Deformations of Lagrangian systems preserving a fixed subalgebra of Noether symmetries

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Abstract. Systems of second-order ordinary differential equations admitting a Lagrangian formulation are deformed requiring that the extended Lagrangian preserves a fixed subalgebra of Noether symmetries of the original system. For the case of the simple Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, this provides non-linear systems with two independent constants of the motion quadratic in the velocities. In the case of scalar differential equations, it is shown that equations of Pinney-type arise as the most general deformation of the time-dependent harmonic oscillator preserving a $\mathfrak{sl}(2,\mathbb{R})$-subalgebra. The procedure is generalized naturally to two dimensions. In particular, it is shown that any deformation of the time-dependent harmonic oscillator in two dimensions that preserves a $\mathfrak{sl}(2,\mathbb{R})$ subalgebra of Noether symmetries is equivalent to a generalized Ermakov-Ray-Reid system that satisfies the Helmholtz conditions of the Inverse Problem of Lagrangian Mechanics. Application of the procedure to other types of Lagrangians is illustrated.

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1. Introduction

The Lie symmetry analysis of differential equations, originally applied to physical problems mainly in the context of (quantum) mechanical systems, constitutes nowadays a standard method in a wide spectrum of physical situations, ranging from quantum phenomena or non-linear optics to cosmological problems (see e.g. [1–5] and references therein). Among the systems of ordinary differential equations (ODEs) analyzed for their symmetry properties and relevant to physical applications, Ermakov systems occupy a distinguished position, because of their various interesting structural properties, such as the existence of a conserved quantity or a non-linear superposition principle. This has motivated that such systems have been intensively studied [6–14]. Besides the classical Ermakov systems, deeply related to the (time-dependent) harmonic oscillator, various types of generalizations have been proposed, nowadays known as Ermakov-Ray-Reid systems or ERR systems in short. Multidimensional analogues of Ermakov systems, which have been proven to be of interest in soliton theory, have also been established and inspected in detail [15]. In the context of symmetry analysis, it has been shown that point symmetries of ERR-systems are closely related to the simple Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ [16, 17]. Further, the existence of a Lagrangian formalism for ERR-systems has been analyzed in [18], showing that the Noether approach is not yet exhausted. All these approaches connect with recent work on generic symmetries of systems of ODEs and their relation to certain types of integrable systems [2, 4, 19–24].

In this work we develop a somewhat inverse procedure, basing on a "symmetry-preservation" procedure applied to Lie algebras of Noether symmetries. More specifically, starting with a Lagrangian L associated to a generic linear homogeneous second order ODE, we determine the most general forcing term $G(t,\mathbf{x})$ such that the extended Lagrangian preserves a subalgebra of Noether symmetries with identical generators. This enables to write a constant of the motion as a combination of the invariant of the original equation and a part corresponding to the forcing term. It follows from this approach that the Pinney-type equation $\ddot{x}+p(t)\dot{x}+q(t)x+C\left(\exp(\int p(t)dt)\right)^{-2}x^{-3}=0$ can be characterized as the most general deformation of the ODE $\ddot{x}+p(t)\dot{x}+q(t)x=0$ that preserves a subalgebra of Noether symmetries isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. For the former non-linear equations, the algebra of point symmetries coincides with that of Noether symmetries. For the special case $g_1(t)=0$, this further provides an additional explanation for the relation between the time-dependent harmonic oscillator and the Pinney equation [6, 12, 25], hence suggesting a connexion with Ermakov systems.

The procedure is generalized naturally to systems in two dimensions, starting with an uncoupled system of damped oscillators $\ddot{x}_i+g_1(t)\dot{x}_i+g_2(t)x_i=0$, $i=1,2$. In contrast to the scalar case, the forcing terms for these systems can depend on the velocities, giving rise to a more ample class of deformed non-linear systems. It is further shown that for the subclass determined by the constraints $g_1(t)=0, g_2(t)=\omega^2(t)$, the most general deformation of the time-dependent harmonic oscillator in two dimensions that preserves a $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries actually corresponds to a Ermakov-Ray-Reid system (with velocity-dependent potential) that satisfies the Helmholtz conditions of the Inverse Problem in Lagrangian Mechanics [26, 27].

The main difference with respect to previous approaches resides in the fact that we do not merely impose the existence of one or more Noether symmetries, but that we require that the symmetry generators are identical for the original and deformed equations, as well as the fact that they generate a copy of $\mathfrak{sl}(2,\mathbb{R})$. This ensures the existence of two independent constants of the motion generally quadratic in the velocities. In the general case, the algebra of point symmetries will also be isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

1.1. Point symmetries of second-order ordinary differential equations

To describe point symmetries of differential equations, we use the standard formulation in terms of differential operators [28, 29]. It is well known that a system of second-order ordinary differential equations

$$\ddot{x}_i = \omega_i\left(t,\mathbf{x},\dot{\mathbf{x}}\right),\ 1\leq i\leq N \tag{1}$$

is formulated in equivalent form in terms of the partial differential equation

$$\mathbf{A}f = \left(\frac{\partial}{\partial t} + \dot{x}_i\frac{\partial}{\partial x_i} + \omega\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\frac{\partial}{\partial \dot{x}_i}\right)f = 0. \tag{2}$$

We call a vector field $X = \xi\left(t,x\right)\frac{\partial}{\partial t} + \eta_j\left(t,x\right)\frac{\partial}{\partial x_j} \in \mathfrak{X}\left(\mathbb{R}^{N+1}\right)$ a Lie point symmetry of the equation(s) (1) if the prolongation $\dot{X} = X + \dot{\eta}_j\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\frac{\partial}{\partial \dot{x}_j}$ satisfies the commutator

$$\left[\dot{X},\mathbf{A}\right] = -\frac{d\xi}{dt}\mathbf{A}, \tag{3}$$

where $\dot{\eta}_j = -\frac{d\xi}{dt}\dot{x}_j + \frac{d\eta_j}{dx}$.

Given arbitrary functions $g_1(t), g_2(t)$, it is straightforward to verify that a second order linear homogeneous differential equation

$$\ddot{x} + g_1\left(t\right)\dot{x} + g_2\left(t\right)x = 0 \tag{4}$$

possesses an algebra of point symmetries $\mathcal{L}$ isomorphic to $\mathfrak{sl}\left(3,\mathbb{R}\right)$ (see e.g. [28, 29]). Three of the symmetry generators of $\mathcal{L}$ are immediate, and can be taken as

$$Y_1 = x\frac{\partial}{\partial x},\ Y_2 = U_1\left(t\right)\frac{\partial}{\partial x},\ Y_3 = U_2\left(t\right)\frac{\partial}{\partial x}, \tag{5}$$

where the general solution of (4) is given by

$$x\left(t\right) = \lambda_1 U_1\left(t\right) + \lambda_2 U_2\left(t\right);\ \lambda_1,\lambda_2\in\mathbb{R}. \tag{6}$$

1.2. Noether symmetries and constants of the motion

As follows from the Inverse Problem in Lagrangian Mechanics, any scalar second-order ODE $\ddot{x} = \omega\left(t,x,\dot{x}\right)$ follows from a variational principle, i.e., there exist functions $f\left(t,x,\dot{x}\right)$ and $L\left(t,x,\dot{x}\right)$ such that the Helmholtz conditions

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = f_i\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\left(\ddot{x}_i - \omega_i\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\right),\ 1\leq i\leq N \tag{7}$$

hold [26, 27]. In the case of the ODE (4), an admissible Lagrangian L is given by

$$L(t,x,\dot{x}) = \mathrm{e}^{\mu(t)}\left(\dot{x}^2 - g_2(t)\,x^2\right)/2, \tag{8}$$

where $\mu(t) = \int g_1(t)\; dt$. The ODE (4) corresponds to the equation of motion of a one-dimensional time-dependent damped oscillator [30].

Recall that a point symmetry X is a Noether symmetry if there exists a function $V(t,\mathbf{x})$ such that the identity

$$\dot{X}(L) + A(\xi)\,L - A(V) = 0 \tag{9}$$

is satisfied. As a consequence, the quantity

$$\psi = \xi(t,\mathbf{x},\dot{\mathbf{x}})\left[\dot{x}_i\frac{\partial L}{\partial \dot{x}_i} - L\right] - \eta_i(t,\mathbf{x},\dot{\mathbf{x}})\,\frac{\partial L}{\partial \dot{x}_i} + V(t,\mathbf{x}) \tag{10}$$

is a constant of the motion of the equation (system) [4, 31, 32].

We now justify how the subalgebra $\mathcal{L}_{NS}$ of Noether symmetries of the equation (4) can be described generically in terms of two arbitrary independent solutions. As the ODE (4) possesses maximal symmetry $\mathfrak{sl}(3,\mathbb{R})$, it has exactly five independent Noether symmetries [29]. It is immediate to verify that the symmetries Y_2 and Y_3 of (5) satisfy the Noether symmetry condition (9) for the function $V(t,x) = \mathrm{e}^{\mu(t)}x\dot{U}_k(t)$, $k = 1, 2$. The conserved quantities deduced from them are $I_k = \mathrm{e}^{\mu(t)}\left(x\dot{U}_k(t) - \dot{x}U_k(t)\right)$.

It is convenient to introduce an auxiliary function $\varphi(t)$ that will allow us, jointly with the general solution (6) of the ODE, to describe generically the remaining Noether symmetry generators of $\mathcal{L}_{NS}$. To this extent, we consider the Wronskian $\mathbf{W} = W\left\{U_1(t), U_2(t)\right\}$ and define the function $\varphi(t)\,\mathbf{W} = -1$. Using the properties of the Wronskian [33], it is straightforward to verify that φ satisfies the first order equation

$$R_2 := \frac{d\varphi}{dt} - g_1(t)\,\varphi(t) = 0. \tag{11}$$

It follows in particular that $\varphi(t) = \exp(\mu(t))$.

Proposition 1 *For arbitrary functions $g_1(t), g_2(t)$, the vector fields*

$$X_k = \varphi(t)\,U_k^2(t)\,\frac{\partial}{\partial t} + x\,\varphi(t)\,U_k(t)\dot{U}_k(t)\,\frac{\partial}{\partial x}, \quad k = 1, 2 \tag{12}$$

are independent Noether symmetries of the linear homogeneous ODE (4).

We prove the assertion by direct computation. For $k = 1, 2$, we define the quantities

$$R_{2+k} := \ddot{U}_k(t) + g_1(t)\dot{U}_k(t) + g_2(t)\,U_k(t), \tag{13}$$

recalling that they are identically zero for the ODE (4). Taking into account equation (11), evaluation of the symmetry condition (9) and reordering with respect to the powers of $\dot{x}$ leads to the equation

$$\left(x\,\varphi^2(t)\left(U_k(t)\,\ddot{U}_k(t) + U_k(t)\,\dot{U}_k(t)g_1(t) + \dot{U}_k(t)^2\right) - \frac{\partial V}{\partial x}\right)\dot{x} - \frac{\partial V}{\partial t} - \frac{x^2\varphi(t)^2}{2}U_k(t)\times$$
$$\left(2g_1(t)\,g_2(t)\,U_k(t) + \left(U_k(t)\,\frac{dg_2}{dt} + 4g_2(t)\,\dot{U}_k(t)\right)\right) = 0. \tag{14}$$

From the term in $\dot{x}$ we immediately obtain $V(t,x)$ as

$$V\left(t,x\right)=\frac{1}{2}x^{2}\varphi(t)^{2}\left(U_{k}\left(t\right)\ddot{U}_{k}(t)+g_{1}(t)U_{k}\left(t\right)\dot{U}_{k}(t)+\dot{U}_{k}(t)^{2}\right). \tag{15}$$

Inserting the latter into the free term of (14) and simplifying gives

$$\left(U_{k}^{2}\left(t\right)\left(1-g_{2}\left(t\right)\right)+2\dot{U}_{k}(t)^{2}\right)R_{2}-\left(2U_{k}\left(t\right)\frac{d\varphi}{dt}+3\varphi\left(t\right)\dot{U}_{k}(t)\right)R_{2+k}$$
$$+U_{k}\left(t\right)\dot{U}_{k}(t)\frac{dR_{2}}{dt}-\varphi\left(t\right)U_{k}\left(t\right)\frac{dR_{2+k}}{dt}=0, \tag{16}$$

proving that X_1 and X_2 are Noether symmetry of (4). As Noether symmetries generate a subalgebra of the Lie algebra $\mathcal{L}$ of point symmetries [29], we conclude that $[X_1, X_2]$ is also a Noether symmetry of the ODE. It is given explicitly by:‡

$$X_{3}:=[X_{1},X_{2}]=-2\varphi(t)U_{1}(t)U_{2}(t)\frac{\partial}{\partial t}-\varphi(t)\left(U_{1}\left(t\right)\dot{U}_{2}(t)+U_{2}\left(t\right)\dot{U}_{1}(t)\right)x\frac{\partial}{\partial x} \tag{17}$$

These three symmetries generate a subalgebra isomorphic to $\mathfrak{sl}\left(2,\mathbb{R}\right)$, with the two remaining Noether symmetries Y_2, Y_3 transforming according to the irreducible representation of $\mathfrak{sl}\left(2,\mathbb{R}\right)$ of dimension two. The complete commutators are given by

$[\cdot,\cdot]$	X_1	X_2	X_3	Y_2	Y_3
X_1	0	X_3	$2X_1$	0	$-Y_2$
X_2		0	$-2X_2$	Y_3	0
X_3			0	$-Y_2$	Y_3

(18)

We observe that the structure constants of $\mathcal{L}_{NS}$ do not depend on the form of the solutions $U_k(t)$ chosen for the general solution of (4).

The constants of the motion associated to the symmetries X_1, X_2 and X_3 are respectively

$$J_{\alpha\beta}=\frac{1}{2}I_{\alpha}I_{\beta},\ \alpha,\beta=1,2; \tag{19}$$

thus they are obtained from the invariants associated to Y_2 and Y_3.

2. Deformations by means of symmetry-preserving forcing terms

The close relation between the time-dependent harmonic oscillator

$$\ddot{x}+\omega^{2}\left(t\right)x=0 \tag{20}$$

and the Pinney equation

$$\ddot{\rho}+\omega^{2}\left(t\right)\rho=\frac{1}{\rho^{3}} \tag{21}$$

has been discussed extensively in the literature (see [7–9, 24, 34] and references therein), from a geometrical point of view, where (20) arises as the projection of a two-dimensional motion and (21) as its radial component, as well as in the context of the physical interpretation of the (generalized) Lewis invariant [6, 12, 35]. Its derivation and

‡ The relations R_2, R_{2+k} are used to simplify the resulting expression.

generalization to equations of type (4), as well as the analysis of higher dimensional systems [30] are also well established facts.

In this section we provide another possible approach to the problem, considering deformed Lagrangians that preserve exactly a fixed subalgebra of Noether symmetries. More specifically, for a differential equation of type (4),§ we determine the most general forcing term $G(t,x)$ such that the $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries is preserved, i.e., we require that the vector fields X_1 and X_2 of (12) are Noether symmetries of the resulting deformed differential equation. This in particular implies that the latter non-linear equation has an algebra of point symmetries that coincides with that of Noether symmetries. As the symmetry generators are identical to those of the homogeneous equation, in some sense we have "broken" the original symmetry algebra to the subalgebra $\mathfrak{sl}(2,\mathbb{R})$. In contrast to the homogeneous equation, two of the constants of the motion quadratic in the velocities will be independent and not obtainable from linear invariants, as these correspond to Noether symmetries that are not preserved.

The starting point for the ansatz is to consider the extended Lagrangian

$$\widetilde{L}(t,x,\dot{x}) = L_0(t,x,\dot{x}) + \Phi(t,x) = \varphi(t)\left(\frac{\dot{x}^2}{2} - \frac{g_2(t)}{2}y^2 - G(t,x)\right), \tag{22}$$

where $L_0(t,x,\dot{x})$ is the Lagrangian given in (8) and $\Phi(t,x) = -\varphi(t)\,G(t,x)$ for some function $G(t,x)$. The corresponding equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial\widetilde{L}}{\partial\dot{x}}\right) - \frac{\partial\widetilde{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} - \frac{\partial\Phi}{\partial x} = \varphi(t)\left(\ddot{x} + g_1(t)\,\dot{x} + g_2(t)\,x + \frac{\partial G}{\partial x}\right) = 0.$$

Discarding the common term, the equation

$$\ddot{x} + g_1(t)\,\dot{x} + g_2(t)\,x + \frac{\partial G}{\partial x} = 0 \tag{23}$$

describes the motion of a particle with both damping and forcing terms.

Having in mind the relations R_2 and (13) defined previously, we impose the conservation of the Noether symmetries X_1 and X_2 for the (generally non-linear) equation (23).

Proposition 2 *For $k = 1,2$ the vector fields*

$$X_k = \varphi(t)\,U_k^2(t)\,\frac{\partial}{\partial t} + x\,\varphi(t)\,U_k(t)\,\dot{U}_k(t)\frac{\partial}{\partial x} \tag{24}$$

are Noether symmetries of the equation of motion (23) only if the forcing term has the form

$$G(x,y) = \frac{\alpha}{\varphi(t)^2\,x^2}, \quad \alpha \in \mathbb{R}. \tag{25}$$

§ More precisely, for the Lagrangian (8) associated to the equation.

We again prove the assertion evaluating directly the symmetry condition (9). For the Lagrangian $\widetilde{L}$ and the prolongation $\dot{Y}_k$, the evaluation of the quantity $\left(\dot{Y}_k\left(\widetilde{L}\right)+\mathbf{A}\left(\xi\right)-\mathbf{A}\left(V\right)\right)$ reduces to the following expression

$$
\begin{aligned}
&\frac{\varphi\left(t\right)}{2}U_k^2\left(t\right)R_2\,\dot{x}^2+\,\dot{x}\left\{x\,\varphi^2\left(t\right)\left(\dot{U}_k(t)^2+\ddot{U}_k(t)U_k\left(t\right)+g_1\left(t\right)U_k\left(t\right)\dot{U}_k(t)\right)-\frac{\partial V}{\partial x}\right\}\\
&-\frac{x^2}{2}U_k\left(t\right)\varphi^2\left(t\right)\left\{4g_2\left(t\right)\dot{U}_k(t)+\left(2g_1\left(t\right)g_2\left(t\right)+\frac{dg_2}{dt}\right)U_k\left(t\right)\right\}-\varphi^2\left(t\right)U_k^2\left(t\right)\frac{\partial G}{\partial t}\\
&-x\,\varphi^2\left(t\right)U_k\left(t\right)\dot{U}_k(t)\frac{\partial G}{\partial x}-\frac{\partial V}{\partial t}-2\varphi^2\left(t\right)U_k\left(t\right)\left(g_1\left(t\right)U_k\left(t\right)+\dot{U}_k(t)\right)G\left(t,x\right)
\end{aligned}
\tag{26}
$$

using the relations (13) and R_2. As the latter is zero, the term in $\dot{x}^2$ vanishes. Next we obtain $V\left(t,x\right)$ from the term in $\dot{x}$ as

$$
V\left(t,x\right)=\frac{x^2}{2}\varphi\left(t\right)^2\left(\ddot{U}_k(t)U_k\left(t\right)+\dot{U}_k(t)^2+g_1(t)U_k\left(t\right)\dot{U}_k(t)\right)+W\left(t\right). \tag{27}
$$

Without loss of generality, we can take $W\left(t\right)=0$. We observe that this function equals identically that obtained in (15) for the homogeneous equation. Inserting this expression of $V\left(t,x\right)$ into (27) and simplifying, the Noether symmetry condition reduces to

$$
\begin{aligned}
&-\frac{\varphi\left(t\right)x^2}{2}\left\{U_k\left(t\right)\left(\varphi\left(t\right)\frac{dR_{2+k}}{dt}-\dot{U}_k(t)\frac{dR_2}{dt}\right)+2\varphi\left(t\right)\left(3\dot{U}_k(t)+2g_1\left(t\right)U_k\left(t\right)\right)R_{2+k}\right\}\\
&+\varphi\left(t\right)\left(x^2\left\{\dot{U}_k^2(t)-\frac{g_2\left(t\right)}{2}U_k^2\left(t\right)\right\}+G\left(t,x\right)\,U_k^2\left(t\right)\right)R_2-\varphi^2\left(t\right)U_k\left(t\right)\times\\
&\quad\left(U_k\left(t\right)\frac{\partial G}{\partial t}+x\dot{U}_k(t)\frac{\partial G}{\partial x}+2\left(U_k\left(t\right)g_1(t)+\dot{U}_k(t)\right)G\left(t,x\right)\right).
\end{aligned}
\tag{28}
$$

Because of $R_2=R_{2+k}=0$ for $U_k\left(t\right)$, the only surviving term is the last one, corresponding to the partial differential equation that must be satisfied by the forcing term $G\left(t,x\right)$ if it preserves the symmetry X_k:

$$
U_k\left(t\right)\frac{\partial G}{\partial t}+x\dot{U}_k(t)\frac{\partial G}{\partial x}+2\left(U_k\left(t\right)g_1\left(t\right)+\dot{U}_k(t)\right)G\left(t,x\right)=0. \tag{29}
$$

As the latter equation should be satisfied simultaneously for the independent solutions $U_1\left(t\right)$ and $U_2\left(t\right)$, in order to obtain a common solution we separate the PDE as follows:

$$
U_k\left(t\right)\left(\frac{\partial G}{\partial t}+2g_1\left(t\right)G\left(t,x\right)\right)=0,\quad \dot{U}_k(t)\left(x\frac{\partial G}{\partial x}+2\,G\left(t,x\right)\right)=0. \tag{30}
$$

The solution to this system is easily found to be

$$
G\left(t,x\right)=\frac{\alpha}{\varphi^2\left(t\right)x^2},\quad \alpha\in\mathbb{R}. \tag{31}
$$

Therefore the nonlinear ODE

$$
\ddot{x}+g_1\left(t\right)\dot{x}+g_2\left(t\right)x-\frac{2\alpha}{\varphi^2\left(t\right)\,x^3}=0 \tag{32}
$$

possesses at least the three Noether symmetries X_1, X_2 and X_3 inherited from the associated homogeneous equation (4).‖ As these symmetries generate a $\mathfrak{sl}(2,\mathbb{R})$ algebra,

‖ Equation (32) should not be confused with the so-called "damped Pinney" equation [36].

the ODE (32) either has an algebra $\mathcal{L}$ of point symmetries isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{sl}(3,\mathbb{R})$.¶

Lemma 1 *For arbitrary functions $g_1(t)$ and $g_2(t)$, the Lie algebra $\mathcal{L}$ of point symmetries of the ODE (32) is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and coincides with the algebra of Noether symmetries.*

Evaluating the symmetry condition (3) for point symmetries, a routine computation shows that a symmetry generator X must have the shape

$$X = \xi(t)\frac{\partial}{\partial t} + \frac{1}{2}\frac{\dot{\xi}(t)\varphi(t) - \xi(t)\dot{\varphi}(t)}{\varphi(t)}x\frac{\partial}{\partial x}. \tag{33}$$

In order to satisfy the symmetry condition, the function $\xi(t)$ must be a solution to the third-order ODE

$$\frac{d^3\xi}{dt^3} + \left(4g_2(t) - g_1^2(t) - 2\frac{dg_1}{dt}\right)\frac{d\xi}{dt} + \left(2\frac{dg_2}{dt} - \left(\frac{dg_1}{dt}\right)^2 - g_1(t)\frac{dg_1}{dt}\right)\xi = 0. \tag{34}$$

Now, as the vector fields X_1, X_2 and $[X_1, X_2]$ are point symmetries of (32) for being Noether symmetries, for any constants $\lambda_1, \lambda_2, \lambda_3$ the function

$$\xi(t) = \varphi(t)\left(\lambda_1 U_1^2(t) + \lambda_2 U_1(t)U_2(t) + \lambda_3 U_2^2(t)\right). \tag{35}$$

is a solution of (34), and since $U_1(t)$ and $U_2(t)$ are independent, it follows that (35) is the general solution of the equation, proving the assertion.

The relevant point in this approach is that the homogeneous ODE (4) and the non-linear equation (32) share the same subalgebra of Noether symmetries with identical generators. In addition, the function $V(t,x)$ of the symmetry condition (9) can be chosen simultaneously for both equations and an arbitrary linear combination of X_1, X_2 and $[X_1, X_2]$, implying that the corresponding constant of the motion satisfies

$$\psi = \xi\left[\dot{x}\frac{\partial\widetilde{L}}{\partial\dot{x}} - \widetilde{L}\right] - \eta\frac{\partial\widetilde{L}}{\partial\dot{x}} + V(t,x) = \xi\left[\dot{x}\frac{\partial L_0}{\partial\dot{x}} - L_0\right] - \eta\frac{\partial L_0}{\partial\dot{x}} + V(t,x) + \frac{\alpha\xi}{\varphi(t)\,x^2}. \tag{36}$$

Now $\psi_0 = \xi\left[\dot{x}\frac{\partial L_0}{\partial\dot{x}} - L_0\right] - \eta\frac{\partial L_0}{\partial\dot{x}} + V(t,x)$ corresponds to the constant of the motion of the homogeneous equation (4) with the Lagrangian (8), while the last term of (36) is the genuine contribution of the forcing term. In particular, we can use the well-known Lewis invariant [6, 12, 34] to express (36) in compact form. Taking into account that x is a solution of equation (32), ψ can be simplified to

$$\psi = \varphi^2(t)\left(\dot{\rho}x - \rho\dot{x}\right)^2 + \frac{\beta x^2}{\rho^2} - \frac{2\alpha\rho^2}{x^2}. \tag{37}$$

In particular, for $g_1(t) = 0$ and $g_2(t) = \omega(t)^2$, this method shows how the Pinney equation arises as the only deformation of the (time-dependent) harmonic oscillator

¶ It should be observed that no forcing terms $G(t,x,\dot{x})$ with $\frac{\partial G}{\partial\dot{x}} \neq 0$ can exist. This follows at once from the symmetry condition (3).

that preserves a $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries. To a certain extent, this fact suggests the origin of the non-linear superposition principle for non-linear equations of the form (32).

It is to be observed that there is a certain ambiguity in the notion of deformation, as the latter depends essentially on the Lagrangian L and not on the resulting equations of motion. An alternative Lagrangian is likely to provide different symmetry-preserving forcing terms and hence, different deformations. In this sense, it would be more precise to speak of "deformations with respect to a fixed Lagrangian" or L-deformations. However, whenever there is no ambiguity on the Lagrangian L used, we will simply use the term deformation.

3. Deformations in $N = 2$ dimensions and Ermakov-Ray-Reid systems

As a natural generalization of the scalar case, we can consider the deformation problem for (linear) systems of ODEs possessing at least a $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries. The novelty with respect to the scalar case will be the possibility of forcing terms depending on the velocities, providing a more ample class of non-linear systems. In this section we derive the most general deformations of the time-dependent harmonic oscillator in two dimensions that preserve a $\mathfrak{sl}\left(2,\mathbb{R}\right)$-subalgebra. It is further shown that these deformations actually correspond to a subclass of (generalized) ERR-systems that admit a Lagrangian formalism [18].

Let $g_1\left(t\right), g_2\left(t\right)$ be arbitrary functions and consider the uncoupled two-dimensional damped oscillator

$$\ddot{x}_i + g_1\left(t\right)\dot{x}_i + g_2\left(t\right)x_i = 0,\ i = 1,2 \tag{38}$$

obtained from the time-dependent Lagrangian

$$L = \frac{1}{2}\varphi\left(t\right)\left(\dot{x}_1^2 + \dot{x}_2^2 - g_2\left(t\right)\left(x_1^2 + x_2^2\right)\right). \tag{39}$$

Using the symmetry condition (9), a routine computation shows that a Noether symmetry $X = \xi\left(t,\mathbf{x}\right)\frac{\partial}{\partial t} + \eta_j\left(t,\mathbf{x}\right)\frac{\partial}{\partial x_j}$ has the following form ($1 \leq j \leq 2$, $k \neq j$):

$$\begin{aligned} &\xi\left(t,\mathbf{x}\right) = \xi\left(t\right),\\ &\eta_j\left(t,\mathbf{x}\right) = \tfrac{1}{2}\left(\dot{\xi}\left(t\right) - g_1\left(t\right)\xi\left(t\right)\right)x_j + \lambda_j^k x_k + \psi_j\left(t\right), \end{aligned} \tag{40}$$

where $\xi\left(t\right)$ satisfies equation (34) and $\psi_j\left(t\right)$ is a solution of (4) for $j = 1,2$. The Lie algebra of Noether symmetries $\mathcal{L}_{NS}$ has thus dimension 8. A basis of $\mathcal{L}_{NS}$ can be easily chosen as

$$\begin{aligned} &X_k = \varphi\left(t\right)U_k^2\left(t\right)\frac{\partial}{\partial t} + \varphi\left(t\right)U_k\left(t\right)\dot{U}_k(t)\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right),\ k = 1,2;\\ &X_3 = \left[X_1, X_2\right];\ X_{12} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1};\ Y_{kj} = U_k\left(t\right)\frac{\partial}{\partial x_j},\ 1 \leq j,k \leq 2. \end{aligned} \tag{41}$$

Clearly the Levi subalgebra of $\mathcal{L}_{NS}$ is isomorphic to $\mathfrak{s} = \mathfrak{sl}\left(2,\mathbb{R}\right) \oplus \mathfrak{so}\left(2\right)$, while the generators Y_{ij} transform according to the representation $\Lambda \otimes \Gamma_{\frac{1}{2}}$ of $\mathfrak{s}$, where Λ is the standard representation of $\mathfrak{so}(N)$ and $\Gamma_{\frac{1}{2}}$ the 2 dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$. From this we easily conclude that $\mathcal{L}_{NS}$ is isomorphic to the unextended Schrödinger algebra $S(2)$.$^{+}$

For systems of this type, we compute the most general forcing term $G(t,\mathbf{x},\dot{\mathbf{x}})$ that can be added to the Lagrangian L in (39) and such that the resulting system preserves the $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries. For technical reasons, it is convenient to separate the case of forcing terms independent and dependent on the velocities.

3.1. Velocity-independent forcing terms

We require that the extended Lagrangian

$$\widetilde{L} = \varphi(t)\left(\frac{1}{2}\left(\dot{x}_1^2+\dot{x}_2^2-g_2(t)\left(x_1^2+x_2^2\right)\right) - G(t,\mathbf{x})\right) \tag{42}$$

preserves the Noether symmetries X_1, X_2, X_3 of (41). In analogy with the scalar case, the symmetry condition (9) for X_k is only satisfied if the forcing term $G(t,\mathbf{x})$ is a solution of the PDE

$$U_k(t)\frac{\partial G}{\partial t}+\dot{U}_k(t)\left(x_1\frac{\partial G}{\partial x_1}+x_2\frac{\partial G}{\partial x_2}\right)+2\left(\dot{U}_k(t)+g_1(t)U_k(t)\right)G(t,\mathbf{x})=0. \tag{43}$$

Imposing that the latter PDE is satisfied simultaneously for $k=1,2$, separation of the equation into two equations independent on $U_k(t)$ leads to a system, the general solution of which is easily seen to be

$$G(t,\mathbf{x}) = F\left(\frac{x_2}{x_1}\right)x_1^{-2}\varphi^{-2}(t). \tag{44}$$

As in the preceding section, a short computation shows that for the non-linear system

$$\ddot{x}_i+g_1(t)\dot{x}_i+g_2(t)x_i+\frac{\partial G}{\partial x_i}=0,\ 1\le i\le 2 \tag{45}$$

the Lie algebra of point symmetries $\mathcal{L}$ coincides with that of Noether symmetries $\mathcal{L}_{NS}$, isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. We further observe that for generic choices of F, the equations are coupled non-trivially.

If we introduce two functions $F_1\left(\frac{x_2}{x_1}\right)$, $F_2\left(\frac{x_1}{x_2}\right)$ that satisfy the constraint $x_1\left(F_1\left(\frac{x_2}{x_1}\right)+F_2\left(\frac{x_1}{x_2}\right)\right)+x_2F\left(\frac{x_2}{x_1}\right)=0$, $F\left(\frac{x_2}{x_1}\right)$ being the generic function from (44), the solution of (43) can be written as

$$G(x_1,x_2) = -\frac{1}{x_1x_2\varphi^2(t)}\left(F_1\left(\frac{x_2}{x_1}\right)+F_2\left(\frac{x_1}{x_2}\right)\right). \tag{46}$$

This enables to write the partial derivatives as

$$\frac{\partial G}{\partial x_1}=\frac{x_1x_2\left(F_1\left(\frac{x_2}{x_1}\right)+F_2\left(\frac{x_1}{x_2}\right)\right)-x_1^2F_2'\left(\frac{x_1}{x_2}\right)+x_2^2F_1'\left(\frac{x_2}{x_1}\right)}{x_1^3x_2^2\varphi^2(t)}=\frac{1}{x_1^2x_2\varphi^2(t)}H_1\left(\frac{x_2}{x_1}\right), \tag{47}$$

$^{+}$ This clearly follows from the fact that the equivalence class of (38) is that of the free particle system and hence possesses $\mathfrak{sl}(4,\mathbb{R})$-symmetry [37].

$$\frac{\partial G}{\partial x_2}=\frac{x_1x_2\left(F_1\left(\frac{x_2}{x_1}\right)+F_2\left(\frac{x_1}{x_2}\right)\right)+x_1^2F_2'\left(\frac{x_1}{x_2}\right)-x_2^2F_1'\left(\frac{x_2}{x_1}\right)}{x_1^2x_2^3\varphi^2\left(t\right)}=\frac{1}{x_1x_2^2\varphi^2\left(t\right)}H_2\left(\frac{x_1}{x_2}\right).\quad(48)$$

The system possesses two independent constants of the motion quadratic in the velocities. Either using the corresponding formula (10) for the Noether symmetries X_i or the preceding equations of motion, a routine computation shows that one invariant is given by

$$J_1=\frac{\varphi^2\left(t\right)}{2}\left(x_2\dot{x}_1-x_1\dot{x}_2\right)^2+\int^{x_2/x_1}H_1\left(z\right)\,dz+\int^{x_1/x_2}H_2\left(z\right)\,dz,\quad(49)$$

while the second can be expressed as

$$J_2=\varphi^2\left(t\right)\left(\dot{x}_1\dot{x}_2+\int^{x_1x_2}g_2(z)dz\right)-\int\left(\varphi^2\left(t\right)g_1\left(t\right)\int^{x_1x_2}\omega(z)^2dz\right)dt+\frac{K}{x_1^2}$$
$$-\frac{1}{x_1^2}\int^{x_2/x_1}\frac{H_1(z)}{z}dz.\quad(50)$$

The independence of J_1 and J_2 is straightforward, taking into account that J_2 depends explicitly on $g_1\left(t\right)$ and $g_2\left(t\right)$.

For the special case $g_1\left(t\right)=0$ and $g_2\left(t\right)=\omega^2\left(t\right)$, $\varphi(t)$ reduces to a constant. The system (47)-(48) thus constitutes a special case of the generalized Ermakov systems introduced in [10].* This shows that the deformations of the two-dimensional time-dependent harmonic oscillator (with respect to the Lagrangian (39)) that preserve the $\mathfrak{sl}\left(2,\mathbb{R}\right)$-subalgebra correspond to a subclass of the generalized Ermakov systems with velocity-independent potentials (hence this subclass admits a Hamiltonian [38]).
The two constants of the motion (49) and (50) reduce to

$$J_1=\frac{1}{2}\left(x_2\dot{x}_1-x_1\dot{x}_2\right)^2+\int^{x_2/x_1}H_1\left(z\right)\,dz+\int^{x_1/x_2}H_2\left(z\right)\,dz\quad(51)$$

and

$$J_2=\dot{x}_1\dot{x}_2+\frac{K}{x_1^2}+\int^{x_1x_2}\omega(z)^2dz-\frac{1}{x_1^2}\int^{x_2/x_1}\frac{H_1(z)}{z}dz.\quad(52)$$

The first integral J_1 corresponds to the well-known ERR-invariant, while J_2 coincides with the additional invariant deduced in [39].

3.2. Velocity-dependent forcing terms

We now analyze the more general case of deformations of the Lagrangian (39) with forcing terms of the form $G\left(t,\mathbf{x},\dot{\mathbf{x}}\right)$. As usual, we start from the extension

$$\widetilde{L}=L-\varphi\left(t\right)\,G\left(t,\mathbf{x},\dot{\mathbf{x}}\right).\quad(53)$$

* In fact, the constraint satisfied by F_1 and F_2 corresponds to the sufficiency condition for the ERR-system to arise from a Lagrangian in 2 dimensions, i.e., to satisfy the Helmholtz conditions (7).

Evaluating the Noether symmetry condition for the vector fields X_1, X_2 leads, after some algebraic manipulation and simplification, to the following PDE

$$U_k(t)\dot{U}_k(t)\left(\sum_{l=1}^{2}\left(x_l\,\frac{\partial G}{\partial x_l}-\dot{x}_l\,\frac{\partial G}{\partial \dot{x}_l}\right)+2G\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\right)+\dot{U}_k^2(t)\sum_{l=1}^{2}x_l\,\frac{\partial G}{\partial \dot{x}_l}+U_k^2\left(t\right)\frac{\partial G}{\partial t}+$$
$$U_k^2\left(t\right)\left(2g_1\left(t\right)\,G\left(t,\mathbf{x},\dot{\mathbf{x}}\right)-g_1\left(t\right)\sum_{l=1}^{2}\dot{x}_l\,\frac{\partial G}{\partial \dot{x}_l}-g_2\left(t\right)\sum_{l=1}^{2}x_l\,\frac{\partial G}{\partial \dot{x}_l}\right)=0. \tag{54}$$

Again, the appropriate strategy to obtain a common solution for $k=1,2$ is to separate the equation into a set of equations not depending explicitly on the $U_k(t)$ functions. This yields the following equations:

$$x_1\,\frac{\partial G}{\partial \dot{x}_1}+x_2\,\frac{\partial G}{\partial \dot{x}_2}=0, \tag{55}$$
$$x_1\,\frac{\partial G}{\partial x_1}+x_2\,\frac{\partial G}{\partial x_2}-\dot{x}_1\,\frac{\partial G}{\partial \dot{x}_1}-\dot{x}_2\,\frac{\partial G}{\partial \dot{x}_2}+2G\left(t,\mathbf{x},\dot{\mathbf{x}}\right)=0, \tag{56}$$
$$\frac{\partial G}{\partial t}+2g_1\left(t\right)\,G\left(t,\mathbf{x},\dot{\mathbf{x}}\right)-g_1\left(t\right)\left(\dot{x}_1\,\frac{\partial G}{\partial \dot{x}_1}+\dot{x}_2\,\frac{\partial G}{\partial \dot{x}_2}\right)=0. \tag{57}$$

We observe that in the third equation, we have skipped the term of (54) involving $g_2\left(t\right)$, as the sum $\sum_{l=1}^{2}x_l\,\frac{\partial G}{\partial \dot{x}_l}$ equals zero by the first equation (55). Solving successively these equations, after a lengthy computation we find the general solution to this system as

$$G\left(t,\mathbf{x},\dot{\mathbf{x}}\right)=F\left(\frac{x_2}{x_1},\varphi\left(t\right)\left(\dot{x}_2x_1-\dot{x}_1x_2\right)\right)x_1^{-2}\varphi^{-2}\left(t\right). \tag{58}$$

We observe that if $g_1\left(t\right)=0$, G simplifies to

$$G\left(\mathbf{x},\dot{\mathbf{x}}\right)=F\left(\frac{x_2}{x_1},\left(\dot{x}_2x_1-\dot{x}_1x_2\right)\right)x_1^{-2}. \tag{59}$$

Introducing the auxiliary variables $r=x_2x_1^{-1}$ and $W=x_1\dot{x}_2-\dot{x}_1x_2$ as in [7], the equations of motion for the deformed Lagrangian

$$L=\frac{\varphi\left(t\right)}{2}\left(\dot{x}_1^2+\dot{x}_2^2-g_2\left(t\right)\left(x_1^2+x_2^2\right)\right)-\frac{1}{x_1^2\varphi\left(t\right)}G\left(r,\varphi\left(t\right)W\right) \tag{60}$$

are explicitly given by

$$\ddot{x}_1+g_1\left(t\right)\dot{x}_1+g_2\left(t\right)x_1+\frac{2\dot{r}}{\varphi\left(t\right)x_1}\frac{\partial G}{\partial W}+\frac{r}{\varphi\left(t\right)^2x_1^3}\left(\varphi\left(t\right)W\frac{\partial^2G}{\partial r\partial W}-\frac{\partial G}{\partial r}\right)-\frac{2\,G}{\varphi\left(t\right)^2x_1^3}$$
$$+\frac{r\left(\dot{W}+g_1\left(t\right)W\right)}{x_1}\frac{\partial^2G}{\partial W^2}=0, \tag{61}$$

$$\ddot{x}_2+g_1\left(t\right)\dot{x}_2+g_2\left(t\right)x_2+\frac{\frac{\partial G}{\partial r}-\varphi\left(t\right)W\frac{\partial^2G}{\partial r\partial W}}{\varphi\left(t\right)^2x_1^3}-\frac{\left(\dot{W}+g_1\left(t\right)W\right)}{x_1}\frac{\partial^2G}{\partial W^2}=0. \tag{62}$$

It is immediate to verify that this system is equivalent to the original 2-dimensional oscillator if G is a linear function of W. For non-linear functions in W, the same argument used for the scalar case shows that the Lie algebra $\mathcal{L}$ of point symmetries of

(61)-(62) coincides with that of Noether symmetries, isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. In this case, as expected from the presence of a damping term $g_1(t)$, the two constants of the motion derived from the Noether symmetries X_1 and X_2 adopt in general a rather complicated integral form, for which reason we skip their explicit expression.

As an elementary example illustrating this situation, let $g_1(t) = 1$ and $g_2(t) = 0$. We consider the forcing term given by the function $G(t, \mathbf{x}, \dot{\mathbf{x}}) = W^2$. The equations of motion of the Lagrangian $L = \frac{1}{2}\mathrm{e}^t\left(\dot{x}_1^2 + \dot{x}_2^2 + W^2 x_1^{-2}\right)$ can be brought to the form

$$\ddot{x}_1 + \dot{x}_1 - \frac{6W^2}{x_1^3\,(3+2r^2)} = 0, \tag{63}$$

$$\ddot{x}_2 + \dot{x}_2 - \frac{4r\,W^2}{x_1^3\,(3+2r^2)} = 0. \tag{64}$$

A first invariant can be found easily. Multiplying the first equation by $x_2 W^{-1}$ and the second by $x_1 W^{-1}$, the difference of the equations leads, after integration, to the conserved quantity

$$I_1 = \mathrm{e}^t\,W\sqrt{(3+2r^2)} = \mathrm{e}^t\left(\dot{x}_2 x_1 - x_2\dot{x}_1\right)x_1^{-2}\sqrt{(3x_1^2 + 2x_2^2)} \tag{65}$$

A second independent invariant is more complicated to find, although for this purpose the explicit Noether symmetries can be used. With this method, and after some computation, the following invariant can be found:

$$I_2 = \frac{\mathrm{e}^t}{2}\left(\dot{x}_1^2 + 3\dot{x}_2^2 + x_1\dot{x}_1 + x_2\dot{x}_2 - \frac{2x_2\dot{x}_1\left(2\dot{x}_2 x_1 - x_2\dot{x}_1\right)}{x_1^2}\right). \tag{66}$$

We further observe that $I_1 I_2^{-1}$ provides an invariant that does not explicitly depend on the time.

At this stage, the natural question that arises in this context is whether for the case $g_1(t) = 0$, the deformed system (61)-(62) also corresponds to a generalized Ermakov-Ray-Reid system (with velocity-dependent potential) that allows a two-dimensional Lagrangian. We prove this assumption to be correct.

Such a generalized ERR system, as first introduced in [11], has the generic form:♯

$$\ddot{x}_1 + \omega^2(t)\,x_1 - \frac{1}{x_2^3}\frac{\partial F_1}{\partial r} + \frac{W}{x_2^3}\frac{\partial^2 F_1}{\partial r\partial W} + \frac{\dot{W}}{x_2}\frac{\partial^2 F_1}{\partial W^2} = 0, \tag{67}$$

$$\ddot{x}_2 + \omega^2(t)\,x_2 + \frac{1}{x_2^2 x_1}\frac{\partial G_1}{\partial r} - \frac{W}{x_2^2 x_1}\frac{\partial^2 G_1}{\partial r\partial W} - \frac{\dot{W}}{x_1}\frac{\partial^2 G_1}{\partial W^2} = 0, \tag{68}$$

where F_1 and G_1 are arbitrary function of r and W. Using the Helmholtz conditions (7) (see also [26, 27]), a long but routine computation shows that (67)-(68) correspond to the equations of motion of a two-dimensional Lagrangian if the following constraints are satisfied:

$$\frac{\partial^2 G_1}{\partial W^2} - r^2\frac{\partial^2 F_1}{\partial W^2} = 0, \tag{69}$$

♯ The only formal difference with respect to [11] is that we have skipped the explicit use of the variable $\widetilde{r} = r^{-1}$ in the equations of motion.

$$3\frac{\partial F_1}{\partial r}+\frac{1}{r^2}\frac{\partial G_1}{\partial r}-3w\frac{\partial^2 F_1}{\partial r\partial W}-\frac{W}{r^2}\frac{\partial^2 G_1}{\partial r\partial W}+r\frac{\partial^2 F_1}{\partial r^2}-\frac{1}{r}\frac{\partial^2 G_1}{\partial r^2}-rw\frac{\partial^3 F_1}{\partial r^2\partial W}$$
$$+\frac{W}{r}\frac{\partial^3 G_1}{\partial r^2\partial W}=0 \tag{70}$$

Integrating the first equation and using the method of characteristics [40], the solution to this system is found to be

$$F_1\left(r,W\right)=\frac{G_1\left(r,w\right)}{r^2}+f_1\left(r\right)w+\frac{C}{r^2}, \tag{71}$$

where $G_1\left(r,W\right)$ is still an arbitrary function. Now, inserting the latter expression into (67)-(68) and simplifying, the equations of motion adopt the form

$$\ddot{x}_1+\omega^2\left(t\right)x_1+\frac{2G_1}{x_1^3}-\frac{2W}{x_1^3}\frac{\partial G_1}{\partial W}-\frac{1}{x_1^2x_2}\frac{\partial G_1}{\partial r}+\frac{W}{x_1^2x_2}\frac{\partial^2 G_1}{\partial r\partial W}+\frac{x_2\dot{W}}{x_1^2}\frac{\partial^2 G_1}{\partial W^2}=0, \tag{72}$$

$$\ddot{x}_2+\omega^2\left(t\right)x_2+\frac{1}{x_2^2x_1}\frac{\partial G_1}{\partial r}-\frac{W}{x_2^2x_1}\frac{\partial^2 G_1}{\partial r\partial W}-\frac{\dot{W}}{x_1}\frac{\partial^2 G_1}{\partial W^2}=0. \tag{73}$$

These equations are rather similar to those in (61)-(62) with $g_1\left(t\right)=0$ and $\varphi\left(t\right)=1$. In fact, if we define the forcing term as $G\left(t,\mathbf{x},\dot{\mathbf{x}}\right)=-G_1\left(\frac{1}{r},-W\right)$ and consider the Lagrangian

$$L=\frac{1}{2}\left(\dot{x}_1^2+\dot{x}_2^2-g_2\left(t\right)\left(x_1^2+x_2^2\right)\right)+\frac{1}{x_1^2}G\left(\frac{1}{r},-W\right), \tag{74}$$

it is immediate to verify that the equations of motion are exactly (72)-(73).

Jointly with the result obtained in the preceding paragraph for the velocity-independent forcing terms, we conclude that deformations of the two-dimensional time-dependent oscillator with respect to the Lagrangian (39) give rise to generalized ERR-systems. This can be formulated in compact form as follows:

Proposition 3 *For $g_1\left(t\right)=0$ and $g_2\left(t\right)=\omega^2\left(t\right)$, any $\mathfrak{sl}(2,\mathbb{R})$-preserving deformation of the two-dimensional time-dependent oscillator (38) corresponds to a generalized ERR-system (67)-(68) satisfying the constraints (69)-(70).*

For $g_1(t)\neq 0$, the systems can be seen as a further possible generalization of ERR-systems, albeit for generic choices of the forcing term, the Lagrangian of the system will generally be explicitly depending on the time. The deformations provide, however, an ample class of dissipative systems with two independent constant of the motion.

4. Conclusions

We have formulated a deformation problem for Lagrangian systems in one and two dimensions with the requirement that the deformed Lagrangian preserves exactly a fixed subalgebra of Noether symmetries. For the scalar case, the deformations of a generic linear homogeneous ordinary differential equation $\ddot{x}+g_1(t)\dot{x}+g_2(t)x=0$ preserving a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ have been obtained. This case is physically justified, as it covers in particular the equations of damped and time-dependent oscillators. It follows in particular that Pinney-type equations constitute the most general deformation of the oscillators that preserve a $\mathfrak{sl}(2,\mathbb{R})$-subalgebra. For the two-dimensional case, more general types of deformations, specifically dependent on the velocities, are possible. In this context, it has been shown that $\mathfrak{sl}(2,\mathbb{R})$-preserving deformations of the time-dependent harmonic oscillator are characterized as generalized Ermakov-Ray-Reid systems, as defined and studied in [11], that admit a Lagrangian (hence Hamiltonian) formalism. Within the classical interpretation of Ermakov systems, this result was to be expected. In the general case with damping terms, no such characterization is given, as the deformed Lagrangians are usually explicitly time-dependent, a characteristic that is carried over to the invariants.

Albeit our analysis has been focused on Euclidean Lagrangians related to the time-dependent (damped) harmonic oscillator in one or two dimensions, the procedure is by no means restricted to these Lagrangians. The symmetry-preserving deformation problem can be applied to any ODE or system that possesses $\mathfrak{sl}(2,\mathbb{R})$ as a subalgebra of Noether symmetries, or even any other fixed algebra of symmetries of this type.
Consider for instance the system

$$\ddot{x}_1 + \frac{\alpha}{x^3} = 0, \quad \ddot{x}_2 - \frac{3\alpha r}{x_1^3} = 0. \tag{75}$$

An admissible Lagrangian is given by

$$L = \dot{x}_1\dot{x}_2 - \alpha x_2 x_1^{-3}. \tag{76}$$

This Lagrangian, which can be seen as a deformation of the free Lagrangian in the pseudo-Euclidean plane, admits a $\mathfrak{sl}\left(2,\mathbb{R}\right)$ Lie algebra of Noether symmetries generated by the vector fields $X = \xi\left(t\right)\frac{\partial}{\partial t} + \frac{1}{2}x_1\dot{\xi}\left(t\right)\frac{\partial}{\partial x_1} + \frac{1}{2}x_2\dot{\xi}\left(t\right)\frac{\partial}{\partial x_2}$, where $\xi^{(3)}\left(t\right) = 0$. In this case, the auxiliary function $V\left(t,\mathbf{x}\right)$ for the symmetry condition (9) is given by $V\left(t,\mathbf{x}\right) = \frac{1}{2}\dot{x}_1\dot{x}_2\ddot{\xi}\left(t\right)$. A routine computation shows that the most general potential that preserves $\mathfrak{sl}\left(2,\mathbb{R}\right)$ is again given by $G\left(t,\mathbf{x}\right) = F\left(x_2x_1^{-1}\right)x_1^{-2} = F\left(r\right)x_1^{-2}$. The equations of motion for the deformed Lagrangian $\widehat{L} = \dot{x}_1\dot{x}_2 - \alpha x_2x_1^{-3} + F\left(r\right)x_1^{-2}$ are

$$\ddot{x}_1 + \frac{\alpha}{x^3} - \frac{F'\left(r\right)}{x_1^3} = 0, \quad \ddot{x}_2 - \frac{3\alpha r}{x_1^3} + \frac{2F\left(r\right)}{x_1^3} + \frac{r\,F'\left(r\right)}{x_1^3} = 0. \tag{77}$$

It is immediate to see that the Hamiltonian $H = \dot{x}_1\dot{x}_2 - \frac{\alpha\,r}{x_1^2} - \frac{F(r)}{x_1^2}$ is a constant of the motion. The second invariant has the form

$$I_1 = \frac{1}{2}W^2 - 2\alpha r^2 + 2rF\left(r\right). \tag{78}$$

We observe that, incidentally, for $F(r) = \lambda_1 + \lambda_2 r$, the (deformed) system is super-integrable, as it admits the additional constant of the motion $I_2 = \dot{x}_1^2 + (\lambda_2 - \alpha)x_1^{-3}$ [41]. Some additional examples of two-dimensional Lagrangians and the most general $\mathfrak{sl}\left(2,\mathbb{R}\right)$-preserving deformations are given the the following table.

There is no formal difficulty in generalizing the results to systems in N dimensions. The system

$$\ddot{x}_i + g_1\left(t\right)\dot{x}_i + g_2\left(t\right)x_i = 0, \; i = 1,\cdots,N \tag{79}$$

Table 1. Most general $\mathfrak{sl}(2,\mathbb{R})$-preserving forcing terms.

Lagrangian L	$G(t,\mathbf{x})$	$G(t,\mathbf{x},\dot{\mathbf{x}})$
$\dot{x}_1\dot{x}_2$	$\Psi\left(x_2x_1^{-2}\right)x_1^{-2}$	$\Psi\left(x_2x_1^{-2},\dot{x}_2x_1-\dot{x}_1x_2\right)x_1^{-2}$
$4\left(x_1^2+x_2^2\right)\left(\dot{x}_1^2+\dot{x}_2^2\right)$	$\Psi\left(x_2x_1^{-2}\right)x_1^{-4}$	$\Psi\left(x_2x_1^{-2},\left(\dot{x}_2x_1-\dot{x}_1x_2\right)x_1^2\right)x_1^{-4}$
$\frac{1}{2}\left(\dot{x}_1^2+x_1^2\dot{x}_2^2\right)$	$\Psi\left(x_2\right)x_1^{-2}$	$\Psi\left(x_2,\dot{x}_2\,x_1^2\right)x_1^{-2}$
$\frac{1}{2}\left(\dot{x}_1^2-x_1^2\dot{x}_2^2\right)$	$\Psi\left(x_2\right)x_1^{-2}$	$\Psi\left(x_1\sqrt{\dot{x}_2},x_2\right)\dot{x}_2$

obtained from the Lagrangian $L=\frac{1}{2}\varphi(t)\sum_{i=1}^{N}\left(\dot{x}_i^2-g_2(t)\,x_i^2\right)$ is linearizable, hence the subalgebra of Noether symmetries is isomorphic to $S(N)$. In particular, the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra is generated by the vector fields

$$X_k=\varphi(t)\,U_k^2(t)\,\frac{\partial}{\partial t}+\sum_{l=1}^{N}\varphi(t)\,U_k(t)\,\dot{U}_k(t)\,x_l\,\frac{\partial}{\partial x_l},\ k=1,2 \tag{80}$$

As expected, the most general forcing term preserving the $\mathfrak{sl}(2,\mathbb{R})$-subalgebra is given by

$$G(t,\mathbf{x},\dot{\mathbf{x}})=F\left(\frac{x_2}{x_1},\cdots,\frac{x_N}{x_1},\varphi(t)\left(\dot{x}_2x_1-\dot{x}_1x_2\right),\cdots,\varphi(t)\left(\dot{x}_Nx_1-\dot{x}_1x_N\right)\right)x_1^{-2}\varphi^{-2}(t)\,.$$

In analogy to the treated cases, for generic choices of F, the point symmetry algebra of the system is also isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

Clearly the deformation problem is heavily dependent on the Lagrangian chosen, as well as the fixed subalgebra of Noether symmetries. The form of symmetry-preserving forcing terms for alternative Lagrangians giving rise to the same equations of motion may differ radically, as can be expected from the existing ambiguities in the Lagrangian formalism [42]. In this sense, it would be of interest to classify these deformations according to their equivalence class as systems of ODEs [37]. The imposition of the generators of the subalgebra to remain unaltered by the deformation, although not explicitly stated, constitutes a constraint that could be used for such a classification. From the physical perspective, however, it seems more relevant to determine a precise interpretation of the conservation laws of the resulting non-linear system. For the case of systems admitting time-dependent invariants, their asymptotic behaviour could provide useful information on the intrinsic structural properties of the system. Both problems present interesting features that deserve to be inspected more closely.

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