On a problem of S.L. Sobolev

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Abstract

In 1930 Sergey L. Sobolev [6, 7] has proposed a construction of the solution of the Cauchy problem for the hyperbolic equation of the second order with variable coefficients in 3-d. Although Sobolev did not construct the fundamental solution, his construction was modified in 1986 by Romanov [4] to obtain that solution. However, these works impose a restrictive assumption of the regularity of geodesic lines in a large domain. In addition, it is unclear how to realize those methods numerically. In this paper a simple construction of a function, which is associated in a clear way with the fundamental solution of the acoustic equation with the variable speed in 3-d, is proposed. Conditions on geodesic lines are not imposed. An important feature of this construction is that it lends itself to effective computations.

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1 Introduction

In 1930 Sergey L. Sobolev, one of the most distinguished mathematicians of the 20^{st} century, has published two papers [6, 7] where he has constructed the solution of the hyperbolic equation of the second order in the 3-d case with variable coefficients in the principle part of the operator. It was assumed that these coefficient depend on spatial variables. This result is readily available in the textbook of Smirnov [5]. Sobolev did not find the fundamental solution. The main reason of this was that the notion of the fundamental solution was unknown in 1930. In his book about inverse problems published in 1986 [4] Romanov has shown how to modify the method of Sobolev to construct the fundamental solution of that equation. However, the constructions of both Sobolev and Romanov impose a quite restrictive assumption on the variable coefficients in the principal part of the hyperbolic operator. Let the time variable $t \in (0,T)$. It is assumed in above cited publications that geodesic lines generated by these coefficient are regular in a domain $Q(T) \subset \mathbb{R}^3$. The larger T is, the larger $Q(T)$ is. So, $Q(\infty) = \mathbb{R}^3$. The regularity of geodesic lines in the domain $Q(T)$ means that for any two points $x, y \in Q(T)$ there exists a single geodesic line connecting them.

Another construction of the fundamental solution of that equation can be found in the book of Vainberg [9]. The construction of [9] imposes the non-trapping condition on variable coefficients in the principal part of the hyperbolic operator. In addition, the technique of [9] relies on the canonical Maslov operator, which is not easy to obtain explicitly. Thus, even though the structure of the fundamental solution in any of above constructions can be seen, formally at least, still many elements of this structure cannot be expressed via explicit formulas. For example, neither the solution of the eikonal equation of the method of Sobolev, nor the Maslov construction cannot be expressed via explicit formulas.

The numerical factor is quite important nowadays. However, it is not immediately clear how to compute numerically fundamental solutions obtained in above references. Therefore, it is also unclear how to compute solutions of Cauchy problems for heterogeneous hyperbolic equations if using those fundamental solutions.

In this paper we propose a simple method of the construction of a function, which is closely associated with the fundamental solution for the acoustic equation in the 3-d case with the variable coefficient. We impose almost minimal assumptions on this coefficient. Geodesic lines are not used. It is important that our function can be both accurately and effectively approximated numerically via the Galerkin method as well as via the Finite Difference Method. Thus, we show that this function can be straightforwardly used for computations of the Cauchy problem for the heterogeneous acoustic equation.

The author was prompted to work on this paper while he was working on publications [1, 2] about reconstruction procedures for phaseless inverse scattering problems. Indeed, in [1, 2] the structure of the fundamental solution of the acoustic equation in time domain with the variable coefficient in its principal part is substantially used.

2 Construction

Below $x \in \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^3$ is a bounded domain. Let c_0 and c_1 be two constants such that $0 < c_0 \leq c_1$. We assume that the function $c(x)$ satisfies the following conditions

$$c \in C^1\left(\mathbb{R}^3\right), c \in [c_0, c_1], \tag{2.1}$$

$$c(x) = 1 \text{ for } x \in \mathbb{R}^3 \diagdown \Omega. \tag{2.2}$$

Consider the following Cauchy problem

$$c(x)\, y_{tt} - \Delta_x y = f(x,t), (x,t) \in \mathbb{R}^3 \times [0,\infty), \tag{2.3}$$

$$y\mid_{t=0} = y_t \mid_{t=0} = 0. \tag{2.4}$$

It is assumed here that f is an appropriate function such that there is a guarantee of the existence of the unique solution $y \in H^2\left(\mathbb{R}^3 \times (0,T)\right), \forall T > 0$ of this problem, see, e.g. Theorem 4.1 in §4 of Chapter 4 of the book of Ladyzhenskaya [3] for sufficient conditions for the latter. We will specify f later. The fundamental solution for the operator $c(x)\,\partial_t^2 - \Delta$ is such a function $P(x,\xi,t,\tau)$ that the solution of the problem (2.3), (2.4) can be represented in the form

$$y(x,t) = \int\limits_0^\infty \int\limits_{\mathbb{R}^3} P(x,\xi,t,\tau)\, f(\xi,\tau)\, d\xi d\tau. \tag{2.5}$$

Below we modify formula (2.5).

Let $\xi \in \mathbb{R}^3$ and $\tau \geq 0$ be parameters. Consider the following Cauchy problem

$$c(x) u_{tt} = \Delta_x u + \delta(x-\xi)\,\delta(t-\tau), \tag{2.6}$$

$$u\mid_{t=0} = u_t \mid_{t=0} = 0. \tag{2.7}$$

2.1 Heuristic part of the construction

It is convenient for us to work in this subsection with a purely heuristic derivation. Represent the solution of the problem (2.6), (2.7) as $u = u_0 + v$, where u_0 is the fundamental solution of the wave equation,

$$u_0 = \frac{\delta(t-\tau-|x-\xi|)}{4\pi|x-\xi|}. \tag{2.8}$$

Thus,

$$\partial_t^2 u_0 = \Delta_x u_0 + \delta(x-\xi)\,\delta(t-\tau), \tag{2.9}$$

$$u_0\mid_{t=0} = u_{0t}\mid_{t=0} = 0. \tag{2.10}$$

Hence, the function v satisfies the following conditions

$$c(x) v_{tt} = \Delta_x v - (c(x)-1)\,\frac{\delta''(t-\tau-|x-\xi|)}{4\pi|x-\xi|}, \tag{2.11}$$

$$v\mid_{t=0} = v_t\mid_{t=0} = 0. \tag{2.12}$$

Consider the operator A,

$$A(f) = \int_0^t f(y)\,dy$$

for appropriate functions f. Purely heuristically again apply the operator A^4 to both sides of equation (2.11). Denote $\widetilde{w}(x,\xi,t,\tau) = A^4(v)$. Then (2.11) and (2.12) imply that

$$c(x)\widetilde{w}_{tt} - \Delta_x \widetilde{w} = -(c(x)-1)\,\frac{(t-\tau-|x-\xi|)}{4\pi|x-\xi|} H(t-\tau-|x-\xi|), \tag{2.13}$$

$$\widetilde{w}\mid_{t=0} = \widetilde{w}_t\mid_{t=0} = 0, \tag{2.14}$$

where $H(z)$ is the Heavyside function,

$$H(z) = \begin{cases} 1, z > 0, \\ 0, z < 0. \end{cases}$$

2.2 Rigorous part of the construction

Starting from this point, we are not acting heuristically anymore. To the contrary, we work rigorously everywhere below.

Consider the Cauchy problem (2.13), (2.14). Denote

$$g(x,\xi,t,\tau) = -\left(c(x)-1\right)\frac{(t-\tau-|x-\xi|)}{4\pi|x-\xi|}H(t-\tau-|x-\xi|).$$

It follows from (2.2) that for any fixed pair $(\xi,\tau) \in \mathbb{R}^3 \times (0,\infty)$ and for any $T > 0$ functions $g, g_t \in L_2(\mathbb{R}^3 \times (0,T))$. Hence, Theorem 4.1 of §4 of Chapter 4 of the book of Ladyzhenskaya [3], (2.1), (2.2) as well as other results of that chapter imply that for any fixed pair $(\xi,\tau) \in \mathbb{R}^3 \times [0,\infty)$ and for any $T > 0$ there exists unique solution $\widetilde{w} \in H^2(\mathbb{R}^3 \times (0,T))$ of the Cauchy problem (2.13), (2.14). Furthermore, it was shown in the proof of that theorem of [3] that this solution can be effectively constructed numerically via the Galerkin method. It is also well known that it can be numerically constructed via the Finite Difference Method. By (2.13) $\widetilde{w} = \widetilde{w}(x,\xi,t-\tau)$. In addition, the energy estimate implies that

$$\widetilde{w}(x,\xi,t,\tau) = 0 \text{ for } t \leq \tau. \tag{2.15}$$

Consider now the function $w(x,\xi,t-\tau) = \widetilde{w}(x,\xi,t-\tau) + A^4(u_0)$. Hence,

$$w(x,\xi,t-\tau) = \widetilde{w}(x,\xi,t-\tau) + \frac{(t-\tau-|x-\xi|)^3}{6\cdot 4\pi|x-\xi|}H(t-\tau-|x-\xi|). \tag{2.16}$$

Using (2.8)-(2.10), (2.13), (2.14) and (2.16) and applying direct calculations, we obtain that the function w satisfies the following conditions

$$c(x)w_{tt} - \Delta_x w = \delta(x-\xi)\frac{(t-\tau)^3}{6}H(t-\tau), \tag{2.17}$$

$$w\mid_{t=0} = w_t\mid_{t=0} = 0. \tag{2.18}$$

Consider now an arbitrary function $f(x,t)$ such that

$$\partial_t^k f(x,t) \in C\left(\mathbb{R}^3 \times [0,\infty)\right), k = 0,1,2,3,4, \tag{2.19}$$

$$\partial_t^k f(x,0) = 0, k = 0,1,2,3,4, \tag{2.20}$$

$$f(x,t) = 0, \forall x \in \mathbb{R}^3 \diagdown G_f, \tag{2.21}$$

where G_f is a bounded domain depending on the function f. Consider the function $p_f(x,t)$ defined as

$$p_f(x,t) = \int_0^t \int_{G_f} w(x,\xi,t-\tau) f(\xi,\tau)\, d\xi d\tau. \tag{2.22}$$

Using the integration by parts, (2.15) and (2.16), we obtain that there exist four derivatives of the function $p_f(x,t)$ with respect to t. In particular,

$$\partial_t^4 p_f(x,t) = \int_0^t \int_{G_f} w(x,\xi,t-\tau)\, \partial_\tau^4 f(\xi,\tau)\, d\xi d\tau. \tag{2.23}$$

Next, applying the operator $c(x)\,\partial_t^2 - \Delta$ to both sides of (2.23) and using (2.15) and (2.17), we obtain

$$\left(c(x)\,\partial_t^2 - \Delta\right)\partial_t^4 p_f(x,t) = \int_0^t \frac{(t-\tau)^3}{6}\partial_\tau^4 f(x,\tau)\,d\tau. \tag{2.24}$$

Using integration by parts in (2.24) as well as (2.20), we obtain

$$\left(c(x)\,\partial_t^2 - \Delta\right)\partial_t^4 p_f(x,t) = f(x,t). \tag{2.25}$$

In addition, it follows from (2.15), (2.16) and (2.23) that

$$\partial_t^4 p_f \mid_{t=0} = \partial_t^5 p_f \mid_{t=0} = 0. \tag{2.26}$$

Since the function $\partial_t^4 p_f \in H^2\left(\mathbb{R}^3 \times (0,T)\right), \forall T > 0$, then the uniqueness theorem for the problem (2.3), (2.4) as well as (2.23), (2.25) and (2.26) imply that

$$y(x,t) = \int_0^t \int_{G_f} w(x,\xi,t-\tau)\,\partial_\tau^4 f(\xi,\tau)\,d\xi d\tau. \tag{2.27}$$

Thus, the solution of the problem (2.3), (2.4) is constructed in (2.27) using the function w. The formula (2.27) is our first analog of the formula (2.5). The integration by parts in (2.27) and the use of (2.15) and (2.16) leads to the second analog of the formula (2.5),

$$y(x,t) = \int_0^t \int_{G_f} \partial_t^2 w(x,\xi,t-\tau)\,\partial_\tau^2 f(\xi,\tau)\,d\xi d\tau. \tag{2.28}$$

2.3 Numerical comments

Recall that the function w can be accurately numerically approximated via either the Galerkin method or the Finite Difference Method. The derivatives of the function f can be found analytically, if f is given by an explicit formula. However, if f is given with a noise, then a regularization method should be applied, see, e.g. the book of Tikhonov and Arsenin [8]. In particular, it is explained in this book how to stably differentiate noisy functions using the regularization. Furthermore, numerical examples of stable computations of first and second derivatives are presented in [8]. The form (2.28) might be sometimes more convenient than the form (2.27) since second derivatives of noisy functions are obviously more stable to calculate than fourth derivatives.

In summary, here is an effective way to calculate the solution $y(x,t)$ of the problem (2.3), (2.4):

Step 1. Given the number $T > 0$, for each pair $(\xi,\tau) \in \mathbb{R}^3 \times (0,T)$ compute the function solution $\widetilde{w}(x,\xi,t,\tau) \in H^2\left(\mathbb{R}^3 \times (0,T)\right)$ of the Cauchy problem (2.13), (2.14) as the function of x and t. To do so, use either the Galerkin method or the Finite Difference method. Next, compute the function $w(x,\xi,t,\tau)$ via (2.16).

Step 2. Let the function $f(x,t)$ satisfies conditions (2.19)-(2.21). Compute up to four corresponding $t-$derivatives of the function f. They should be computed either analytically, if f is given by an explicit formula, or numerically using the regularization, if f is given with a noise.

Step 3. Apply one of formulas (2.27) or (2.28). If applying (2.28), then it is sufficient to assume that $k = 0, 1, 2$ in (2.19), (2.20).

Since integrals (2.27) and (2.28) are calculated via a discretization in practical computations, then it is sufficient in these computations to calculate the function $\widetilde{w}$ only for a corresponding finite set of parameters (ξ_i, τ_i), $i = 1, ..., n$ for a corresponding number n. A good analogy of the above procedure is the well known process of solving linear algebraic systems with many different right hand sides and the same matrix. Computations in this case are done rapidly if that matrix is factorized only once for all right hand sides.

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