ON PERIODIC SOLUTIONS TO SOME LAGRANGIAN SYSTEM WITH TWO DEGREES OF FREEDOM

OLEG ZUBELEVICH

DEPT. OF THEORETICAL MECHANICS,
MECHANICS AND MATHEMATICS FACULTY,
M. V. LOMONOSOV MOSCOW STATE UNIVERSITY
RUSSIA, 119899, MOSCOW, MGU

Abstract. A Lagrangian system with two degrees of freedom is considered. The configuration space of the system is a cylinder. A large class of periodic solutions has been found. The solutions are not homotopy equivalent to each other.

1. Statement of the Problem and Main Result

This short note is devoted to the following dynamical system.

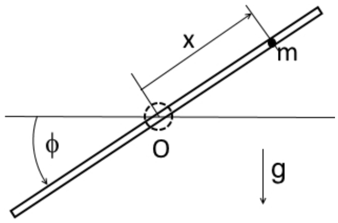

Figure 1. the tube and the ball

A thin tube can rotate freely in the vertical plane about a fixed horizontal axis passing through its centre O. A moment of inertia of the tube about this axis is equal to J. The mass of the tube is distributed symmetrically such that tube's centre of mass is placed at the point O.

Inside the tube there is a small ball which can slide without friction. The mass of the ball is m. The ball can pass by the point O and fall out from the ends of the tube.

The system undergoes the standard gravity field $\boldsymbol{g}$.

It seems to be evident that for typical motion the ball reaches an end of the tube and falls down out the tube. It is surprisingly, at least for the first

2000 *Mathematics Subject Classification.* 34C25, 70F20 .
Key words and phrases. Lagrangian systems, periodic solutions.
Partially supported by grants RFBR 12-01-00441., Science Sch.-2964.2014.1.

glance, that this system has very many periodic solutions such that the tube turns around several times during the period.

The sense of generalised coordinates ϕ, x is clear from Figure 1.

A kinetic energy and a potential of the system are given by the formulas

$$T=\frac{1}{2}\Big(mx^2+J\Big)\dot\phi^2+\frac{1}{2}m\dot x^2,\quad V=mgx\sin\phi.$$

By the suitable choice of dimension of units we obtain

$$J=1,\quad g=1,\quad m=1.$$

So that a Lagrangian of the system is

$$L(x,\phi,\dot x,\dot\phi)=\frac{1}{2}\Big(x^2+1\Big)\dot\phi^2+\frac{1}{2}\dot x^2-x\sin\phi. \tag{1.1}$$

Theorem 1.1. *For any constants $\omega>0$, $k\in\mathbb{N}$ system (1.1) has a solution $\phi(t), x(t)$, $t\in\mathbb{R}$ such that*
1) $x(t)=-x(-t)$, $\phi(t)=-\phi(-t)$;
2) $x(t+\omega)=x(t)$, $\phi(t+\omega)=\phi(t)+2\pi k$.

This theorem means that if ω and k are given and the tube is long enough then the system has an $\omega-$periodic motion and the tube turns around k times during the period.

2. Proof of Theorem 1.1

2.1. **Preliminary Remarks.** Introduce a space

$$H^1_o(-a,a)=\{u\in H^1(-a,a)\mid u(-t)=-u(t)\},\quad a\in(0,\infty).$$

Recall that the Sobolev space $H^1(-a,a)$ is compactly embedded to $C[-a,a]$.

Lemma 2.1. *Let $u\in H^1_o(-a,a)$ then the following inequalities hold*

$$\|u\|^2_{L^2(0,a)}\le\frac{a^2}{2}\|\dot u\|^2_{L^2(0,a)},\quad \|u\|^2_{C[0,a]}\le a\|\dot u\|^2_{L^2(0,a)}.$$

This Lemma is absolutely standard, we bring its proof just for completeness of exposition.

Remark 1. *Lemma 2.1 implies that the function $u\mapsto\|\dot u\|_{L^2(0,a)}$ is a norm of $H^1_o(-a,a)$ and this norm is equivalent to the standard norm of $H^1(-a,a)$.*

Proof of Lemma 2.1. We prove only the first inequality the second one goes in the same way. First assume that a function $u\in H^1(-a,a)$ is smooth. From the formula

$$u(t)=\int_0^t\dot u(s)ds$$

it follows that

$$\int_0^a u^2(s)ds=\int_0^a\Big(\int_0^t\dot u(s)ds\Big)^2dt.$$

It remains to observe that by the Cauchy inequality

$$\Big|\int_0^t \dot u(s)ds\Big|\le\int_0^t|\dot u(s)|ds\le\|\dot u\|_{L^2(0,a)}\Big(\int_0^t ds\Big)^{1/2},\quad t\in[0,a].$$

Since the space of smooth functions is dense in $H^1(-a,a)$, the inequality under consideration holds for all $u\in H^1(-a,a)$.

The Lemma is proved.

Lemma 2.2. *Being endowed with a collection of seminorms*

$$\|u\|_n=\|u\|_{H^1(-n,n)},\quad n\in\mathbb{N} \tag{2.1}$$

the space $H^1_{\mathrm{loc}}(\mathbb{R})$ turns to a reflexive Fréchet space.

Remark 2. *It would be more accurate to write formula (2.1) as follows $\|u\|_n=\|u\mid_{[-n,n]}\|_{H^1(-n,n)}$, where $|_{[-n,n]}$ is the operation of restriction to the interval $[-n,n]$. Nevertheless here and in the sequel we will hold this little bit informal notation. It will not generate a misleading.*

Surely Lemma 2.2 is a trivial and well-known fact. Nevertheless, we did not encounter it in the textbooks, so we present its proof.

Proof of Lemma 2.2. It is clear that the space $H^1_{\mathrm{loc}}(\mathbb{R})$ is compete, thus it is a barrelled space [3].

The space $H^1_{\mathrm{loc}}(\mathbb{R})$ is a projective limit of the spaces $H^1(-n,n)$ with respect to the restriction operators

$$H^1_{\mathrm{loc}}(\mathbb{R})\to H^1(-n,n).$$

The projective limit of reflexive spaces is a semi-reflexive space [2]. A barreled semi-reflexive space is a reflexive space [3]. Consequently, $H^1_{\mathrm{loc}}(\mathbb{R})$ is a reflexive space.

The Lemma is proved.

Determine the following subspaces

$$H^1_o(\mathbb{R})=\{u\in H^1_{\mathrm{loc}}(\mathbb{R})\mid u(-t)=-u(t)\}$$

and

$$X_\omega=\{x\in H^1_o(\mathbb{R})\mid x(t+\omega)=x(t)\}.$$

They both are closed. Moreover, from Lemma 2.1 it follows that a function $x\mapsto\|\dot x\|_{L^2(0,\omega)}$ is a norm in X_ω and the topology of this norm is equivalent to the one inherited of $H^1_{\mathrm{loc}}(\mathbb{R})$. So X_ω is a Banach space.

Lemma 2.3. *The spaces $H^1_o(\mathbb{R}),X_\omega$ are reflexive.*

The proof of this lemma almost literally repeats the proof of Lemma 2.2. Just note that the space $H^1_o(-n,n)$ is a reflexive space because it is a real Hilbert space with standard inner product

$$(u,v)=\int_{(-n,n)}u(t)v(t)dt+\int_{(-n,n)}\dot u(t)\dot v(t)dt.$$

Introduce a set

$$\Phi_{k,\omega}=\{\phi\in H^1_o(\mathbb{R})\mid \phi(t+\omega)=\phi(t)+2\pi k\}.$$

The set $\Phi_{k,\omega}$ is closed and convex in $H^1_o(\mathbb{R})$.

With the help of Lemma 2.1 it is not hard to show that the function $\rho(u,v)=\|\dot u-\dot v\|_{L^2(0,\omega)}$ determines a metric on $\Phi_{k,\omega}$ and this metric endows $\Phi_{k,\omega}$ with the same topology as the space $H^1_o(\mathbb{R})$ does.

2.2. **The Action.** Our goal is to prove that a functional

$$S:X_\omega\times H^1_o(\mathbb{R})\to\mathbb{R},\quad S(x,\phi)=\int_0^\omega L\big(x(t),\phi(t),\dot x(t),\dot\phi(t)\big)dt$$

attains a minimum in a set $E_{k,\omega}=X_\omega\times\Phi_{k,\omega}$.

Lemma 2.4. *For any $(x,\phi)\in H^1_o(\mathbb{R})\times H^1_o(\mathbb{R})$ the following inequality holds*

$$S(x,\phi)\ge\frac12\|\dot\phi\|^2_{L^2(0,\omega)}+\frac12\|\dot x\|^2_{L^2(0,\omega)}-\frac{\omega^{3/2}}{\sqrt2}\|\dot x\|_{L^2(0,\omega)}.$$

Proof. Indeed, with the help of Cauchy inequality it immediately follows that

$$\begin{aligned}S(x,\phi)&\ge\frac12\|\dot\phi\|^2_{L^2(0,\omega)}+\frac12\|\dot x\|^2_{L^2(0,\omega)}-\|x\|_{L^1(0,\omega)}\\&\ge\frac12\|\dot\phi\|^2_{L^2(0,\omega)}+\frac12\|\dot x\|^2_{L^2(0,\omega)}-\|x\|_{L^2(0,\omega)}\sqrt{\omega}.\end{aligned}$$

It remains to apply Lemma 2.1.

The Lemma is proved.

2.3. **Minimization of the Action Functional.** Let $\{(x_n,\phi_n)\}_{n\in\mathbb{N}}\subset E_{k,\omega}$ be a minimizing sequence for the functional S that is

$$S(x_n,\phi_n)\to\alpha,\quad n\to\infty,\quad \alpha=\inf_{E_{k,\omega}}S.$$

From Lemma 2.4 it follows that the sequence $\{(x_n,\phi_n)\}_{n\in\mathbb{N}}$ is bounded in $X_\omega\times H^1_o(\mathbb{R})$ and $\alpha>-\infty$.

Thus the sequence $\{(x_n,\phi_n)\}$ contains a weakly convergent subsequence, we denote this subsequence by the same letters:

$$x_n\to x_*\in X_\omega,\quad \phi_n\to\phi_*\in H^1_o(\mathbb{R}).$$

Since a convex set of a locally convex space is closed iff it is weakly closed [1], we have $\phi_*\in\Phi_{k,\omega}$.

We also know from analysis that the sequence $\{(x_n,\phi_n)\}$ contains a subsequence that is convergent in $C[0,\omega]\times C[0,\omega]$. (See Remark 2.) So we accept that $(x_n,\phi_n)\to(x_*,\phi_*)$ in $C[0,\omega]\times C[0,\omega]$.

Our next goal is to prove that $\alpha=S(x_*,\phi_*)$.

Observe the following evident estimates

$$\int_0^\omega x_n^2\dot\phi_n^2dt \geq \int_0^\omega (x_n^2-x_*^2)\dot\phi_n^2dt + \int_0^\omega x_*^2\dot\phi_*^2dt + 2\int_0^\omega \dot\phi_* x_*^2(\dot\phi_n-\dot\phi_*)dt, \tag{2.2}$$

$$\int_0^\omega \dot\phi_n^2dt \geq \int_0^\omega \dot\phi_*^2dt + 2\int_0^\omega \dot\phi_*(\dot\phi_n-\dot\phi_*)dt. \tag{2.3}$$

Since $x_n\to x_*$ in $C[0,\omega]$ and the sequence $\{\dot\phi_n\}$ is bounded in $L^2(0,\omega)$ the first term in the right side of formula (2.2) vanishes as $n\to\infty$.

The last terms in the right sides of formulas (2.2) and (2.3) are vanished as $n\to\infty$ because $\phi_n\to\phi_*$ weakly in $H^1_o(\mathbb{R})$.

Note also that

$$\int_0^\omega x_n\sin\phi_n dt\to\int_0^\omega x_*\sin\phi_* dt$$

it is because $\{(x_n,\phi_n)\}$ converges in $C[0,\omega]\times C[0,\omega]$.

Gathering all these observations we get $\alpha\geq S(x_*,\phi_*)$. So that

$$\alpha=S(x_*,\phi_*),\quad (x_*,\phi_*)\in E_{k,\omega}.$$

2.4. **Weak Solutions to the Lagrange Equations.** Take any two functions $x,\phi\in X_\omega$ and put

$$f(\xi,\eta)=S(x_*+\xi x,\phi_*+\eta\phi),\quad \xi,\eta\in\mathbb{R}.$$

From previous section it follows that a point $\xi=\eta=0$ is a minimum of f. This implies

$$\frac{\partial f}{\partial\xi}\Big|_{\xi=\eta=0}=\frac{\partial f}{\partial\eta}\Big|_{\xi=\eta=0}=0,$$

or in the detailed form

$$\int_0^\omega\Big(\dot x_*(t)\dot x(t)dt+\frac{\partial L}{\partial x}\big(x_*(t),\phi_*(t),\dot x_*(t),\dot\phi_*(t)\big)x(t)\Big)dt=0, \tag{2.4}$$

$$\int_0^\omega\Big((1+x_*^2(t))\dot\phi_*(t)\dot\phi(t)dt+\frac{\partial L}{\partial\phi}\big(x_*(t),\phi_*(t),\dot x_*(t),\dot\phi_*(t)\big)\phi(t)\Big)dt=0. \tag{2.5}$$

Equations (2.4) and (2.5) imply that the functions x_*,ϕ_* are the weak solutions to the Lagrange equations and $x,\phi\in X_\omega$ are the test functions.

2.5. **Regularization.** From the theory developed above we know that x_*,ϕ_* belong to $H^1_{\mathrm{loc}}(\mathbb{R})$ end by the Sobolev embedding theorem $x_*,\phi_*\in C(\mathbb{R})$.

Our aim is to show that $x_*,\phi_*\in C^2(\mathbb{R})$. Let us check this for ϕ_*.

Introduce a space

$$Y_\omega=\{u\in L^2_{\mathrm{loc}}(\mathbb{R})\mid u(-t)=u(t),\quad u(t+\omega)=u(t)\}.$$

In this definition the equalities hold almost everywhere.

Assume that a function y belongs to Y_ω. If in addition this function satisfies equality

$$\int_0^\omega y(s)ds = 0$$

then

$$\phi(t) = \int_0^t y(s)ds \in X_\omega.$$

Moreover, it is clear that every function from X_ω can be presented in this way.

Let us put

$$a(t) = (1 + x_*^2(t))\dot\phi_*(t) \in Y_\omega, \quad l(t) = -x_*(t)\cos\phi_*(t) \in X_\omega.$$

Introduce the following linear functionals

$$p(y) = \int_0^\omega y(s)ds,$$
$$h(y) = \int_0^\omega \Big(a(t)y(t) + l(t)\int_0^t y(s)ds\Big)dt.$$

Them both belong to Y'_ω. By Fubini's theorem we can rewrite the last functional in the form

$$h(y) = \int_0^\omega a(t)y(t)dt + \int_0^\omega y(s)\int_s^\omega l(t)dtds.$$

From equation (2.5) we know that $\ker p \subset \ker h$. Therefore there exists a constant λ such that

$$h = \lambda p. \tag{2.6}$$

Since $y \in Y_\omega$ is an arbitrary function, and the functions $a(t)$, $\int_t^\omega l(s)ds$ are even, equation (2.6) takes the form

$$(1 + x_*^2(t))\dot\phi_*(t) + \int_t^\omega l(s)ds = \lambda. \tag{2.7}$$

Since $\{l, x_*\} \subset X_\omega \subset C(\mathbb{R})$ we obtain $\phi_* \in C^1(\mathbb{R})$.

By the same argument from equation (2.4) we get

$$\dot x_*(t) + \int_t^\omega \big(x_*(s)\dot\phi_*^2(s) - \sin\phi_*(s)\big)ds = \text{const}. \tag{2.8}$$

From equation (2.8) it follows that $x_* \in C^2(\mathbb{R})$. Then we go back to equation (2.7) and yield $\phi_* \in C^2(\mathbb{R})$.

The Theorem is proved.

References

[1] R. Edwards: Functional Analysis. New York, 1965.
[2] H. Schaefer: Topological Vector Spaces, London, 1966.
[3] K. Yosida: Functional Analysis. Springer, Berlin, 1965.