On the effective size of a non-Weyl graph

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Abstract. We show how to find the coefficient by the leading term of the resonance asymptotics using the method of pseudo orbit expansion for quantum graphs which do not obey the Weyl asymptotics. For a non-Weyl graph we develop a method how to reduce the number of edges of a corresponding directed graph. With this method we prove bounds on the above coefficient depending on the structure of the graph for graphs with the same lengths of the internal edges. We explicitly find the positions of the resolvent resonances.

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1. Introduction

Quantum graphs have been intensively studied mainly in the last thirty years. There is a large bibliography, we refer e.g. to the book [BK13] and the proceedings [AGA08] and the references therein. One of the studied models are quantum graphs with attached semiinfinite leads. For this model, the notion of resonances can be defined; there are two main definitions – resolvent resonances (singularities of the resolvent) and scattering resonances (singularities of the scattering matrix). For the study of resonances in quantum graphs we refer e.g. to [EŠ94, Exn13, Exn97, KS04, BSS10, EL07, EL10].

The problem of finding resonance asymptotics in quantum graphs has been addressed in [DP11, DEL10, EL11]. A surprising observation by Davies and Pushnitski [DP11] shows that the graph has in some cases fewer resonances than expected by the Weyl asymptotics. Criteria, which can distinguish these non-Weyl graphs from the graphs with regular Weyl asymptotics, have been presented in [DP11] (graphs with standard coupling) and [DEL10] (graphs with general coupling). Although distinguishing these two cases is quite easy (in depends on the vertex properties of the graph), finding the constant by the leading term of the asymptotics (which is closely related to the "effective size" of the graph) is more difficult since it uses the structure of the whole graph.

In the present paper we expressed the first term of non-Weyl asymptotics using the method of pseudo orbits and found bounds on the effective size which depend on the structure of the graph. The paper is structured as follows. In the second section we introduce the model, in the third section we state known theorems on the asymptotics of the resonances. In section 4 we develop the method of pseudo orbit expansion for the resonance condition. In the fifth section we state how the effective size can be found for an equilateral graph (graph with the same lengths of the internal edges). In section 6 we develop a method which allows us to delete some of the edges of a non-Weyl graph. In section 7 we state the main theorems on the bounds on the effective size and position of the resonances for equilateral graphs. Section 8 illustrates found results and developed method in two examples.

2. Preliminaries

First, we describe the model and introduce the main notions. We consider a metric graph Γ consisting of N finite edges of length ℓ_j and M infinite edges, which can be parametrized as halflines $(0,\infty)$. On this graph we define a second order differential operator H acting as $-\mathrm{d}^2/\mathrm{d}x^2$ with the domain consisting of functions with edge components in Sobolev space $W^{2,2}(e_j)$ which satisfy the coupling conditions at the vertices

$$(U_j - I)\Psi_j + i(U_j + I)\Psi'_j = 0\,, \tag{1}$$

where U_j is $d_j \times d_j$ unitary matrix (d_j is the degree of the given vertex), I is a unit matrix, Ψ_j is the vector of limits of functional values from various edges to the given vertex and, similarly, Ψ'_j is the vector of outgoing derivatives. This coupling condition was independently found by Kostrykin and Schrader [KS00] and Harmer [Har00].

As it was shown in [EL10, Kuc08], one can describe the set of equations (1) by only one equation using a big $(2N+M)\times(2N+M)$ unitary matrix U, which is in certain basis block diagonal and consists of blocks U_j. This matrix encodes not only the coupling but also the topology of the whole graph.

$$(U - I)\Psi + i(U + I)\Psi' = 0\,. \tag{2}$$

Here, I is $(2N+M)\times(2N+M)$ unit matrix and the vectors Ψ and Ψ' consist of entries of Ψ_j and Ψ'_j, respectively. This coupling condition corresponds to a graph where all the vertices are joint into one vertex.

One can also describe the coupling on the compact part of the graph using energy-dependent $2N\times 2N$ coupling matrix $\tilde{U}(k)$. It can be straightforwardly proven [EL10] by a standard method (Schur, etc.) that the effective coupling matrix is

$$\tilde{U}(k) = U_1 - (1-k)U_2[(1-k)U_4 - (1+k)I]^{-1}U_3\,, \tag{3}$$

where the matrices $U_1, \dots, U_4$ are blocks of the matrix $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$; U_1 corresponds to the coupling between internal edges, U_4 between external edges and U_2 and U_3 correspond to the mixed coupling. The coupling condition has a similar form to (2) with U replaced by $\tilde{U}(k)$ and I meaning now $2N \times 2N$ unit matrix.

3. Asymptotics of the number of resonances

We are interested in the asymptotical behaviour of the number of resolvent resonances of the system. By a *resolvent resonance* we mean the pole of meromorphic continuation of the the resolvent $(H - \lambda \mathrm{id})^{-1}$ into the second sheet. The resolvent resonances can be obtained by the method of complex scaling (more on that in [EL07]). We will use a simpler but equivalent definition.

Definition 3.1. *We call $\lambda = k^2$ a resolvent resonance if there exists a generalized eigenfunction $f \in L^2_{\mathrm{loc}}(\Gamma)$, $f \not\equiv 0$ satisfying $-f''(x) = k^2 f(x)$ and the coupling conditions (1), which on all external edges behaves as $c_j\, \mathrm{e}^{ikx}$.*

This definition is equivalent to the previously mentioned one because these generalized eigenfunctions (which, of course, are not square integrable), become after complex scaling square integrable. It was proven in [EL07] that the set of resolvent resonances is equal to the union of the set of scattering resonances and the eigenvalues with the eigenfunctions supported only on the internal part of the graph.

We define a counting function $N(R)$, which counts the number of resolvent resonances including their multiplicities which are contained in the circle of radius R in the k-plane. One should note that by this method we find twice more resonances than in the energy plane. This is clear for the case of a compact graph, because the k and $-k$ correspond to the same eigenvalue.

Most of the quantum graphs have the asymptotics of the resolvent resonances obeying Weyl law. However, there exist such graphs for which the constant by the leading term of the asymptotics is smaller than expected; we call these graphs *non-Weyl*. The problem was studied for graphs with standard coupling (in the literature also called Kirchhoff, free or Neumann coupling) in [DP11] and for graphs with general coupling in [DEL10]. We state main results of these papers.

Theorem 3.2. *(Davies and Pushnitski)*
Graphs with standard coupling (functional values are continuous in the vertex and the sum of the outgoing derivatives is zero) with the sum of the lengths of the internal edges equal to $\mathrm{vol}\,\Gamma$ *have the following asymptotics of resonances*

$$N(R) = \frac{2}{\pi} W R + \mathcal{O}(1)\,, \quad \text{as } R \to \infty\,,$$

where $0 \le W \le \mathrm{vol}\,\Gamma$. One has $W < \mathrm{vol}\,\Gamma$ iff there exist at least one balanced vertex, i.e. the vertex which connects the same number of internal and external edges.

Theorem 3.3. *(Davies, Exner and Lipovský)*
The graph with general coupling (2) with the sum of the internal edges equal to $\operatorname{vol}\Gamma$ *has the asymptotics*

$$N(R) = \frac{2}{\pi} W R + \mathcal{O}(1)\,, \quad \text{as } R \to \infty\,,$$

where $0 \le W \le \operatorname{vol}\Gamma$*. One has* $W < \operatorname{vol}\Gamma$ *(the graph is non-Weyl) iff the effective coupling matrix* $\tilde U(k)$ *has the eigenvalue either* $\frac{1+k}{1-k}$ *or* $\frac{1-k}{1+k}$*.*

While finding whether the graph is Weyl or non-Weyl is quite easy (it depends only on the vertex properties of the graph), finding the coefficient W (the effective size of the graph) is more complicated, because it depends on the properties of the whole graph. It is illustrated e.g. in the theorem 7.3 in [DEL10].

4. Pseudo orbit expansion for the resonance condition

There is a theory developed how to find the spectrum of a compact quantum graph by a pseudo orbit expansion (see e.g. [BHJ12, KS99]). In this section we adjust this method to finding the resonance condition. The idea is to find the effective vertex-scattering matrix which acts only on the compact part of the graph. The vertex-scattering matrix maps the vector of amplitudes of the incoming waves to the vertex into the vector of amplitudes of the outgoing waves. We will show that in the case of standard coupling this matrix is not energy dependent and has a nice form.

Definition 4.1. *For a compact quantum graph we can express the solution of the Schrödinger equation on each edge as a combination of two waves* $f_j(x) = \alpha_{e_j}^{\mathrm{in}}\,\mathrm{e}^{-ikx} + \alpha_{e_j}^{\mathrm{out}}\,\mathrm{e}^{ikx}$*. Let us consider a vertex of the graph and let all the edges emanating from this vertex be parametrized by* $(0,\ell_j)$*, where* $x = 0$ *corresponds to this vertex. By a vertex-scattering matrix we mean a matrix* $\sigma^{(v)}$ *for which holds* $\vec\alpha_v^{\mathrm{out}} = \sigma^{(v)}\vec\alpha_v^{\mathrm{in}}$*, where* $\vec\alpha_v^{\mathrm{in}}$ *and* $\vec\alpha_v^{\mathrm{out}}$ *are the vectors of coefficients of the incoming and outgoing waves on the edges which emanate from vertex* v*, respectively. For a non-compact quantum graph the effective coupling matrix* $\tilde\sigma^{(v)}$ *is defined similarly; its size is given by the number of finite edges which emanate from vertex* v *and maps the vector of coefficients of incoming waves on these finite edges on the vector of coefficients of outgoing waves.*

For the next theorem we drop the superscript or subscript v denoting the vertex in coupling and vertex-scattering matrices.

Theorem 4.2. *(general form of the effective vertex-scattering matrix)*
Let us consider a non-compact graph Γ *with the coupling at the vertex* v *given by (1) and the matrix* U*. Let the vertex* v *connect* n *internal edges and* m *external edges. Let the matrix* I *be* $(n+m)\times(n+m)$ *unit matrix and by* I_n *we denote the* $n\times n$ *unit matrix and by* I_m *the* $m\times m$ *unit matrix. Then the effective vertex-scattering matrix is given by* $\tilde\sigma(k) = -[(1-k)\tilde U(k) - (1+k)I_n]^{-1}[(1+k)\tilde U(k) - (1-k)I_n]$*. The inverse relation is* $\tilde U(k) = [(1+k)\tilde\sigma(k) + (1-k)I_n][(1-k)\tilde\sigma(k) + (1+k)I_n]^{-1}$*.*

Proof. Let the solution on the internal edges emanating from the vertex v be $f_j(x) = \alpha_j^{\mathrm{in}} \mathrm{e}^{-ikx} + \alpha_j^{\mathrm{out}} \mathrm{e}^{ikx}$ and on the external edges $g_j(x) = \beta_j \mathrm{e}^{ikx}$. Vector of functional values therefore is $\Psi = \begin{pmatrix} \vec{\alpha}^{\mathrm{in}} + \vec{\alpha}^{\mathrm{out}} \\ \vec{\beta} \end{pmatrix}$ and the vector of outgoing derivatives is $\Psi' = ik \begin{pmatrix} -\vec{\alpha}^{\mathrm{in}} + \vec{\alpha}^{\mathrm{out}} \\ \vec{\beta} \end{pmatrix}$. The coupling condition (1) is

$$(U - I) \begin{pmatrix} \vec{\alpha}^{\mathrm{in}} + \vec{\alpha}^{\mathrm{out}} \\ \vec{\beta} \end{pmatrix} + iik(U + I) \begin{pmatrix} -\vec{\alpha}^{\mathrm{in}} + \vec{\alpha}^{\mathrm{out}} \\ \vec{\beta} \end{pmatrix} = 0\,.$$

Hence we have the set of equations

$$\begin{aligned} &[U_1 - I_n - k(U_1 + I_n)]\vec{\alpha}^{\mathrm{out}} + [(U_1 - I_n) + k(U_1 + I_n)]\vec{\alpha}^{\mathrm{in}} + (1-k)U_2\vec{\beta} = 0\,,\\ &(1-k)U_3\vec{\alpha}^{\mathrm{out}} + (1+k)U_3\vec{\alpha}^{\mathrm{in}} + [(U_4 - I_m) - k(U_4 + I_m)]\vec{\beta} = 0\,. \end{aligned}$$

Expressing $\vec{\beta}$ from the second equation and substituting into the first one one has

$$\begin{aligned} &\{(1-k)U_1 - (1+k)I_n - (1-k)U_2[(1-k)U_4 - (1+k)I_m]^{-1}(1-k)U_3\}\vec{\alpha}^{\mathrm{out}} + \\ &+\{(1+k)U_1 - (1-k)I_n - (1-k)U_2[(1-k)U_4 - (1+k)I_m](1+k)U_3\}\vec{\alpha}^{\mathrm{in}} = 0\,, \end{aligned}$$

from which using (3) the claim follows. Expressing the inverse relation is straightforward. □

Using the previous theorem one can straightforwardly compute the effective vertex-scattering matrix for standard condition.

Corollary 4.3. *Let v be the vertex connecting n internal and m external edges and let there be a standard coupling condition in v (i.e. $U = \frac{2}{n+m} J_{n+m} - I_{n+m}$, where J_n denotes $n \times n$ matrix with all entries equal to one). Then the effective vertex-scattering matrix is $\tilde{\sigma}(k) = \frac{2}{n+m} J_n - I_n$, in particular, for a balanced vertex we have $\tilde{\sigma}(k) = \frac{1}{n} J_n - I_n$.*

Proof. There are two ways how to prove this corollary. First is to compute the effective vertex-scattering matrix from the definition in a similar way to the proof of theorem 4.2. Second, a little bit longer, but straightforward, uses the theorem. Using the formula $(aJ_n + bI_n)^{-1} = \frac{1}{b}\left(-\frac{a}{an+b} J_n + I_n\right)$ we compute the effective coupling matrix and obtain $\tilde{U}(k) = \frac{2}{km+n} J_n - I_n$. Substituting it into the formula in theorem 4.2 and using the above expression of the inverse matrix and the fact that $J_n \cdot J_n = nJ_n$ one obtains the result. □

Having found the effective vertex-scattering matrix one can proceed in the lines of the method shown in [BHJ12]. We replace the compact part of the graph Γ by an oriented graph Γ_2, each finite edge e_j of Γ is replaced by two oriented edges b_j, $\hat{b}_j$ of the same length, each of them with different orientation. On these edges we use the ansatz

$$\begin{aligned} f_{b_j}(x) &= \alpha_{b_j}^{\mathrm{in}} \mathrm{e}^{-ikx} + \alpha_{b_j}^{\mathrm{out}} \mathrm{e}^{ikx}\,,\\ f_{\hat{b}_j}(x) &= \alpha_{\hat{b}_j}^{\mathrm{in}} \mathrm{e}^{-ikx} + \alpha_{\hat{b}_j}^{\mathrm{out}} \mathrm{e}^{ikx}\,. \end{aligned}$$

Since we have relation $f_{b_j}(x) = f_{\hat{b}_j}(\ell_j - x)$ (functional values in both directions must be the same), we obtain the following relations between coefficients

$$\alpha_{b_j}^{\mathrm{in}} = \mathrm{e}^{ik\ell_j}\alpha_{\hat{b}_j}^{\mathrm{out}}\,, \qquad \alpha_{\hat{b}_j}^{\mathrm{in}} = \mathrm{e}^{ik\ell_j}\alpha_{b_j}^{\mathrm{out}}\,. \tag{4}$$

Now we define the matrix $\tilde{\Sigma}(k)$ (which is in general energy-dependent) as a block-diagonalizable matrix written in the basis corresponding to

$$\vec{\alpha} = (\alpha_{b_1}^{\mathrm{in}}, \dots, \alpha_{b_N}^{\mathrm{in}}, \alpha_{\hat{b}_1}^{\mathrm{in}}, \dots, \alpha_{\hat{b}_N}^{\mathrm{in}})^{\mathrm{T}}$$

which is block diagonal with blocks $\tilde{\sigma}_v(k)$ if transformed to the basis

$$(\alpha_{b_{v_1 1}}^{\mathrm{in}}, \dots, \alpha_{b_{v_1 d_1}}^{\mathrm{in}}, \alpha_{b_{v_2 1}}^{\mathrm{in}}, \dots, \alpha_{b_{v_2 d_2}}^{\mathrm{in}}, \dots)^{\mathrm{T}}\,,$$

where $b_{v_1 j}$ is the j-th edge emanating from the vertex v_1.

Furthermore, we define $2N \times 2N$ matrices $Q = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}$, scattering matrix $S(k) = Q\tilde{\Sigma}(k)$ and

$$L = \mathrm{diag}\,(\ell_1, \dots, \ell_N, \ell_1, \dots, \ell_N)\,.$$

Using these matrices we can state the following theorem.

Theorem 4.4. *The resonance condition is given by*

$$\det\,(\mathrm{e}^{ikL}Q\tilde{\Sigma}(k) - I_{2N}) = 0\,.$$

Proof. If we define as $\vec{\alpha}_b^{\mathrm{in}} = (\alpha_{b_1}^{\mathrm{in}}, \dots \alpha_{b_N}^{\mathrm{in}})^{\mathrm{T}}$ and similarly for the outgoing amplitudes and bonds in the opposite direction $\hat{b}_j$, we can subsequently obtain

$$\begin{pmatrix} \vec{\alpha}_b^{\mathrm{in}} \\ \vec{\alpha}_{\hat{b}}^{\mathrm{in}} \end{pmatrix} = \mathrm{e}^{ikL}\begin{pmatrix} \vec{\alpha}_{\hat{b}}^{\mathrm{out}} \\ \vec{\alpha}_b^{\mathrm{out}} \end{pmatrix} = \mathrm{e}^{ikL}Q\begin{pmatrix} \vec{\alpha}_b^{\mathrm{out}} \\ \vec{\alpha}_{\hat{b}}^{\mathrm{out}} \end{pmatrix} = \mathrm{e}^{ikL}Q\tilde{\Sigma}(k)\begin{pmatrix} \vec{\alpha}_b^{\mathrm{in}} \\ \vec{\alpha}_{\hat{b}}^{\mathrm{in}} \end{pmatrix}.$$

We have first used relations (4), then definitions of matrices Q and $\tilde{\Sigma}$. Since the vectors on the lhs and rhs are the same, the equation of the solvability of the system gives the resonance condition. □

Now we define orbits on the graph, we use the same notation as in [BHJ12].

Definition 4.5. *A* periodic orbit γ *on the graph* Γ_2 *is a closed path on the graph* Γ_2 *which begins and ends in the same vertex. A* pseudo orbit *is a collection of periodic orbits* ($\tilde{\gamma} = \{\gamma_1, \gamma_2, \dots, \gamma_m\}$)*. An* irreducible pseudo orbit $\bar{\gamma}$ *is a pseudo orbit, which contains no directed bond more than once. The* metric length *of a periodic orbit is defined as* $\ell_\gamma = \sum_{b_j \in \gamma} \ell_{b_j}$*; the length of a pseudo orbit is the sum of the lengths of all periodic orbits from which it is composed. The product of scattering amplitudes along the periodic orbit* $\gamma = (b_1, b_2, \dots b_n)$ *we denote as* $A_\gamma = S_{b_2 b_1} S_{b_3 b_2} \dots S_{b_1 b_n}$*; for a pseudo orbit we define* $A_{\tilde{\gamma}} = \prod_{\gamma_j \in \tilde{\gamma}} A_{\gamma_j}$*. By* $m_{\tilde{\gamma}}$ *we denote the number of periodic orbits in the pseudo orbit* $\tilde{\gamma}$ *and by* $B_{\tilde{\gamma}}$ *its total topological length (number of bonds in the pseudo orbit).*

We give without a proof (which can be found in [BHJ12]) a theorem on finding the resonance condition using pseudo orbits.

Theorem 4.6. *The resonance condition is given by*

$$\sum_{\bar{\gamma}} (-1)^{m_{\bar{\gamma}}} A_{\bar{\gamma}}(k)\, \mathrm{e}^{ik\ell_{\bar{\gamma}}} = 0\,.$$

5. Effective size of an equilateral graph

As we stated in section 3, the effective size of a graph is defined as $\frac{\pi}{2}$-multiple of the constant by the leading term of the asymptotics. In this section we will find the effective size of an equilateral graph (graph with the same lengths of the internal edges) from matrices Q and $\tilde{\Sigma}$.

First, we will show a general criterion whether the graph is non-Weyl using the notions of theorem 4.4.

Theorem 5.1. *The graph is non-Weyl iff* $\det \tilde{\Sigma}(k) = 0$*. In other words, the graph is non-Weyl iff there exists a vertex for which* $\det \tilde{\sigma}_v(k) = 0$*.*

Proof. The first term of the determinant in the theorem 4.4 is $\det [Q\tilde{\Sigma}(k)]\, \mathrm{e}^{2ik\sum_{j=1}^{N} \ell_j}$ and the last term is 1. The theorem 3.1 in [DEL10] shows that the number of zeros of this determinant in the circle of radius R is asymptotically equal to $\frac{2}{\pi}\mathrm{vol}\,\Gamma$ iff the constant by the first term is nonzero. Since multiplying by Q means only rearranging the rows, $\det Q\tilde{\Sigma}(k) = 0$ iff $\det \tilde{\Sigma}(k) = 0$. Since $\tilde{\Sigma}(k)$ consists in certain basis of blocks $\tilde{\sigma}_v$, the second part of the theorem follows. □

From this theorem and corollary 4.3 the theorem 1.2 from [DP11] follows, which states that a graph with standard coupling is non-Weyl iff there is a vertex with the same number of finite and infinite edges.

Now we will state a theorem which gives the effective size of an equilateral graph.

Theorem 5.2. *Let us assume an equilateral graph (graph which has the same lengths of the internal edges* ℓ*). Then the effective size of this graph is* $\frac{\ell}{2} n_{\mathrm{nonzero}}$*, where* n_{nonzero} *is the number of nonzero eigenvalues of the matrix* $Q\tilde{\Sigma}(k)$*.*

Proof. We will use the theorem 4.4 again. First, we notice that the matrix $Q\tilde{\Sigma}(k)$ can be in this theorem replaced by its Jordan form $D(k) = VQ\tilde{\Sigma}(k)V^{-1}$ with V unitary. The matrix L is for an equilateral graph a multiple of unit matrix and therefore V commutes with e^{ikL}. Unitary transformation does not change the determinant of a matrix, hence we have the resonance condition $\det (\mathrm{e}^{ik\ell} D(k) - I_{2N}) = 0$. Since the matrix under the determinant is upper triangular, the determinant is equal to multiplication of its diagonal elements. The first nozero term of the determinant, which contains exponential to the highest power, is $\mathrm{e}^{ik\ell n_{\mathrm{nonzero}}}$. The last term is 1. The claim follows from the theorem 3.1 in [DEL10]. □

Clearly, if there is $n_{\rm bal}$ balanced vertices with standard coupling, then there is at least $n_{\rm bal}$ zeros in the eigenvalues of the matrix $Q\tilde{\Sigma}$ and hence the effective size is bounded $W \leq \mathrm{vol}\,\Gamma - \frac{\ell}{2}n_{\rm bal}$. The following corollary of the previous theorem gives a criterion when this bound can be improved.

Corollary 5.3. *Let* Γ *be an equilateral graph with standard coupling and with* $n_{\rm bal}$ *balanced vertices. Then the effective size* $W < \mathrm{vol}\,\Gamma - \frac{\ell}{2}n_{\rm bal}$ *iff* $\mathrm{rank}\,(Q\tilde{\Sigma}Q\tilde{\Sigma}) < \mathrm{rank}(\mathrm{Q}\tilde{\Sigma}) = 2N - n_{\rm bal}$.

Proof. For the vertex of degree d the rank of $\tilde{\sigma}_v$ is either $d-1$ for a balanced vertex or d otherwise. Hence $\mathrm{rank}(\mathrm{Q}\tilde{\Sigma}) = \mathrm{rank}(\tilde{\Sigma}) = 2N - n_{\rm bal}$. The effective size is smaller than $\mathrm{vol}\,\Gamma - \frac{\ell}{2}n_{\rm bal}$ iff there is at least one 1 just above the diagonal in the block with zeros on the diagonal in the Jordan form of $Q\tilde{\Sigma}$. Each Jordan block consisting of n zeros on the diagonal and $n-1$ ones above the diagonal has rank $n-1$; its square has rank $n-2$. The blocks with eigenvalues different than zero have (as well as their squares) rank maximal. Hence rank of $Q\tilde{\Sigma}Q\tilde{\Sigma}$ is smaller than the rank of $Q\tilde{\Sigma}$. □

6. Deleting edges of the oriented graph

In this section we will develop a method how to "delete" some edges of the graph Γ_2 for a non-Weyl graph. We will restrain to equilateral graphs with standard coupling. For each balanced vertex we delete one directed edge which ends in this vertex and replace it by one or several "ghost edges". These edges allow for the pseudo orbits to hop from a vertex to a directed edge which is not connected with the vertex. The "ghost edges" do not contribute to the resonance condition with the coefficient $\mathrm{e}^{ik\ell}$, they only change the product of scattering amplitudes. The idea under this deleting is a unitary transformation of the matrix $Q\tilde{\Sigma}$, after which one of its rows consists of all zeros. We will use this method in the following section to prove the main theorems. We will explain this method in the following theorem; for better understanding the example in section 8.1 might be helpful.

Theorem 6.1. *Let us assume an equilateral graph* Γ *(all internal edges of length* ℓ*) with standard coupling. Let us assume that there is no edge which starts and ends in one vertex and that no two vertices are connected by two or more edges. Let the vertex* v_2 *be balanced and let the directed bonds* $b_1, \dots, b_d$ *end in the vertex* v_2 *(part of the corresponding directed graph* Γ_2 *is shown in the figure 1). Then the following construction does not change the resonance condition. We delete the directed edge* b_1*, which starts at the vertex* v_1 *and ends at* v_2*. There are introduced new directed "ghost edges"* $b_1', b_1'', \dots, b^{(d-1)}$ *which start in the vertex* v_1 *and are connected to the edges* $b_2, b_3, \dots b_d$*, respectively (see figure 2). One can use pseudo orbits to obtain a resonance condition from the new graph similarly to the theorem 4.6. If the "ghost edge"* b_1' *is contained in the irreducible pseudo orbit* $\bar{\gamma}$*, the length of this "ghost edge" does not contribute to* $\ell_{\bar{\gamma}}$*. Let e.g. the "ghost edge"* b_1' *be included in* $\bar{\gamma}$*. Then the scattering amplitude from a bond* b *ending in* v_1 *and the bond* b_2 *is the scattering amplitude in the*

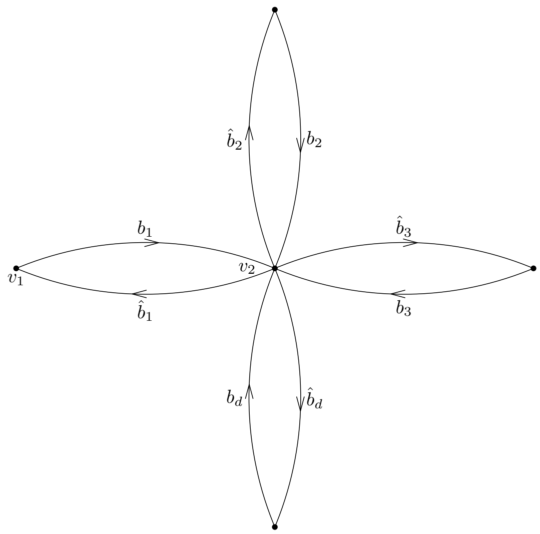

Figure 1. Part of the graph Γ_2. To vertex v_1 and undenoted vertices which neighbour v_2 other bonds can be attached.

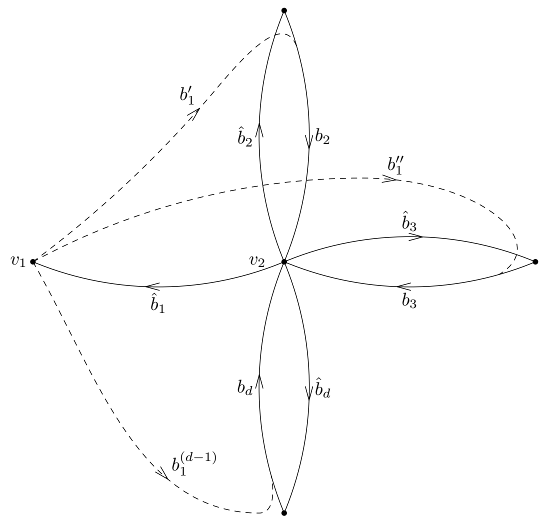

Figure 2. Part of the graph Γ_2 after deleting the bond b_1 and introducing "ghost edges".

graph Γ_2 between b and b_1 taken with the opposite sign. In the irreducible pseudo orbit each "ghost edge" can be used only once. The above procedure can be repeated; for each balanced vertex one can delete one directed edge which ends in this vertex and replace it by the "ghost edges".

Proof. Unitary transformation of the matrix $Q\tilde{\Sigma}$ does not change the determinant in theorem 4.4, since for an equilateral graph e^{ikL} is replaced by $\mathrm{e}^{ik\ell} I_{2N}$ and

$$\det(\mathrm{e}^{ik\ell} V_1 Q\tilde{\Sigma} V_1^{-1} - I_{2N}) = \det[V_1(\mathrm{e}^{ik\ell} Q\tilde{\Sigma} - I_{2N}) V_1^{-1}] = \det(\mathrm{e}^{ik\ell} Q\tilde{\Sigma} - I_{2N})\,.$$

Let v_2 be balanced vertex, we want to delete edge b_1 and let the other edges ending in v_2 be $b_2, b_3, \dots, b_d$. We will choose as V_1 the $2N \times 2N$ matrix with 1 on the diagonal, the entries which are at once in the row corresponding to b_1 and in the columns corresponding $b_2, b_3, \dots b_d$ are also 1, the other entries are zero. One can easily show that V_1^{-1} is equal to V_1, only the nondiagonal entries have negative sign.

The matrix $\tilde{\sigma}_{v_2} = \frac{1}{d} J_d - I_d$ has linearly dependent rows, hence if one multiplies it from the left by a matrix with ones on the diagonal and ones in one of its rows, otherwise zeros, one obtains a matrix which has one row with all zeros and the other rows the same as $\tilde{\sigma}_{v_2}$. The matrix $Q\tilde{\Sigma}$ has entries of $\tilde{\sigma}_{v_2}$ in the rows corresponding to bonds ending at v_2 and columns corresponding to bonds starting from v_2. By the same reasoning as for $\tilde{\sigma}_{v_2}$ the row of $V_1 Q\tilde{\Sigma}$ corresponding to b_1 has all entries equal to zero and other rows are unchanged.

Now it remains to show how multiplying from the right by V_1^{-1} changes the matrix. Since nondiagonal entries are only in the columns of V_1^{-1} corresponding to bonds $b_2, b_3, \dots b_d$, multiplying by V_1^{-1} changes only these columns. Since nondiagonal entries are only in the row corresponding to b_1, the change can happen only in rows in which there is nontrivial entry in the column corresponding to b_1. These rows correspond to bonds which end in the vertex v_1. We have to multiply these rows by columns of V_1^{-1} which have 1 in the b_j-th position, $j = 2, 3, \ldots, d$ and -1 in the b_1-th position. Since no two vertices are connected by two or more edges, the only bond starting at v_1 and ending at v_2 is b_1 and the entries of $V_1 Q\tilde{\Sigma}$ in the row corresponding to the edges ending at v_1 and in the column corresponding to the edges $b_2, b_3, \ldots b_d$ are zero (the edges in the row cannot be followed in an orbit by edges in the column). Hence 1 in the above column of V_1^{-1} is multiplied by 0 and -1 is multiplied by the scattering amplitude between the bonds ending in v_1 and b_1. Therefore, the only change is that there is this negatively taken scattering amplitude in the rows corresponding to the bonds which end in v_1 and in the column corresponding to bonds $b_2, b_3, \ldots, b_d$. These entries are represented by the "ghost edges".

It is clear now that one has to take the entry of I_{2N} in the row corresponding to b_1 in the determinant in theorem 4.4 with $Q\tilde{\Sigma}$ replaced by $V_1 Q\tilde{\Sigma} V_1^{-1}$, therefore this edge effectively does not exist. The "ghost edge" does not contribute to $\ell_{\bar{\gamma}}$, it only says which bonds are connected in the pseudo orbit. Similar arguments can be used for other balanced vertices, for each of them we delete one edge which ends in it. Note that this method does not delete edges to which a "ghost edge" leads. □

7. Main results

In this section we give two main theorems on bounds on the effective size for equilateral graphs with standard coupling and a theorem which gives the positions of the resonances.

Theorem 7.1. *Let us assume an equilateral graph with N internal edges of lengths ℓ, with standard coupling, n_{bal} balanced vertices and n_{nonneig} balanced vertices which do not neighbour any other balanced vertex. Then the effective size is bounded by $W \leq N\ell - \frac{\ell}{2}n_{\mathrm{bal}} - \frac{\ell}{2}n_{\mathrm{nonneig}}$.*

Proof. Clearly, for each balanced vertex, we can delete one directed edge of the graph Γ_2, the size of the graph is reduced by $\frac{\ell}{2}n_{\mathrm{bal}}$. In the balanced vertex of the degree d which do not neighbour any other balanced vertex we have $d-1$ incoming directed bonds and d outgoing directed bonds. No outgoing bond is deleted and no "ghost edge" ends in the outgoing edge, because there is no balanced vertex which neighbours the given vertex. Hence we cannot use one of the outgoing directed edges in the irreducible pseudo orbit (there is no way how to get to this vertex for d-th time). The longest irreducible pseudo orbit does not include n_{nonneig} bonds and the effective size of the graph must be reduced by $\frac{\ell}{2}n_{\mathrm{nonneig}}$. □

Theorem 7.2. *Let us assume an equilateral graph (N internal edges of the lengths ℓ) with standard coupling. Let there be a square of balanced vertices v_1, v_2, v_3 and v_4 without diagonals, i.e. v_1 neighbours v_2, v_2 neighbours v_3, v_3 neighbours v_4, v_4 neighbours v_1, v_1*

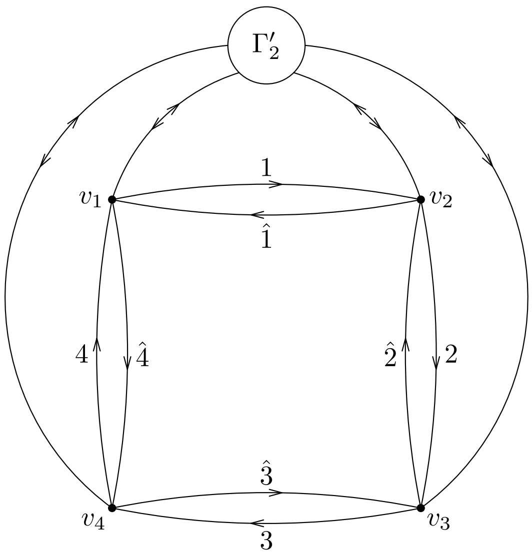

Figure 3. Figure to theorem 7.2. The bonds between Γ_2' and vertices $v_1, \ldots, v_4$ represent possible directed edges between this subgraph and vertices in both directions.

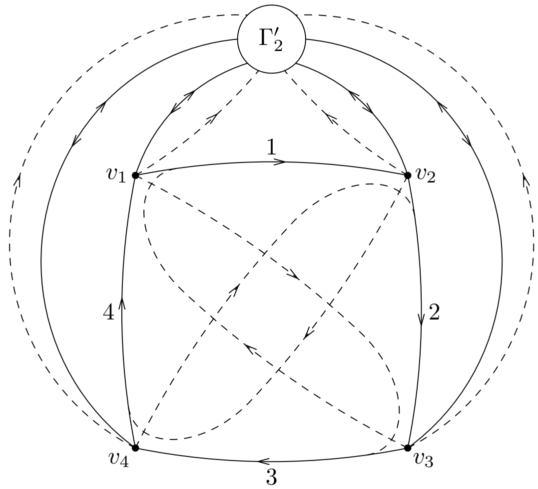

Figure 4. Figure to theorem 7.2. The bonds and "ghost edges" between Γ_2' and vertices $v_1, \ldots, v_4$ represent possible directed edges between this subgraph and vertices in both directions and "ghost edges" from the vertices $v_1, \ldots, v_4$ to the subgraph Γ_2'.

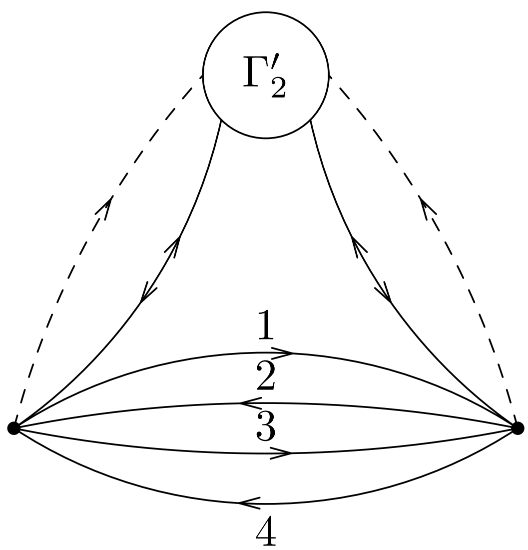

Figure 5. Figure to theorem 7.2. The effective graph with "ghost edges" taken into account.

does not neighbour v_3 *and* v_2 *does not neighbour* v_4*. Then the effective size is bounded by* $W \leq (N-3)\ell$.

Proof. Let us denote the bond from v_1 to v_2 by 1, the bond from v_2 to v_3 by 2, the bond from v_3 to v_4 by 3 and the bond from v_4 to v_1 by 4, the bonds in the opposite directions by $\hat{1}, \hat{2}, \hat{3}$ and $\hat{4}$ (see figure 3). Let Γ_2' be the rest of the graph Γ_2; it can be connected with the square by bonds in both directions, we denote them in figure 3 by edges with arrows in both directions. Now we delete bonds $\hat{1}, \hat{2}, \hat{3}$ and $\hat{4}$ (see figure 4), there arise "ghost edges" in the square (explicitly shown) and there may arise "ghost edges" from vertices of the square to edges of the rest of the graph (represented by dashed edges between the square and Γ_2'). Since the pseudo orbit can continue from the bond 1 to the bond 4 (with scattering amplitude equal to the scattering amplitude from 1 to $\hat{1}$ with the opposite sign), from bond 2 to 1, etc., one can effectively represent the oriented graph by figure 5.

It is clear that the effective size has been reduced by 2ℓ, because four edges have been deleted. The highest term of the resonance condition corresponds to the contribution of irreducible pseudo orbits on all remaining "non-ghost" bonds (the pseudo orbits may or may not use the "ghost edges"). The contribution of the pseudo orbits (1234) and (12)(34) cancels out, because both pseudo orbits differ only in the number of orbits, hence there is a factor of -1. Similar argument holds also for the pair of pseudo orbits (1432) and (14)(32) and all irreducible pseudo orbits which include these pseudo orbits.

Now we show why also the second highest term is zero. It includes contribution of all bond but one. If the non-included bond is not 1, 2, 3 or 4, the contributions cancel due to the previous argument. If e.g. the bond 4 is not included, it would mean that one must go from one of the lower vertices in the figure 5 to the other through Γ_2'. This is not possible, because the irreducible pseudo orbit has to include all the bonds but 4, none of the bonds in the part Γ_2' is deleted and there is no "ghost edge" ending in the bond 1, 2, 3 or 4. If there exists a path through Γ_2', from one vertex to another, then the path in the opposite direction cannot be covered by the irreducible pseudo orbit.

Therefore, at least 6 directed edges of the former graph Γ_2 are not used and the effective size is reduced by 3ℓ. □

Finally, we state what the positions of the resolvent resonances are.

Theorem 7.3. *Let us assume an equilateral graph (lengths ℓ) with standard coupling. Let the eigenvalues of $Q\tilde{\Sigma}$ be $c_j = r_j \mathrm{e}^{i\varphi_j}$. Then the resolvent resonances are $\lambda = k^2$ with $k = \frac{1}{\ell}(-\varphi_j + 2n\pi + i \ln r_j)$, $n \in \mathbb{Z}$. Moreover, $|c_j| \leq 1$ and for a graph with no edge starting and ending at one vertex also $\sum_{j=1}^{2N} c_j = 0$.*

Proof. The resonance condition is

$$\prod_{j=1}^{2N} (\mathrm{e}^{ik\ell} c_j - 1) = 0\,,$$

hence we have for $k = k_{\mathrm{R}} + ik_{\mathrm{I}}$

$$r_j \mathrm{e}^{-k_{\mathrm{I}}\ell} \mathrm{e}^{ik_{\mathrm{R}}\ell} \mathrm{e}^{i\varphi_j} = 1\,,$$

from which the claim follows. For $r_j > 1$ we would have positive imaginary part of k which would contradict the fact that eigenvalues of the selfadjoint Hamiltonian are real (the corresponding generalized eigenfunction would be square integrable). If the graph does not have any edge starting and ending in one vertex, then there are zeros on the diagonal of $Q\tilde{\Sigma}$, hence its trace (the sum of its eigenvalues) is zero. □

8. Examples

In this section we show two particular examples, which illustrate the general behaviour. In the first example the method of deleting the directed bonds is explained. The second example shows that the symmetry of the graph is not sufficient to obtain the effective size smaller than it is expected from the bound in theorem 7.1.

8.1. Square with the diagonal and all vertices balanced

We assume equilateral graph with four internal edges in the square and fifth edge as diagonal of this square. All four vertices are balanced, i.e. to two of them two halflines are attached and to the other two vertices three halflines are attached. There is standard coupling in all vertices. The oriented graph Γ_2 is shown in figure 6.

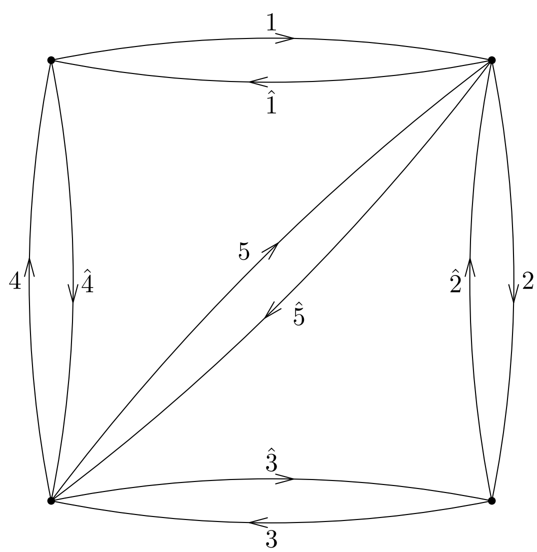

Figure 6. The graph Γ_2 for the graph in example 8.1.

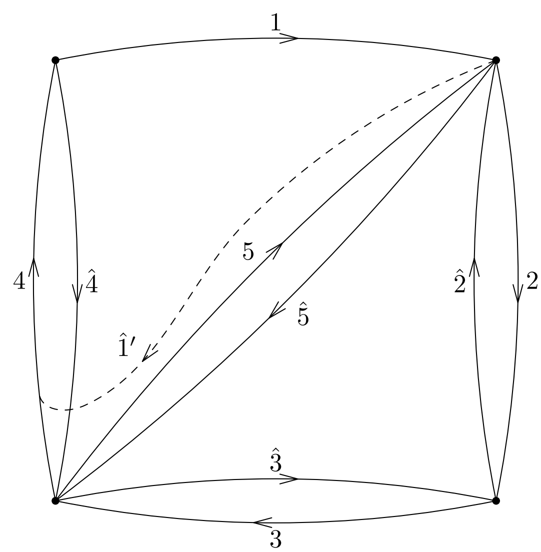

Figure 7. The graph Γ_2 after deleting the bond $\hat{1}$.

Let us denote the vertex from which the edge 1 starts by v_1 and the vertex, where 1 ends by v_2. Then the effective vertex-scattering matrices are

$$\tilde{\sigma}_{v_1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \tilde{\sigma}_{v_2} = \frac{1}{3}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix},$$

similarly for the other two vertices. Hence the matrix $Q\tilde{\Sigma}$ is equal to

	1	2	3	4	5	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$
1	0	1/3	0	0	0	−2/3	0	0	0	1/3
2	0	0	1/2	0	0	0	−1/2	0	0	0
3	0	0	0	1/3	1/3	0	0	−2/3	0	0
4	1/2	0	0	0	0	0	0	0	−1/2	0
5	0	1/3	0	0	0	1/3	0	0	0	−2/3
$\hat{1}$	−1/2	0	0	0	0	0	0	0	1/2	0
$\hat{2}$	0	−2/3	0	0	0	1/3	0	0	0	1/3
$\hat{3}$	0	0	−1/2	0	0	0	1/2	0	0	0
$\hat{4}$	0	0	0	−2/3	1/3	0	0	1/3	0	0
$\hat{5}$	0	0	0	1/3	−2/3	0	0	1/3	0	0

.

The edges, to which the rows and columns correspond, are denoted on the left and on the top of the matrix.

Now we delete the bond $\hat{1}$ (figure 7). Deleting this edge is equivalent to the unitary transformation $V_1 Q\tilde{\Sigma}V_1^{-1}$, where V_1 has 1 on the diagonal, 1 in the sixth row (corresponding to the bond $\hat{1}$) and fourth column (corresponding to the bond 4), other entries of this matrix are zero. Its inverse has 1 on the diagonal and -1 in the sixth row and fourth column. We obtain the matrix

	1	2	3	4	5	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$
1	0	1/3	0	**2/3**	0	−2/3	0	0	0	1/3
2	0	0	1/2	0	0	0	−1/2	0	0	0
3	0	0	0	1/3	1/3	0	0	−2/3	0	0
4	1/2	0	0	0	0	0	0	0	−1/2	0
5	0	1/3	0	**−1/3**	0	1/3	0	0	0	−2/3
$\hat{1}$	0	0	0	0	0	0	0	0	0	0
$\hat{2}$	0	−2/3	0	**−1/3**	0	1/3	0	0	0	1/3
$\hat{3}$	0	0	−1/2	0	0	0	1/2	0	0	0
$\hat{4}$	0	0	0	−2/3	1/3	0	0	1/3	0	0
$\hat{5}$	0	0	0	1/3	−2/3	0	0	1/3	0	0

.

In this matrix the sixth column consist of all zeros and there are three other new entries in the fourth columns which are printed in bold. Since the new entries are in the fourth column, there must be a "ghost edge" $\hat{1}'$ from the vertex v_2 pointing to the bond 4. We can use it in pseudo orbits containing one of the bonds 1, 5, $\hat{2}$ continuing then to the bond 4 with the scattering amplitudes in bold above.

Now we delete the bond $\hat{2}$ (figure 8). Since two other bonds end in the vertex v_2 (bonds 1 and 5), there will be two "ghost edges". We will use the unitary transformation $V_2 V_1 Q\tilde{\Sigma}V_1^{-1}V_2^{-1}$. V_2 has 1 on the diagonal and 1 in the seventh row (corresponding to the edge $\hat{2}$) and columns 1 and 5. Its inverse has nondiagonal terms with opposite signs.

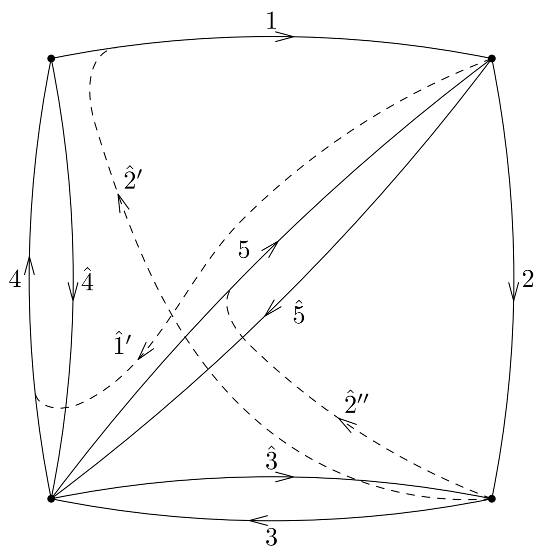

Figure 8. The graph Γ_2 after deleting the bonds $\hat{1}$ and $\hat{2}$.

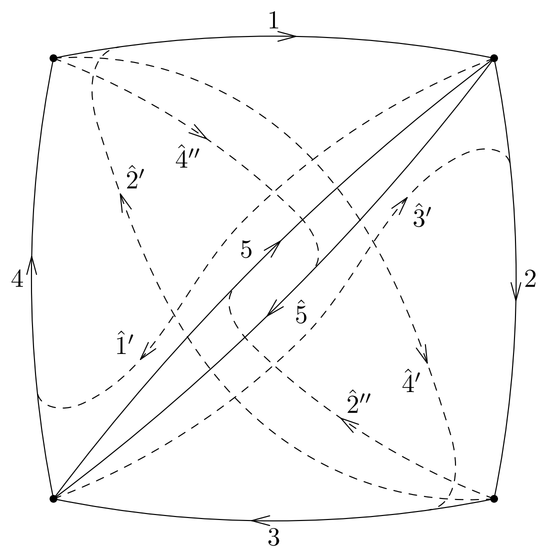

Figure 9. The graph Γ_2 after deleting the bonds $\hat{1}$, $\hat{2}$, $\hat{3}$ and $\hat{4}$.

After the transformation we obtain

	1	2	3	4	5	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$
1	0	1/3	0	2/3	0	−2/3	0	0	0	1/3
2	**1/2**	0	1/2	0	**1/2**	0	−1/2	0	0	0
3	0	0	0	1/3	1/3	0	0	−2/3	0	0
4	1/2	0	0	0	0	0	0	0	−1/2	0
5	0	1/3	0	−1/3	0	1/3	0	0	0	−2/3
$\hat{1}$	0	0	0	0	0	0	0	0	0	0
$\hat{2}$	0	0	0	0	0	0	0	0	0	0
$\hat{3}$	**−1/2**	0	−1/2	0	**−1/2**	0	1/2	0	0	0
$\hat{4}$	0	0	0	−2/3	1/3	0	0	1/3	0	0
$\hat{5}$	0	0	0	1/3	−2/3	0	0	1/3	0	0

.

The seventh row (corresponding to $\hat{2}$) has all entries equal to zero; other new entries are printed in bold. These entries correspond to the new "ghost edges" $\hat{2}'$ and $\hat{2}''$. Similarly, we delete edges $\hat{3}$ and $\hat{4}$ (figure 9); the matrix $Q\tilde{\Sigma}$ after these transformations is

	1	2	3	4	5	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$
1	0	1/3	0	2/3	0	−2/3	0	0	0	1/3
2	1/2	0	1/2	0	1/2	0	−1/2	0	0	0
3	0	2/3	0	1/3	1/3	0	0	−2/3	0	0
4	1/2	0	1/2	0	0	0	0	0	−1/2	1/2
5	0	1/3	0	−1/3	0	1/3	0	0	0	−2/3 .
$\hat{1}$	0	0	0	0	0	0	0	0	0	0
$\hat{2}$	0	0	0	0	0	0	0	0	0	0
$\hat{3}$	0	0	0	0	0	0	0	0	0	0
$\hat{4}$	0	0	0	0	0	0	0	0	0	0
$\hat{5}$	0	−1/3	0	1/3	−2/3	0	0	1/3	0	0

The eigenvalues of the matrix $Q\tilde{\Sigma}$ are $-2/3$, $-1/3$, -1, 1 with multiplicity 2, 0 with multiplicity 5. There is one Jordan block $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the Jordan form of $Q\tilde{\Sigma}$. The resonance condition can be obtained from the eigenvalues by the equation at the beginning of the proof of the theorem 7.3

$$1 - \frac{16}{9}\mathrm{e}^{2ik\ell} - \frac{2}{9}\mathrm{e}^{3ik\ell} + \frac{7}{9}\mathrm{e}^{4ik\ell} + \frac{2}{9}\mathrm{e}^{5ik\ell} = 0\,.$$

By the theorem 7.3 we obtain that the positions of the resonances are such $\lambda = k^2$ with $k = \frac{1}{\ell}[(2n+1)\pi - i\ln 3]$, $k = \frac{1}{\ell}[(2n+1)\pi - i\ln\frac{3}{2}]$, $k = \frac{1}{\ell}(2n+1)\pi$ and $k = \frac{1}{\ell}2n\pi$ with multiplicity 2, $n \in \mathbb{Z}$.

8.2. Fully connected graph on four vertices with all vertices balanced

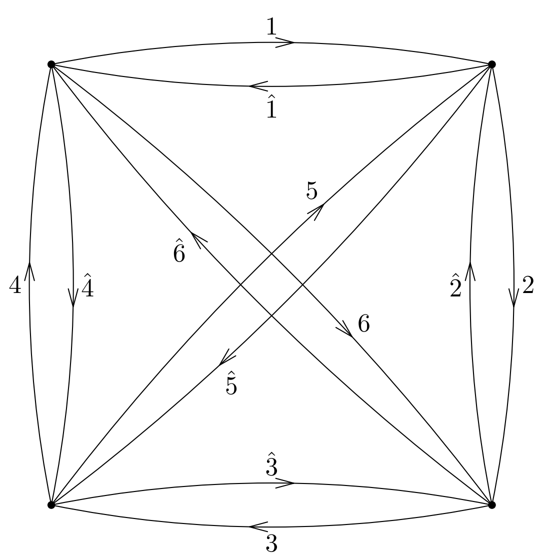

Figure 10. Graph Γ_2 for fully connected graph on four vertices.

In this subsection we assume a graph on four vertices, every two vertices are connected by one edge of length ℓ and there are three halflines attached at each vertex, hence each vertex is balanced. The directed graph Γ_2 is shown in figure 10. For each vertex we have the effective vertex-scattering matrix

$$\tilde{\sigma}_v = \frac{1}{3}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

The matrix $Q\tilde{\Sigma}$ is

	1	2	3	4	5	6	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$
1	0	1/3	0	0	0	0	−2/3	0	0	0	1/3	0
2	0	0	1/3	0	0	0	0	−2/3	0	0	0	1/3
3	0	0	0	1/3	1/3	0	0	0	−2/3	0	0	0
4	1/3	0	0	0	0	1/3	0	0	0	−2/3	0	0
5	0	1/3	0	0	0	0	1/3	0	0	0	−2/3	0
6	0	0	1/3	0	0	0	0	1/3	0	0	0	−2/3
$\hat{1}$	−2/3	0	0	0	0	1/3	0	0	0	1/3	0	0
$\hat{2}$	0	−2/3	0	0	0	0	1/3	0	0	0	1/3	0
$\hat{3}$	0	0	−2/3	0	0	0	0	1/3	0	0	0	1/3
$\hat{4}$	0	0	0	−2/3	1/3	0	0	0	1/3	0	0	0
$\hat{5}$	0	0	0	1/3	−2/3	0	0	0	1/3	0	0	0
$\hat{6}$	1/3	0	0	0	0	−2/3	0	0	0	1/3	0	0

Its eigenvalues are -1 with multiplicity 2, 1 with multiplicity 3, $-1/3$ with multiplicity 3 and 0 with multiplicity 4. There is no Jordan block. The resonance condition is

$$1 - \frac{8}{3}\mathrm{e}^{2ik\ell} - \frac{8}{27}\mathrm{e}^{3ik\ell} + \frac{62}{27}\mathrm{e}^{4ik\ell} + \frac{16}{27}\mathrm{e}^{5ik\ell} - \frac{16}{27}\mathrm{e}^{6ik\ell} - \frac{8}{27}\mathrm{e}^{7ik\ell} - \frac{1}{27}\mathrm{e}^{8ik\ell} = 0\,.$$

Similarly to the previous example we can find the positions of the resolvent resonances $\lambda = k^2$ with $k = \frac{1}{\ell}(2n+1)\pi$ with multiplicity 2, $k = \frac{1}{\ell}2n\pi$ with multiplicity 3 and $k = \frac{1}{\ell}[(2n+1)\pi - i\ln 3]$ with multiplicity 3, $n \in \mathbb{Z}$.

This example shows that the symmetry of the graph does not assure that it will have the effective size smaller than the bound in theorem 7.1. This graph has four 0 as eigenvalues of $Q\tilde{\Sigma}$, hence the effective size is 4ℓ, as we expect from four balanced vertices. Although this graph is very symmetric, we do not have smaller effective size in contrary to the example in theorem 7.3 in [DEL10].

Acknowledgements

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