\xi_0)) - \lambda\big|^2 = 0$ due to (3.4) and (3.5). The proof is completed. $\square$

3. Let, this time, $\mathfrak{H} = \ell^2(\mathbb{Z}^d)$. Let $\mathbb{T}^d$ be the torus $\mathbb{R}^d/(2\pi\mathbb{Z}^d)$ and let $\Phi : \mathfrak{H} \to L^2(\mathbb{T}^d)$ be the unitary operator defined by

$$(\Phi u)(\xi) = (2\pi)^{-d/2} \sum_{n\in\mathbb{Z}^d} e^{-i\xi n} u(n).$$

Instead of defining the operator A by its quadratic form, we set

$$A = \Phi^*[\alpha]\Phi, \tag{3.6}$$

where $[\alpha]$ denotes the operator of multiplication by a continuous real-valued function $\alpha : \mathbb{T}^d \to \mathbb{R}$. Let now V be a real-valued bounded potential on $\mathbb{Z}^d$. The theorem below is an analogue of Theorem 1.2 formulated for the lattice operators $H_\pm = A \pm V$.

Theorem 3.3. **(see** [4]**)** *Assume that $\alpha \in C(\mathbb{T}^d)$ appearing in* (3.6) *is a continuous function. Let V be a real-valued bounded function on $\mathbb{Z}^d$. Suppose that the spectra of $H_+ = A + V$ and $H_- = A - V$ are discrete below the point $\gamma_0 = \min_\xi \alpha(\xi)$. Then the spectrum of each operator H_+, H_- contains the spectrum of A.*

Proof. One should simply repeat the arguments of the proof of Theorem 3.2. No substantial change is needed. □

As a consequence of Theorem 3.3, we obtain the following result formulated for the Schrödinger operator $A+V$, where A is defined by

$$(Au)(n) = \sum_{|m-n|=1} u(m), \qquad n, m \in \mathbb{Z}^d.$$

Corollary 3.4. *Let α appearing in the definition* (3.6) *of the operator A be the function*

$$\alpha(\xi) = \sum_{j=1}^{d} 2\cos(\xi_j).$$

Let V be a real-valued bounded potential on $\mathbb{Z}^d$. Assume that the spectrum of $H_+ = A+V$ is discrete outside of the interval $[-2d, 2d]$. Then the spectrum of H_+ contains the interval $[-2d, 2d]$.

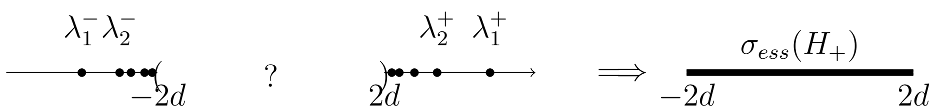

Fig. 4. The left picture implies the right one

Proof. It is enough to mention that the operator $H_- = A - V$ is unitary equivalent to the operator $-H_+ = -A - V$. The latter fact is very well known: the corresponding unitary operator $\mathcal{U}$ is defined by

$$\mathcal{U}\psi(n) = (-1)^{n_1+\cdots+n_d}\psi(n), \qquad n \in \mathbb{Z}^d.$$

□

The latter corollary could be also viewed as a particular case of a more general statement from [4], formulated and proved for arbitrary continuous symbols $\alpha(\xi)$. It says that if the spectrum of $A+V$ is discrete outside of $\alpha[\mathbb{T}^d]$, then $\alpha[\mathbb{T}^d] \subset \sigma(A+V)$.

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Oleg Safronov, Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA

E-mail address: osafrono@uncc.edu