EXISTENCE OF STATIONARY SOLUTIONS FOR SOME SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH SUPERDIFFUSION

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Abstract: In the article we establish the existence of solutions of a system of integro- differential equations arising in population dynamics in the case of anomalous diffusion. The proof of existence of solutions is based on a fixed point technique. Solvability conditions for elliptic operators without Fredholm property in unbounded domains are being used.

AMS Subject Classification: 35J05, 35P30, 47F05
Key words: integro-differential equations, non Fredholm operators, Sobolev spaces

1. Introduction

In the present work we address the existence of stationary solutions of the system of integro-differential equations

$$\frac{\partial u_s}{\partial t} = -D_s\sqrt{-\Delta}u_s + \int_{\mathbb{R}^d} K_s(x-y)g_s(u(y,t))dy + f_s(x), \quad 1 \le s \le N \quad (1.1)$$

appearing in cell population dynamics. The space variable x here corresponds to the cell genotype, $u_s(x,t)$ are densities for different groups of cells as functions of their genotype and time and $u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T$. The right side of system (1.1) describes the evolution of cell densities by means of cell proliferation, mutations and cell influx. The anomalous diffusion terms here correspond to the change of genotype via small random mutations, and the nonlocal terms describe large mutations. Functions $g_s(u)$ are the rates of cell birth dependent on u (density dependent proliferation), and the functions $K_s(x-y)$ show the proportion of newly born cells changing their genotype from y to x. Let us assume here that they depend on the distance between the genotypes. The last term in the right side of (1.1) describes the influxes of cells for different genotypes.

The square root of minus Laplacian in system (1.1) represents a particular example of superdiffusion actively studied in relation with different applications in plasma physics and turbulence [11], [12], surface diffusion [13], [14], semiconductors [15] and so on. The physical meaning of superdiffusion is that the random process occurs with longer jumps in comparison with normal diffusion. The moments of jump length distribution is finite in the case of normal diffusion, but this is not the case for superdiffusion. The operator $\sqrt{-\Delta}$ is defined via the spectral calculus. A similar problem in the presence of the standard Laplacian in the diffusion term was treated recently in [27] and [29]. Work [28] deals with a single equation analogous to system (1.1).

Let us set all $D_s = 1$ and study the existence of solutions of the system of equations

$$-\sqrt{-\Delta}u_s + \int_{\mathbb{R}^d} K_s(x-y)g_s(u(y))dy + f_s(x) = 0, \quad 1 \le s \le N. \tag{1.2}$$

We consider the case when the linear part of the operator above does not satisfy the Fredholm property, such that conventional methods of nonlinear analysis may not be applicable. Solvability conditions for non Fredholm operators along with the method of contraction mappings will be used.

Let us consider the equation

$$-\Delta u + V(x)u - au = f \tag{1.3}$$

with $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either vanishing in the whole space or converging to 0 at infinity. For $a \ge 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (1.3) contains the origin. As a consequence, this operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. In the present article we study some properties of the operators of this kind. Recall that elliptic equations with non Fredholm operators were treated extensively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The Schrödinger type operators without Fredholm property were studied via the methods of the spectral and the scattering theory in [16], [18], [19], [20], [22]. The Laplacian operator with drift from the point of view of non Fredholm operators was treated in [21] and linearized Cahn-Hilliard problems in [23] and [25]. Articles [24] and [26] were devoted to the studies of nonlinear non Fredholm elliptic problems. Significant applications to the theory of reaction-diffusion equations were developed in [8], [9]. Non Fredholm operators appear also when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, in the case of $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the situation when $a \neq 0$ is significantly different and the approach developed in these articles cannot be applied. Front propagation problems with superdiffusion were treated extensively in recent years (see e.g. [30], [31]).

We set $K_s(x) = \varepsilon_s \mathcal{K}_s(x)$ with $\varepsilon_s \geq 0$ and

$$\varepsilon := \max_{1 \leq s \leq N} \varepsilon_s$$

and suppose that the following assumption holds.

Assumption 1. *Let $1 \leq s \leq N$, such that $f_s(x) : \mathbb{R}^3 \to \mathbb{R}$, $f_s(x) \in L^1(\mathbb{R}^3)$ and $\nabla f_s(x) \in L^2(\mathbb{R}^3)$. Furthermore, $f_s(x)$ is nontrivial for a certain s. Assume also that $\mathcal{K}_s(x) : \mathbb{R}^3 \to \mathbb{R}$, such that $\mathcal{K}_s(x) \in L^1(\mathbb{R}^3)$ and $\nabla \mathcal{K}_s(x) \in L^2(\mathbb{R}^3)$. Moreover,*

$$\mathcal{K}^2 := \sum_{s=1}^{N} \|\mathcal{K}_s\|^2_{L^1(\mathbb{R}^3)} > 0$$

and

$$Q^2 := \sum_{s=1}^{N} \|\nabla \mathcal{K}_s\|^2_{L^2(\mathbb{R}^3)} > 0.$$

Let us choose the space dimension $d = 3$, which is related to the solvability conditions for the linear Poisson equation (3.1) stated in Lemma 5. The results obtained below can be generalized to $d > 3$. From the point of view of applications, the space dimension is not limited to $d = 3$ because the space variable is correspondent to cell genotype but not to the usual physical space.

By virtue of the standard Sobolev inequality (see e.g. p.183 of [10]) under the assumption stated above we have

$$f_s(x) \in L^2(\mathbb{R}^3), \quad 1 \leq s \leq N.$$

We use the Sobolev space of vector functions

$$H^2(\mathbb{R}^3, \mathbb{R}^N) := \{u(x) : \mathbb{R}^3 \to \mathbb{R}^N \mid u_s(x) \in L^2(\mathbb{R}^3),\ \Delta u_s \in L^2(\mathbb{R}^3),\ 1 \leq s \leq N\}$$

equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^3, \mathbb{R}^N)} := \sum_{s=1}^{N} \|u_s\|^2_{H^2(\mathbb{R}^3)} = \sum_{s=1}^{N} \{\|u_s\|^2_{L^2(\mathbb{R}^3)} + \|\Delta u_s\|^2_{L^2(\mathbb{R}^3)}\}. \tag{1.4}$$

Also,

$$\|u\|^2_{L^2(\mathbb{R}^3, \mathbb{R}^N)} := \sum_{s=1}^{N} \|u_s\|^2_{L^2(\mathbb{R}^3)}.$$

The Sobolev embedding implies

$$\|\phi\|_{L^\infty(\mathbb{R}^3)} \le c_e \|\phi\|_{H^2(\mathbb{R}^3)}, \tag{1.5}$$

where $c_e > 0$ is the constant of the embedding. The hat symbol will denote the standard Fourier transform, such that

$$\widehat{\phi}(p) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \phi(x) e^{-ipx} dx. \tag{1.6}$$

When all the nonnegative parameters $\varepsilon_s = 0$, we obtain the linear Poisson equations

$$\sqrt{-\Delta} u_s = f_s(x), \quad 1 \le s \le N. \tag{1.7}$$

By virtue of Lemma 5 below along with Assumption 1 problem (1.7) has a unique solution $u_{0,s}(x) \in H^1(\mathbb{R}^3)$, such that no orthogonality conditions are required. Lemmas 5 gives us that in dimensions $d < 3$ we need specific orthogonality relations for the solvability of (1.7) in $H^1(\mathbb{R}^d)$. We will not study the problem in dimensions $d > 3$ to avoid extra technicalities because the proof will be based on similar ideas (see Lemma 5). By means of Assumption 1, using that

$$\|\Delta u_s\|^2_{L^2(\mathbb{R}^3)} = \|\nabla f_s(x)\|^2_{L^2(\mathbb{R}^3)},$$

we derive for the unique solution of (1.7) that $u_{0,s}(x) \in H^2(\mathbb{R}^3)$, such that

$$u_0(x) = (u_{0,1}(x), u_{0,2}(x), ..., u_{0,N}(x))^T \in H^2(\mathbb{R}^3, \mathbb{R}^N).$$

We look for the resulting solution of the nonlinear system of equations (1.2) as

$$u(x) = u_0(x) + u_p(x) \tag{1.8}$$

with

$$u_p(x) = (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T.$$

Apparently, we derive the perturbative system of equations

$$\sqrt{-\Delta} u_{p,s} = \varepsilon_s \int_{\mathbb{R}^3} \mathcal{K}_s(x-y) g_s(u_0(y) + u_p(y)) dy, \quad 1 \le s \le N. \tag{1.9}$$

We introduce a closed ball in our Sobolev space as

$$B_\rho := \{u(x) \in H^2(\mathbb{R}^3, \mathbb{R}^N) \mid \|u\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} \le \rho\}, \quad 0 < \rho \le 1. \tag{1.10}$$

Let us look for the solution of (1.9) as the fixed point of the auxiliary nonlinear system

$$\sqrt{-\Delta} u_s = \varepsilon_s \int_{\mathbb{R}^3} \mathcal{K}_s(x-y) g_s(u_0(y) + v(y)) dy, \quad 1 \le s \le N \tag{1.11}$$

in ball (1.10). For a given vector function $v(y)$ this is a system of equations with respect to $u(x)$. The left side of (1.11) contains the non Fredholm operator $\sqrt{-\Delta}$: $H^1(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$. Since its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$, this operator has no bounded inverse. The analogous situation appeared in articles [24] and [26] but as distinct from the present work, the problems treated there required orthogonality relations. The fixed point technique was applied in [17] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear problem there had the Fredholm property (see Assumption 1 of [17], also [7]). We define a closed ball in the space of N dimensions

$$I := \{z \in \mathbb{R}^N \mid |z| \le c_e \|u_0\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + c_e\}. \tag{1.12}$$

Let us introduce for technical goals the following quantities with $1 \le s, j \le N$

$$a_{2,s,j} := \text{sup}_{z \in I} \Big|\nabla \frac{\partial g_s}{\partial z_j}\Big|, \quad a_{2,s} := \sqrt{\sum_{j=1}^N a_{2,s,j}^2}, \quad a_2 := \text{max}_{1 \le s \le N} a_{2,s}.$$

Also,

$$a_{1,s} := \text{sup}_{z \in I} |\nabla g_s(z)|, \quad a_1 := \text{max}_{1 \le s \le N} a_{1,s}.$$

Let us make the following assumption on the nonlinear part of problem (1.2).

Assumption 2. *Let $1 \le s \le N$, such that $g_s(z) : \mathbb{R}^N \rightarrow \mathbb{R}$ with $g_s(z) \in C_2(\mathbb{R}^N)$. We also assume that $g_s(0) = 0,\ \nabla g_s(0) = 0$ and $a_2 > 0$.*

Apparently, a_1 defined above is also positive, otherwise all the functions $g_s(z)$ will be constants in the ball I and then $a_2 = 0$. For example, $g_s(z) = z^2,\ z \in \mathbb{R}^N$ obviously satisfies the assumption above.

We introduce the operator T_g, such that $u = T_g v$, where u is a solution of the system of equations (1.11). Our main result is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then system (1.11) defines the map $T_g : B_\rho \rightarrow B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon*$ for some $\varepsilon* > 0$. The unique fixed point $u_p(x)$ of the map T_g is the only solution of the system of equations (1.9) in B_ρ.*

Obviously, the resulting solution of system (1.2) given by (1.8) will be nontrivial due to the fact that the source terms $f_s(x)$ are nontrivial for a certain $s = 1, ..., N$ and all $g_s(0) = 0$ as assumed. We will apply the following trivial lemma.

Lemma 4. *Consider the function* $\varphi(R) := \alpha R + \dfrac{\beta}{R^2}$ *for* $R \in (0, +\infty)$ *with the constants* $\alpha, \beta > 0$. *It achieves the minimal value at* $R^* = \left(\dfrac{2\beta}{\alpha}\right)^{\frac{1}{3}}$, *which is given by* $\varphi(R^*) = \dfrac{3}{2^{\frac{2}{3}}} \alpha^{\frac{2}{3}} \beta^{\frac{1}{3}}$.

Let us proceed to the proof of our main statement.

2. The existence of the perturbed solution

Proof of Theorem 3. We choose arbitrarily $v(x) \in B_\rho$ and denote the terms involved in the integral expressions in right side of system (1.11) as

$$G_s(x) := g_s(u_0(x) + v(x)), \quad 1 \leq s \leq N.$$

Let us apply the standard Fourier transform (1.6) to both sides of system (1.11), which yields

$$\widehat{u}_s(p) = \varepsilon_s (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}_s(p) \widehat{G}_s(p)}{|p|}, \quad 1 \leq s \leq N.$$

Thus for the norm we arrive at

$$\|u_s\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^3 \varepsilon_s^2 \int_{\mathbb{R}^3} \frac{|\widehat{\mathcal{K}}_s(p)|^2 |\widehat{G}_s(p)|^2}{p^2} dp. \tag{2.1}$$

Obviously, for any $\phi(x) \in L^1(\mathbb{R}^3)$

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{(2\pi)^{\frac{3}{2}}} \|\phi(x)\|_{L^1(\mathbb{R}^3)}. \tag{2.2}$$

As distinct from articles [24] and [26] involving the standard Laplacian operator in the diffusion term, here we do not try to control the norms

$$\left\| \frac{\widehat{\mathcal{K}}_s(p)}{|p|} \right\|_{L^\infty(\mathbb{R}^3)}.$$

We estimate the right side of (2.1) applying (2.2) with $R > 0$ as

$$(2\pi)^3 \varepsilon_s^2 \int_{|p| \leq R} \frac{|\widehat{\mathcal{K}}_s(p)|^2 |\widehat{G}_s(p)|^2}{p^2} dp + (2\pi)^3 \varepsilon_s^2 \int_{|p| > R} \frac{|\widehat{\mathcal{K}}_s(p)|^2 |\widehat{G}_s(p)|^2}{p^2} dp \leq$$

$$\leq \varepsilon_s^2 \|\mathcal{K}_s\|^2_{L^1(\mathbb{R}^3)} \left\{ \frac{1}{2\pi^2} \|G_s(x)\|^2_{L^1(\mathbb{R}^3)} R + \frac{1}{R^2} \|G_s(x)\|^2_{L^2(\mathbb{R}^3)} \right\}. \tag{2.3}$$

Since $v(x) \in B_\rho$, we obtain

$$\|u_0 + v\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)} \le \|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1$$

and by means of the Sobolev embedding (1.5)

$$|u_0 + v| \le c_e \|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + c_e.$$

We will use the identity formula

$$G_s(x) = \int_0^1 \nabla g_s(t(u_0(x) + v(x))).(u_0(x) + v(x))dt, \quad 1 \le s \le N.$$

Here and further down the dot symbol stands for the scalar product of two vectors in $\mathbb{R}^N$. With the ball I defined in (1.12) we derive

$$|G_s(x)| \le sup_{z\in I}|\nabla g_s(z)||u_0(x) + v(x)| \le a_1|u_0(x) + v(x)|.$$

Hence

$$\|G_s(x)\|_{L^2(\mathbb{R}^3)} \le a_1\|u_0 + v\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)} \le a_1(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1).$$

Evidently, for $t \in [0, 1]$ and $1 \le j \le N$

$$\frac{\partial g_s}{\partial z_j}(t(u_0(x) + v(x))) = \int_0^t \nabla \frac{\partial g_s}{\partial z_j}(\tau(u_0(x) + v(x))).(u_0(x) + v(x))d\tau.$$

This implies

$$\left|\frac{\partial g_s}{\partial z_j}(t(u_0(x) + v(x)))\right| \le sup_{z\in I}\left|\nabla\frac{\partial g_s}{\partial z_j}\right||u_0(x) + v(x)| = a_{2,s,j}|u_0(x) + v(x)|.$$

Hence by virtue of the Schwarz inequality

$$|G_s(x)| \le |u_0(x) + v(x)|\sum_{j=1}^N a_{2,s,j}|u_{0,j}(x) + v_j(x)| \le a_2|u_0(x) + v(x)|^2,$$

such that

$$\|G_s(x)\|_{L^1(\mathbb{R}^3)} \le a_2\|u_0(x) + v(x)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)} \le a_2(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^2.$$

Then, we arrive at the upper bound for the right side of (2.3) given by

$$\varepsilon_s^2\|\mathcal{K}_s\|^2_{L^1(\mathbb{R}^3)}(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^2\left\{\frac{a_2^2}{2\pi^2}(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^2 R + \frac{a_1^2}{R^2}\right\}$$

with $R \in (0, +\infty)$. Lemma 4 gives us the minimal value of the expression above. Therefore,

$$\|u_s\|^2_{L^2(\mathbb{R}^3)} \leq \frac{3}{2^{\frac{4}{3}}\pi^{\frac{4}{3}}}\varepsilon^2 \|\mathcal{K}_s\|^2_{L^1(\mathbb{R}^3)}(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^{3\frac{1}{3}} a_1^{\frac{2}{3}} a_2^{\frac{4}{3}},$$

such that

$$\|u\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)} \leq \frac{3}{2^{\frac{4}{3}}\pi^{\frac{4}{3}}}\varepsilon^2 \mathcal{K}^2(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^{3\frac{1}{3}} a_1^{\frac{2}{3}} a_2^{\frac{4}{3}}. \tag{2.4}$$

Obviously, by means of (1.11)

$$-\Delta u_s = \varepsilon_s \sqrt{-\Delta} \int_{\mathbb{R}^3} \mathcal{K}_s(x-y) G_s(y) dy, \quad 1 \leq s \leq N.$$

Using (2.2) we easily obtain

$$\|\Delta u_s\|^2_{L^2(\mathbb{R}^3)} \leq \varepsilon^2 a_2^2 (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^4 \|\nabla \mathcal{K}_s\|^2_{L^2(\mathbb{R}^3)}.$$

Hence

$$\sum_{s=1}^{N} \|\Delta u_s\|^2_{L^2(\mathbb{R}^3)} \leq \varepsilon^2 a_2^2 (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^4 Q^2. \tag{2.5}$$

The definition of the norm (1.4) along with inequalities (2.4) and (2.5) give us

$$\|u\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \leq \varepsilon(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^2 a_2^{\frac{2}{3}} \sqrt{\frac{3}{2^{\frac{4}{3}}\pi^{\frac{4}{3}}} \mathcal{K}^2 a_1^{\frac{2}{3}} + a_2^{\frac{2}{3}} Q^2} \leq \rho$$

for all $\varepsilon > 0$ sufficiently small. Therefore, $u(x) \in B_\rho$ as well. If for a certain $v(x) \in B_\rho$ there are two solutions $u_{1,2}(x) \in B_\rho$ of system (1.11), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3, \mathbb{R}^N)$ solves

$$\sqrt{-\Delta} w = 0.$$

Due to the fact that the operator $\sqrt{-\Delta}$ does not possess nontrivial square integrable zero modes, $w(x) = 0$ a.e. in $\mathbb{R}^3$. Hence, system (1.11) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ small enough.

Our goal is to show that this map is a strict contraction. Let us choose arbitrarily $v_{1,2}(x) \in B_\rho$. By means of the argument above we have $u_{1,2} = T_g v_{1,2} \in B_\rho$ as well. By virtue of system (1.11)

$$\sqrt{-\Delta} u_{1,s} = \varepsilon_s \int_{\mathbb{R}^3} \mathcal{K}_s(x-y) g_s(u_0(y) + v_1(y)) dy, \quad 1 \leq s \leq N, \tag{2.6}$$

$$\sqrt{-\Delta} u_{2,s} = \varepsilon_s \int_{\mathbb{R}^3} \mathcal{K}_s(x-y) g_s(u_0(y) + v_2(y)) dy, \quad 1 \leq s \leq N. \tag{2.7}$$

We define

$$G_{1,s}(x) := g_s(u_0(x) + v_1(x)), \quad G_{2,s}(x) := g_s(u_0(x) + v_2(x)), \quad 1 \le s \le N$$

and apply the standard Fourier transform (1.6) to both sides of systems (2.6) and (2.7). We arrive at

$$\widehat{u_{1,s}}(p) = \varepsilon_s (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}_s(p)\widehat{G_{1,s}}(p)}{|p|}, \quad \widehat{u_{2,s}}(p) = \varepsilon_s (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}_s(p)\widehat{G_{2,s}}(p)}{|p|}.$$

Obviously

$$\|u_{1,s}(x) - u_{2,s}(x)\|^2_{L^2(\mathbb{R}^3)} = \varepsilon_s^2 (2\pi)^3 \int_{\mathbb{R}^3} \frac{|\widehat{\mathcal{K}}_s(p)|^2 |\widehat{G_{1,s}}(p) - \widehat{G_{2,s}}(p)|^2}{|p|^2} dp.$$

Apparently, it can be estimated from above using (2.2) by

$$\varepsilon_s^2 \|\mathcal{K}_s\|^2_{L^1(\mathbb{R}^3)} \left\{ \frac{1}{2\pi^2} \|G_{1,s}(x) - G_{2,s}(x)\|^2_{L^1(\mathbb{R}^3)} R + \|G_{1,s}(x) - G_{2,s}(x)\|^2_{L^2(\mathbb{R}^3)} \frac{1}{R^2} \right\},$$

where $R \in (0, +\infty)$. For $1 \le s \le N$ we use the identity

$$G_{1,s}(x) - G_{2,s}(x) = \int_0^1 \nabla g_s(u_0(x) + tv_1(x) + (1-t)v_2(x)).(v_1(x) - v_2(x))dt.$$

Clearly, for $v_{1,2}(x) \in B_\rho$ and $t \in [0,1]$ we easily obtain the upper bound for $\|v_2(x) + t(v_1(x) - v_2(x))\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)}$ as

$$t\|v_1(x)\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} + (1-t)\|v_2(x)\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)} \le \rho.$$

Thus, $v_2(x) + t(v_1(x) - v_2(x)) \in B_\rho$ as well. We derive

$$|G_{1,s}(x) - G_{2,s}(x)| \le \text{sup}_{z \in I} |\nabla g_s(z)| |v_1(x) - v_2(x)| = a_{1,s} |v_1(x) - v_2(x)|.$$

Therefore,

$$\|G_{1,s}(x) - G_{2,s}(x)\|_{L^2(\mathbb{R}^3)} \le a_{1,s} \|v_1(x) - v_2(x)\|_{H^2(\mathbb{R}^3, \mathbb{R}^N)}.$$

Evidently, for $1 \le j \le N$

$$\frac{\partial g_s}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x)) = \int_0^1 \nabla \frac{\partial g_s}{\partial z_j}(\tau[u_0(x) + tv_1(x) + (1-t)v_2(x)]).$$

$$.(u_0(x) + tv_1(x) + (1-t)v_2(x))d\tau.$$

Hence

$$\left| \frac{\partial g_s}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x)) \right| \le$$

$$\leq \sup_{z \in I} \Big| \nabla \frac{\partial g_s}{\partial z_j} \Big| \{|u_0(x)| + t|v_1(x)| + (1-t)|v_2(x)|\},$$

where $t \in [0,1]$. Clearly, by virtue of the Schwarz inequality we estimate $|G_{1,s}(x) - G_{2,s}(x)|$ from above by

$$\sum_{j=1}^{N} a_{2,s,j} \{|u_0(x)| + \frac{1}{2}|v_1(x)| + \frac{1}{2}|v_2(x)|\} |v_{1,j}(x) - v_{2,j}(x)| \leq$$

$$\leq a_{2,s} \{|u_0(x)| + \frac{1}{2}|v_1(x)| + \frac{1}{2}|v_2(x)|\} |v_1(x) - v_2(x)|.$$

The Schwarz inequality gives us the upper bound for $\|G_{1,s}(x) - G_{2,s}(x)\|_{L^1(\mathbb{R}^3)}$ as

$$a_{2,s} \{\|u_0(x)\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)} + \frac{1}{2}\|v_1(x)\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)} + \frac{1}{2}\|v_2(x)\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)}\} \times$$

$$\times \|v_1(x) - v_2(x)\|_{L^2(\mathbb{R}^3,\mathbb{R}^N)} \leq a_2 \{\|u_0(x)\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1\} \|v_1(x) - v_2(x)\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}.$$

This allows us to estimate from above the norm $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)}$ by

$$\varepsilon^2 \mathcal{K}^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \Big\{ \frac{{a_2}^2}{2\pi^2} (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^2 R + \frac{{a_1}^2}{R^2} \Big\}.$$

By virtue of Lemma 4 we minimize the expression above over $R > 0$ to obtain that $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^N)}$ is bounded from above by

$$\varepsilon^2 \mathcal{K}^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)} \frac{3}{2^{\frac{4}{3}}} \frac{a_2^{\frac{4}{3}}}{\pi^{\frac{4}{3}}} (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^{\frac{4}{3}} a_1^{\frac{2}{3}}. \tag{2.8}$$

By means of (2.6) and (2.7) we have

$$-\Delta(u_{1,s} - u_{2,s}) = \varepsilon_s \sqrt{-\Delta} \int_{\mathbb{R}^3} \mathcal{K}_s(x-y)[G_{1,s}(y) - G_{2,s}(y)]dy, \quad 1 \leq s \leq N.$$

Hence via (2.2) we arrive at

$$\|\Delta(u_{1,s} - u_{2,s})\|^2_{L^2(\mathbb{R}^3)} \leq \varepsilon^2 \|\nabla \mathcal{K}_s\|^2_{L^2(\mathbb{R}^3)} \|G_{1,s}(x) - G_{2,s}(x)\|^2_{L^1(\mathbb{R}^3)} \leq$$

$$\leq \varepsilon^2 \|\nabla \mathcal{K}_s\|^2_{L^2(\mathbb{R}^3)} a_2^2 (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)},$$

such that $\sum_{s=1}^{N} \|\Delta(u_{1,s} - u_{2,s})\|^2_{L^2(\mathbb{R}^3)}$ is estimated from above by

$$\varepsilon^2 Q^2 a_2^2 (\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)} + 1)^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3,\mathbb{R}^N)}. \tag{2.9}$$

By virtue of inequalities (2.8) and (2.9) the norm $\|u_1 - u_2\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}$ is bounded from above by

$$\varepsilon a_2^{\frac{2}{3}}(\|u_0\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}+1)\Big[\frac{3}{2^{\frac{4}{3}}}\frac{\mathcal{K}^2 a_1^{\frac{2}{3}}}{\pi^{\frac{4}{3}}}+Q^2 a_2^{\frac{2}{3}}\Big]^{\frac{1}{2}}\|v_1-v_2\|_{H^2(\mathbb{R}^3,\mathbb{R}^N)}.$$

Hence, the map $T_g : B_\rho \to B_\rho$ defined by system (1.11) is a strict contraction for all values of $\varepsilon > 0$ small enough. Its unique fixed point $u_p(x)$ is the only solution of system (1.9) in B_ρ. The resulting $u(x) \in H^2(\mathbb{R}^3, \mathbb{R}^N)$ given by (1.8) is a solution of the system of equations (1.2). ■

3. Auxiliary results

Let us recall the solvability conditions for the linear Poisson type equation

$$\sqrt{-\Delta}\phi = f(x), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N} \tag{3.1}$$

established in [28]. We denote the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx, \tag{3.2}$$

with a slight abuse of notations when the functions involved in (3.2) are not square integrable, like for example the ones present in orthogonality conditions (3.3) and (3.4) below. Indeed, if $f(x) \in L^1(\mathbb{R}^d)$ and $g(x) \in L^\infty(\mathbb{R}^d)$, then the integral in the right side of (3.2) makes sense. The technical result below was easily proved in [28] by applying the standard Fourier transform to problem (3.1).

Lemma 5. *Let $f(x) \in L^2(\mathbb{R}^d), d \in \mathbb{N}$.*

1) When $d = 1$ and in addition $|x|f(x) \in L^1(\mathbb{R})$, equation (3.1) admits a unique solution $\phi(x) \in H^1(\mathbb{R})$ if and only if the orthogonality condition

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{3.3}$$

holds.

2) When $d = 2$ and additionally $|x|f(x) \in L^1(\mathbb{R}^2)$, problem (3.1) possesses a unique solution $\phi(x) \in H^1(\mathbb{R}^2)$ if and only if the orthogonality relation

$$(f(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{3.4}$$

holds.

3) When $d \geq 3$ and in addition $f(x) \in L^1(\mathbb{R}^d)$, equation (3.1) has a unique solution $\phi(x) \in H^1(\mathbb{R}^d)$.

Note that in dimensions $d \geq 3$ under the assumptions stated above no orthogonality relations are needed to solve the linear Poisson equation (3.1) in $H^1(\mathbb{R}^d)$.

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