ON THE EXISTENCE OF STATIONARY SOLUTIONS FOR SOME SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH ANOMALOUS DIFFUSION

Vitali Vougalter[1], **Vitaly Volpert**[2]

[1] Department of Mathematics, University of Toronto
Toronto, Ontario, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

[2] Institute Camille Jordan, UMR 5208 CNRS, University Lyon 1
Villeurbanne, 69622, France
e-mail: volpert@math.univ-lyon1.fr

Abstract: The article is devoted to the proof of the existence of solutions of a system of integro-differential equations appearing in population dynamics in the case of anomalous diffusion when the negative Laplacian is raised to some fractional power. The argument relies on a fixed point technique. Solvability conditions for elliptic operators without Fredholm property in unbounded domains along with the Sobolev inequality for a fractional Laplace operator are being used.

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1. Introduction

In the present article we study the existence of stationary solutions of the following system of $N \geq 2$ integro-differential equations

$$\frac{\partial u_m}{\partial t} = -D_m(-\Delta)^s u_m + \int_{\mathbb{R}^3} K_m(x-y) g_m(u(y,t))dy + f_m(x), \qquad (1.1)$$

with $1 \leq m \leq N$ and $\frac{1}{4} < s < \frac{3}{4}$, which appears in cell population dynamics. The space variable x corresponds to the cell genotype, $u_s(x,t)$ denote the cell densities for various groups of cells as functions of their genotype and time, such that

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The right side of this system of equations describes the evolution of cell densities due to cell proliferation, mutations and cell influx. In this context the anomalous diffusion terms correspond to the change of genotype via small random mutations, and the nonlocal terms describe large mutations. Here $g_m(u)$ denote the rates of cell birth which depend on u (density dependent proliferation), and the functions $K_m(x-y)$ express the proportions of newly born cells which change their genotype from y to x. We assume here that they dependent on the distance between the genotypes. Finally, the last term in the right side of (1.1) denotes the influxes of cells for different genotypes.

The operator $(-\Delta)^s$, $\frac{1}{4} < s < \frac{3}{4}$ in system (1.1) represents a particular case of the anomalous diffusion actively studied in relation with various applications in plasma physics and turbulence [13], [14], surface diffusion [15], [16], semiconductors [17] and so on. The physical meaning of the anomalous diffusion is that the random process occurs with longer jumps in comparison with normal diffusion. The moments of jump length distribution are finite in the case of normal diffusion, but this is not the case for the anomalous diffusion. The operator $(-\Delta)^s$, $\frac{1}{4} < s < \frac{3}{4}$ is defined by virtue of the spectral calculus. A similar problem in the case of the standard Laplacian in the diffusion term was treated recently in [28].

We set all $D_m = 1$ and prove the existence of solutions of the system

$$-(-\Delta)^s u_m + \int_{\mathbb{R}^3} K_m(x-y) g_m(u(y)) dy + f_m(x) = 0, \quad \frac{1}{4} < s < \frac{3}{4}, \tag{1.2}$$

with $1 \le m \le N$. Let us consider the case when the linear part of this operator fails to satisfy the Fredholm property. Hence, conventional methods of nonlinear analysis may not be applicable. We use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

with $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ tends to 0 at infinity. For $a \ge 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (1.3) contains the origin. Consequently, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article is devoted to the studies of some properties of such operators. Note that elliptic equations with non Fredholm operators were treated actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The non Fredholm Schrödinger type operators were studied with the methods of the spectral and the scattering theory in [18], [21], [23]. The Laplacian with drift from the point of view of non Fredholm operators was treated in [22] and linearized Cahn-Hilliard equations in [24] and

[26]. Nonlinear non Fredholm elliptic problems were studied in [25] and [27]. Important applications to the theory of reaction-diffusion equations were developed in [8], [9]. Non Fredholm operators arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is significantly different and the approach developed in these articles cannot be appplied. Front propagation problems with anomalous diffusion were treated actively in recent years (see e.g. [30], [31]).

Let us set $K_m(x) = \varepsilon_m \mathcal{K}_m(x)$ for $1 \leq m \leq N$ with $\varepsilon_m \geq 0$, such that

$$\varepsilon := max_{1 \leq m \leq N} \varepsilon_m$$

and suppose that the following assumption is satisfied.

Assumption 1. *Let* $1 \leq m \leq N$ *and consider* $\frac{1}{4} < s < \frac{3}{4}$. *Let* $f_m(x) : \mathbb{R}^3 \to \mathbb{R}$, *such that* $f_m(x) \in L^1(\mathbb{R}^3)$ *and* $(-\Delta)^{1-s} f_m(x) \in L^2(\mathbb{R}^3)$. *For some* $1 \leq m \leq N$, $f_m(x)$ *is nontrivial. Assume also that* $\mathcal{K}_m(x) : \mathbb{R}^3 \to \mathbb{R}$, *such that* $\mathcal{K}_m(x) \in L^1(\mathbb{R}^3)$ *and* $(-\Delta)^{1-s} \mathcal{K}_m(x) \in L^2(\mathbb{R}^3)$. *Furthermore,*

$$\mathcal{K}^2 := \sum_{m=1}^{N} \|\mathcal{K}_m(x)\|^2_{L^1(\mathbb{R}^3)} > 0$$

and

$$Q^2 := \sum_{m=1}^{N} \|(-\Delta)^{1-s} \mathcal{K}_m(x)\|^2_{L^2(\mathbb{R}^3)} > 0.$$

We choose the space dimension $d = 3$, which is related to the solvability conditions for the linear Poisson type equation (3.24) established in Lemma 5 below. From the point of view of applications, the space dimension is not restricted to $d = 3$ because the space variable corresponds to the cell genotype but not to the usual physical space.

Let us use the Sobolev inequality for the fractional Laplace operator (see e.g. Lemma 2.2 of [10], also [11])

$$\|f_m(x)\|_{L^{\frac{6}{4s-1}}(\mathbb{R}^3)} \leq c_s \|(-\Delta)^{1-s} f_m(x)\|_{L^2(\mathbb{R}^3)}, \quad \frac{1}{4} < 1 - s < \frac{3}{4} \tag{1.4}$$

along with the assumption above and the standard interpolation argument. Hence,

$$f_m(x) \in L^2(\mathbb{R}^3), \quad 1 \leq m \leq N.$$

We use the Sobolev spaces

$$H^{2s}(\mathbb{R}^3) := \{\phi(x) : \mathbb{R}^3 \to \mathbb{R} \mid \phi(x) \in L^2(\mathbb{R}^3), \ (-\Delta)^s \phi \in L^2(\mathbb{R}^3)\}, \quad 0 < s \leq 1$$

equipped with the norm

$$\|\phi\|^2_{H^{2s}(\mathbb{R}^3)} := \|\phi\|^2_{L^2(\mathbb{R}^3)} + \|(-\Delta)^s\phi\|^2_{L^2(\mathbb{R}^3)}. \tag{1.5}$$

For a vector function $u(x) = (u_1(x), u_2(x), ..., u_N(x))^T$ we use the norms

$$\|u\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)} := \sum_{m=1}^{N} \|u_m\|^2_{H^2(\mathbb{R}^3)} = \sum_{m=1}^{N}\{\|u_m\|^2_{L^2(\mathbb{R}^3)} + \|\Delta u_m\|^2_{L^2(\mathbb{R}^3)}\}, \tag{1.6}$$

$$\|u\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)} := \sum_{m=1}^{N} \|u_m\|^2_{L^2(\mathbb{R}^3)}.$$

The standard Sobolev embedding tells that

$$\|\phi\|_{L^\infty(\mathbb{R}^3)} \le c_e\|\phi\|_{H^2(\mathbb{R}^3)}, \tag{1.7}$$

where $c_e > 0$ is the constant of the embedding. When all the nonnegative parameters ε_m vanish, we arrive at the linear Poisson type equations

$$(-\Delta)^s u_m = f_m(x), \quad 1 \le m \le N. \tag{1.8}$$

By means of part 1) of Lemma 5 below along with Assumption 1 each equation (1.8) admits a unique solution

$$u_{0,m}(x) \in H^{2s}(\mathbb{R}^3), \quad \frac{1}{4} < s < \frac{3}{4},$$

and no orthogonality relations are required. By virtue of part 2) of Lemma 5, for $\frac{3}{4} \le s < 1$, a specific orthogonality condition (3.28) is needed to be able to solve problem (1.8) in $H^{2s}(\mathbb{R}^3)$. On the other hand, one needs $s > \dfrac{1}{4}$ to be able to use the Sobolev type inequality (1.4). By virtue of Assumption 1, using that

$$-\Delta u_m(x) = (-\Delta)^{1-s} f_m(x) \in L^2(\mathbb{R}^3),$$

we obtain for the unique solution of the linear problem (1.8) that $u_{0,m}(x) \in H^2(\mathbb{R}^3)$. Therefore,

$$u_0(x) := (u_{0,1}(x), u_{0,2}(x), ..., u_{0,N}(x))^T \in H^2(\mathbb{R}^3, \mathbb{R}^N).$$

Let us look for the resulting solution of the nonlinear system (1.2) as

$$u(x) = u_0(x) + u_p(x), \tag{1.9}$$

where

$$u_p(x) := (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T.$$

Evidently, we derive the perturbative system of equations

$$(-\Delta)^s u_{p,m}(x) = \varepsilon_m \int_{\mathbb{R}^3} \mathcal{K}_m(x-y) g_m(u_0(y) + u_p(y)) dy, \quad \frac{1}{4} < s < \frac{3}{4}, \quad (1.10)$$

where $1 \leq m \leq N$. Let us introduce a closed ball in the Sobolev space of vector functions

$$B_\rho := \{u(x) \in H^2(\mathbb{R}^3, \mathbb{R}^N) \mid \|u\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \leq \rho\}, \quad 0 < \rho \leq 1. \quad (1.11)$$

We look for the solution of system (1.10) as the fixed point of the auxiliary nonlinear problem

$$(-\Delta)^s u_m(x) = \varepsilon_m \int_{\mathbb{R}^3} \mathcal{K}_m(x-y) g_m(u_0(y) + v(y)) dy, \quad \frac{1}{4} < s < \frac{3}{4}, \quad (1.12)$$

with $1 \leq m \leq N$ in ball (1.11). For a given vector function $v(y)$ this is a system of equations with respect to $u(x)$. The left side of (1.12) involves the non Fredholm operator $(-\Delta)^s : H^{2s}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Thus, this operator has no bounded inverse. The similar situation appeared in works [25] and [27] but as distinct from the present article, the equations treated there required orthogonality conditions. The fixed point technique was involved in [20] to evaluate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in such nonlinear problem possessed the Fredholm property (see Assumption 1 of [20], also [7]). We define the closed ball in the space of N dimensions as

$$I := \{z \in \mathbb{R}^N \mid |z| \leq c_e \|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + c_e\}. \quad (1.13)$$

Let us introduce the following quantities for $1 \leq m, j \leq N$ as

$$a_{2,m,j} := sup_{z \in I} \left| \nabla \frac{\partial g_m}{\partial z_j} \right|, \quad a_{2,m} := \sqrt{\sum_{j=1}^N a_{2,m,j}^2}, \quad a_2 := max_{1 \leq m \leq N} a_{2,m}$$

and

$$a_{1,m} := sup_{z \in I} |\nabla g_m(z)|, \quad a_1 := max_{1 \leq m \leq N} a_{1,m}.$$

We make the following assumption on the nonlinear part of system (1.2).

Assumption 2. *Let $1 \leq m \leq N$. Assume that $g_m(z) : \mathbb{R}^N \to \mathbb{R}$, such that $g_m(z) \in C_2(\mathbb{R}^N)$ with $g_m(0) = 0$ and $\nabla g_m(0) = 0$. Let $a_2 > 0$.*

Obviously, $a_1 > 0$ as well, otherwise the functions $g_m(z)$ will be constants in the ball I, such that a_2 will vanish. For instance, $g_m(z) = z^2$, $z \in \mathbb{R}^N$ clearly satisfy this assumption above.

Let us introduce the operator T_g, such that $u = T_g v$, where u is a solution of system (1.12). Our main statement is as follows.

Theorem 3. *Let Assumptions 1 and 2 be fulfilled. Then system (1.12) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon*$ for a certain $\varepsilon* > 0$. The unique fixed point $u_p(x)$ of this map T_g is the only solution of system (1.10) in B_ρ.*

Apparently, the resulting solution of system (1.2) given by (1.9) will be nontrivial because for some $1 \le m \le N$ the source term $f_m(x)$ is nontrivial and all $g_m(0) = 0$ as assumed. We will make use of the following elementary lemma.

Lemma 4. *For $R \in (0, +\infty)$ consider the function*

$$\varphi(R) := \alpha R^{3-4s} + \frac{\beta}{R^{4s}}, \quad \frac{1}{4} < s < \frac{3}{4}, \quad \alpha, \beta > 0.$$

It attains the minimal value at $R^* = \left(\frac{4\beta s}{\alpha(3-4s)}\right)^{\frac{1}{3}}$, which is given by

$$\varphi(R^*) = 3(3-4s)^{\frac{4s}{3}-1}(4s)^{-\frac{4s}{3}}\alpha^{\frac{4s}{3}}\beta^{1-\frac{4s}{3}}.$$

We proceed to the proof of our main proposition.

2. The existence of the perturbed solution

Proof of Theorem 3. Let us choose an arbitrary $v(x) \in B_\rho$ and denote the terms involved in the integral expression in the right side of system (1.12) as

$$G_m(x) := g_m(u_0(x) + v(x)), \quad 1 \le m \le N.$$

We apply the standard Fourier transform (3.25) to both sides of system (1.12), which gives us

$$\widehat{u}_m(p) = \varepsilon_m (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}_m(p)\widehat{G}_m(p)}{|p|^{2s}}, \quad 1 \le m \le N.$$

Hence for the norm we obtain

$$\|u_m\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^3 \varepsilon_m^2 \int_{\mathbb{R}^3} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G}_m(p)|^2}{|p|^{4s}} dp. \tag{2.14}$$

As distinct from works [25] and [27] containing the standard Laplace operator in the diffusion term, here we do not try to control the norms

$$\left\| \frac{\widehat{\mathcal{K}}_m(p)}{|p|^{2s}} \right\|_{L^\infty(\mathbb{R}^3)}.$$

Let us estimate the right side of (2.14) applying inequality (3.26) with $R \in (0, +\infty)$ as

$$(2\pi)^3 \varepsilon_m^2 \int_{|p|\le R} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G}_m(p)|^2}{|p|^{4s}} dp + (2\pi)^3 \varepsilon_m^2 \int_{|p|>R} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G}_m(p)|^2}{|p|^{4s}} dp \le$$

$$\le \varepsilon_m^2 \|\mathcal{K}_m\|_{L^1(\mathbb{R}^3)}^2 \left\{ \frac{1}{2\pi^2} \|G_m(x)\|_{L^1(\mathbb{R}^3)}^2 \frac{R^{3-4s}}{3-4s} + \frac{1}{R^{4s}} \|G_m(x)\|_{L^2(\mathbb{R}^3)}^2 \right\}. \quad (2.15)$$

Due to the fact that $v(x) \in B_\rho$, we get

$$\|u_0 + v\|_{L^2(\mathbb{R}^3, \mathbb{R}^N)} \le \|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + 1.$$

The Sobolev embedding (1.7) yields

$$|u_0 + v| \le c_e \|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + c_e.$$

By virtue of the representation

$$G_m(x) = \int_0^1 \nabla g_m(t(u_0(x) + v(x))).(u_0(x) + v(x))dt, \quad 1 \le m \le N,$$

where the dot denotes the scalar product of two vectors in $\mathbb{R}^N$, we arrive at

$$|G_m(x)| \le sup_{z \in I} |\nabla g_m(z)||u_0(x) + v(x)| \le a_1 |u_0(x) + v(x)|,$$

with the ball I defined in (1.13). Thus

$$\|G_m(x)\|_{L^2(\mathbb{R}^3)} \le a_1 \|u_0 + v\|_{L^2(\mathbb{R}^3, \mathbb{R}^N)} \le a_1(\|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + 1).$$

Apparently, for $t \in [0, 1]$ and $1 \le j \le N$, we have

$$\frac{\partial g_m}{\partial z_j}(t(u_0(x) + v(x))) = \int_0^t \nabla \frac{\partial g_m}{\partial z_j}(\tau(u_0(x) + v(x))).(u_0(x) + v(x))d\tau.$$

This implies

$$\left| \frac{\partial g_m}{\partial z_j}(t(u_0(x) + v(x))) \right| \le sup_{z \in I} \left| \nabla \frac{\partial g_m}{\partial z_j} \right| |u_0(x) + v(x)| = a_{2,m,j} |u_0(x) + v(x)|.$$

We use the Schwarz inequality to estimate

$$|G_m(x)| \le |u_0(x) + v(x)| \sum_{j=1}^N a_{2,m,j} |u_{0,j}(x) + v_j(x)| \le a_2 |u_0(x) + v(x)|^2,$$

such that for $1 \le m \le N$

$$\|G_m(x)\|_{L^1(\mathbb{R}^3)} \le a_2 \|u_0 + v\|^2_{L^2(\mathbb{R}^3, \mathbb{R}^N)} \le a_2(\|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + 1)^2. \tag{2.16}$$

Thus we obtain the upper bound for the right side of (2.15) as

$$\varepsilon_m{}^2 \|\mathcal{K}_m\|^2_{L^1(\mathbb{R}^3)} (\|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + 1)^2 \left\{ \frac{a_2^2(\|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + 1)^2}{2\pi^2(3-4s)} R^{3-4s} + \frac{a_1^2}{R^{4s}} \right\},$$

with $R \in (0, +\infty)$. By virtue of Lemma 4, we obtain the minimal value of the expression above. Thus,

$$\|u\|^2_{L^2(\mathbb{R}^3, \mathbb{R}^N)} \le \varepsilon^2 \mathcal{K}^2 (\|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + 1)^{2 + \frac{8s}{3}} \frac{3 a_2^{\frac{8s}{3}} a_1^{2 - \frac{8s}{3}}}{(3-4s) s^{\frac{4s}{3}} \pi^{\frac{8s}{3}} 2^{4s}}. \tag{2.17}$$

Evidently, (1.12) gives us for $1 \le m \le N$

$$-\Delta u_m(x) = \varepsilon_m (-\Delta)^{1-s} \int_{\mathbb{R}^3} \mathcal{K}_m(x-y) G_m(y) dy.$$

By means of (3.26) along with (2.16)

$$\|\Delta u_m\|^2_{L^2(\mathbb{R}^3)} \le \varepsilon^2 \|G_m\|^2_{L^1(\mathbb{R}^3)} \|(-\Delta)^{1-s} \mathcal{K}_m\|^2_{L^2(\mathbb{R}^3)} \le$$

$$\le \varepsilon^2 a_2^2 (\|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + 1)^4 \|(-\Delta)^{1-s} \mathcal{K}_m\|^2_{L^2(\mathbb{R}^3)},$$

such that

$$\sum_{m=1}^N \|\Delta u_m\|^2_{L^2(\mathbb{R}^3)} \le \varepsilon^2 a_2^2 (\|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + 1)^4 Q^2. \tag{2.18}$$

Hence, by virtue of the definition of the norm (1.6) along with inequalities (2.17) and (2.18), we obtain the estimate from above for $\|u\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)}$ as

$$\varepsilon (\|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + 1)^2 a_2 \times$$

$$\times \left[\mathcal{K}^2 \left(\frac{a_2(\|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + 1)}{a_1} \right)^{\frac{8s}{3} - 2} \frac{3}{(3-4s) s^{\frac{4s}{3}} \pi^{\frac{8s}{3}} 2^{4s}} + Q^2 \right]^{\frac{1}{2}} \le \rho$$

for all $\varepsilon > 0$ small enough. Hence, $u(x) \in B_\rho$ as well. If for some $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of system (1.12), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3, \mathbb{R}^N)$ satisfies

$$(-\Delta)^s w_m = 0, \quad 1 \le m \le N.$$

Since the operator $(-\Delta)^s$ considered in $\mathbb{R}^3$ does not have nontrivial square integrable zero modes, $w(x) = 0$ a.e. in the whole space. Therefore, system (1.12) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ sufficiently small.

Our goal is to prove that such map is a strict contraction. We choose arbitrarily $v_{1,2}(x) \in B_\rho$. By means of the argument above $u_{1,2} = T_g v_{1,2} \in B_\rho$ as well. By virtue of system (1.12), we have

$$(-\Delta)^s u_{1,m}(x) = \varepsilon_m \int_{\mathbb{R}^3} \mathcal{K}_m(x-y) g_m(u_0(y) + v_1(y)) dy, \quad 1 \le m \le N, \tag{2.19}$$

$$(-\Delta)^s u_{2,m}(x) = \varepsilon_m \int_{\mathbb{R}^3} \mathcal{K}_m(x-y) g_m(u_0(y) + v_2(y)) dy, \quad 1 \le m \le N, \tag{2.20}$$

with $\frac{1}{4} < s < \frac{3}{4}$. Let us denote

$$G_{1,m}(x) := g_m(u_0(x) + v_1(x)), \quad G_{2,m}(x) := g_m(u_0(x) + v_2(x)), \quad 1 \le m \le N$$

and apply the standard Fourier transform (3.25) to both sides of systems (2.19) and (2.20). This gives us

$$\widehat{u_{1,m}}(p) = \varepsilon_m (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}_m(p) \widehat{G_{1,m}}(p)}{|p|^{2s}}, \quad \widehat{u_{2,m}}(p) = \varepsilon_m (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}_m(p) \widehat{G_{2,m}}(p)}{|p|^{2s}}.$$

Clearly,

$$\|u_{1,m} - u_{2,m}\|^2_{L^2(\mathbb{R}^3)} = \varepsilon_m^2 (2\pi)^3 \int_{\mathbb{R}^3} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)|^2}{|p|^{4s}} dp.$$

Apparently, such expression can be bounded above using (3.26) by $\varepsilon_m^2 \|\mathcal{K}_m\|^2_{L^1(\mathbb{R}^3)} \times$

$$\times \left\{ \frac{\|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^1(\mathbb{R}^3)}}{2\pi^2} \frac{R^{3-4s}}{3-4s} + \frac{\|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^2(\mathbb{R}^3)}}{R^{4s}} \right\},$$

with $R \in (0, +\infty)$. For $t \in [0,1]$ and $1 \le m \le N$, let us use the representation

$$G_{1,m}(x) - G_{2,m}(x) = \int_0^1 \nabla g_m(u_0(x) + t v_1(x) + (1-t) v_2(x)).(v_1(x) - v_2(x)) dt.$$

Since $\|v_2(x) + t(v_1(x) - v_2(x))\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} \le$

$$\le t \|v_1(x)\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + (1-t) \|v_2(x)\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} \le \rho,$$

we have $v_2(x) + t(v_1(x) - v_2(x)) \in B_\rho$. We estimate

$$|G_{1,m}(x) - G_{2,m}(x)| \le sup_{z \in I} |\nabla g_m(z)| |v_1(x) - v_2(x)| = a_{1,m} |v_1(x) - v_2(x)|.$$

Hence

$$\|G_{1,m}(x) - G_{2,m}(x)\|_{L^2(\mathbb{R}^3)} \le a_{1,m} \|v_1 - v_2\|_{L^2(\mathbb{R}^3, \mathbb{R}^N)} \le a_{1,m} \|v_1 - v_2\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)}.$$

Evidently, for $1 \le m, j \le N$ we have $\dfrac{\partial g_m}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x)) =$

$$= \int_0^1 \nabla \frac{\partial g_m}{\partial z_j}(\tau[u_0(x) + tv_1(x) + (1-t)v_2(x)]).[u_0(x) + tv_1(x) + (1-t)v_2(x)]d\tau,$$

such that $\left| \dfrac{\partial g_m}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x)) \right| \le$

$$\le sup_{z\in I} \left| \nabla \frac{\partial g_m}{\partial z_j} \right| (|u_0(x)| + t|v_1(x)| + (1-t)|v_2(x)|), \quad t \in [0,1].$$

By means of the Schwarz inequality we obtain the upper bound for $|G_{1,m}(x) - G_{2,m}(x)|$ as

$$\sum_{j=1}^N a_{2,m,j} |v_{1,j}(x) - v_{2,j}(x)| \Big(|u_0(x)| + \frac{1}{2}|v_1(x)| + \frac{1}{2}|v_2(x)| \Big) \le$$

$$\le a_{2,m} |v_1(x) - v_2(x)| \Big(|u_0(x)| + \frac{1}{2}|v_1(x)| + \frac{1}{2}|v_2(x)| \Big).$$

By virtue of the Schwarz inequality the norm $\|G_{1,m}(x) - G_{2,m}(x)\|_{L^1(\mathbb{R}^3)} \le$

$$\le a_{2,m} \|v_1 - v_2\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)} \Big(\|u_0\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)} + \frac{1}{2}\|v_1\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)} + \frac{1}{2}\|v_2\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)} \Big) \le$$

$$\le a_2 \|v_1 - v_2\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1). \tag{2.21}$$

Hence we obtain the upper bound for the norm $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)}$ given by

$$\varepsilon^2 \mathcal{K}^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \Big\{ \frac{{a_2}^2}{2\pi^2} (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^2 \frac{R^{3-4s}}{3-4s} + \frac{{a_1}^2}{R^{4s}} \Big\}.$$

Lemma 4 gives us the minimum of the expression above over $R \in (0, +\infty)$. Thus we arrive at the estimate from above for $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)}$ as

$$\varepsilon^2 \mathcal{K}^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \frac{3a_1^{2-\frac{8s}{3}}}{(3-4s)2^{4s}s^{\frac{4s}{3}}} \left[\frac{a_2(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)}{\pi} \right]^{\frac{8s}{3}}. \tag{2.22}$$

Formulas (2.19) and (2.20) yield

$$(-\Delta)(u_{1,m}(x) - u_{2,m}(x)) = \varepsilon_m (-\Delta)^{1-s} \int_{\mathbb{R}^3} \mathcal{K}_m(x-y)[G_{1,m}(y) - G_{2,m}(y)]dy.$$

By virtue of inequalities (3.26) and (2.21) we derive

$$\|\Delta(u_{1,m} - u_{2,m})\|^2_{L^2(\mathbb{R}^3)} \le \varepsilon^2 \|(-\Delta)^{1-s}\mathcal{K}_m\|^2_{L^2(\mathbb{R}^3)} \|G_{1,m} - G_{2,m}\|^2_{L^1(\mathbb{R}^3)} \le$$

$$\leq \varepsilon^2 a_2^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)} (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^2 \|(-\Delta)^{1-s}\mathcal{K}_m\|^2_{L^2(\mathbb{R}^3)}.$$

Therefore, $\sum_{m=1}^N \|\Delta(u_{1,m} - u_{2,m})\|^2_{L^2(\mathbb{R}^3)} \leq$

$$\leq \varepsilon^2 a_2^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)} (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^2 Q^2. \tag{2.23}$$

Inequalities (2.22) and (2.23) yield $\|u_1 - u_2\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \leq \varepsilon a_2 (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1) \times$

$$\times \left\{ \frac{3\mathcal{K}^2}{(3-4s)2^{4s} s^{\frac{4s}{3}} \pi^{\frac{8s}{3}}} \left[\frac{a_2(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)}{a_1} \right]^{\frac{8s}{3}-2} + Q^2 \right\}^{\frac{1}{2}} \|v_1 - v_2\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}.$$

Thus, the map $T_g : B_\rho \to B_\rho$ defined by system (1.12) is a strict contraction for all values of $\varepsilon > 0$ sufficiently small. Its unique fixed point $u_p(x)$ is the only solution of system (1.10) in the ball B_ρ. The resulting $u(x) \in H^2(\mathbb{R}^3, \mathbb{R}^N)$ given by (1.9) is a solution of system (1.2). ■

3. Auxiliary results

We recall the solvability conditions for the linear Poisson type equation with a square integrable right side

$$(-\Delta)^s \phi = f(x), \quad x \in \mathbb{R}^3, \quad 0 < s < 1, \tag{3.24}$$

easily obtained in [29] by using the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} f(x) e^{-ipx} dx. \tag{3.25}$$

Evidently, we have the estimate for it as

$$\|\widehat{f}(p)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \|f(x)\|_{L^1(\mathbb{R}^3)}. \tag{3.26}$$

Let us denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R}^3)} := \int_{\mathbb{R}^3} f(x) \bar{g}(x) dx, \tag{3.27}$$

with a slight abuse of notations when the functions involved in (3.27) fail to be square integrable, like for instance the ones involved in orthogonality relation (3.28) below. Indeed, if $f(x) \in L^1(\mathbb{R}^3)$ and $g(x) \in L^\infty(\mathbb{R}^3)$, then the integral in the right side of (3.27) makes sense. The technical result easily derived in [29] by virtue of (3.25) is formulated as follows.

Lemma 5. *Let* $f(x) \in L^2(\mathbb{R}^3)$.

1) When $0 < s < \frac{3}{4}$ and additionally $f(x) \in L^1(\mathbb{R}^3)$, problem (3.24) has a unique solution $\phi(x) \in H^{2s}(\mathbb{R}^3)$.

2) When $\frac{3}{4} \leq s < 1$ and in addition $|x|f(x) \in L^1(\mathbb{R}^3)$, equation (3.24) admits a unique solution $\phi(x) \in H^{2s}(\mathbb{R}^3)$ if and only if the orthogonality condition

$$(f(x), 1)_{L^2(\mathbb{R}^3)} = 0 \tag{3.28}$$

holds.

Note that for the lower values of the power of the negative Laplacian $0 < s < \frac{3}{4}$ under the assumptions stated above no orthogonality relations are needed to solve the linear Poisson type equation (3.24) in $H^{2s}(\mathbb{R}^3)$.

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