A Note on the Definitions of Discrete Symmetries Operators

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Abstract. On the basis of the Silagadze research[1], we investigate the question of the definitions of the discrete symmetry operators both on the classical level, and in the secondary-quantization scheme [2,3]. We studied the physical content within several bases: light-front form formulation [4], helicity basis, angular momentum basis, and so on, on several practical examples. The conclusion is that we have ambiguities in the definitions of the corresponding operators P, C; T, which lead to different physical consequences [5, 6].

1. Introduction.

In his paper of 1992 Z. Silagadze claimed: "It is shown that the usual situation when boson and its antiparticle have the same internal parity, while, fermion and its antiparticle have opposite particles, assumes a kind of locality of the theory. In general, when a quantum-mechanical parity operator is defined by means of the group extension technique, the reversed situation is also possible", Ref. [1].

Then, Ahluwalia *et al* proposed [5] the so-called "Bargmann-Wightman-Wigner-type" quantum field theory, where, as they claimed, boson and antiboson have oposite intrinsic parities (see also [6]). Actually, this type of theories has been first proposed by Gelfand and Tsetlin [7]. In fact, it is based on the two-dimensional representation of the inversion group. They indicated applicability of this theory to the description of the set of K-mesons and possible relations to the Lee-Yang paper. The comutativity/anticommutativity of the discrete symmetry operations has also been investigated by Foldy and Nigam [8]. The relations of the Gelfand-Tsetlin construct to the representations of the anti-de Sitter $SO(3,2)$ group and the general relativity theory (including continuous and discrete transformations) have also been discussed in subsequent papers of Sokolik. E. Wigner [9] presented somewhat related results at the Istanbul School on Theoretical Physics in 1962. Later, Fushchich *et al* discussed corresponding wave equations. Actually, the theory presented by Ahluwalia, Goldman and Johnson is the Dirac-like generalization of the Weinberg $2(2J+1)$-theory for the spin 1. The equations have already been presented in the Sankaranarayanan and Good paper of 1965, Ref. [10]. In Ref. [11] the theory in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation based on the chiral helicity 4-eigenspinors was proposed. The corresponding equations have been obtained in [3] and in several less known articles. However, later we found the papers by Ziino and Barut [12] and the Markov papers [13], which also have connections with the subject under consideration.

However, the question of definitions of the discrete symmetries operators raised by Silagadze, has not been clarified in detail. In the next sections several explicit examples are presented. The paper has been adapted to the proceedings style.

2. Helicity Basis and Parity.

The 4-spinors have been studied well when the basis has been chosen in such a way that they are eigenstates of the $\hat{\mathbf{S}}_3$ operator. And, oppositely, the helicity basis case has not been studied almost at all (see, however, Refs. [14, 15]). Let me remind that the boosted 4-spinors in the "common-used" basis are the parity eigenstates with the eigenvalues of ± 1. In the helicity spin basis the 2-eigenspinors of the helicity operator [16] can be defined as follows [17]:

$$\phi_{\frac{1}{2}\uparrow} \sim \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{+i\phi/2} \end{pmatrix} , \quad \phi_{\frac{1}{2}\downarrow} \sim \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi/2} \\ -\cos\frac{\theta}{2} e^{+i\phi/2} \end{pmatrix} , \tag{1}$$

for $h = \pm 1/2$ eigenvalues, respectively. We start from the Klein-Gordon equation, generalized for describing the spin-1/2 particles (i. e., two degrees of freedom), $c = \hbar = 1$: $(E+\sigma\cdot\mathbf{p})(E-\sigma\cdot\mathbf{p})\phi = m^2\phi$. It can be re-written in the form of the system of two first-order equations for 2-spinors. At the same time, we observe that they may be chosen as the eigenstates of the helicity operator:

$$(E - (\sigma\cdot\mathbf{p}))\phi_\uparrow = (E-p)\phi_\uparrow = m\chi_\uparrow\,, (E + (\sigma\cdot\mathbf{p}))\chi_\uparrow = (E+p)\chi_\uparrow = m\phi_\uparrow\,, \tag{2}$$

$$(E - (\sigma\cdot\mathbf{p}))\phi_\downarrow = (E+p)\phi_\downarrow = m\chi_\downarrow\,, (E + (\sigma\cdot\mathbf{p}))\chi_\downarrow = (E-p)\chi_\downarrow = m\phi_\downarrow\,. \tag{3}$$

If the ϕ spinors are defined by the equation (1) then we can construct the corresponding $u-$ and $v-$ 4-spinors[1]

$$u_\uparrow = N_\uparrow^+ \begin{pmatrix} \phi_\uparrow \\ \frac{E-p}{m}\phi_\uparrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{E+p}{m}}\phi_\uparrow \\ \sqrt{\frac{m}{E+p}}\phi_\uparrow \end{pmatrix}, u_\downarrow = N_\downarrow^+ \begin{pmatrix} \phi_\downarrow \\ \frac{E+p}{m}\phi_\downarrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{m}{E+p}}\phi_\downarrow \\ \sqrt{\frac{E+p}{m}}\phi_\downarrow \end{pmatrix} \tag{4}$$

$$v_\uparrow = N_\uparrow^- \begin{pmatrix} \phi_\uparrow \\ -\frac{E-p}{m}\phi_\uparrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{E+p}{m}}\phi_\uparrow \\ -\sqrt{\frac{m}{E+p}}\phi_\uparrow \end{pmatrix}, v_\downarrow = N_\downarrow^- \begin{pmatrix} \phi_\downarrow \\ -\frac{E+p}{m}\phi_\downarrow \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{m}{E+p}}\phi_\downarrow \\ -\sqrt{\frac{E+p}{m}}\phi_\downarrow \end{pmatrix} \tag{5}$$

where the normalization to the unit (± 1) was used. One can prove that the matrix $P = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ can be used in the parity operator as in the original Dirac basis. Indeed, the 4-spinors (4,5) satisfy the Dirac equation in the spinorial representation of the γ-matrices. Hence, the parity-transformed function $\Psi'(t,-\mathbf{x}) = P\Psi(t,\mathbf{x})$ must satisfy $[i\gamma^\mu\partial'_\mu - m]\Psi'(t,-\mathbf{x}) = 0$ with $\partial'_\mu = (\partial/\partial t, -\nabla_i)$. This is possible when $P^{-1}\gamma^0 P = \gamma^0$ and $P^{-1}\gamma^i P = -\gamma^i$. The P-matrix above satisfies these requirements, as in the textbook case [18].

Next, it is easy to prove that one can form the projection operators $P_+ = +\sum_h u_h(\mathbf{p})\bar{u}_h(\mathbf{p}) = \frac{p_\mu\gamma^\mu + m}{2m}$, $P_- = -\sum_h v_h(\mathbf{p})\bar{v}_h(\mathbf{p}) = \frac{m - p_\mu\gamma^\mu}{2m}$, with the properties $P_+ + P_- = 1$ and $P_\pm^2 = P_\pm$. This permits us to expand the 4-spinors defined in the parity basis in linear superpositions of the helicity basis 4-spinors and to find corresponding coefficients of the expansion:

$$u_\sigma(\mathbf{p}) = A_{\sigma h} u_h(\mathbf{p}) + B_{\sigma h} v_h(\mathbf{p}),\ v_\sigma(\mathbf{p}) = C_{\sigma h} u_h(\mathbf{p}) + D_{\sigma h} v_h(\mathbf{p}). \tag{6}$$

[1] One can also try to construct yet another theory differing from the ordinary Dirac theory. The 4-spinors might be *not* the eigenspinors of the helicity operator of the $(1/2,0)\oplus(0,1/2)$ representation space, cf. [11]. They might be the eigenstates of the *chiral* helicity operator introduced in [11].

Multiplying the above equations by $\bar{u}_{h'}$, $\bar{v}_{h'}$ and using the normalization conditions, we obtain $A_{\sigma h} = D_{\sigma h} = \bar{u}_h u_\sigma$, $B_{\sigma h} = C_{\sigma h} = -\bar{v}_h u_\sigma$. Thus, the transformation matrix from the common-used basis to the helicity basis is

$$\begin{pmatrix} u_\sigma \\ v_\sigma \end{pmatrix} = \mathcal{U} \begin{pmatrix} u_h \\ v_h \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \tag{7}$$

Neither A nor B are unitary:

$$A = (a_{++} + a_{+-})(\sigma_\mu a^\mu) + (-a_{-+} + a_{--})(\sigma_\mu a^\mu)\sigma_3\,, \tag{8}$$
$$B = (-a_{++} + a_{+-})(\sigma_\mu a^\mu) + (a_{-+} + a_{--})(\sigma_\mu a^\mu)\sigma_3\,, \tag{9}$$

where

$$a^0 \;=\; -i\cos(\theta/2)\sin(\phi/2) \in \Im m\,,\; a^1 = \sin(\theta/2)\cos(\phi/2) \in \Re e, \tag{10}$$
$$a^2 \;=\; \sin(\theta/2)\sin(\phi/2) \in \Re e\,,\; a^3 = \cos(\theta/2)\cos(\phi/2) \in \Re e, \tag{11}$$

and

$$a_{++} \;=\; \frac{\sqrt{(E+m)(E+p)}}{2\sqrt{2}m}\,, \quad a_{+-} = \frac{\sqrt{(E+m)(E-p)}}{2\sqrt{2}m}\,, \tag{12}$$
$$a_{-+} \;=\; \frac{\sqrt{(E-m)(E+p)}}{2\sqrt{2}m}\,, \quad a_{--} = \frac{\sqrt{(E-m)(E-p)}}{2\sqrt{2}m}\,. \tag{13}$$

However, $A^\dagger A + B^\dagger B = I$, so the matrix $\mathcal{U}$ is unitary. Please note that this matrix acts on the *spin* indices (σ, h), and not on the spinorial indices; it is the 4×4 matrix.

We now investigate the properties of the helicity-basis 4-spinors with respect to the discrete symmetry operations P, C and T. It is expected that $h \to -h$ under parity, as Berestetskiĭ, Lifshitz and Pitaevskiĭ claimed [19]. Indeed, if $\mathbf{x} \to -\mathbf{x}$, then the vector $\mathbf{p} \to -\mathbf{p}$, but the axial vector $\mathbf{S} \to \mathbf{S}$, that implies the above statement. The helicity 2-eigenspinors transform $\phi_{\uparrow\downarrow} \Rightarrow -i\phi_{\downarrow\uparrow}$ with respect to $\mathbf{p} \to -\mathbf{p}$, Ref. [17]. Hence,

$$Pu_\uparrow(-\mathbf{p}) \;=\; -iu_\downarrow(\mathbf{p})\,, Pv_\uparrow(-\mathbf{p}) = +iv_\downarrow(\mathbf{p})\,, \tag{14}$$
$$Pu_\downarrow(-\mathbf{p}) \;=\; -iu_\uparrow(\mathbf{p})\,, Pv_\downarrow(-\mathbf{p}) = +iv_\uparrow(\mathbf{p})\,. \tag{15}$$

Thus, on the level of classical fields, we observe that the helicity 4-spinors transform to the 4-spinors of the opposite helicity.

$$Cu_\uparrow(\mathbf{p}) \;=\; -v_\downarrow(\mathbf{p})\,, Cv_\uparrow(\mathbf{p}) = +u_\downarrow(\mathbf{p})\,, \tag{16}$$
$$Cu_\downarrow(\mathbf{p}) \;=\; +v_\uparrow(\mathbf{p})\,, Cv_\downarrow(\mathbf{p}) = -u_\uparrow(\mathbf{p})\,. \tag{17}$$

due to the properties of the Wigner operator $\Theta\phi^*_\uparrow = -\phi_\downarrow$ and $\Theta\phi^*_\downarrow = +\phi_\uparrow$. Similar conclusions can be drawn in the Fock space. We define the field operator as follows:

$$\Psi(x^\mu) = \sum_h \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\sqrt{m}}{2E} [u_h(\mathbf{p}) a_h(\mathbf{p}) e^{-ip_\mu x^\mu} + v_h(\mathbf{p}) b_h^\dagger(\mathbf{p}) e^{+ip_\mu x^\mu}]\,. \tag{18}$$

The commutation relations are assumed to be the standard ones [21, 22, 18, 20] (compare with Refs. [3, 11]). If one defines $U_P \Psi(x^\mu) U_P^{-1} = \gamma^0 \Psi(x^{\mu'})$, $U_C \Psi(x^\mu) U_C^{-1} = C\Psi^\dagger(x^\mu)$ and the anti-unitary operator of time reversal $(V_T \Psi(x^\mu) V_T^{-1})^\dagger = T\Psi^\dagger(x^{\mu''})$, then it is easy to obtain the corresponding transformations of the creation/annihilation operators:

$$U_P a_h(\mathbf{p}) U_P^{-1} \;=\; -ia_{-h}(-\mathbf{p}), U_P b_h(\mathbf{p}) U_P^{-1} = -ib_{-h}(-\mathbf{p}), \tag{19}$$
$$U_C a_h(\mathbf{p}) U_C^{-1} \;=\; (-1)^{\frac{1}{2}+h} b_{-h}(\mathbf{p})\,, U_C b_h(\mathbf{p}) U_C^{-1} = (-1)^{\frac{1}{2}-h} a_{-h}(-\mathbf{p}). \tag{20}$$

As a consequence, we obtain (provided that $U_P|0>=|0>$, $U_C|0>=|0>$)

$$U_P a_h^\dagger(\mathbf{p})|0>= U_P a_h^\dagger U_P^{-1}|0>= i a_{-h}^\dagger(-\mathbf{p})|0>= i|-\mathbf{p},-h>^+\,, \qquad (21)$$
$$U_P b_h^\dagger(\mathbf{p})|0>= U_P b_h^\dagger U_P^{-1}|0>= i b_{-h}^\dagger(-\mathbf{p})|0>= i|-\mathbf{p},-h>^-\,, \qquad (22)$$

and

$$U_C a_h^\dagger(\mathbf{p})|0>= U_C a_h^\dagger U_C^{-1}|0>= (-1)^{\frac{1}{2}+h} b_{-h}^\dagger(\mathbf{p})|0>= (-1)^{\frac{1}{2}+h}|\mathbf{p},-h>^-\,,$$
$$U_C b_h^\dagger(\mathbf{p})|0>= U_C b_h^\dagger U_C^{-1}|0>= (-1)^{\frac{1}{2}-h} a_{-h}^\dagger(\mathbf{p})|0>= (-1)^{\frac{1}{2}-h}|\mathbf{p},-h>^+\,.$$

Finally, for the CP operation one should obtain:

$$\begin{aligned}
& U_P U_C a_h^\dagger(\mathbf{p})|0>= -U_C U_P a_h^\dagger(\mathbf{p})|0>= (-1)^{\frac{1}{2}+h} U_P b_{-h}^\dagger(\mathbf{p})|0>= \\
= \; & i(-1)^{\frac{1}{2}+h} b_h^\dagger(-\mathbf{p})|0>= i(-1)^{\frac{1}{2}+h}|-\mathbf{p},h>^-\,, \qquad (23)\\
& U_P U_C b_h^\dagger(\mathbf{p})|0>= -U_C U_P b_h^\dagger(\mathbf{p}) = (-1)^{\frac{1}{2}-h} U_P a_{-h}^\dagger(\mathbf{p})|0>= \\
= \; & i(-1)^{\frac{1}{2}-h} a_h^\dagger(-\mathbf{p})|0>= i(-1)^{\frac{1}{2}-h}|-\mathbf{p},-h>^+\,. \qquad (24)
\end{aligned}$$

As in the classical case, the P and C operations anticommutes in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ quantized case. This opposes to the theory based on 4-spinor eigenstates of chiral helicity (cf. [3]), where other definition was used, cf. [8] and below.

Since the V_T is an anti-unitary operator the problem must be solved after taking into account that in this case the c-numbers should be put outside the hermitian conjugation *without* complex conjugation:

$$[V_T h A V_T^{-1}]^\dagger = [h^* V_T A V_T^{-1}]^\dagger = h[V_T A^\dagger V_T^{-1}]\,. \qquad (25)$$

After applying this definition we obtain:[2] $V_T a_h^\dagger(\mathbf{p}) V_T^{-1} = +i(-1)^{\frac{1}{2}-h} a_h^\dagger(-\mathbf{p})\,, V_T b_h(\mathbf{p})_T^{-1} = +i(-1)^{\frac{1}{2}-h} b_h(-\mathbf{p})\,.$ Furthermore, we observed that the question of whether a particle and an antiparticle have the same or opposite parities depend on the phase factor in the following definition:

$$U_P \Psi(t,\mathbf{x}) U_P^{-1} = e^{i\alpha}\gamma^0 \Psi(t,-\mathbf{x})\,. \qquad (26)$$

Indeed,

$$U_P a_h(\mathbf{p}) U_P^{-1} = -i e^{i\alpha} a_{-h}(-\mathbf{p})\,, \qquad (27)$$
$$U_P b_h^\dagger(\mathbf{p}) U_P^{-1} = +i e^{i\alpha} b_{-h}^\dagger(-\mathbf{p})\,. \qquad (28)$$

From this, if $\alpha=\pi/2$ we obtain *opposite* parity properties of creation/annihilation operators for particles and anti-particles:

$$U_P a_h(\mathbf{p}) U_P^{-1} = +a_{-h}(-\mathbf{p})\,, \qquad (29)$$
$$U_P b_h(\mathbf{p}) U_P^{-1} = -b_{-h}(-\mathbf{p})\,. \qquad (30)$$

However, the difference with the Dirac case still preserves (h transforms to $-h$). We find somewhat similar situation with the question of constructing the neutrino field operator (cf. with the Goldhaber-Kayser creation phase factor).

Next, we find the explicit form of the parity operator U_P and prove that it commutes with the Hamiltonian operator. We prefer to use the method described in [20, §10.2-10.3]. It is

[2] T should be chosen in such a way in order to fulfill $T^{-1}\gamma_0^T T = \gamma_0$, $T^{-1}\gamma_i^T T = \gamma_i$ and $T^T = -T$, as in Ref. [21].

based on the *anzatz* that $U_P = \exp[i\alpha\hat{A}]\exp[i\hat{B}]$ with $\hat{A} = \sum_s \int d^3\mathbf{p}[a^\dagger_{\mathbf{p},s}a_{-\mathbf{p}s} + b^\dagger_{\mathbf{p}s}b_{-\mathbf{p}s}]$ and $\hat{B} = \sum_s \int d^3\mathbf{p}[\beta a^\dagger_{\mathbf{p},s}a_{\mathbf{p}s} + \gamma b^\dagger_{\mathbf{p}s}b_{\mathbf{p}s}]$. On using the known operator identity

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}]_- + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots \tag{31}$$

and $[\hat{A},\hat{B}\hat{C}]_- = [\hat{A},\hat{B}]_+\hat{C} - \hat{B}[\hat{A},\hat{C}]_+$ one can fix the parameters α, β, γ such that one satisfies the physical requirements that a Dirac particle and its anti-particle have opposite intrinsic parities.

In our case, we need to satisfy the requirement that the operator should invert not only the sign of the momentum, but the sign of the helicity too. We may achieve this goal by the analogous postulate $U_P = e^{i\alpha\hat{A}}$ with

$$\hat{A} = \sum_h \int \frac{d^3\mathbf{p}}{2E}[a^\dagger_h(\mathbf{p})a_{-h}(-\mathbf{p}) + b^\dagger_h(\mathbf{p})b_{-h}(-\mathbf{p})]\,. \tag{32}$$

By direct verification, the requirement is satisfied provided that $\alpha = \pi/2$. Cf. this parity operator with that given in [18, 20] for Dirac fields:[3]

$$\begin{aligned} U_P = \exp\Bigg[i\frac{\pi}{2}\int d^3\mathbf{p}\sum_s \Big(& a(\mathbf{p},s)^\dagger a(\tilde{\mathbf{p}},s) + b(\mathbf{p},s)^\dagger b(\tilde{\mathbf{p}},s) - \\ & -a(\mathbf{p},s)^\dagger a(\mathbf{p},s) + b(\mathbf{p},s)^\dagger b(\mathbf{p},s)\Big)\Bigg]\,, \quad (10.69) \text{ of Ref. [20]}\,. \end{aligned} \tag{33}$$

By direct verification one can also come to the conclusion that our new U_P commutes with the Hamiltonian:

$$\mathcal{H} = \int d^3\mathbf{x}\Theta^{00} = \int d^3\mathbf{k}\sum_h [a^\dagger_h(\mathbf{k})a_h(\mathbf{k}) - b_h(\mathbf{k})b^\dagger_h(\mathbf{k})]\,, \tag{34}$$

i.e.

$$[U_P, \mathcal{H}]_- = 0\,. \tag{35}$$

Alternatively, we can try to choose another set of commutation relations [3, 11] for the bi-orthonormal states. As it was said, in the cited papers and preprints I presented a theory based on 6-component Weinberg-like equations in the $(1,0)\oplus(0,1)$ representation. The theory in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation based on the chiral helicity 4-eigenspinors was also proposed. The papers by Ziino and Barut [12] and the Markov papers [13] have connections with the subject under consideration.

3. The Chiral Helicity Construct and the Different Definition of the Charge Conjugate Operator on the Secondary Quantization Level.

In the chiral representation one can choose the spinorial basis (zero-momentum spinors) in the following way:

$$\begin{aligned} \lambda^S_\uparrow(\mathbf{0}) &= \sqrt{\frac{m}{2}}\begin{pmatrix}0\\ i\\ 1\\ 0\end{pmatrix}, \lambda^S_\downarrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}-i\\ 0\\ 0\\ 1\end{pmatrix}, \lambda^A_\uparrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}0\\ -i\\ 1\\ 0\end{pmatrix}, \lambda^A_\downarrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}i\\ 0\\ 0\\ 1\end{pmatrix}, \\ \rho^S_\uparrow(\mathbf{0}) &= \sqrt{\frac{m}{2}}\begin{pmatrix}1\\ 0\\ 0\\ -i\end{pmatrix}, \rho^S_\downarrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}0\\ 1\\ i\\ 0\end{pmatrix}, \rho^A_\uparrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}1\\ 0\\ 0\\ i\end{pmatrix}, \rho^A_\downarrow(\mathbf{0}) = \sqrt{\frac{m}{2}}\begin{pmatrix}0\\ 1\\ -i\\ 0\end{pmatrix}. \end{aligned}$$

[3] Greiner used the following anticommutation relations $\left[a(\mathbf{p},s), a^\dagger(\mathbf{p}',s')\right]_+ = \left[b(\mathbf{p},s), b^\dagger(\mathbf{p}',s')\right]_+ = \delta^3(\mathbf{p}-\mathbf{p}')\delta_{ss'}$. One should also note that the Greiner form of the parity operator is not the unique one. Itzykson and Zuber [18] proposed another one differing by the phase factors from (10.69) of [20]. In order to find relations between those two forms of the parity operator one should apply additional rotation in the Fock space.

The indices $\uparrow\downarrow$ should be referred to the chiral helicity quantum number introduced in Ref. [11]. Using the boost the reader would immediately find the 4-spinors of the second kind $\lambda^{S,A}_{\uparrow\downarrow}(p^\mu)$ and $\rho^{S,A}_{\uparrow\downarrow}(p^\mu)$ in an arbitrary frame:

$$
\lambda^S_\uparrow(p^\mu) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} ip_l\\ i(p^-+m)\\ p^-+m\\ -p_r\end{pmatrix}, \lambda^S_\downarrow(p^\mu) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} -i(p^++m)\\ -ip_r\\ -p_l\\ (p^++m)\end{pmatrix}, \tag{36}
$$

$$
\lambda^A_\uparrow(p^\mu) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} -ip_l\\ -i(p^-+m)\\ (p^-+m)\\ -p_r\end{pmatrix}, \lambda^A_\downarrow(p^\mu) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} i(p^++m)\\ ip_r\\ -p_l\\ (p^++m)\end{pmatrix}, \tag{37}
$$

$$
\rho^S_\uparrow(p^\mu) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} p^++m\\ p_r\\ ip_l\\ -i(p^++m)\end{pmatrix}, \rho^S_\downarrow(p^\mu) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} p_l\\ (p^-+m)\\ i(p^-+m)\\ -ip_r\end{pmatrix}, \tag{38}
$$

$$
\rho^A_\uparrow(p^\mu) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} p^++m\\ p_r\\ -ip_l\\ i(p^++m)\end{pmatrix}, \rho^A_\downarrow(p^\mu) = \frac{1}{2\sqrt{E+m}}\begin{pmatrix} p_l\\ (p^-+m)\\ -i(p^-+m)\\ ip_r\end{pmatrix}. \tag{39}
$$

Some of the 4-spinors are connected each other. The normalization of the spinors $\lambda^{S,A}_{\uparrow\downarrow}(p^\mu)$ and $\rho^{S,A}_{\uparrow\downarrow}(p^\mu)$ are the following ones:

$$
\overline{\lambda}^S_\uparrow(p^\mu)\lambda^S_\downarrow(p^\mu) = -im \quad , \quad \overline{\lambda}^S_\downarrow(p^\mu)\lambda^S_\uparrow(p^\mu) = +im \quad , \tag{40}
$$

$$
\overline{\lambda}^A_\uparrow(p^\mu)\lambda^A_\downarrow(p^\mu) = +im \quad , \quad \overline{\lambda}^A_\downarrow(p^\mu)\lambda^A_\uparrow(p^\mu) = -im \quad , \tag{41}
$$

$$
\overline{\rho}^S_\uparrow(p^\mu)\rho^S_\downarrow(p^\mu) = +im \quad , \quad \overline{\rho}^S_\downarrow(p^\mu)\rho^S_\uparrow(p^\mu) = -im \quad , \tag{42}
$$

$$
\overline{\rho}^A_\uparrow(p^\mu)\rho^A_\downarrow(p^\mu) = -im \quad , \quad \overline{\rho}^A_\downarrow(p^\mu)\rho^A_\uparrow(p^\mu) = +im \quad . \tag{43}
$$

All other conditions are equal to zero.

Implying that $\lambda^S(p^\mu)$ (and $\rho^A(p^\mu)$) answer for positive-frequency solutions; $\lambda^A(p^\mu)$ (and $\rho^S(p^\mu)$), for negative-frequency solutions, one can deduce the dynamical coordinate-space equations [3]:

$$
i\gamma^\mu\partial_\mu\lambda^S(x) - m\rho^A(x) = 0\,, \tag{44}
$$

$$
i\gamma^\mu\partial_\mu\rho^A(x) - m\lambda^S(x) = 0\,, \tag{45}
$$

$$
i\gamma^\mu\partial_\mu\lambda^A(x) + m\rho^S(x) = 0\,, \tag{46}
$$

$$
i\gamma^\mu\partial_\mu\rho^S(x) + m\lambda^A(x) = 0\,. \tag{47}
$$

They can be written in the 8-component form. This is just another representation of the Dirac-like equation in the appropriate Clifford Algebra. One can also re-write the equations into the two-component form.

In the Fock space operators of the charge conjugation and space inversions can be defined as above. We imply the bi-orthonormal system of the anticommutation relations. As a result we have the following properties of creation (annihilation) operators in the Fock space:

$$
\begin{aligned}
U^s_{[1/2]}a_\uparrow(\mathbf{p})(U^s_{[1/2]})^{-1} &= -ia_\downarrow(-\mathbf{p}), U^s_{[1/2]}a_\downarrow(\mathbf{p})(U^s_{[1/2]})^{-1} = +ia_\uparrow(-\mathbf{p}),\\
U^s_{[1/2]}b^\dagger_\uparrow(\mathbf{p})(U^s_{[1/2]})^{-1} &= +ib^\dagger_\downarrow(-\mathbf{p}), U^s_{[1/2]}b^\dagger_\downarrow(\mathbf{p})(U^s_{[1/2]})^{-1} = -ib_\uparrow(-\mathbf{p}),
\end{aligned}
$$

that signifies that the states created by the operators $a^\dagger(\mathbf{p})$ and $b^\dagger(\mathbf{p})$ have very different properties with respect to the space inversion operation, comparing with Dirac states (the case also regarded in [12]):

$$U^s_{[1/2]}|\mathbf{p},\,\uparrow>^+ \;=\; +i|-\mathbf{p},\,\downarrow>^+, U^s_{[1/2]}|\mathbf{p},\,\uparrow>^- = +i|-\mathbf{p},\,\downarrow>^-, \tag{48}$$

$$U^s_{[1/2]}|\mathbf{p},\,\downarrow>^+ \;=\; -i|-\mathbf{p},\,\uparrow>^+, U^s_{[1/2]}|\mathbf{p},\,\downarrow>^- = -i|-\mathbf{p},\,\uparrow>^- . \tag{49}$$

For the charge conjugation operation in the Fock space we have two physically different possibilities. The first one, *e.g.*,

$$U^c_{[1/2]} a_\uparrow(\mathbf{p}) (U^c_{[1/2]})^{-1} \;=\; +b_\uparrow(\mathbf{p}), U^c_{[1/2]} a_\downarrow(\mathbf{p}) (U^c_{[1/2]})^{-1} = +b_\downarrow(\mathbf{p}), \tag{50}$$

$$U^c_{[1/2]} b^\dagger_\uparrow(\mathbf{p}) (U^c_{[1/2]})^{-1} \;=\; -a^\dagger_\uparrow(\mathbf{p}), U^c_{[1/2]} b^\dagger_\downarrow(\mathbf{p}) (U^c_{[1/2]})^{-1} = -a^\dagger_\downarrow(\mathbf{p}), \tag{51}$$

in fact, has some similarities with the Dirac construct. The action of this operator on the physical states are

$$U^c_{[1/2]}|\mathbf{p},\,\uparrow>^+ \;=\; +\,|\mathbf{p},\,\uparrow>^- , \quad U^c_{[1/2]}|\mathbf{p},\,\downarrow>^+ = +\,|\mathbf{p},\,\downarrow>^- , \tag{52}$$

$$U^c_{[1/2]}|\mathbf{p},\,\uparrow>^- \;=\; -\,|\mathbf{p},\,\uparrow>^+ , \quad U^c_{[1/2]}|\mathbf{p},\,\downarrow>^- = -\,|\mathbf{p},\,\downarrow>^+ . \tag{53}$$

But, one can also construct the charge conjugation operator in the Fock space which acts, *e.g.*, in the following manner:

$$\widetilde{U}^c_{[1/2]} a_\uparrow(\mathbf{p}) (\widetilde{U}^c_{[1/2]})^{-1} \;=\; -b_\downarrow(\mathbf{p}), \widetilde{U}^c_{[1/2]} a_\downarrow(\mathbf{p}) (\widetilde{U}^c_{[1/2]})^{-1} = -b_\uparrow(\mathbf{p}), \tag{54}$$

$$\widetilde{U}^c_{[1/2]} b^\dagger_\uparrow(\mathbf{p}) (\widetilde{U}^c_{[1/2]})^{-1} \;=\; +a^\dagger_\downarrow(\mathbf{p}), \widetilde{U}^c_{[1/2]} b^\dagger_\downarrow(\mathbf{p}) (\widetilde{U}^c_{[1/2]})^{-1} = +a^\dagger_\uparrow(\mathbf{p}), \tag{55}$$

and, therefore,

$$\widetilde{U}^c_{[1/2]}|\mathbf{p},\,\uparrow>^+ \;=\; -\,|\mathbf{p},\,\downarrow>^- \quad , \quad \widetilde{U}^c_{[1/2]}|\mathbf{p},\,\downarrow>^+ = -\,|\mathbf{p},\,\uparrow>^- \quad , \tag{56}$$

$$\widetilde{U}^c_{[1/2]}|\mathbf{p},\,\uparrow>^- \;=\; +\,|\mathbf{p},\,\downarrow>^+ \quad , \quad \widetilde{U}^c_{[1/2]}|\mathbf{p},\,\downarrow>^- = +\,|\mathbf{p},\,\uparrow>^+ \quad . \tag{57}$$

Next, by straightforward verification one can convince ourselves about correctness of the assertions made in [8,11b] that it is possible a situation when the operators of the space inversion and charge conjugation commute each other in the Fock space. For instance,

$$U^c_{[1/2]} U^s_{[1/2]}|\mathbf{p},\,\uparrow>^+ \;=\; +iU^c_{[1/2]}|-\mathbf{p},\,\downarrow>^+ = +i|-\mathbf{p},\,\downarrow>^-, \tag{58}$$

$$U^s_{[1/2]} U^c_{[1/2]}|\mathbf{p},\,\uparrow>^+ \;=\; U^s_{[1/2]}|\mathbf{p},\,\uparrow>^- = +i|-\mathbf{p},\,\downarrow>^- . \tag{59}$$

The second choice of the charge conjugation operator answers for the case when the $\widetilde{U}^c_{[1/2]}$ and $U^s_{[1/2]}$ operations anticommute:

$$\widetilde{U}^c_{[1/2]} U^s_{[1/2]}|\mathbf{p},\,\uparrow>^+ \;=\; +i\widetilde{U}^c_{[1/2]}|-\mathbf{p},\,\downarrow>^+ = -i\,|-\mathbf{p},\,\uparrow>^-, \tag{60}$$

$$U^s_{[1/2]} \widetilde{U}^c_{[1/2]}|\mathbf{p},\,\uparrow>^+ \;=\; -U^s_{[1/2]}|\mathbf{p},\,\downarrow>^- = +i\,|-\mathbf{p},\,\uparrow>^- . \tag{61}$$

Finally, the time reversal *anti-unitary* operator in the Fock space should be defined in such a way the formalism to be compatible with the CPT theorem. If we wish the Dirac states to transform as $V(T)|\mathbf{p},\pm 1/2> = \pm\,|-\mathbf{p},\mp 1/2>$ we have to choose (within a phase factor), Ref. [18]:

$$S(T) = \begin{pmatrix} \Theta_{[1/2]} & 0 \\ 0 & \Theta_{[1/2]} \end{pmatrix} \quad . \tag{62}$$

Thus, in the first relevant case we obtain for the $\Psi(x^\mu)$ field, composed of $\lambda^{S,A}$ or $\rho^{A,S}$ 4-spinors

$$V^{^T} a^\dagger_\uparrow(\mathbf{p}) (V^{^T})^{-1} \;=\; a^\dagger_\downarrow(-\mathbf{p}),\; V^{^T} a^\dagger_\downarrow(\mathbf{p}) (V^{^T})^{-1} = -a^\dagger_\uparrow(-\mathbf{p}), \tag{63}$$

$$V^{^T} b_\uparrow(\mathbf{p}) (V^{^T})^{-1} \;=\; b_\downarrow(-\mathbf{p}),\; V^{^T} b_\downarrow(\mathbf{p}) (V^{^T})^{-1} = -b_\uparrow(-\mathbf{p})\,, \tag{64}$$

that is not surprising.

4. The Conclusions.

Thus, we proceeded as in the textbooks and defined the parity matrix as $P = \gamma_0$, because the helicity 4-spinors satisfy the Dirac equation, and the 2nd-type λ-spinors satisfy the coupled Dirac equations. Nevertheless, the properties of the corresponding spinors appear to be different with respect to the parity both on the first and the second quantization level. The result is compatible with the statement made by Berestetskii, Lifshitz and Pitaevskii. We also defined another charge conjugation operator in the Fock space, which also transforms the positive-energy 4-spinors to the negative-energy ones.

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