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On resonances and bound states of Smilansky Hamiltonian

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In memory of B. S. Pavlov (1936–2016)

We consider the self-adjoint Smilansky Hamiltonian H_ε in $L^2(\mathbb{R}^2)$ associated with the formal differential expression $-\partial_x^2 - \frac{1}{2}(\partial_y^2 + y^2) - \sqrt{2}\varepsilon y\delta(x)$ in the sub-critical regime, $\varepsilon \in (0,1)$. We demonstrate the existence of resonances for H_ε on a countable subfamily of sheets of the underlying Riemann surface whose distance from the physical sheet is finite. On such sheets, we find resonance free regions and characterise resonances for small $\varepsilon > 0$. In addition, we refine the previously known results on the bound states of H_ε in the weak coupling regime ($\varepsilon \to 0+$). In the proofs we use Birman-Schwinger principle for H_ε, elements of spectral theory for Jacobi matrices, and the analytic implicit function theorem.

Keywords: Smilansky Hamiltonian, resonances, resonance free region, weak coupling asymptotics, Riemann surface, bound states.

1. Introduction

In this paper we investigate resonances and bound states of the self-adjoint Hamiltonian H_ε acting in the Hilbert space $L^2(\mathbb{R}^2)$ and corresponding to the formal differential expression

$$-\partial_x^2 - \frac{1}{2}(\partial_y^2 + y^2) - \sqrt{2}\varepsilon y\delta(x) \qquad \text{on } \mathbb{R}^2, \tag{1.1}$$

in the sub-critical regime, $\varepsilon \in (0,1)$. The operator H_ε will be rigorously introduced in Section 1.1 below. Operators of this type were suggested by U. Smilansky in [30] as a model of *irreversible quantum system*. His aim was to demonstrate that the 'heat bath' need not have an infinite number of degrees of freedom. On a physical level of rigour he showed that the spectrum undergoes an abrupt transition at the critical value $\varepsilon = 1$. A mathematically precise spectral analysis of these operators and their generalisations has been performed by M. Solomyak and his collaborators in [15, 16, 24, 27, 31–33]. Time-dependent Schrödinger equation generated by Smilansky-type Hamiltonian is considered in [21].

By now many of the spectral properties of H_ε are understood. On the other hand, little attention has been paid so far to the fact that such a system can also exhibit resonances. The main aim of this paper is to initiate investigation of these resonances starting from demonstration of their existence. One of the key difficulties is that this model belongs to a class wherein the resolvent extends to a *Riemann surface* having uncountably many sheets. The same complication appears *e.g.* in studying resonances for quantum waveguides [2, 13, 14, 17], [18, §3.4.2] and for general manifolds with cylindrical ends [10, 11].

In this paper we prove the existence and obtain a characterisation of resonances of H_ε on a countable subfamily of sheets whose distance from the physical sheet is finite in the sense explained below. On any such sheet we characterise a region which is free of resonances. As $\varepsilon \to 0+$, the resonances on such sheets are localized in the vicinities of the thresholds $\nu_n = n + 1/2$, $n \in \mathbb{N}$. We obtain a description of the subset of the thresholds in the vicinities of which a resonance exists for all sufficiently small $\varepsilon > 0$ and derive asymptotic expansions of these resonances in the limit $\varepsilon \to 0+$. No attempt has been made here to define and study resonances on the sheets whose distance from the physical sheet is infinite.

As a byproduct, we obtain refined properties of the bound states of H_ε using similar methods as for resonances. More precisely, we obtain a lower bound on the first eigenvalue of H_ε and an asymptotic expansion of the weakly coupled bound state of H_ε in the limit $\varepsilon \to 0+$.

Methods developed in this paper can also be useful to tackle resonances for the analogue of Smilansky model with regular potential which is suggested in [3] and further investigated in [4, 5].

Notations

We use notations $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for the sets of positive and natural integers, respectively. We denote the complex plane by $\mathbb{C}$ and define its commonly used subdomains: $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$, $\mathbb{C}_\pm := \{\lambda \in \mathbb{C} \colon \pm \operatorname{Im} \lambda > 0\}$ and $\mathbb{D}_r(\lambda_0) := \{\lambda \in \mathbb{C} \colon |\lambda - \lambda_0| < r\}$, $\mathbb{D}_r^\times(\lambda_0) := \{\lambda \in \mathbb{C} \colon 0 < |\lambda - \lambda_0| < r\}$, $\mathbb{D}_r := \mathbb{D}_r(0)$, $\mathbb{D}_r^\times := \mathbb{D}_r^\times(0)$ with $r > 0$. The principal value of the argument for $\lambda \in \mathbb{C}^\times$ is denoted by $\arg \lambda \in (-\pi, \pi]$. The branches of the square root are defined by

$$\mathbb{C}^\times \ni \lambda \mapsto (\lambda)_j^{1/2} := |\lambda|^{1/2} e^{\mathrm{i}((1/2)\arg\lambda + j\pi)}, \qquad j = 0, 1.$$

If the branch of the square root is not explicitly specified we understand the branch $(\cdot)_0^{1/2}$ by default. We also set $\mathbf{0} = (0, 0) \in \mathbb{C}^2$.

The L^2-space over $\mathbb{R}^d$, $d = 1, 2$, with the usual inner product is denoted by $(L^2(\mathbb{R}^d), (\cdot, \cdot)_{\mathbb{R}^d})$ and the L^2-based first order Sobolev space by $H^1(\mathbb{R}^d)$, respectively. The space of square-summable sequences of vectors in a Hilbert space $\mathcal{G}$ is denoted by $\ell^2(\mathbb{N}_0; \mathcal{G})$. In the case that $\mathcal{G} = \mathbb{C}$ we simply write $\ell^2(\mathbb{N}_0)$ and denote by $(\cdot, \cdot)$ the usual inner product on it.

For $\xi = \{\xi_n\} \in \ell^2(\mathbb{N}_0)$ we adopt the convention that $\xi_{-1} = 0$. *Kronecker symbol* is denoted by δ_{nm}, $n, m \in \mathbb{N}_0$, we set $\mathsf{e}_n := \{\delta_{nm}\}_{m \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$, $n \in \mathbb{N}_0$, and adopt the convention that $\mathsf{e}_{-1} := \{0\}$. We understand by $\operatorname{diag}(\{q_n\})$ the *diagonal matrix* in $\ell^2(\mathbb{N}_0)$ with entries $\{q_n\}_{n \in \mathbb{N}_0}$ and by $\mathsf{J}(\{a_n\}, \{b_n\})$ the *Jacobi matrix* in $\ell^2(\mathbb{N}_0)$ with diagonal entries $\{a_n\}_{n \in \mathbb{N}_0}$ and off-diagonal entries $\{b_n\}_{n \in \mathbb{N}}$[1]. We also set $\mathsf{J}_0 := \mathsf{J}(\{0\}, \{1/2\})$.

By $\sigma(\mathsf{K})$ we denote the spectrum of a closed (not necessarily self-adjoint) operator K in a Hilbert space. An isolated eigenvalue $\lambda \in \mathbb{C}$ of K having finite algebraic multiplicity is a point of the *discrete spectrum* for K; see [26, §XII.2] for details. The set of all the points of the discrete spectrum for K is denoted by $\sigma_{\mathrm{d}}(\mathsf{K})$ and the *essential spectrum* of K is defined by $\sigma_{\mathrm{ess}}(\mathsf{K}) := \sigma(\mathsf{K}) \setminus \sigma_{\mathrm{d}}(\mathsf{K})$.

For a self-adjoint operator T in a Hilbert space we set $\lambda_{\mathrm{ess}}(\mathsf{T}) := \inf \sigma_{\mathrm{ess}}(\mathsf{T})$ and, for $k \in \mathbb{N}$, $\lambda_k(\mathsf{T})$ denotes the k-th eigenvalue of T in the interval $(-\infty, \lambda_{\mathrm{ess}}(\mathsf{T}))$. These eigenvalues are

[1] We do not distinguish between Jacobi matrices and operators in the Hilbert space $\ell^2(\mathbb{N}_0)$ induced by them, since in our considerations all the Jacobi matrices are bounded, closed, and everywhere defined in $\ell^2(\mathbb{N}_0)$.

ordered non-decreasingly with multiplicities taken into account. The number of eigenvalues with multiplicities of the operator T lying in a closed, open, or half-open interval $\Delta \subset \mathbb{R}$ satisfying $\sigma_{\mathrm{ess}}(\mathsf{T}) \cap \Delta = \varnothing$ is denoted by $\mathcal{N}(\Delta; \mathsf{T})$. For $\lambda \leqslant \lambda_{\mathrm{ess}}(\mathsf{T})$ the *counting function* of T is defined by $\mathcal{N}_\lambda(\mathsf{T}) := \mathcal{N}((-\infty, \lambda); \mathsf{T})$.

1.1. Smilansky Hamiltonian

Define the *Hermite functions*

$$\chi_n(y) := e^{-y^2/2} H_n(y), \qquad n \in \mathbb{N}_0. \tag{1.2}$$

Here, $H_n(y)$ is the *Hermite polynomial* of degree $n \in \mathbb{N}_0$ normalized by the condition $\|\chi_n\|_{\mathbb{R}} = 1$[2]. For more details on Hermite polynomials see [1, Chap. 22] and also [9, Chap. 5]. As it is well-known, the family $\{\chi_n\}_{n\in\mathbb{N}_0}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$. Note also that the functions χ_n satisfy the three-term recurrence relation

$$\sqrt{n+1}\chi_{n+1}(y) - \sqrt{2}y\chi_n(y) + \sqrt{n}\chi_{n-1}(y) = 0, \qquad n \in \mathbb{N}_0, \tag{1.3}$$

where we adopt the convention $\chi_{-1} \equiv 0$. The relation (1.3) can be easily deduced from the recurrence relation [1, eq. 22.7.13] for Hermite polynomials. By a standard argument any function $U \in L^2(\mathbb{R}^2)$ admits unique expansion

$$U(x,y) = \sum_{n\in\mathbb{N}_0} u_n(x)\chi_n(y), \qquad u_n(x) := \int_{\mathbb{R}} U(x,y)\chi_n(y)\mathrm{d}y, \tag{1.4}$$

where $\{u_n\} \in \ell^2(\mathbb{N}_0; L^2(\mathbb{R}))$. Following the presentation in [32], we identify the function $U \in L^2(\mathbb{R}^2)$ and the sequence $\{u_n\}$ and write $U \sim \{u_n\}$. This identification defines a natural unitary transform between the Hilbert spaces $L^2(\mathbb{R}^2)$ and $\mathcal{H} := \ell^2(\mathbb{N}_0; L^2(\mathbb{R}))$. For the sake of brevity, we denote the inner product on $\mathcal{H}$ by $\langle\cdot,\cdot\rangle$. Note that the Hilbert space $\mathcal{H}$ can also be viewed as the tensor product $\ell^2(\mathbb{N}_0) \otimes L^2(\mathbb{R})$.

For any $\varepsilon \in \mathbb{R}$ we define the subspace $\mathcal{D}_\varepsilon$ of $\mathcal{H}$ as follows: an element $U \sim \{u_n\} \in \mathcal{H}$ belongs to $\mathcal{D}_\varepsilon$ *if, and only if*

(i) $u_n \in H^1(\mathbb{R})$ for all $n \in \mathbb{N}_0$;

(ii) $\{-(u_{n,+}'' \oplus u_{n,-}'') + \nu_n u_n\} \in \mathcal{H}$ with $u_{n,\pm} := u_n|_{\mathbb{R}_\pm}$ and $\nu_n = n + 1/2$ for $n \in \mathbb{N}_0$;

(iii) the boundary conditions

$$u_n'(0+) - u_n'(0-) = \varepsilon\big(\sqrt{n+1}u_{n+1}(0) + \sqrt{n}u_{n-1}(0)\big)$$

are satisfied for all $n \in \mathbb{N}_0$. For $n = 0$ only the first term is present on the right-hand side.

By [32, Thm. 2.1], the operator

$$\operatorname{dom}\mathsf{H}_\varepsilon := \mathcal{D}_\varepsilon, \qquad \mathsf{H}_\varepsilon\{u_n\} := \{-(u_{n,+}'' \oplus u_{n,-}'') + \nu_n u_n\}, \tag{1.5}$$

is self-adjoint in $\mathcal{H}$. It corresponds to the formal differential expression (1.1). Further, we provide another way of defining H_ε which makes the correspondence between the operator H_ε and the formal differential expression (1.1) more transparent. To this aim we define the straight line $\Sigma := \{(0,y) \in \mathbb{R}^2 \colon y \in \mathbb{R}\}$. Then the Hamiltonain H_ε, $\varepsilon \in (-1,1)$, can be alternatively introduced as the unique self-adjoint operator in $L^2(\mathbb{R}^2)$ associated via the first

[2]This normalization means that $H_n(y)$ is, in fact, a product of what is usually called the *Hermite polynomial* of degree $n \in \mathbb{N}_0$ with a normalization constant which depends on n.

representation theorem [22, Thm. VI.2.1] with a closed, densely defined, symmetric, and semi-bounded quadratic form

$$\begin{aligned} \mathfrak{h}_\varepsilon[u] &:= \|\partial_x u\|^2_{\mathbb{R}^2} + \frac{1}{2}\|\partial_y u\|^2_{\mathbb{R}^2} + \frac{1}{2}(yu, yu)_{\mathbb{R}^2} + \varepsilon\sqrt{2}\left(\operatorname{sign}(y)|y|^{1/2}u|_\Sigma, |y|^{1/2}u|_\Sigma\right)_{\mathbb{R}}, \\ \operatorname{dom}\mathfrak{h}_\varepsilon &:= \left\{u \in H^1(\mathbb{R}^2)\colon yu \in L^2(\mathbb{R}^2), |y|^{1/2}(u|_\Sigma) \in L^2(\mathbb{R})\right\}. \end{aligned} \tag{1.6}$$

For more details and for the proof of equivalence between the two definitions of H_ε see [32, §9]. Since H_ε commutes with the parity operator in y-variable, it is unitarily equivalent to $\mathsf{H}_{-\varepsilon}$. We remark that the case $\varepsilon = 0$ admits separation of variables. Thus, it suffices to study H_ε with $\varepsilon > 0$.

In the following proposition we collect fundamental spectral properties of H_ε, $\varepsilon \in (0,1)$, which are of importance in the present paper.

Proposition 1.1. *Let the self-adjoint operator H_ε, $\varepsilon \in (0,1)$, be as in (1.5). Then the following claims hold:*

(i) $\sigma_{\mathrm{ess}}(\mathsf{H}_\varepsilon) = [1/2, +\infty)$;

(ii) $\inf\sigma(\mathsf{H}_\varepsilon) \geqslant \frac{1-\varepsilon}{2}$;

(iii) $1 \leqslant \mathcal{N}_{1/2}(\mathsf{H}_\varepsilon) < \infty$;

(iv) $\mathcal{N}_{1/2}(\mathsf{H}_\varepsilon) = 1$ *for all sufficiently small* $\varepsilon > 0$.

Items (i)-(iii) follow from [31, Lem 2.1] and [32, Thm. 3.1 (1),(2)]. Item (iv) is a consequence of [31, Thm. 3.2] and [32, §10.1]. Although we only deal with the sub-critical case, $\varepsilon \in (0,1)$, we remark that in the critical case, $\varepsilon = 1$, the spectrum of H_1 equals to $[0, +\infty)$ and that in the sup-critical case, $\varepsilon > 1$, the spectrum of H_ε covers the whole real axis. Finally, we mention that in most of the existing literature on the subject not $\varepsilon > 0$ itself but $\alpha = \sqrt{2}\varepsilon$ is chosen as the coupling parameter. We choose another normalization of the coupling parameter in order to simplify formulae in the proofs of the main results.

1.2. Main results

While we are primarily interested in the resonances, as indicated in the introduction, we have also a claim to make about the discrete spectrum which we present here as our first main result and which complements the results listed in Proposition 1.1.

Theorem 1.2. *Let the self-adjoint operator H_ε, $\varepsilon \in (0,1)$, be as in (1.5). Then the following claims hold.*

(i) $\lambda_1(\mathsf{H}_\varepsilon) \geqslant 1 - \sqrt{\frac{1}{4} + \varepsilon^4}$ *for all* $\varepsilon \in (0,1)$.

(ii) $\lambda_1(\mathsf{H}_\varepsilon) = \nu_0 - \frac{\varepsilon^4}{16} + \mathcal{O}(\varepsilon^5)$ *as* $\varepsilon \to 0+$.

Theorem 1.2 (i) is proven by means of Birman-Schwinger principle. The bound in Theorem 1.2 (i) is non-trivial for $\varepsilon^4 < 3/4$. This bound is better than the one in Proposition 1.1 (ii) for small $\varepsilon > 0$.

For the proof of Theorem 1.2 (ii) we combine Birman-Schwinger principle and the analytic implicit function theorem. We expect that the error term $\mathcal{O}(\varepsilon^5)$ in Theorem 1.2 (ii) can be replaced by $\mathcal{O}(\varepsilon^6)$ because the operator H_ε has the same spectral properties as $\mathsf{H}_{-\varepsilon}$ for any $\varepsilon \in (0,1)$. Therefore, the expansion of $\lambda_1(\mathsf{H}_\varepsilon)$ must be invariant with respect to interchange between ε and $-\varepsilon$. In Lemma 4.1 given in Section 4 we derive an implicit scalar equation

on $\lambda_1(\mathsf{H}_\varepsilon)$. This equation gives analyticity of $\varepsilon \mapsto \lambda_1(\mathsf{H}_\varepsilon)$ for small ε. It can also be used to compute higher order terms in the expansion of $\lambda_1(\mathsf{H}_\varepsilon)$. However, these computations might be quite tedious.

Our second main result concerns the resonances of H_ε. Before formulating it, we need to define the resonances rigorously. Let us consider the sequence of functions

$$r_n(\lambda) := (\nu_n - \lambda)^{1/2}, \qquad n \in \mathbb{N}_0. \tag{1.7}$$

Each of them has two branches $r_n(\lambda, l) := (\nu_n - \lambda)_l^{1/2}$, $l = 0, 1$. The vector-valued function $R(\lambda) = (r_0(\lambda), r_1(\lambda), r_2(\lambda), \dots)$ naturally defines the Riemann surface $\widehat{Z}$ with uncountably many sheets. With each sheet of $\widehat{Z}$ we associate the set $E \subset \mathbb{N}_0$ and the *characteristic vector* l^E defined as

$$l^E := \{l_0^E, l_1^E, l_2^E, \dots\}, \qquad l_n^E := \begin{cases} 0, & n \notin E, \\ 1, & n \in E. \end{cases} \tag{1.8}$$

We adopt the convention that $l_{-1}^E = 0$. The respective sheet of $\widehat{Z}$ is convenient to denote by Z_E. Each sheet Z_E of $\widehat{Z}$ can be identified with the set $\mathbb{C} \setminus [\nu_0, +\infty)$ and we denote by $Z_E^\pm$ the parts of Z_E corresponding to $\mathbb{C}_\pm$. With the notation settled we define the realization of $R(\cdot)$ on Z_E as

$$R_E(\lambda) := (r_0(\lambda, l_0^E), r_1(\lambda, l_1^E), r_2(\lambda, l_2^E), \dots). \tag{1.9}$$

The sheets Z_E and Z_F are *adjacent* through the interval $(\nu_n, \nu_{n+1}) \subset \mathbb{R}$, $n \in \mathbb{N}_0$, ($Z_E \sim_n Z_F$), if their characteristic vectors l^E and l^F satisfy

$$\begin{aligned} l_k^F &= 1 - l_k^E, \qquad \text{for } k = 0, 1, 2, \dots, n \\ l_k^F &= l_k^E, \qquad \text{for } k > n. \end{aligned}$$

We set $\nu_{-1} = -\infty$ and note that any sheet Z_E is adjacent to itself through (ν_{-1}, ν_0). In particular, the function $\lambda \mapsto R_E(\lambda)$ turns out to be componentwise analytic on the Riemann surface $\widehat{Z}$.

The sequence $\mathfrak{E} = \{E_1, E_2, \dots, E_N\}$ of subsets of $\mathbb{N}_0$ is called a *path* if for any $k = 1, 2, \dots, N-1$ the sheets Z_{E_k} and $Z_{E_{k+1}}$ are adjacent. The following discrete metric

$$\rho(E, F) := \inf\{N \in \mathbb{N}_0 \colon \mathfrak{E} = \{E_1, E_2, \dots, E_N\}, E_1 = E, E_N = F\}, \tag{1.10}$$

turns out to be convenient. The value $\rho(E, F)$ equals the number of sheets in the shortest path connecting Z_E and Z_F. Note that for some sheets Z_E and Z_F a path between them does not exist and in this case we have $\rho(E, F) = \infty$. We identify the *physical sheet* with the sheet $Z_\varnothing$ (for $E = \varnothing$). A sheet Z_E of $\widehat{Z}$ is *adjacent* to the physical sheet $Z_\varnothing$ if $\rho(E, \varnothing) = 1$ and it can be characterised by existence of $N \in \mathbb{N}_0$ such that $l_n^E = 1$ *if, and only if* $n \leqslant N$. Further, we define the component

$$\widetilde{Z} := \cup_{E \in \mathcal{E}} Z_E \subset \widehat{Z}, \qquad \mathcal{E} := \{E \subset \mathbb{N}_0 \colon \rho(E, \varnothing) < \infty\}, \tag{1.11}$$

of $\widehat{Z}$ which plays a distinguished role in our considerations. Any sheet in $\widetilde{Z}$ is located on a finite distance from the physical sheet with respect to the metric $\rho(\cdot, \cdot)$. The component $\widetilde{Z}$ of $\widehat{Z}$ in (1.11) can be alternatively characterised as

$$\widetilde{Z} = \cup_{F \in \mathcal{F}} Z_F, \qquad \mathcal{F} := \{F \subset \mathbb{N}_0 \colon \sup\{n \in \mathbb{N}_0 \colon l_n^F = 1\} < \infty\}. \tag{1.12}$$

The number of the sheets in $\widetilde{Z}$ is easily seen to be countable. In order to define the resonances of H_ε on $\widetilde{Z}$ we show that the resolvent of H_ε admits an extension to $\widetilde{Z}$ in a certain weak sense.

Proposition 1.3. *For any $u \in L^2(\mathbb{R})$ and $n \in \mathbb{N}_0$ the function*

$$\lambda \mapsto \mathfrak{r}^{\varnothing}_{n,\varepsilon}(\lambda; u) := \big\langle (\mathsf{H}_\varepsilon - \lambda)^{-1} u \otimes \mathsf{e}_n, u \otimes \mathsf{e}_n \big\rangle \tag{1.13}$$

admits unique meromorphic continuation $\mathfrak{r}^{E}_{n,\varepsilon}(\cdot; u)$ from the physical sheet $Z_\varnothing$ to any sheet $Z_E \subset \widetilde{Z}$.

The proof of Proposition 1.3 is postponed until Appendix A. Now we have all the tools to define resonances of H_ε on $\widetilde{Z}$.

Definition 1.4. *Each resonance of H_ε on $Z_E \subset \widetilde{Z}$ is identified with a pole of $\mathfrak{r}^{E}_{n,\varepsilon}(\cdot; u)$ for some $u \in L^2(\mathbb{R})$ and $n \in \mathbb{N}_0$. The set of all the resonances for H_ε on the sheet Z_E is denoted by $\mathcal{R}_E(\varepsilon)$.*

Our definition of resonances for H_ε is consistent with [26, §XII.6], see also [18, Chap. 2] and [23] for multi-threshold case. It should be emphasized that by the spectral theorem for self-adjoint operators eigenvalues of H_ε are also regarded as resonances in the sense of Definition 1.4 lying on the physical sheet $Z_\varnothing$. This allows us to treat the eigenvalues and 'true' resonances on the same footing. Needless to say, bound states and true resonances correspond to different physical phenomena and their equivalence in this paper is merely a useful mathematical abstraction.

According to Remark 2.5 below the set of the resonances for H_ε on Z_E is symmetric with respect to the real axis. Thus, it suffices to analyse resonances on Z_E^-. Now we are prepared to formulate the main result on resonances.

Theorem 1.5. *Let the self-adjoint operator H_ε, $\varepsilon \in (0,1)$, be as in (1.5). Let the sheet $Z_E \subset \widetilde{Z}$ of the Riemann surface $\widehat{Z}$ be fixed. Define the associated set by*

$$\mathcal{S}(E) := \big\{ n \in \mathbb{N} \colon (l^E_{n-1}, l^E_n, l^E_{n+1}) \in \{(1,0,0),(0,1,1)\} \big\}.$$

Let $\mathcal{R}_E(\varepsilon)$ be as in Definition 1.4 and set $\mathcal{R}^-_E(\varepsilon) := \mathcal{R}_E(\varepsilon) \cap \mathbb{C}_-$. Then the following claims hold.

(i) $\mathcal{R}^-_E(\varepsilon) \subset \mathcal{U}(\varepsilon) := \{\lambda \in \mathbb{C}_- \colon |\nu_{n-1} - \lambda||\nu_n - \lambda| \leqslant \varepsilon^4 n^2,\ \forall n \in \mathbb{N}\}$.

(ii) *For any $n \in \mathcal{S}(E)$ and sufficiently small $\varepsilon > 0$ there is exactly one resonance $\lambda^E_n(\mathsf{H}_\varepsilon) \in \mathbb{C}_-$ of H_ε on Z^-_E lying in a neighbourhood of ν_n, with the expansion*

$$\lambda^E_n(\mathsf{H}_\varepsilon) = \nu_n - \frac{\varepsilon^4}{16}\big[(2n+1) + 2n(n+1)\mathrm{i}\big] + \mathcal{O}(\varepsilon^5), \qquad \varepsilon \to 0+. \tag{1.14}$$

(iii) *For any $n \in \mathbb{N} \setminus \mathcal{S}(E)$ and all sufficiently small $\varepsilon, r > 0$*

$$\mathcal{R}^-_E(\varepsilon) \cap \mathbb{D}_r(\nu_n) = \varnothing.$$

In view of Theorem 1.5 (i) for sufficiently small $\varepsilon > 0$ the resonances of H_ε on any sheet of $\widetilde{Z}$ are located in some neighbourhoods of the thresholds ν_n (see Figure 1.1). Such behaviour is typical for problems with many thresholds; see *e.g.* [13,17] and [18, §2.4, 3.4.2]. Note also that the estimate in Theorem 1.5 (i) reflects the correct order in ε in the weak coupling limit $\varepsilon \to 0+$ given in Theorem 1.5 (ii). However, the coefficient of ε^4 in the definition of $\mathcal{U}(\varepsilon)$ can be probably improved. Observe also that $\mathcal{R}^-_E(\varepsilon) \subset \mathcal{U}(1)$ for any $\varepsilon \in (0,1)$.

According to Theorem 1.5 (ii)-(iii) existence of a resonance near the threshold ν_n, $n \in \mathbb{N}$, on a sheet Z_E for small $\varepsilon > 0$ depends only on the branches chosen for $r_{n-1}(\lambda)$, $r_n(\lambda)$, $r_{n+1}(\lambda)$ on Z_E. Although, one can not exclude that higher order terms in the asymptotic expansion (1.14) depend on the branches chosen for other square roots. By exactly the same

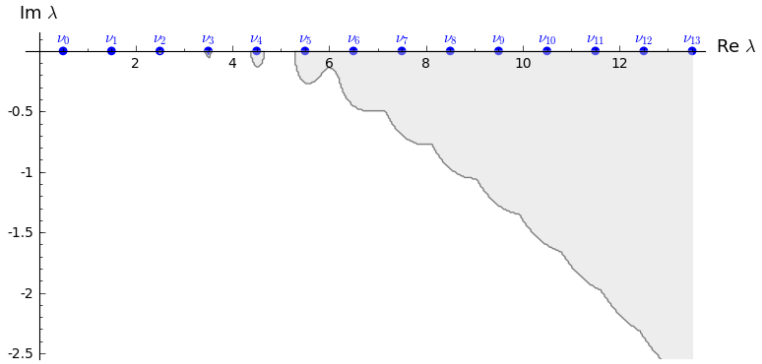

FIG. 1.1. The region $\mathcal{U}(0.12)$ (for $\varepsilon = 0.12$) from Theorem 1.5 (i) (in grey) consists of 6 connected components. The components located in the neighbourhoods of the points $\nu_0, \nu_1, \nu_2, \nu_3$, are not visible because of being too small. The plot is performed with the aid of *Sagemath*.

reason as in Theorem 1.2 (ii), we expect that the error term $\mathcal{O}(\varepsilon^5)$ in Theorem 1.5 (ii) can be replaced by $\mathcal{O}(\varepsilon^6)$. Theorem 1.5 (ii)-(iii) are proven by means of the Birman-Schwinger principle and the analytic implicit function theorem. The implicit scalar equation on resonances derived in Lemma 4.1 gives analyticity of $\varepsilon \mapsto \lambda_n^E(\mathsf{H}_\varepsilon)$ for small $\varepsilon > 0$ and, as in the bound state case, it can be used to compute further terms in the expansion of $\lambda_n^E(\mathsf{H}_\varepsilon)$.

We point out that according to numerical tests in [19] some resonances emerge from the inner points of the intervals (ν_n, ν_{n+1}), $n \in \mathbb{N}_0$, as $\varepsilon \to 1-$. The mechanism of creation for these resonances is unclear at the moment.

Example 1.6. *Let $E = \{1, 2, 4, 5\}$. In this case $l^E = \{0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, \dots\}$ and we get that $\mathcal{S}(E) = \{1, 4, 6\}$. By Theorem 1.5 (ii)-(iii) for all sufficiently small $\varepsilon > 0$ there will be exactly one resonance on Z_E^- near ν_1, ν_4, ν_6 and no resonances near the thresholds ν_n with $n \in \mathbb{N} \setminus \{1, 4, 6\}$. We confirm this result by numerical tests whose outcome is shown in Figures 1.2 and 1.3.*

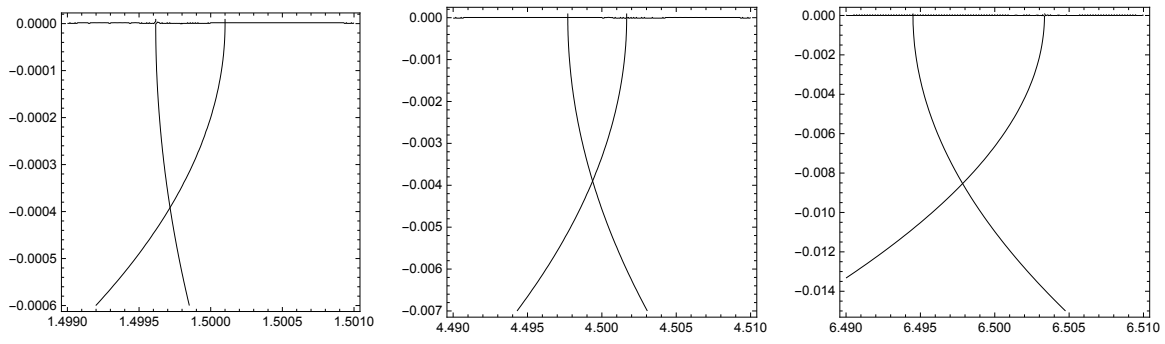

FIG. 1.2. Resonances of H_ε with $\varepsilon = 0.2$ lying on Z_E^- with $E = \{1, 2, 4, 5\}$ are computed numerically with the help of *Mathematica*. Unique weakly coupled resonances near the thresholds $\nu_1 = 1.5$, $\nu_4 = 4.5$, $\nu_6 = 6.5$ are located at the intersections of the curves.

To plot Figure 1.2 we used the condition on resonances in Theorem 2.4 below. The infinite Jacobi matrix in this condition was truncated up to a reasonable finite size. Along the curves, respectively, the real and the imaginary part of the determinant of the truncated

matrix vanishes. At the points of intersection of the curves the determinant itself vanishes. These points are expected to be close to true resonances[3]*. We have also verified numerically that resonances do not exist near other low-lying thresholds* ν_n *with* $n \in \mathbb{N} \setminus \{1,4,6\}$ *which corresponds well to Theorem 1.5. In Figure 1.3 we summarise the results of all the numerical tests.*

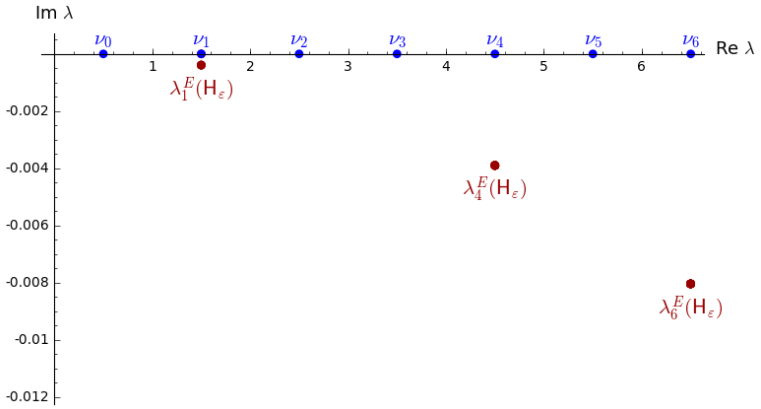

FIG. 1.3. Resonances of H_ε with $\varepsilon = 0.2$ lying on Z_E^- with $E = \{1,2,4,5\}$.

Finally, we mention that no attempt has been made here to analyse the multiplicities of the resonances and to investigate resonances lying on $\widehat{Z} \setminus \widetilde{Z}$.

Structure of the paper

Birman-Schwinger-type principles for characterisation of eigenvalues and resonances of H_ε are provided in Section 2. Theorem 1.2 (i) on a lower bound for the first eigenvalue and Theorem 1.5 (i) on resonance free region are proven in Section 3. The aim of Section 4 is to prove Theorem 1.2 (ii) and Theorem 1.5 (ii)-(iii) on weakly coupled bound states and resonances. The proofs of technical statements formulated in Proposition 1.3 and Theorem 2.4 are postponed until Appendix A.

2. Birman-Schwinger-type conditions

Birman-Schwinger principle is a powerful tool for analysis of the discrete spectrum of a perturbed operator in the spectral gaps of the unperturbed one. This principle has also various other applications. Frequently, it can be generalized to detect resonances, defined as the poles of a meromorphic continuation of the (sandwiched) resolvent from the physical sheet to non-physical sheet(s) of the underlying Riemann surface. In the model under consideration we encounter yet another manifestation of this principle.

In order to formulate a Birman-Schwinger-type condition on the bound states for H_ε we introduce the sequence of functions

$$b_n(\lambda) := \frac{n^{1/2}}{2(\nu_n - \lambda)^{1/4}(\nu_{n-1} - \lambda)^{1/4}}, \qquad n \in \mathbb{N}, \tag{2.1}$$

and the off-diagonal Jacobi matrix

$$\mathsf{J}(\lambda) = \mathsf{J}\left(\{0\}, \{b_n(\lambda)\}\right), \qquad \lambda \in (0, \nu_0)\,. \tag{2.2}$$

[3]The analysis of convergence of the numerical method is beyond our scope.

Recall that we use the same symbol $\mathsf{J}(\lambda)$ for the operator in $\ell^2(\mathbb{N}_0)$ generated by this matrix. It is straightforward to check that the operator $\mathsf{J}(\lambda)$ is bounded and self-adjoint. It can be easily verified that the difference $\mathsf{J}(\lambda) - \mathsf{J}_0$ is a compact operator. Therefore, one has $\sigma_{\mathrm{ess}}(\mathsf{J}(\lambda)) = \sigma_{\mathrm{ess}}(\mathsf{J}_0) = [-1, 1]$. Moreover, the operator $\mathsf{J}(\lambda)$ has simple eigenvalues $\pm\mu_n$, $\mu_n > 1$, with the only possible accumulation points at $\mu = \pm 1$.

Theorem 2.1. *[31, Thm. 3.1] Let the self-adjoint operator H_ε, $\varepsilon \in (0, 1)$, be as in* (1.5) *and let the Jacobi matrix $\mathsf{J}(\lambda)$ be as in* (2.2)*. Then the relation*

$$\mathcal{N}((0, \lambda); \mathsf{H}_\varepsilon) = \mathcal{N}((1/\varepsilon, +\infty); \mathsf{J}(\lambda)) \tag{2.3}$$

holds for all $\lambda \in (0, \nu_0)$.

Remark 2.2. A careful inspection of the proof of [31, Thm 3.1] yields that Theorem 2.1 can also be modified, replacing (2.3) by

$$\mathcal{N}((0, \lambda]; \mathsf{H}_\varepsilon) = \mathcal{N}([1/\varepsilon, +\infty); \mathsf{J}(\lambda)). \tag{2.4}$$

In other words, the right endpoint of the interval $(0, \lambda)$ and the left endpoint of the interval $(1/\varepsilon, +\infty)$ can be simultaneously included.

The following consequence of Theorem 2.1 and of the above remark will be useful further.

Corollary 2.3. *Let the assumptions be as in Theorem 2.1. Then the following claims hold:*

(i) $\varepsilon \mapsto \lambda_k(\mathsf{H}_\varepsilon)$ *are continuous non-increasing functions;*

(ii) $\dim\ker(\mathsf{H}_\varepsilon - \lambda) = \dim\ker\big(\mathsf{I} + \varepsilon\mathsf{J}(\lambda)\big)$ *for all $\lambda \in (0, \nu_0)$. In particular, since the eigenvalues of $\mathsf{J}(\lambda)$ are simple, the eigenvalues of H_ε are simple as well.*

Proof. (i) Let $\varepsilon_1 \in (0, 1)$. For $\lambda = \lambda_k(\mathsf{H}_{\varepsilon_1})$, $k \in \mathbb{N}$, we have by Theorem 2.1 and Remark 2.2

$$\mathcal{N}([1/\varepsilon_1, +\infty); \mathsf{J}(\lambda)) = \mathcal{N}((0, \lambda]; \mathsf{H}_{\varepsilon_1}) \geqslant k.$$

Hence, for any $\varepsilon_2 \in (\varepsilon_1, 1)$ we obtain

$$\mathcal{N}((0, \lambda]; \mathsf{H}_{\varepsilon_2}) = \mathcal{N}([1/\varepsilon_2, +\infty); \mathsf{J}(\lambda)) \geqslant \mathcal{N}([1/\varepsilon_1, +\infty); \mathsf{J}(\lambda)) \geqslant k.$$

Therefore, we get $\lambda_k(\mathsf{H}_{\varepsilon_2}) \leqslant \lambda = \lambda_k(\mathsf{H}_{\varepsilon_1})$.

Recall that H_ε represents the quadratic form $\mathfrak{h}_\varepsilon$ defined in (1.6). Continuity of the eigenvalues follows from [22, Thms. VI.3.6, VIII.1.14] and from the fact that the quadratic form

$$\operatorname{dom}\mathfrak{h}_\varepsilon \ni u \mapsto \varepsilon\sqrt{2}\left(\operatorname{sign} y|y|^{1/2}u|_\Sigma, |y|^{1/2}u|_\Sigma\right)_{\mathbb{R}}, \qquad \varepsilon \in (0, 1),$$

is relatively bounded with respect to

$$\operatorname{dom}\mathfrak{h}_\varepsilon \ni u \mapsto \|\partial_x u\|^2_{\mathbb{R}^2} + \frac{1}{2}\|\partial_y u\|^2_{\mathbb{R}^2} + \frac{1}{2}(yu, yu)_{\mathbb{R}^2}$$

with a bound less than one; *cf.* [31, Lem. 2.1].

(ii) By Theorem 2.1, Remark 2.2, and using symmetry of $\sigma(\mathsf{J}(\lambda))$ with respect to the origin we get

$$\begin{aligned}\dim\ker(\mathsf{H}_\varepsilon - \lambda) &= \mathcal{N}((0, \lambda]; \mathsf{H}_\varepsilon) - \mathcal{N}((0, \lambda); \mathsf{H}_\varepsilon) \\ &= \mathcal{N}([1/\varepsilon, +\infty); \mathsf{J}(\lambda)) - \mathcal{N}((1/\varepsilon, +\infty); \mathsf{J}(\lambda)) = \dim\ker\big(\mathsf{I} + \varepsilon\mathsf{J}(\lambda)\big). \quad \square\end{aligned}$$

For resonances of H_ε one can derive a Birman-Schwinger-type condition analogous to the one in Corollary 2.3 (ii). For the sheet $Z_E \subset \widetilde{Z}$ of the Riemann surface $\widehat{Z}$ we define the Jacobi matrix

$$\mathsf{J}_E(\lambda) := \mathsf{J}(\{0\}, \{b_n^E(\lambda)\}), \qquad \lambda \in \mathbb{C} \setminus [\nu_0, +\infty), \tag{2.5}$$

where

$$b_n^E(\lambda) := \frac{1}{2}\left(\frac{n}{r_n(\lambda, l_n^E) r_{n-1}(\lambda, l_{n-1}^E)}\right)^{1/2}, \qquad n \in \mathbb{N}. \tag{2.6}$$

The Jacobi matrix $\mathsf{J}_E(\lambda)$ in (2.5) is closed, bounded, and everywhere defined in $\ell^2(\mathbb{N}_0)$, but in general non-selfadjoint. For $E = \varnothing$ and $\lambda \in (0, \nu_0)$ the Jacobi matrix $\mathsf{J}_\varnothing(\lambda)$ coincides with $\mathsf{J}(\lambda)$ in (2.2). In what follows it is also convenient to set $b_0^E(\lambda) = 0$. In the next theorem we characterise resonances of H_ε lying on the sheet Z_E.

Theorem 2.4. *Let the self-adjoint operator H_ε, $\varepsilon \in (0,1)$, be as in (1.5). Let the sheet $Z_E \subset \widetilde{Z}$ be fixed, let $\mathcal{R}_E(\varepsilon)$ be as in Definition 1.4 and the associated operator-valued function $\mathsf{J}_E(\lambda)$ be as in (2.5). Then the following equivalence holds*

$$\lambda \in \mathcal{R}_E(\varepsilon) \qquad \Longleftrightarrow \qquad \ker\left(\mathsf{I} + \varepsilon \mathsf{J}_E(\lambda)\right) \neq \{0\}. \tag{2.7}$$

For $E = \varnothing$ the claim of Theorem 2.4 follows from Corollary 2.3 (ii). The proof of the remaining part of Theorem 2.4 is postponed until Appendix A. The argument essentially relies on Krein-type resolvent formula [32] for H_ε and on the analytic Fredholm theorem [29, Thm. 3.4.2].

Remark 2.5. Thanks to compactness of the difference $\mathsf{J}_E(\lambda) - \mathsf{J}_0$ we get by [26, Lem. XIII.4.3] that $\sigma_{\mathrm{ess}}(\varepsilon \mathsf{J}_E(\lambda)) = \sigma_{\mathrm{ess}}(\varepsilon \mathsf{J}_0)) = [-\varepsilon, \varepsilon]$. Therefore, the equivalence (2.7) can be rewritten as

$$\lambda \in \mathcal{R}_E(\varepsilon) \qquad \Longleftrightarrow \qquad -1 \in \sigma_{\mathrm{d}}(\varepsilon \mathsf{J}_E(\lambda)).$$

Identity $\mathsf{J}_E(\lambda)^* = \mathsf{J}_E(\overline{\lambda})$ combined with [22, Rem. III.6.23] and with Theorem 2.4 yields that the set $\mathcal{R}_E(\varepsilon)$ is symmetric with respect to the real axis.

3. Localization of bound states and resonances

In this section we prove Theorem 1.2 (i) and Theorem 1.5 (i). The idea of the proof is to estimate the norm of $\mathsf{J}_E(\lambda)$ and to apply Corollary 2.3 (ii) and Theorem 2.4.

Proof of Theorem 1.2 (i) and Theorem 1.5 (i). The square of the norm of the operator $\mathsf{J}_E(\lambda)$ in (2.5) can be estimated from above by

$$\begin{aligned}
\|\mathsf{J}_E(\lambda)\|^2 &\leqslant \sup_{\xi \in \ell^2(\mathbb{N}_0), \|\xi\|=1} \|\mathsf{J}_E(\lambda)\xi\|^2 \\
&\leqslant \sup_{\xi \in \ell^2(\mathbb{N}_0), \|\xi\|=1} \left(\sum_{n \in \mathbb{N}_0} |b_n^E(\lambda)\xi_{n-1} + b_{n+1}^E(\lambda)\xi_{n+1}|^2 \right) \\
&\leqslant \sup_{\xi \in \ell^2(\mathbb{N}_0), \|\xi\|=1} \left(2 \sum_{n \in \mathbb{N}_0} \left(|b_n^E(\lambda)|^2 |\xi_{n-1}|^2 + |b_{n+1}^E(\lambda)|^2 |\xi_{n+1}|^2 \right) \right) \\
&\leqslant 4 \sup_{n \in \mathbb{N}_0} |b_n^E(\lambda)|^2 \sup_{\xi \in \ell^2(\mathbb{N}_0), \|\xi\|=1} \|\xi\|^2 = 4 \sup_{n \in \mathbb{N}} |b_n^E(\lambda)|^2,
\end{aligned} \tag{3.1}$$

where $b_n^E(\lambda)$, $n \in \mathbb{N}_0$, are defined as in (2.6).

If $\|\varepsilon \mathsf{J}_E(\lambda)\| < 1$ holds for a point $\lambda \in \mathbb{C}_-$ then the condition $\ker\left(\mathsf{I} + \varepsilon \mathsf{J}_E(\lambda)\right) \neq \{0\}$ is not satisfied. Thus, λ cannot by Theorem 2.4 be a resonance of H_ε lying on Z_E^- in the sense

of Definition 1.4. In view of estimate (3.1) and of (2.6) to fulfil $\|\varepsilon \mathsf{J}_E(\lambda)\| < 1$ it suffices to satisfy

$$\frac{n}{|\nu_{n-1}-\lambda|^{1/2}|\nu_n-\lambda|^{1/2}} < \frac{1}{\varepsilon^2}, \qquad \forall\, n \in \mathbb{N},$$

or, equivalently,

$$|\nu_n-\lambda| \cdot |\nu_{n-1}-\lambda| > \varepsilon^4 n^2, \qquad \forall\, n \in \mathbb{N}.$$

Thus, the claim of Theorem 1.5 (i) is proven.

If $\|\varepsilon \mathsf{J}_\varnothing(\lambda)\| < 1$ holds for a point $\lambda \in (0,1/2)$ then the condition $\ker\big(\mathsf{I}+\varepsilon \mathsf{J}_\varnothing(\lambda)\big) \neq \{0\}$ is not satisfied. Thus, by Corollary 2.3 (ii), λ is not an eigenvalue of H_ε. In view of (3.1) and (2.6) to fulfil $\|\varepsilon \mathsf{J}_\varnothing(\lambda)\| < 1$ it suffices to satisfy

$$\big(\nu_{n-1}-\lambda\big)\big(\nu_n-\lambda\big) = \lambda^2 - 2n\lambda + n^2 - 1/4 > n^2\varepsilon^4, \qquad \forall\, n \in \mathbb{N}. \tag{3.2}$$

The roots of the equation $\lambda^2-2n\lambda+n^2-1/4-n^2\varepsilon^4=0$ are given by $\lambda_n^{\pm}(\varepsilon) = n \pm \sqrt{1/4+n^2\varepsilon^4}$. Since $\lambda_n^+(\varepsilon) > 1/2$ for all $n \in \mathbb{N}$, the condition (3.2) yields $\lambda_1(\mathsf{H}_\varepsilon) \geqslant \min_{n\in\mathbb{N}} \lambda_n^-(\varepsilon)$. For $n \in \mathbb{N}$ we have

$$\lambda_{n+1}^-(\varepsilon) - \lambda_n^-(\varepsilon) = 1 - \frac{(2n+1)\varepsilon^4}{\big(\frac{1}{4}+n^2\varepsilon^4\big)^{1/2} + \big(\frac{1}{4}+(n+1)^2\varepsilon^4\big)^{1/2}} \geqslant 1 - \frac{(2n+1)\varepsilon^4}{(2n+1)\varepsilon^2} = 1-\varepsilon^2 > 0.$$

Hence, $\min_{n\in\mathbb{N}} \lambda_n^-(\varepsilon) = \lambda_1^-(\varepsilon)$ and the claim of Theorem 1.2 (i) follows. □

4. The weak coupling regime: $\varepsilon \to 0+$

In this section we prove Theorem 1.2 (ii) and Theorem 1.5 (ii)-(iii). Intermediate results of this section given in Lemmata 4.1 and 4.3 are of an independent interest.

First, we introduce some auxiliary operators and functions. Let $n \in \mathbb{N}_0$ and the sheet $Z_E \subset \widetilde{Z}$ be fixed. We make use of notation $\mathsf{P}_{kl} := \mathsf{e}_{n+k-2}(\cdot, \mathsf{e}_{n+l-2})$ with $k,l \in \{1,2,3\}$. Note that for $n=0$ we have $\mathsf{P}_{k1} = \mathsf{P}_{1k} = 0$ for $k=1,2,3$. It will also be convenient to decompose the Jacobi matrix $\mathsf{J}_E(\lambda)$ in (2.5) as

$$\mathsf{J}_E(\lambda) = \mathsf{S}_{n,E}(\lambda) + \mathsf{T}_{n,E}(\lambda), \tag{4.1}$$

where the operator-valued functions $\lambda \mapsto \mathsf{T}_{n,E}(\lambda), \mathsf{S}_{n,E}(\lambda)$ are defined by

$$\mathsf{T}_{n,E}(\lambda) := b_{n+1}^E(\lambda)\,[\mathsf{P}_{23}+\mathsf{P}_{32}] + b_n^E(\lambda)\,[\mathsf{P}_{21}+\mathsf{P}_{12}]\,, \qquad \mathsf{S}_{n,E}(\lambda) := \mathsf{J}_E(\lambda) - \mathsf{T}_{n,E}(\lambda). \tag{4.2}$$

Clearly, the operator-valued function $\mathsf{S}_{n,E}(\cdot)$ is uniformly bounded on $\mathbb{D}_{1/2}(\nu_n)$. Moreover, for sufficiently small $r = r(n) \in (0,1/2)$ the bounded operator $\mathsf{I}+\varepsilon \mathsf{S}_{n,E}(\lambda)$ is at the same time boundedly invertible for all $(\varepsilon,\lambda) \in \Omega_r(n) := \mathbb{D}_r \times \mathbb{D}_r(\nu_n)$. Thus, the operator-valued function

$$\mathsf{R}_{n,E}(\varepsilon,\lambda) := \big(\mathsf{I}+\varepsilon \mathsf{S}_{n,E}(\lambda)\big)^{-1}, \tag{4.3}$$

is well-defined and analytic on $\Omega_r(n)$ and, in particular, $\mathsf{R}_{n,E}(0,\nu_n) = \mathsf{I}$. Furthermore, we introduce auxiliary scalar functions $\Omega_r(n) \ni (\varepsilon,\lambda) \mapsto f_{kl}^E(\varepsilon,\lambda)$ by

$$f_{kl}^E(\varepsilon,\lambda) := \big(\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{e}_{n+k-2}, \mathsf{e}_{n+l-2}\big), \qquad k,l \in \{1,2,3\}. \tag{4.4}$$

Thanks to $\mathsf{R}_{n,E}(0,\nu_n) = \mathsf{I}$ we have $f_{kl}^E(0,\nu_n) = \delta_{kl}$. Finally, we introduce 3×3 matrix-valued function

$$\mathbb{D}_r \times \mathbb{D}_r^{\times}(\nu_n) \ni (\varepsilon,\lambda) \mapsto \mathsf{A}_{n,E}(\varepsilon,\lambda) := \big(a_{kl}^E(\varepsilon,\lambda)\big)_{k,l=1}^{3,3} \tag{4.5}$$

with the entries given for $k,l=1,2,3$ by

$$a_{kl}^E(\varepsilon,\lambda) := b_n^E(\lambda)\big(f_{1k}^E(\varepsilon,\lambda)\delta_{2l} + f_{2k}^E(\varepsilon,\lambda)\delta_{1l}\big) + b_{n+1}^E(\lambda)\big(f_{2k}^E(\varepsilon,\lambda)\delta_{3l} + f_{3k}^E(\varepsilon,\lambda)\delta_{2l}\big). \tag{4.6}$$

We remark that $\operatorname{rank}\mathsf{A}_{n,E}(\varepsilon,\lambda)\leqslant 2$ due to linear dependence between the first and the third columns in $\mathsf{A}_{n,E}(\varepsilon,\lambda)$.

In the first lemma we derive an implicit scalar equation which characterises those points $\lambda\in\mathbb{C}\setminus[\nu_0,+\infty)$ near ν_n for which the condition $\ker\left(\mathsf{I}+\varepsilon\mathsf{J}_E(\lambda)\right)\neq\{0\}$ is satisfied under additional assumption that $\varepsilon>0$ is small enough. This equation can be used to characterise the 'true' resonances for H_ε as well as the weakly coupled bound state if $n=0$ and $E=\varnothing$.

Lemma 4.1. *Let the self-adjoint operator H_ε, $\varepsilon\in(0,1)$, be as in (1.5). Let $n\in\mathbb{N}_0$ and the sheet $Z_E\subset\widetilde{Z}$ be fixed. Let $r=r(n)>0$ be chosen as above. Then for all $\varepsilon\in(0,r)$ a point $\lambda\in\mathbb{D}_r(\nu_n)\setminus[\nu_0,\infty)$ is a resonance of H_ε on Z_E if, and only if*

$$\det\left(\mathsf{I}+\varepsilon\mathsf{A}_{n,E}(\varepsilon,\lambda)\right)=0.$$

Proof. Using the decomposition (4.1) of $\mathsf{J}_E(\lambda)$ and the auxiliary operator in (4.3) we find

$$\begin{aligned}\dim\ker\left(\mathsf{I}+\varepsilon\mathsf{J}_E(\lambda)\right)&=\dim\ker\left(\mathsf{I}+\varepsilon\mathsf{S}_{n,E}(\lambda)+\varepsilon\mathsf{T}_{n,E}(\lambda)\right)\\&=\dim\ker\left(\mathsf{I}+\varepsilon\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{T}_{n,E}(\lambda)\right).\end{aligned}\tag{4.7}$$

Note that

$$\operatorname{rank}\left(\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{T}_{n,E}(\lambda)\right)\leqslant\operatorname{rank}\left(\mathsf{T}_{n,E}(\lambda)\right)\leqslant 3$$

and, hence, using [28, Thm. 3.5 (b)], we get

$$\dim\ker\left(\mathsf{I}+\varepsilon\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{T}_{n,E}(\lambda)\right)\geqslant 1\qquad\Longleftrightarrow\qquad\det\left(\mathsf{I}+\varepsilon\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{T}_{n,E}(\lambda)\right)=0.\tag{4.8}$$

For the orthogonal projector $\mathsf{P}:=\mathsf{P}_{11}+\mathsf{P}_{22}+\mathsf{P}_{33}$ the identity $\mathsf{T}_{n,E}(\lambda)=\mathsf{T}_{n,E}(\lambda)\mathsf{P}$ is straightforward. Hence, employing [20, IV.1.5] we find

$$\begin{aligned}\det\left(\mathsf{I}+\varepsilon\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{T}_{n,E}(\lambda)\right)&=\det\left(\mathsf{I}+\varepsilon\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{T}_{n,E}(\lambda)\mathsf{P}\right)\\&=\det\left(\mathsf{I}+\varepsilon\mathsf{P}\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{T}_{n,E}(\lambda)\right).\end{aligned}\tag{4.9}$$

For $k,l\in\{1,2,3\}$ we can write the following identities

$$\begin{aligned}\mathsf{P}_{kk}\mathsf{P}\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{T}_{n,E}(\lambda)\mathsf{P}_{ll}&=\mathsf{P}_{kk}\mathsf{R}_{n,E}(\varepsilon,\lambda)\Big(b_n^E(\lambda)\left[\mathsf{P}_{21}+\mathsf{P}_{12}\right]+b_{n+1}^E(\lambda)\left[\mathsf{P}_{23}+\mathsf{P}_{32}\right]\Big)\mathsf{P}_{ll}\\&=\mathsf{P}_{kk}\mathsf{R}_{n,E}(\varepsilon,\lambda)\Big(b_n^E(\lambda)\left[\mathsf{P}_{2l}\delta_{1l}+\mathsf{P}_{1l}\delta_{2l}\right]+b_{n+1}^E(\lambda)\left[\mathsf{P}_{2l}\delta_{3l}+\mathsf{P}_{3l}\delta_{2l}\right]\Big)\\&=\mathsf{P}_{kl}b_n^E(\lambda)\left[f_{2k}^E(\varepsilon,\lambda)\delta_{1l}+f_{1k}^E(\varepsilon,\lambda)\delta_{2l}\right]\\&\qquad+\mathsf{P}_{kl}b_{n+1}^E(\lambda)\left[f_{2k}^E(\varepsilon,\lambda)\delta_{3l}+f_{3k}^E(\varepsilon,\lambda)\delta_{2l}\right]\\&=a_{kl}^E(\varepsilon,\lambda)\mathsf{P}_{kl}\end{aligned}$$

with f_{kl}^E as in (4.4), and as a result we get

$$\mathsf{P}\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{T}_{n,E}(\lambda)=\sum_{k=1}^{3}\sum_{l=1}^{3}a_{kl}^E(\varepsilon,\lambda)\mathsf{P}_{kl},$$

with $a_{kl}^E(\varepsilon,\lambda)$ as in (4.6). Hence, the determinant in (4.9) can be expressed as

$$\det\left(\mathsf{I}+\varepsilon\mathsf{R}_{n,E}(\varepsilon,\lambda)\mathsf{T}_{n,E}(\lambda)\right)=\det(\mathsf{I}+\varepsilon\mathsf{A}_{n,E}(\varepsilon,\lambda))$$

where on the right-hand side we have the determinant of the 3×3 matrix $\mathsf{I}+\varepsilon\mathsf{A}_{n,E}(\varepsilon,\lambda)$; *cf.* (4.5). The claim of lemma follows then from (4.7), (4.8), and Theorem 2.4. □

In the second lemma we establish the existence and investigate properties of solutions of the scalar equation in Lemma 4.1. To this aim it is natural to try to apply the *analytic implicit function theorem*. The main obstacle that makes a direct application of the implicit function theorem difficult lies in the fact that $\lambda \mapsto \det(\mathsf{I} + \varepsilon \mathsf{A}_{n,E}(\varepsilon, \lambda))$ is not analytic near ν_n due to the cut on the real axis. We circumvent this obstacle by applying the analytic implicit function theorem to an auxiliary function which is analytic in the disc and has values in different sectors of this disc that are in direct correspondence with the values of $\lambda \mapsto \det(\mathsf{I} + \varepsilon \mathsf{A}_{n,E}(\varepsilon, \lambda))$ on the four different sheets in $\widetilde{Z}$ which are mutually adjacent in a proper way.

Assumption 4.2. *Let $n \in \mathbb{N}_0$ and the sheet $Z_E \subset \widetilde{Z}$ be fixed. Let the sheets Z_F, Z_G and Z_H be such that $Z_E \sim_{n-1} Z_F$, $Z_F \sim_n Z_G$ and $Z_G \sim_{n-1} Z_H$. For $r > 0$ let the matrix-valued function $\mathbb{D}_r \times \mathbb{D}_r^\times \ni (\varepsilon, \kappa) \mapsto \mathsf{B}_{n,E}(\varepsilon, \kappa)$ be defined by*

$$\mathsf{B}_{n,E}(\varepsilon, \kappa) := \begin{cases} \mathsf{A}_{n,E}(\varepsilon, \nu_n - \kappa^4), & \arg \kappa \in \Phi_E := (-\pi, -\frac{3\pi}{4}] \cup (0, \frac{\pi}{4}], \\ \mathsf{A}_{n,F}(\varepsilon, \nu_n - \kappa^4), & \arg \kappa \in \Phi_F := (-\frac{3\pi}{4}, -\frac{\pi}{2}] \cup (\frac{\pi}{4}, \frac{\pi}{2}], \\ \mathsf{A}_{n,G}(\varepsilon, \nu_n - \kappa^4), & \arg \kappa \in \Phi_G := (-\frac{\pi}{2}, -\frac{\pi}{4}] \cup (\frac{\pi}{2}, \frac{3\pi}{2}], \\ \mathsf{A}_{n,H}(\varepsilon, \nu_n - \kappa^4), & \arg \kappa \in \Phi_H := (-\frac{\pi}{4}, 0] \cup (\frac{3\pi}{2}, \pi]. \end{cases}$$

Tracing the changes in the characteristic vector along the path $Z_E \sim_{n-1} Z_F \sim_n Z_G \sim_{n-1} Z_H$ one easily verifies that $Z_H \sim_n Z_E$. Thanks to that $\mathsf{B}_{n,E}$ is analytic on $\mathbb{D}_r \times \mathbb{D}_r^\times$ for sufficiently small $r > 0$ which is essentially a consequence of componentwise analyticity in $\mathbb{D}_r$ of vector-valued function

$$\kappa \mapsto R_\bullet(\nu_n - \kappa^4), \qquad \bullet \in \{E, F, G, H\} \quad \text{for} \quad \arg \kappa \in \Phi_\bullet,$$

where $R_\bullet$ is as in (1.9).

Lemma 4.3. *Let $n \in \mathbb{N}_0$ and the sheet $Z_E \subset \widetilde{Z}$ be fixed. Set $(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) := (l_{n-1}^E, l_n^E, l_{n+1}^E)$. Let the matrix-valued function $\mathsf{B}_{n,E}$ be as in Assumption 4.2. Then the implicit scalar equation*

$$\det\big(\mathsf{I} + \varepsilon \mathsf{B}_{n,E}(\varepsilon, \kappa)\big) = 0$$

has exactly two solutions $\kappa_{n,E,j}(\cdot)$ analytic near $\varepsilon = 0$ such that $\kappa_{n,E,j}(0) = 0$, satisfying $\det(\mathsf{I} + \varepsilon \mathsf{B}_{n,E}(\varepsilon, \kappa_{n,E,j}(\varepsilon))) = 0$ pointwise for sufficiently small $\varepsilon > 0$, and having asymptotic expansions

$$\kappa_{n,E,j}(\varepsilon) = \varepsilon \frac{(z_{n,E})_j^{1/2}}{2} + \mathcal{O}(\varepsilon^2), \qquad \varepsilon \to 0+, \tag{4.10}$$

where $z_{n,E} = (-1)^{\mathfrak{q}+\mathfrak{r}}(n+1) + (-1)^{\mathfrak{p}+\mathfrak{q}+1} n\mathrm{i}$.

Proof. First, we introduce the shorthand notations

$$u(\kappa) := b_n^\bullet(\nu_n - \kappa^4), \quad v(\kappa) := b_{n+1}^\bullet(\nu_n - \kappa^4), \qquad \bullet \in \{E, F, G, H\} \quad \text{for} \quad \arg \kappa \in \Phi_\bullet.$$

Let b_{kl} with $k, l \in \{1, 2, 3\}$ be the entries of the matrix-valued function $\mathsf{B}_{n,E}$. Furthermore, define the scalar functions $X = X(\varepsilon, \kappa)$, $Y = Y(\varepsilon, \kappa)$, and $Z = Z(\varepsilon, \kappa)$ by

$$\begin{aligned} X &:= b_{11} + b_{22} + b_{33}, \\ Y &:= b_{11}b_{22} + b_{22}b_{33} + a_{11}a_{33} - b_{13}b_{31} - b_{12}b_{21} - b_{23}b_{32}, \\ Z &:= b_{11}b_{22}b_{33} + b_{13}b_{32}b_{21} + b_{12}b_{23}b_{31} - b_{13}b_{31}b_{22} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32}. \end{aligned} \tag{4.11}$$

Employing an elementary formula for the determinant of 3×3 matrix, the equation $\det(\mathsf{I} + \varepsilon \mathsf{B}_{n,E}(\varepsilon, \kappa)) = 0$ can be equivalently written as

$$1 + \varepsilon X(\varepsilon, \kappa) + \varepsilon^2 Y(\varepsilon, \kappa) + \varepsilon^3 Z(\varepsilon, \kappa) = 0. \tag{4.12}$$

By purely algebraic argument one can derive from (4.6) that $Z = 0$. Hence, (4.12) simplifies to $1 + \varepsilon X(\varepsilon,\kappa) + \varepsilon^2 Y(\varepsilon,\kappa) = 0$. Introducing new parameter $t := \varepsilon/\kappa$ we can further rewrite this equation as

$$1 + t\kappa X(\varepsilon,\kappa) + t^2\kappa^2 Y(\varepsilon,\kappa) = 0. \tag{4.13}$$

Note also that the coefficients $(\varepsilon,\kappa) \mapsto \kappa X(\varepsilon,\kappa), \kappa^2 Y(\varepsilon,\kappa)$ of the quadratic equation (4.13) are analytic in $\mathbb{D}_r^2$. For each fixed pair (ε,κ) the equation (4.13) has (in general) two distinct roots $t_j(\varepsilon,\kappa)$, $j = 0, 1$. The condition $\det(\mathsf{I} + \varepsilon \mathsf{B}_{n,E}(\varepsilon,\kappa)) = 0$ with $\kappa \neq 0$ holds *if, and only if* at least one of the two conditions

$$f_j(\varepsilon,\kappa) := \varepsilon - \kappa t_j(\varepsilon,\kappa) = 0, \qquad j = 0, 1, \tag{4.14}$$

is satisfied. Using analyticity of $u(\cdot)$ and $v(\cdot)$ near $\kappa = 0$, we compute

$$\begin{aligned}
\lim_{\kappa\to 0} \kappa u &= \lim_{r\to 0+} re^{\mathrm{i}\pi/8} u(re^{\mathrm{i}\pi/8}) \\
&= \lim_{r\to 0+} \frac{n^{1/2}}{2} \frac{re^{\mathrm{i}\pi/8}}{((-1+\mathrm{i}r^4)_{\mathfrak{p}}^{1/2}(\mathrm{i}r^4)_{\mathfrak{q}}^{1/2})^{1/2}} = \frac{n^{1/2}e^{\mathrm{i}\pi/8}}{2((-1)^{\mathfrak{p}+\mathfrak{q}}\mathrm{i}e^{\mathrm{i}\pi/4})^{1/2}}, \\
\lim_{\kappa\to 0} \kappa v &= \lim_{r\to 0+} re^{\mathrm{i}\pi/8} v(re^{\mathrm{i}\pi/8}) \\
&= \lim_{r\to 0+} \frac{(n+1)^{1/2}}{2} \frac{re^{\mathrm{i}\pi/8}}{((\mathrm{i}r^4)_{\mathfrak{q}}^{1/2}(1+\mathrm{i}r^4)_{\mathfrak{r}}^{1/2})^{1/2}} = \frac{(n+1)^{1/2}e^{\mathrm{i}\pi/8}}{2((-1)^{\mathfrak{q}+\mathfrak{r}}e^{\mathrm{i}\pi/4})^{1/2}}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\lim_{\varepsilon,r\to 0+} re^{\mathrm{i}\pi/8} b_{kl}(\varepsilon, re^{\mathrm{i}\pi/8}) &= \lim_{r\to 0+} re^{\mathrm{i}\pi/8} u(re^{\mathrm{i}\pi/8}) \big(f_{2k}^E(\mathbf{0})\delta_{3l} + f_{3k}^E(\mathbf{0})\delta_{2l}\big) \\
&\quad + \lim_{r\to 0+} re^{\mathrm{i}\pi/8} v(re^{\mathrm{i}\pi/8}) \big(f_{1k}^E(\mathbf{0})\delta_{2l} + f_{2k}^E(\mathbf{0})\delta_{1l}\big) \\
&= \frac{n^{1/2}e^{\mathrm{i}\pi/8}\big(\delta_{2k}\delta_{3l} + \delta_{3k}\delta_{2l}\big)}{2((-1)^{\mathfrak{p}+\mathfrak{q}}\mathrm{i}e^{\mathrm{i}\pi/4})^{1/2}} + \frac{(n+1)^{1/2}e^{\mathrm{i}\pi/8}\big(\delta_{1k}\delta_{2l} + \delta_{2k}\delta_{1l}\big)}{2((-1)^{\mathfrak{q}+\mathfrak{r}}e^{\mathrm{i}\pi/4})^{1/2}}.
\end{aligned}$$

Combining this with (4.11) we end up with

$$\begin{aligned}
\lim_{(\varepsilon,\kappa)\to\mathbf{0}} \kappa X &= \lim_{\varepsilon,r\to 0+} re^{\mathrm{i}\pi/8} X(\varepsilon, re^{\mathrm{i}\pi/8}) = \lim_{\varepsilon,r\to 0+} re^{\mathrm{i}\pi/8}\big[b_{11} + b_{22} + b_{33}\big](\varepsilon, re^{\mathrm{i}\pi/8}) = 0, \\
\lim_{(\varepsilon,\kappa)\to\mathbf{0}} \kappa^2 Y &= \lim_{\varepsilon,r\to 0+} r^2 e^{\mathrm{i}\pi/4} Y(\varepsilon, re^{\mathrm{i}\pi/8}) \\
&= \lim_{\varepsilon,r\to 0+} r^2 e^{\mathrm{i}\pi/4}\big[b_{11}b_{22} + b_{22}b_{33} + b_{11}b_{33} - b_{13}b_{31} - b_{12}b_{21} - b_{23}b_{32}\big](\varepsilon, re^{\mathrm{i}\pi/8}) \\
&= \lim_{\varepsilon,r\to 0+} r^2 e^{\mathrm{i}\pi/4}\big[- b_{12}b_{21} - b_{23}b_{32}\big](\varepsilon, re^{\mathrm{i}\pi/8}) \\
&= -\left(\frac{n^{1/2}e^{\mathrm{i}\pi/8}}{2((-1)^{\mathfrak{p}+\mathfrak{q}}\mathrm{i}e^{\mathrm{i}\pi/4})^{1/2}}\right)^2 - \left(\frac{(n+1)^{1/2}e^{\mathrm{i}\pi/8}}{2((-1)^{\mathfrak{q}+\mathfrak{r}}e^{\mathrm{i}\pi/4})^{1/2}}\right)^2 \\
&= -\frac{(-1)^{\mathfrak{p}+\mathfrak{q}+1}n\mathrm{i}}{4} - \frac{(-1)^{\mathfrak{q}+\mathfrak{r}}(n+1)}{4} = -\frac{z_{n,E}}{4}.
\end{aligned}$$

Hence, the roots $t_j(\varepsilon,\kappa)$ of (4.13) converge in the limit $(\varepsilon,\kappa) \to \mathbf{0}$ to the roots $2\big[(z_{n,E})_j^{1/2}\big]^{-1}$, $j = 0, 1$, of the quadratic equation $z_{n,E}t^2 - 4 = 0$. Moreover, analyticity of the coefficients in equation (4.13), the above limits, and the formula for the roots of a quadratic equation imply analyticity of the functions $(\varepsilon,\kappa) \mapsto t_j(\varepsilon,\kappa)$ near $\mathbf{0}$.

Step 2. The partial derivatives of f_j in (4.14) with respect to ε and κ are given by $\partial_\varepsilon f_j = 1 - \kappa\partial_\varepsilon t_j$ and $\partial_\kappa f_j = -t_j - \kappa\partial_\kappa t_j$. Analyticity of t_j near $\mathbf{0}$ implies $(\partial_\varepsilon f_j)(\mathbf{0}) = 1$ and

$(\partial_\kappa f_j)(\mathbf{0}) = -t_j$. In particular, we have shown that $(\partial_\kappa f_j)(\mathbf{0}) \neq 0$. Since the functions $f_j(\cdot)$ are analytic near $\mathbf{0}$ and satisfy $f_j(\mathbf{0}) = 0$, we can apply the analytic implicit function theorem [29, Thm. 3.4.2] which yields existence of a unique function $\kappa_j(\cdot)$, analytic near $\varepsilon = 0$ such that $\kappa_j(0) = 0$ and that $f_j(\varepsilon, \kappa_j(\varepsilon)) = 0$ holds pointwise. Moreover, the derivative of κ_j at $\varepsilon = 0$ can be expressed as

$$\kappa_j'(0) = -\frac{(\partial_\varepsilon f_j)(\mathbf{0})}{(\partial_\kappa f_j)(\mathbf{0})} = \frac{1}{t_j(\mathbf{0})}. \tag{4.15}$$

Hence, we obtain Taylor expansion for κ_j near $\varepsilon = 0$

$$\kappa_j(\varepsilon) = \kappa_j(0) + \kappa_j'(0)\varepsilon + \mathcal{O}(\varepsilon^2) = \frac{\varepsilon}{t_j(\mathbf{0})} + \mathcal{O}(\varepsilon^2) = \varepsilon\frac{\left(z_{n,E}\right)_j^{1/2}}{2} + \mathcal{O}(\varepsilon^2) \qquad \varepsilon \to 0+ .$$

The functions κ_j, $j = 0, 1$, satisfy all the requirements in the claim of the lemma. □

Now we are prepared to prove Theorem 1.2 (ii) and Theorem 1.5 (ii)-(iii) from the introduction.

Proof of Theorem 1.2 (ii). By Proposition 1.1 (iv) we have $\mathcal{N}_{1/2}(\mathsf{H}_\varepsilon) = 1$ for all sufficiently small $\varepsilon > 0$. Recall that we denote by $\lambda_1(\mathsf{H}_\varepsilon)$ the corresponding unique eigenvalue. Thus, we have by Lemma 4.1

$$\det(\mathsf{I} + \varepsilon\mathsf{A}_{0,\varnothing}(\varepsilon, \lambda_1(\mathsf{H}_\varepsilon))) = 0.$$

Using the construction of Assumption 4.2 for the physical sheet and $n = 0$, we obtain

$$\det(\mathsf{I} + \varepsilon\mathsf{B}_{0,\varnothing}(\varepsilon, (\lambda_1(\mathsf{H}_\varepsilon) - \nu_0)^{1/4})) = \det(\mathsf{I} + \varepsilon\mathsf{A}_{0,\varnothing}(\varepsilon, \lambda_1(\mathsf{H}_\varepsilon))) = 0,$$

where we have chosen the principal branch for $(\cdot)^{1/4}$. Thus, by Lemma 4.3 we get

$$(\lambda_1(\mathsf{H}_\varepsilon) - \nu_0)^{1/4} = \pm\frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2), \qquad \varepsilon \to 0+,$$

where we have used the fact that $z_{0,\varnothing} = 1$. Hence, taking the fourth power of the left and right hand sides in the above equation we arrive at

$$\lambda_1(\mathsf{H}_\varepsilon) = \nu_0 - \frac{\varepsilon^4}{16} + \mathcal{O}(\varepsilon^5), \qquad \varepsilon \to 0+ . \qquad \square$$

Proof of Theorem 1.5 (ii)-(iii). Let $n \in \mathbb{N}$ and the sheet $Z_E \subset \widetilde{Z}$ be fixed. Let us repeat the construction of Assumption 4.2. By Lemma 4.3 we infer that there exist exactly two analytic solutions $\kappa_{n,E,j}$, $j = 0, 1$ of the implicit scalar equation $\det(\mathsf{I} + \varepsilon\mathsf{B}_{n,E}(\varepsilon, \kappa)) = 0$ such that $\kappa_{n,E,j}(0) = 0$. It can be checked that both solutions correspond to the same resonance and it suffices to analyse the solution $\kappa_{n,E} := \kappa_{n,E,0}$ only.

For all small enough $\varepsilon > 0$ the asymptotics (4.10) yields

$$\arg(\kappa_{n,E}(\varepsilon)) = \frac{1}{2}\arg(z_{n,E}) \in \Phi_E, \quad \textit{if, and only if} \quad n \in \mathcal{S}(E).$$

Hence, if $n \in \mathbb{N} \setminus \mathcal{S}(E)$, Lemmata 4.1 and 4.3 imply that there will be no resonances in a neighbourhood of the point $\lambda = \nu_n$ lying on Z_E^- for sufficiently small $\varepsilon > 0$. Thus, we have proven Theorem 1.5 (iii). While if $n \in \mathcal{S}(E)$ we get by Lemmata 4.1 and 4.3 that there will be exactly one resonance

$$\lambda_n^E(\mathsf{H}_\varepsilon) = \nu_n - (\kappa_{n,E}(\varepsilon))^4,$$

in a neighbourhood of the point $\lambda = \nu_n$ lying on Z_E^- for sufficiently small $\varepsilon > 0$ and its asymptotic expansion is a direct consequence of the asymptotic expansion (4.10) given in Lemma 4.3. Thus, the claim of Theorem 1.5 (ii) follows. □

A. Krein's formula, meromorphic continuation of resolvent, and condition on resonances

In this appendix we use Krein's resolvent formula for Smilansky Hamiltonian to prove Proposition 1.3 and Theorem 2.4 on meromorphic continuation of $(\mathsf{H}_\varepsilon - \lambda)^{-1}$ to $\widetilde{Z}$. The proposed continuation procedure is of an iterative nature wherein we, first, extend $(\mathsf{H}_\varepsilon-\lambda)^{-1}$ to the sheets adjacent to the physical sheet, then to the sheets which are adjacent to the sheets being adjacent to the physical sheet and so on.

To this aim we define for $n \in \mathbb{N}_0$ the scalar functions $\mathbb{C} \setminus [\nu_0, +\infty) \mapsto y_n(\lambda)$ and $(\mathbb{C} \setminus [\nu_0, +\infty)) \times \mathbb{R} \mapsto \eta_n(\lambda; x)$ by

$$y_n(\lambda) := r_n(\lambda)\sqrt{\nu_n}, \qquad \eta_n(\lambda; x) := \nu_n^{1/4} \exp(-r_n(\lambda)|x|), \tag{A.1}$$

where $r_n(\cdot)$, $n \in \mathbb{N}_0$, is as in (1.7). Next, we introduce the following operator-valued function

$$\mathsf{T}(\lambda)\colon \ell^2(\mathbb{N}_0) \to \mathcal{H}, \qquad \mathsf{T}(\lambda)\{c_n\} := \{c_n \eta_n(\lambda; x)\}.$$

For each fixed $\lambda \in \mathbb{C} \setminus [\nu_0, +\infty)$ the operator $\mathsf{T}(\lambda)$ is bounded and everywhere defined and the adjoint of $\mathsf{T}(\overline{\lambda})$ acts as

$$\mathsf{T}(\overline{\lambda})^*\{u_n\} \sim \{I_n(\lambda; u_n)\}_{n\in\mathbb{N}_0}, \qquad I_n(\lambda; u) := \int_{\mathbb{R}} \eta_n(\lambda; x) u(x) \mathrm{d}x.$$

With these preparations, the resolvent difference of H_ε and H_0 can be expressed by [32, Thm. 6.1] (see also [24, Sec. 6]) as follows

$$(\mathsf{H}_\varepsilon-\lambda)^{-1} = (\mathsf{H}_0-\lambda)^{-1} + \mathsf{T}(\lambda)\mathsf{Y}(\lambda)\big[\big(\mathsf{I}+\varepsilon \mathsf{J}_\varnothing(\lambda)\big)^{-1} - \mathsf{I}\big]\mathsf{Y}(\lambda)\mathsf{T}(\overline{\lambda})^*, \quad \lambda \in \mathbb{C}\setminus[\nu_0, +\infty), \tag{A.2}$$

where H_0 is the Smilansky Hamiltonian with $\varepsilon = 0$, $\mathsf{Y}(\lambda) := \operatorname{diag}\{(y_n(\lambda))^{-1/2}\}$, and $\mathsf{J}_\varnothing(\lambda)$ is as in (2.5). The formula (A.2) can be viewed as a particular case of abstract Krein's formula (see *e.g.* [6, 8, 12]) for the resolvent difference of two self-adjoint extensions of their common densely defined symmetric restriction.

Proof of Proposition 1.3 and Theorem 2.4. Let us fix $n \in \mathbb{N}_0$ and a sheet $Z_E \subset \widetilde{Z}$. We denote by $\mathsf{R}_n(\lambda)$ the resolvent of the self-adjoint operator $H^2(\mathbb{R}) \ni f \mapsto -f'' + \nu_n f$ in the Hilbert space $L^2(\mathbb{R})$. We can express the function $\mathfrak{r}^{\varnothing}_{n,\varepsilon}(\cdot; u)$ in (1.13) using Krein's formula (A.2) as

$$\begin{aligned}
\mathfrak{r}^{\varnothing}_{n,\varepsilon}(\lambda; u) &= \big\langle (\mathsf{H}_\varepsilon - \lambda)^{-1} u \otimes \mathsf{e}_n, u \otimes \mathsf{e}_n \big\rangle \\
&= \big\langle (\mathsf{H}_0 - \lambda)^{-1} u \otimes \mathsf{e}_n, u \otimes \mathsf{e}_n \big\rangle \\
&\qquad + \Big(\mathsf{Y}(\lambda) \Big[\big(\mathsf{I} + \varepsilon \mathsf{J}_\varnothing(\lambda)\big)^{-1} - \mathsf{I} \Big] \mathsf{Y}(\lambda) \mathsf{T}(\overline{\lambda})^* u \otimes \mathsf{e}_n, \mathsf{T}(\lambda)^* u \otimes \mathsf{e}_n \Big) \\
&= (\mathsf{R}_n(\lambda) u, u)_{\mathbb{R}} + I_n(\lambda; u)\overline{I_n(\overline{\lambda}; u)} \left(\big[\big(\mathsf{I} + \varepsilon \mathsf{J}_\varnothing(\lambda)\big)^{-1} - \mathsf{I} \big] \mathsf{Y}(\lambda) \mathsf{e}_n, \mathsf{Y}(\lambda)^* \mathsf{e}_n \right) \\
&= (\mathsf{R}_n(\lambda) u, u)_{\mathbb{R}} + \frac{I_n(\lambda; u) I_n(\lambda; \overline{u})}{y_n(\lambda)} \left[\Big(\big(\mathsf{I} + \varepsilon \mathsf{J}_\varnothing(\lambda)\big)^{-1} \mathsf{e}_n, \mathsf{e}_n \Big) - 1 \right].
\end{aligned}$$

Since $(\mathsf{R}_n(\lambda)u, u)_{\mathbb{R}}$, $y_n(\lambda)$, $I_n(\lambda; u)$, and $I_n(\lambda; \overline{u})$ can be easily analytically continued to $\widetilde{Z}$, to extend $\mathfrak{r}^{\varnothing}_{n,\varepsilon}(\cdot; u)$ meromorphically to the other sheets of the component $\widetilde{Z}$ it suffices to extend

$$\mathfrak{s}^{\varnothing}_{n,\varepsilon}(\lambda) := \left(\big(\mathsf{I} + \varepsilon \mathsf{J}_\varnothing(\lambda)\big)^{-1} \mathsf{e}_n, \mathsf{e}_n \right),$$

meromorphically from $Z_\varnothing$ to $\widetilde{Z}$. The poles of the meromorphic extension of $\mathfrak{s}^{\varnothing}_{n,\varepsilon}(\cdot)$ can be identified with the resonances of H_ε in the sense of Definition 1.4.

To this aim we set by definition

$$\mathfrak{s}^E_{n,\varepsilon}(\lambda) := \left(\left(\mathsf{I} + \varepsilon \mathsf{J}_E(\lambda)\right)^{-1} \mathsf{e}_n, \mathsf{e}_n\right),$$

for any $\lambda \in \mathbb{C} \setminus [\nu_0, +\infty)$ such that $-1 \notin \sigma(\varepsilon \mathsf{J}_E(\lambda))$. In what follows let Z_E and Z_F be two sheets of $\widetilde{Z}$ such that $Z_E \sim_{n-1} Z_F$ with $n \in \mathbb{N}_0$[4]. Suppose that $\lambda \mapsto \mathfrak{s}^E_{n,\varepsilon}(\cdot)$ is well defined and meromorphic either on Z_E^+ or on Z_E^-. Next, we extend $\lambda \mapsto \mathfrak{s}^E_{n,\varepsilon}(\cdot)$ meromorphically from $Z_E^\pm$ to $Z_F^\mp$. Without loss of generality we restrict our attention to the case that $\lambda \mapsto \mathfrak{s}^E_{n,\varepsilon}(\cdot)$ is meromorphic on Z_E^+ and extend it meromorphically to Z_F^-. On the open set $\Omega_n := \mathbb{C}_+ \cup \mathbb{C}_- \cup (\nu_{n-1}, \nu_n)$ the operator-valued function

$$\mathsf{J}_{EF}(\lambda) := \begin{cases} \mathsf{J}_E(\lambda), & \lambda \in \mathbb{C}_+, \\ \mathsf{J}_F(\lambda), & \lambda \in \Omega_n \setminus \mathbb{C}_+, \end{cases}$$

is analytic which is essentially a consequence of analyticity on Ω_n of the entries $b^\bullet_m(\lambda)$ (with $\bullet = E$ for $\lambda \in \mathbb{C}_+$ and $\bullet = F$ for $\lambda \in \mathbb{C}_-$) for the underlying Jacobi matrix. Thus, the operator-valued function

$$\Omega_n \ni \lambda \mapsto \mathsf{A}^{EF}_\varepsilon(\lambda) := \varepsilon \left(\mathsf{I} + \varepsilon \mathsf{J}_0\right)^{-1} \left(\mathsf{J}_{EF}(\lambda) - \mathsf{J}_0\right)$$

is also analytic on Ω_n because of the analyticity of $\mathsf{J}_{EF}(\lambda)$. Furthermore, the values of $\mathsf{A}^{EF}_\varepsilon(\cdot)$ are compact operators thanks to compactness of the difference $\mathsf{J}_{EF}(\lambda) - \mathsf{J}_0$. Taking into account that

$$\left(\left(\mathsf{I} + \mathsf{A}^{EF}_\varepsilon(\lambda)\right)^{-1} \mathsf{e}_n, \left(\mathsf{I} + \varepsilon \mathsf{J}_0\right)^{-1} \mathsf{e}_n\right) = \begin{cases} \mathfrak{s}^E_{n,\varepsilon}(\lambda), & \lambda \in \mathbb{C}_+, \\ \mathfrak{s}^F_{n,\varepsilon}(\lambda), & \lambda \in \Omega_n \setminus \mathbb{C}_+, \end{cases}$$

we obtain from the analytic Fredholm theorem [25, Thm. VI.14] that $\mathbb{C}_- \ni \lambda \mapsto \mathfrak{s}^F_{n,\varepsilon}(\lambda)$ is a meromorphic continuation of $\mathbb{C}_+ \ni \lambda \mapsto \mathfrak{s}^E_{n,\varepsilon}(\lambda)$ across the interval (ν_{n-1}, ν_n) and that the poles of $\mathbb{C}_- \ni \lambda \mapsto \mathfrak{s}^F_{n,\varepsilon}(\lambda)$ satisfy the condition

$$\ker\left(\mathsf{I} + \varepsilon \mathsf{J}_F(\lambda)\right) \neq \{0\}, \qquad \lambda \in \mathbb{C}_-.$$

Starting from the physical sheet $Z_\varnothing$ we use the above procedure iteratively to extend $\mathfrak{s}^\varnothing_{n,\varepsilon}(\cdot)$ meromorphically to the whole of $\widetilde{Z}$ thus proving Proposition 1.3 and Theorem 2.4. □

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