Chaotic dynamics for 2-D tent maps

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Abstract

For a 2D-extension of the classical one-dimensional family of tent maps, we prove the existence of an open set of parameters for which the respective transformation presents a strange attractor with two positive Lyapounov exponents. Moreover, periodic orbits are dense on this attractor and the attractor supports a unique ergodic invariant probability measure.

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1 Introduction

We recover from [6] the family of bidimensional maps $\Lambda_{t,s}$ defined on the triangle $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$, with

$$\mathcal{T}_0 = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq x\}, \qquad \mathcal{T}_1 = \{(x,y) : 1 \leq x \leq 2,\ 0 \leq y \leq 2-x\}$$

by

$$\Lambda_{t,s}(x,y) = \begin{cases} (tx+sy, t(x-y)), & \text{if} \quad (x,y) \in \mathcal{T}_0 \\ (t(2-x)+sy, t(2-x-y)) & \text{if} \quad (x,y) \in \mathcal{T}_1 \end{cases} \tag{1}$$

It is easy to see that the critical set for any $\Lambda_{t,s}$ is the line $\mathcal{C} = \{(x,y) \in \mathbb{R}^2 : x = 1\}$. Furthermore, the triangle $\mathcal{T}$ is invariant for the map $\Lambda_{t,s}$ whenever (t,s) belongs to the set $\Omega = \{(t,s) : 0 \leq t \leq 1, 0 \leq s \leq 2-t\}$. Therefore, an attractor for $\Lambda_{t,s}$ arises inside the

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triangle $\mathcal{T}$. By an attractor for a transformation f defined in a compact manifold M, we mean a transitive attracting set. An attracting set is a f-invariant set A whose stable set

$$W^s(A) = \{z \in M : d(f^n(z), A) \to 0 \text{ as } n \to \infty\}$$

has nonempty interior. An attractor is said to be strange if it contains a dense orbit $\{f^n(z_1) : n \geq 0\}$ displaying exponential growth of the derivative: there exists some constant $c > 0$ such that, for every $n \geq 0$,

$$\|Df^n(z_1)\| \geq \exp(cn).$$

In this paper we restrict ourselves to the case $\{(t,s) \in \Omega : t = s\}$. Let us denote, for simplicity, Λ_t=$\Lambda_{t,t}$, for $0 \leq t \leq 1$. Hence, let us write

$$\Lambda_t(x,y) = \begin{cases} (t(x+y), t(x-y)), & \text{if} \qquad (x,y) \in \mathcal{T}_0 \\ (t(2-x+y), t(2-x-y)) & \text{if} \qquad (x,y) \in \mathcal{T}_1 \end{cases} \tag{2}$$

As was proved in [4] the map Λ_1 displays the same properties of the one-dimensional tent map $\lambda_2(x) = 1 - 2|x|$. Among them, the consecutive pre-images $\{\Lambda_1^{-n}(\mathcal{C})\}_{n\in\mathbb{N}}$ of the critical line $\mathcal{C}$ define a sequence of partitions (whose diameter tends to zero as n goes to infinity) of $\mathcal{T}$ leading us to conjugate Λ_1 to a one sided shift with two symbols. Hence, it easily follows that Λ_1 is transitive in $\mathcal{T}$. Furthermore, for every point $(x_0, y_0) \in \mathcal{T}$ whose orbit never visits the critical line the Lyapounov exponent of Λ_1 along the orbit of (x_0, y_0) is positive (and coincides with $\frac{1}{2}\log 2$) in all nonzero direction. Finally, it can be constructed an absolutely continuous ergodic invariant measure for Λ_1, see again [4]. These were the main reasons why the authors called Λ_1 the *bidimensional tent map.* Since the parameter t in (2) essentially gives the rate of expansion for Λ_t (playing the same roll as the parameter a does for $\lambda_a(x) = 1 - a|x|$), the family Λ_t can be consider a natural extension of the one-dimensional family of tent maps. So, let us call Λ_t a family of $2-D$ *tent maps.*

The first objective will be to analytically obtain the maximal attracting set for Λ_t in $\mathcal{T}$, which will be called $\mathcal{R}_t$, for any $t \in (\frac{\sqrt{2}}{2}, 1]$. The dynamics of Λ_t for $t \in [0, \frac{\sqrt{2}}{2}]$ was completely described in [6]. Moreover, for $t \in (t_0, 1]$ with $t_0 = \frac{1}{\sqrt{2}}(\sqrt{2}+1)^{\frac{1}{4}} \approx 0.882$, we are going to prove that the transformation Λ_t is transitive on this set $\mathcal{R}_t$ (see Theorem 1.1 for details) and, furthermore, that on any Λ_t-dense orbit $\{\Lambda_t^n(x_0, y_0) : n \in \mathbb{N}\}$ not intersecting the critical line $\mathcal{C}$, we have exponential grow of the derivatives (in fact, this is the easiest part of the work, because one easily obtains that the Lyapounov exponent of Λ_t along any orbit not visiting the critical line coincides with $\frac{1}{2}\log(2t)$ in all nonzero direction). Henceforth, once we construct a dense orbit of Λ_t not intersecting the critical set $\mathcal{C}$, an example of two-dimensional robust strange attractors is given in the two-dimensional piecewise linear setting. Moreover, we will also prove that the set of periodic orbits is dense on $\mathcal{R}_t$ and that there exists a unique ergodic absolutely continuous invariant measure supported in $\mathcal{R}_t$. In fact, as we may see in Theorem 1.1, we will prove that, for any $t \in (t_0, 1]$, Λ_t is strongly transitive in $\mathcal{R}_t$; i. e., for any open set U contained in $\mathcal{R}_t$, there exists a natural number n depending on U such that $\Lambda_t^n(U) = \mathcal{R}_t$. This fact, in particular, implies that not only Λ_t but also Λ_t^n is transitive in $\mathcal{R}_t$, for every $n \in \mathbb{N}$.

Theorem 1.1 *For every $t \in (t_0, 1]$ with $t_0 = \frac{1}{\sqrt{2}}(\sqrt{2}+1)^{\frac{1}{4}} \approx 0.882$, the map Λ_t exhibits a strange attractor $\mathcal{R}_t \subset \mathcal{T}$. Moreover the map Λ_t is strongly transitive in $\mathcal{R}_t$, the periodic orbits are dense in $\mathcal{R}_t$, and $\mathcal{R}_t$ supports a unique absolutely continuous invariant and ergodic measure. Furthermore, $\mathcal{R}_t$ is a two dimensional strange attractor: There exists a dense orbit of Λ_t in $\mathcal{R}_t$ with two positive Lyapounov exponents.*

The relationship between the family of piecewise linear maps given in (1) and the return maps emerging when certain kind of homoclinic bifurcations are unfolded by a two-parameter family of three-dimensional diffeomorphisms has been explained in a recient paper by the authors, see [6]. Let us only make a short summary about this relationship. In [7], the author studies certain 3-D homoclinic tangencies where the unstable manifold of the saddle point involved in the homoclinic tangency has dimension two. A two parameter family, $T_{a,b}$, of limit return maps are obtained for these scenarios in [7]. Later, in [4] and [5], a curve $(a(t), b(t))$ in the space of parameters (a, b) was chosen in order to make a numerical study on the geometry of the respective attractors. Finally, in [6], see Proposition 2, and the comments below it, it is stated that the family Λ_t given at (2), becomes the best choice in the piecewise linear setting for describing the dynamics of the limit return maps $T_{a(t),b(t)}$ studied in [4] and [5].

This paper is organized as follows: In Section 2 the maximal attracting set $\mathcal{R}_t$ of Λ_t is described for every $t \in (\frac{\sqrt{2}}{2}, 1]$. In Section 3 an useful partition on $\mathcal{R}_t$ is constructed. In Section 4 we prove Theorem 1.1. The proof of Theorem 1.1 strongly depends on two Lemmas whose proofs are given in Section 5. Section 6 contains some conclusions and final remarks.

2 The invariant set $\mathcal{R}_t$

In this section we will obtain the maximal attracting set for Λ_t, for any $t \in (\frac{\sqrt{2}}{2}, 1]$. To this end it will be very useful Figure 1.

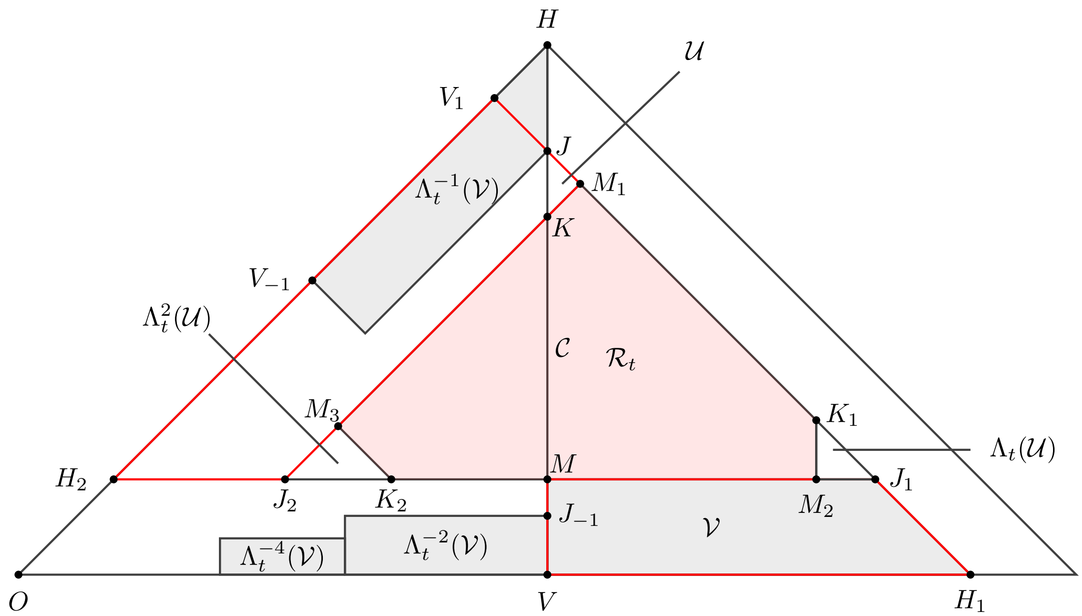

Figure 1: The maximal attracting set for Λ_t

Given two different points in $\mathbb{R}^2$, let us denote by $\overline{AB}$ the straight segment joining A and B; moreover, we will denote by A_j the j-th iterate of a point A by Λ_t. Let $V = (1, 0)$, and $H = (1, 1)$, then the critical set of Λ_t coincides with $\overline{VH}$. We consider $\overline{V_1 H_1}$ the first image of the critical set, with $V_1 = (t, t)$ and $H_1 = (2t, 0)$. This straight segment has slope -1 and intersects with the critical set at the point $J=(1,2t-1)$. Let us consider $J_1=(2t^2,2t(1-t)) \in \overline{V_1H_1}$ and compute the image of the segment $\overline{JJ_1}$. Since $\overline{JJ_1}$ is contained in $\mathcal{T}_1$, its image is again a straight segment $\overline{J_1J_2}$, with $J_2=(2t(1+t-2t^2),2t(1-t))$. This straight segment has null slope and the point J_2 belongs to $\mathcal{T}_0$, so we denote by $M=(1,2t(1-t))$ the intersection between $\overline{J_1J_2}$ and the critical set $\mathcal{C}$. It is easy to see that the symmetric by $\mathcal{C}$ of $\overline{J_2M}$ is contained in $\overline{MJ_1}$. Hence, since Λ_t is symmetric with respect to $\mathcal{C}$, $\Lambda_t(\overline{J_1J_2}) = \Lambda_t(\overline{MJ_1}) = \overline{M_1J_2}$, with $M_1=(t+2t^2(1-t),t-2t^2(1-t))$. This last straight segment has slope $+1$ and one can check that $M_1 \in \mathcal{T}_1$. Hence, $\overline{M_1J_2}$ intersects with the critical set at the point $K=(1,1-4t^2+4t^3)$. Since $K \in \overline{MJ}$, $K_1=(2t-4t^3(1-t),4t^3(1-t)) \in \overline{M_1J_1}$ and $\overline{K_1M_2}=\Lambda_t(\overline{KM_1})$ is a vertical straight segment with $M_2=(2t-4t^3(1-t),2t(1-t)) \in \overline{MJ_1}$. Finally, let us take $\overline{K_2M_3}=\Lambda_t(\overline{K_1M_2})$. This is a straight segment with slope -1 contained in $\mathcal{T}_0$, being $M_3=(2t-2t^3+4t^4(1-t),2t-4t^2+2t^3+4t^4(1-t))$ and $K_2=(2t-2t^2+8t^4(1-t),2t(1-t))$.

We now denote by $\mathcal{C}_1=\overline{M_1K_1}$, $\mathcal{C}_2=\overline{M_2K_2}$, $\mathcal{C}_3=\overline{M_3M_1}$, $\mathcal{C}_4=\overline{K_1M_2}$ and $\mathcal{C}_5=\overline{K_2M_3}$. One has that $\mathcal{C}_i \subset \Lambda_t^i(\mathcal{C})$ and the union of these five segments bound a compact and connected pentagonal domain of $\mathcal{T}$ which will be called $\mathcal{R}_t$, see also Figure 2.

Remark 2.1 *For $t=1$ one has $M_1=J=K=H$, $M_2=J_1=K_1=H_1$, and so on. Therefore, in this case we have that $\mathcal{R}_t$ is the whole triangle $\mathcal{T}$. We refer the reader to a previous paper, see [4], where statements of Theorem 1.1 were already obtained for the case $t=1$.*

Henceforth, we will divide $\mathcal{R}_t$ into the sets $\mathcal{R}_{t,0}=\mathcal{R}_t \cap \mathcal{T}_0$ and $\mathcal{R}_{t,1}=\mathcal{R}_t \cap \mathcal{T}_1$. We also define $\mathcal{C}_0=\overline{MK}=\mathcal{C} \cap \mathcal{R}_t$, being as usual $\mathcal{C}$ the critical set for Λ_t (see Figure 2).

Lemma 2.2 *For every $\frac{\sqrt{2}}{2} < t \leq 1$, $\mathcal{R}_t$ is invariant by Λ_t. Moreover, if $\frac{\sqrt{2}}{2} < t < 1$, then the stable set of $\mathcal{R}_t$ coincides with the interior of $\mathcal{T}$.*

Proof. First, we will prove that $\Lambda_t(\mathcal{R}_{t,0}) \subset \mathcal{R}_t$. In fact, we will demonstrate something stronger (which will be used along this proof) by checking that the image of the triangle with vertices M, J_2 and K is contained in $\mathcal{R}_t$. To this end, since Λ_t is linear on this triangle

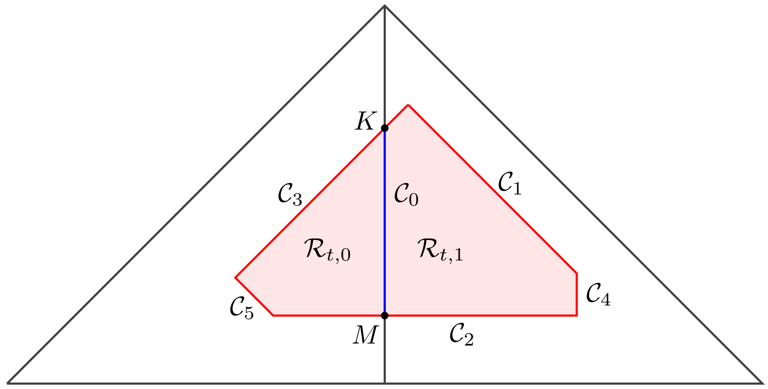

Figure 2: The attracting set $\mathcal{R}_t$

and $\mathcal{R}_t$ is a convex set, it is enough to prove that the image of its vertices belong to $\mathcal{R}_t$. This follows by taking into account that $\Lambda_t(\overline{KJ_2})$ is a horizontal segment joining K_1 with certain point J_3 of $\mathcal{C}_3$. Then, it is enough to obtain that $J_3 \in \overline{M_3K}$ or, in other words, that the ordinate of K_1 is greater or equal than the ordinate of M_3. Consequently, we must check that $\eta(t) = (2t^2-1)(t-1)^2 \geq 0$ for $t \in [\frac{\sqrt{2}}{2}, 1]$. Therefore, the result is proved.

On the other hand, the equality $\Lambda_t(\mathcal{R}_{t,1}) = \mathcal{R}_t$ directly follows from the fact that Λ_t is linear also in $\mathcal{R}_{t,1}$. Therefore, $\Lambda_t(\mathcal{R}_t) = \mathcal{R}_t$.

Now, let us take $\frac{\sqrt{2}}{2} < t < 1$, (x,y) any point in $int(\mathcal{T})$ and prove that for some natural number n, one has $\Lambda_t^n(x,y) \in \mathcal{R}_t$. To this end let us first observe that, for every t, the boundary of $\mathcal{T}$ is invariant and, moreover, since $t \neq 1$ then the interior of $\mathcal{T}$ is also invariant. Therefore, if $(x,y) \in int(\mathcal{T})$, then for an infinite number of iterates n_k one must have $\Lambda_t^{n_k}(x,y) \in \mathcal{T}_1$ because the orbit of (x,y) can not visit the boundary of $\mathcal{T}$ and it can not remain for an infinite number of successive iterates in $\mathcal{T}_0$. Let us divide the set $(\mathcal{T}_1 \cap \Lambda_t(\mathcal{T})) \setminus \mathcal{R}_t$ into three different zones: The triangle $\mathcal{U}$ with vertices K, J and M_1, the triangle $\Lambda_t(\mathcal{U})$ and the polygonal open region $\mathcal{V}$ with vertices J_1, H_1, V and M. See again Figure 1. In the first part of this proof we have demonstrated that $\Lambda_t^3(\mathcal{U}) \subset \mathcal{R}_t$ hence, now, it is enough to prove that every orbit passing through $\mathcal{V}$ converges to $\mathcal{R}_t$. We may assume that if $\Lambda_t^n(x,y) \in \mathcal{V}$, then $\Lambda_t^{n-1}(x,y) \in \mathcal{T}_0$ because the pre-image of $\mathcal{V}$ in $\mathcal{T}_1$ (the symmetric of the pre-image of $\mathcal{V}$ in $\mathcal{T}_0$ with respect to the critical line) is outside $\Lambda_t(\mathcal{T})$. Moreover, $\Lambda_t(\mathcal{V})$ is the polygonal region with vertices J_2, H_2, V_1 and M_1. Hence, $\Lambda_t(\mathcal{V}) \subset \mathcal{T}_0 \cup \mathcal{U}$.

Now the result is consequence of the following three claims:

CLAIM 1.- If some point (x, y) satisfies (x, y), $\Lambda_t(x, y)$ and $\Lambda_t^2(x, y)$ belong to $int(\mathcal{T}_0)$ then, since (x, y) and $\Lambda_t^2(x, y) = (x_2, y_2)$ belong to the same line $y = mx$, $0 < m < 1$, with $x_2 > x$, one has $dist(\Lambda_t^2(x, y), \overline{0H}) > dist((x, y), \overline{0H})$.

CLAIM 2.- If some point (x, y) satisfies $(x, y) \in \mathcal{T}_0$, $\Lambda_t(x, y) \in \mathcal{V}$, $\Lambda_t^2(x, y) \in \mathcal{T}_0$ then, for every $t \in (\frac{\sqrt{2}}{2}, 1]$, $dist(\Lambda_t^2(x, y), \overline{0H}) > dist((x, y), \overline{0H})$. This fact is due to the expansiveness of Λ_t for every $t \in (\frac{\sqrt{2}}{2}, 1]$. In fact, Λ_t expands all the segments not intersecting the critical line by a factor $\sqrt{2}t$.

CLAIM 3.- If some point $(x, y) \in \mathcal{V}$ satisfies $\Lambda_t^i(x, y) \in \mathcal{T}_0$, $i = 1, \ldots, k$ and $\Lambda_t^{k+1}(x, y) \in \mathcal{V}$, then k is odd, and $dist(\Lambda_t^{k+2}(x, y), \overline{0H}) > dist(\Lambda_t(x, y), \overline{0H})$. This last claim easily follows if we take into account that, since $J \in \overline{V_1M_1}$ then $J_{-1} \in \overline{VM}$ and therefore $\Lambda_t^{-2j}(\mathcal{V}) \cap \Lambda_t(\mathcal{V}) = \emptyset$, for every $j \in \mathbb{N}$.

Therefore, for every point (x, y) in the interior of $\mathcal{T}$ one has $\Lambda_t^n(x, y) \in (\mathcal{T}_1 \cap \Lambda_t(\mathcal{T})) \setminus \mathcal{V}$, for some $n \in \mathbb{N}$ and thus the result is proved. ∎

3 A partition of $\mathcal{R}_t$

We are going to construct a partition for the maximal attracting set $\mathcal{R}_t = \mathcal{R}_{t,0} \cup \mathcal{R}_{t,1}$, see Figure 4. To this end, let us define the sets, $\Theta_0 = \mathcal{R}_{t,0}$, $\Theta_{-1} = (\Lambda_t^{-1}(\Theta_0)) \cap \mathcal{T}_1 = (\Lambda_t^{-1}(\Theta_0)) \cap \mathcal{R}_{t,1}$, and, successively, $\Theta_{-k} = (\Lambda_t^{-1}(\Theta_{-(k-1)})) \cap \mathcal{R}_{t,1}$. Let us observe that, for every $t \in (\frac{\sqrt{2}}{2}, 1]$, there exists a fixed point of Λ_t in $\mathcal{R}_{t,1}$ given by

$$P = P_t = (\frac{2t(2t+1)}{2t^2+2t+1}, \frac{2t}{2t^2+2t+1}). \tag{3}$$

From the fact that $\Lambda_{t|\mathcal{R}_{t,1}}$ is a linear expansion, the Λ_t-orbit of any point in $\mathcal{R}_t$, except for the fixed point P, must visits $\mathcal{R}_{t,0} = \Theta_0$. Hence,

$$\mathcal{R}_t \setminus \{P\} = \bigcup_{k=0}^{\infty} \Theta_{-k}.$$

Moreover, for any $k = 1, 2, \ldots$, we define the set

$$\mathcal{A}_k = \mathcal{R}_t \setminus \bigcup_{j=0}^{k-1} \Theta_{-j} \tag{4}$$

which is a pentagonal domain containing P such that $\Lambda_t(\mathcal{A}_k) = \mathcal{A}_{k-1}$ and $\Lambda_t^k(\mathcal{A}_k) = \mathcal{R}_t$. The family of sets $\{\mathcal{A}_k\}_{k=1}^{\infty}$ is a basis of neighborhoods of the fixed point P.

Remark 3.1 *The map $\Lambda^k_{t|\mathcal{A}_k}$ linearly sends $\mathcal{A}_k$ into $\mathcal{R}_t$ and moreover, $\Lambda_t^k(\Theta_{-j}) = \Theta_{-j+k}$, for every $j \geq k$. Furthermore, since Λ_t rotates (under an angle $\rho = -\frac{3\pi}{4}$) and expands (uniformly in any direction) any set contained in $\mathcal{R}_{t,1}$, it easily follows that $\{\Theta_{-j}\}_{j=k}^{\infty}$ is a partition of $\mathcal{A}_k$ which coincides (of course up to a linear change in coordinates) with the original partition $\{\Theta_{-j}\}_{j=0}^{\infty}$ of $\mathcal{R}_t$. See Figure 3 where the set $\mathcal{R}_t$ is rotated by an angle $\rho = -\frac{3\pi}{4}$ and observe the shape of $\{\Theta_{-j}\}_{j=1}^{\infty}$ as a partition of $\mathcal{A}_1$ compared with the shape of $\{\Theta_{-j}\}_{j=0}^{\infty}$ as a partition of $\mathcal{R}_t$.*

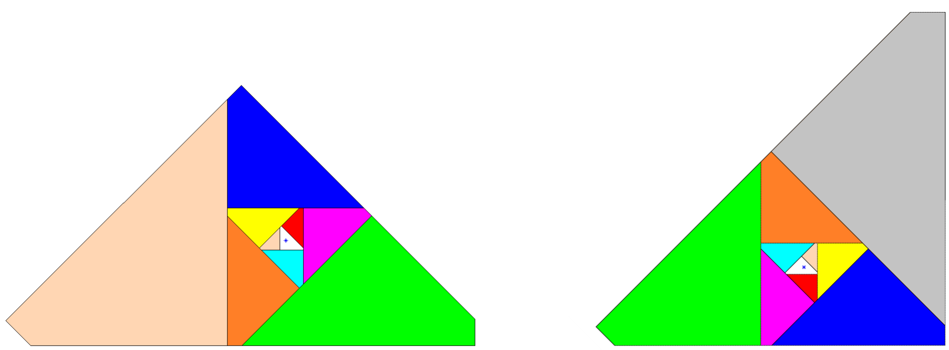

Figure 3: Self-similarities of the partition of $\mathcal{R}_t$

Now, let us consider the straight segments given by $\mathcal{C}_{-1} = (\Lambda_t^{-1}(\mathcal{C}_0)) \cap \mathcal{R}_{t,1}$ and, successively, $\mathcal{C}_{-k} = (\Lambda_t^{-1}(\mathcal{C}_{-(k-1)})) \cap \mathcal{R}_{t,1}$ (see Figure 4). Recall that the sets $\mathcal{C}_j$, for $j = 0, \ldots, 5$ were also defined in the last section. If, for some $i, j \in \mathbb{Z}$ with $i < j \leq 5$, the sets $\mathcal{C}_i$ and $\mathcal{C}_j$ intersect, let us denote by $P_{i,j} = \mathcal{C}_i \cap \mathcal{C}_j$ (see Figure 4). In this way we have $\Lambda_t(P_{i,j}) = P_{i+1,j+1}$.

It is not hard to see that, for every $j \leq 2$, $\mathcal{C}_j$ is a straight segment joining the point $P_{j,j+2}$ of $\mathcal{C}_{j+2}$ with the point $P_{j,j+3}$ of $\mathcal{C}_{j+3}$. Hence, we may write $\mathcal{C}_j = \overline{P_{j,j+2}P_{j,j+3}}$, for

$j \leq 2$. Moreover, for every $j \leq 2$, $\mathcal{C}_j$ also intersects $\mathcal{C}_{j-3}$ and $\mathcal{C}_{j-2}$ in $P_{j-3,j}$ and $P_{j-2,j}$, respectively.

Henceforth, for every point $(x, y) \in \Theta_{-k}$ we will consider $\mathbb{S}_k(x, y)$ the symmetric of (x, y) with respect to $\mathcal{C}_{-k}$.

Lemma 3.2 *For every $k = 0, 1, 2 \ldots$ the following statements hold:*

i) $\mathbb{S}_k(\Theta_{-k}) \subset \mathcal{A}_{k+1} = \mathcal{R}_t \setminus \bigcup_{j=0}^{k} \Theta_{-j} \subset \mathcal{R}_{t,1}$.

ii) For every $(x, y) \in \Theta_{-k}$, $\Lambda_t^j(x, y) = \Lambda_t^j(\mathbb{S}_k(x, y))$ for every $j > k$.

iii) For every $t \in (\sqrt[3]{\frac{1}{2}}, 1]$, $P \in \mathbb{S}_k(\Theta_{-k})$. Consequently, the fixed point P has a pre-image $P_k^\star$ in Θ_{-k}. This point $P_k^\star$ satisfies $\Lambda_t(P_{k+1}^\star) = P_k^\star$, $\Lambda_t(P_0^\star) = P$ and $\mathbb{S}_k(P_k^\star) = P$.

Proof. From Remark 3.1, it is enough to prove the first statement for $k = 0$. But, for proving that $\mathbb{S}_0(\Theta_0) \subset \mathcal{A}_1 = \mathcal{R}_{t,1}$ it suffices to observe that the point K belongs to the segment $\overline{M_3 M_1}$ and therefore $\mathbb{S}_0(\overline{M_3 K}) \subset \mathcal{A}_1$ (see Figure 1). The second statement easily follows because for each $(x, y) \in \Theta_{-k}$, the points (x, y) and $\mathbb{S}_k(x, y)$ are symmetric with respect to $\mathcal{C}_{-k}$ and, since $\Lambda_t^j(\overline{(x, y)\mathbb{S}_k(x, y)}) \cap \mathcal{C}_0 = \emptyset$, for $j = 0, 1, \ldots, k-1$, we have that $\Lambda^k(x, y)$ and $\Lambda^k(\mathbb{S}_k(x, y))$ are symmetric with respect to $\mathcal{C}_0$. Hence, $\Lambda^{k+1}(x, y) = \Lambda^{k+1}(\mathbb{S}_k(x, y))$. Once again due to Remark 3.1, it is enough to prove the third statement only for $k = 0$; i. e., that $P \in \mathbb{S}_0(\Theta_0)$, for every $t \in (\sqrt[3]{\frac{1}{2}}, 1]$. Let us first compute the pre-image of P in $\mathcal{T}_0$ (observe that this pre-image has to be the symmetric of P with respect to the critical line):

$$P_0^\star = \left(\frac{2t+2}{2t^2+2t+1}, \frac{2t}{2t^2+2t+1}\right). \tag{5}$$

Then, we must obtain the value of the parameter t for which this point $P_0^\star$ belongs to the straight segment $\mathcal{C}_3 = \overline{M_3 M_1}$. As this segment takes part of the line given by $L_3 = \{(x, y) \in \mathbb{R}^2 : \ y = x + 4t^2(t-1)\}$ we must solve the equation $\sigma(t) = 0$, with $\sigma(t) = 4t^5 - 2t^3 - 2t^2 + 1$. It holds that $\sigma(\sqrt[3]{\frac{1}{2}}) = 0$ and $\sigma(t) \neq 0$ for every $\sqrt[3]{\frac{1}{2}} < t \leq 1$. The fact that each $P_k^\star$ is a pre-image of P is now consequence of the second statement. ∎

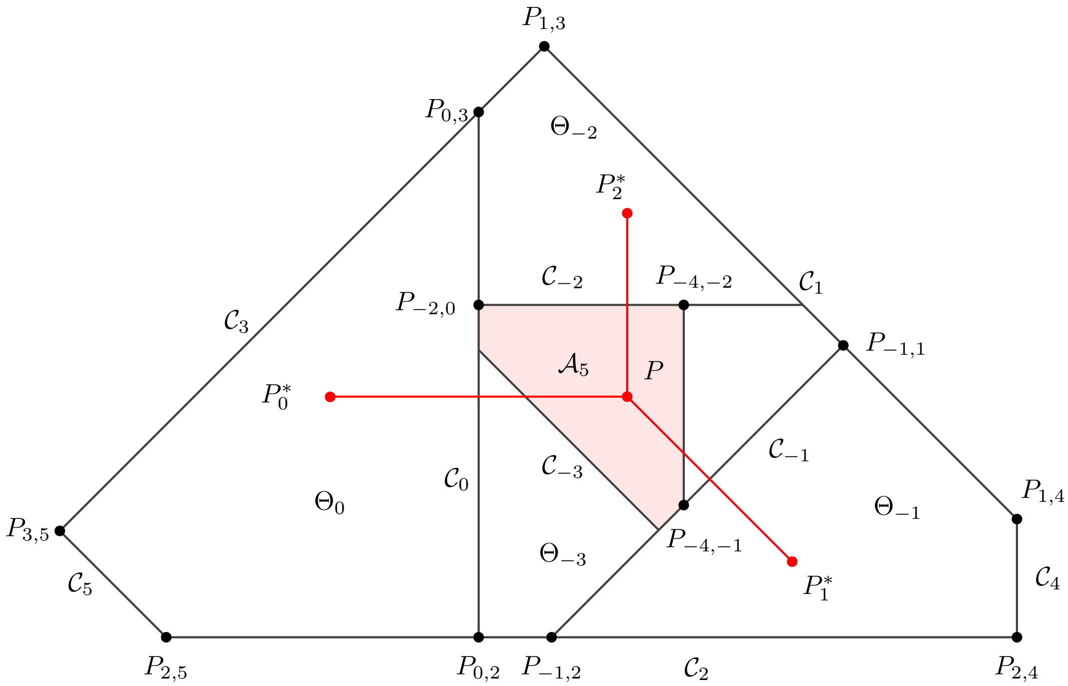

Figure 4: The partition of $\mathcal{R}_t$

For proving the main theorem it will be useful to control the maximal distance between each point $P_k^\star$ and the boundary of the respective Θ_{-k}. Let us denote by $\partial(\Theta_{-k})$ the boundary of Θ_{-k} and define

$$D_k = \max_{(x,y)\in\partial(\Theta_{-k})} dist(P_k^\star,(x,y)). \tag{6}$$

To bound this distance it is enough to control D_0, the maximal distance between $P_0^\star$ to the boundary of Θ_0. This is due to the fact that, given two points A and B in Θ_{-k}, then $\Lambda_t^k(A)$ and $\Lambda_t^k(B)$ belong to Θ_0 and $dist(A,B) = (\sqrt{2}t)^{-k} dist(\Lambda_t^k(A),\Lambda_t^k(B))$. Therefore since $\Lambda_t^k(P_k^\star) = P_0^\star$, $D_0 = (\sqrt{2}t)^k D_k$. Moreover, since Θ_0 is a convex polygonal domain one has that D_0 coincides with the maximum distance between $P_0^\star$ and the four vertices of the boundary of Θ_0, see Figure 4. That is, denoting by

$$D_{0,3} = dist(P_0^\star, P_{0,3}) \quad , \quad D_{2,5} = dist(P_0^\star, P_{2,5})$$

$$D_{0,2} = dist(P_0^\star, P_{0,2}) \quad , \quad D_{3,5} = dist(P_0^\star, P_{3,5}).$$

then it follows that $D_0 = \max\{D_{0,3}, D_{0,2}, D_{2,5}, D_{3,5}\}$.

Then, in the proof of Lemma 4.7 we will use the following result:

Lemma 3.3 *There exist $t_{0,2} \approx 0.8478$, $t_{0,3} \approx 0.9112$ such that*

i) $D_0 = D_{0,2}$ *if* $t \in [1/\sqrt[3]{2}, t_{0,2}]$.

ii) $D_0 = D_{0,3}$ *if* $t \in [t_{0,2}, t_{0,3}]$.

iii) $D_0 = D_{3,5}$ *if* $t \in [t_{0,3}, 1]$.

Proof. The coordinates of $P_{0,3}(= K)$, $P_{0,2}(= M)$, $P_{2,5}(= K_2)$ and $P_{3,5}(= M_3)$, see Figures 1 and 4, were given in Section 2. The coordinates of $P_0^\star$ were given at (5). Then, one may check that

$$\begin{aligned} D_{0,3}^2 &= \frac{(1-2t^2)^2}{1+2t+2t^2}(2-4t+4t^2-8t^3+8t^4) \\ D_{0,2}^2 &= \frac{(1-2t^2)^2}{1+2t+2t^2}(1-2t+2t^2) \\ D_{2,5}^2 &= \frac{(1-2t^2)^2}{1+2t+2t^2}(4-8t+16t^2-16t^3+8t^4-16t^5+16t^6-32t^7+32t^8) \\ D_{3,5}^2 &= \frac{(1-2t^2)^2}{1+2t+2t^2}(4-8t+8t^2-8t^3+8t^6-16t^7+16t^8) \end{aligned} \tag{7}$$

Then, a numerical analysis in the interval of parameters $[1/\sqrt[3]{2}, 1]$, allows us to conclude the statement of the lemma. ∎

4 Proof of Theorem 1.1

Along the rest of the paper we will denote by $B(q, r)$ the ball in $\mathbb{R}^2$ centered at the point q with radius r.

Let us begin by stating the following result whose proof, see the expression of Λ_t in (2), is trivial:

Lemma 4.1 *For every $t \in [0, 1]$ if $\mathcal{B} = B(q, r)$ is a ball in $\mathcal{R}_t$ with $\mathcal{B} \cap \mathcal{C}_0 = \emptyset$, then $\Lambda_t(\mathcal{B})$ is also a ball with $\Lambda_t(\mathcal{B}) = B(\Lambda_t(q), \sqrt{2}tr)$.*

In order to prove the strong transitivity of Λ_t it will be necessary to demonstrate the next result, whose proof will be given in Section 4.1. Let us recall that $t_0 \approx 0.882$.

Theorem 4.2 *Let $t \in (t_0, 1]$ and $\mathcal{B} = B(q, r)$ be a ball with $\mathcal{B} \subset \mathcal{R}_t$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. Then, at least one of the two following situations holds:*

A) There exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$, or

B) There exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \widetilde{r})$ with $\widetilde{\mathcal{B}} \subset \mathcal{B}$ such that:

i) $\widetilde{\mathcal{B}} \cap \mathcal{C}_0 = \emptyset$.

ii) If n is the first natural number with $\Lambda_t^n(\widetilde{\mathcal{B}}) \cap \mathcal{C}_0 \neq \emptyset$, then $\Lambda_t^n(\widetilde{\mathcal{B}})$ is again a ball, $\Lambda_t^n(\widetilde{\mathcal{B}}) = B(\Lambda_t^n(\widetilde{q}), (\sqrt{2}t)^n \widetilde{r})$, with $(\sqrt{2}t)^n \widetilde{r} > r$.

Of course, since item B in the above theorem can not occur up to the infinity, the previous theorem easily implies the following:

Lemma 4.3 *Let $t \in (t_0, 1]$. If U is any open set in $\mathcal{R}_t$, then there exists $n \in \mathbb{N}$ such that $\Lambda_t^n(U) = \mathcal{R}_t$; i. e., Λ_t is strongly transitive in $\mathcal{R}_t$.*

Proof. Given any open set U in $\mathcal{R}_t$, let us take any ball $\mathcal{B}$ contained in U. Let n_1 be the first time for which $\Lambda_t^{n_1}(\mathcal{B})$ intersects the critical line $\mathcal{C}_0$. Observe that, from Lemma 4.1, $\mathcal{B}_1 = \Lambda_t^{n_1}(\mathcal{B})$ is again a ball. Then we apply Theorem 4.2 to $\mathcal{B}_1$ in order to obtain either a natural number $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}_1) = \mathcal{R}_t$ (in this case we have finished) or a ball $\widetilde{\mathcal{B}_1} = B(\widetilde{q_1}, \widetilde{r_1}) \subset \mathcal{B}_1$ satisfying that $\widetilde{\mathcal{B}_1} \cap \mathcal{C}_0 = \emptyset$ and, if n_2 is the first natural number for which $\Lambda_t^{n_2}(\widetilde{\mathcal{B}_1}) \cap \mathcal{C}_0 \neq \emptyset$, then $\mathcal{B}_2 = \Lambda_t^{n_2}(\widetilde{\mathcal{B}_1})$ is again a ball and $area(\mathcal{B}_2) > area(\mathcal{B}_1)$. The finiteness of the diameter of $\mathcal{R}_t$ allows us to conclude (after a finite number of steps) the existence of a natural number $n \in \mathbb{N}$ such that $\Lambda_t^n(\mathcal{B}) = \mathcal{R}_t$. ■

Let us assume by the moment that Theorem 4.2 is proved and let us finish the proof of the Main Theorem, Theorem 1.1. The denseness of the periodic points of Λ_t in $\mathcal{R}_t$ is easily obtained from Lemma 4.3: Namely, if U is any open set in $\mathcal{R}_t$, then we may construct a compact set $\mathcal{D}$ contained in U and a natural number n such that Λ_t^n is linear in $\mathcal{D}$ and $\Lambda_t^n(\mathcal{D}) = \mathcal{R}_t$. Therefore, there must exist a periodic point in $\mathcal{D}$ for Λ_t.

Moreover, we recalling that in [6], Section 7, we have already announced that, applying the results of M. Tsujii, see [8], or J. Buzzi, see [2], there exist absolutely continuous invariant measures (a.c.i.m.'s for short) for the family of bidimensional tent maps given in (2) whenever $t \in (\frac{\sqrt{2}}{2}, 1]$. Since for $t \in (\frac{\sqrt{2}}{2}, 1]$, in Lemma 2.2 we have proved that the dynamics converges to the invariant set $\mathcal{R}_t$, it follows that the support of any a.c.i.m. has to be contained in $\mathcal{R}_t$. Moreover, see also Main Theorem in [3], any a.c.i.m. can be written as a convex combination of a fixed, finite collection of ergodic ones. But, since for $t \in (t_0, 1]$ we have demonstrated (ones Theorem 4.2 is proved) that Λ_t is (strongly) transitive on $\mathcal{R}_t$ then there exists only one a.c.i.m and, henceforth, this a.c.i.m. is supported in $\mathcal{R}_t$ and it must be ergodic.

Finally, let us show the existence of at least one dense Λ_t-orbit on $\mathcal{R}_t$ not visiting the critical set $\mathcal{C}_0$ (this is enough to conclude that both Lyapounov exponents along this orbit are positive). To this end, we denote by μ_t the (unique) ergodic a.c.i.m. described in the last paragraph and by *Leb* the usual Lebesgue measure in $\mathbb{R}^2$. Then the existence of such orbit directly follows from the next result:

Lemma 4.4 *For every $t \in (t_0, 1]$ the following statements hold:*

i) $\mu_t(\widetilde{\mathcal{C}}) = 0$, *where* $\widetilde{\mathcal{C}} = \{(x, y) \in \mathcal{R}_t : \Lambda_t^j(x, y) \in \mathcal{C}_0$, *for some* $j \in \mathbb{N}\}$.

ii) There exists a set $\mathcal{S} \in \mathcal{R}_t$ such that $\mu_t(\mathcal{S}) = 1$ and if (x_0, y_0) belongs to $\mathcal{S}$ then its Λ_t-orbit is dense in $\mathcal{R}_t$.

Proof. The first statement follows taking into account that μ_t is absolutely continuous with respect to the Lebesgue measure and that $Leb(\widetilde{\mathcal{C}}) = 0$. The second statement can be proved in the same way as Lemma 4 in [4]. ■

Now, in order to finish the proof of the Main Theorem, see Theorem 1.1, it is enough to prove Theorem 4.2.

4.1 Proof of Theorem 4.2

Let q be a point in $\mathbb{R}^2$, r a positive real number and α_1, α_2 real numbers with $0 \leq \alpha_1 < \alpha_2 \leq 2\pi$. We denote by $CS(q, r, \alpha_1, \alpha_2)$ the circular sector defined by

$$CS(q, r, \alpha_1, \alpha_2) = \{(x, y) : (x, y) = q + r' \exp{(i\alpha)}, (r', \alpha) \in [0, r] \times [\alpha_1, \alpha_2]\}$$

Then, we are interested in obtain the ball of maximum radius inside certain circular sectors. The proof of next lemma is easy.

Lemma 4.5 *Let $\mathcal{B} = B(q, r)$ be a ball and $\Gamma = CS(q, r, \alpha_1, \alpha_2)$ be a circular sector. Then,*

i) If $\alpha_2 - \alpha_1 = \pi$, then there exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \frac{r}{2})$ contained in Γ.

ii) If $\alpha_2 - \alpha_1 = \frac{\pi}{2}$, then there exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, (\sqrt{2} - 1)r)$ contained in Γ.

iii) If $\alpha_2 - \alpha_1 = \frac{\pi}{4}$, then there exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \delta_0 r)$ contained in Γ, being $\delta_0 = \frac{\sin\frac{\pi}{8}}{1+\sin\frac{\pi}{8}} \approx 0.2767$.

Let us take a ball $\mathcal{B} = B(q, r)$ with $\mathcal{B} \subset \mathcal{R}_t$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. We may assume that the center q belongs to $\mathcal{R}_{t,1}$, because in the case in which $q \in \mathcal{R}_{t,0} = \Theta_0$ we may work with $\mathcal{B}^*$, being $\mathcal{B}^*$ the symmetric of $\mathcal{B}$ with respect to the critical line $\mathcal{C}_0$.

Remark 4.6 *Of course, $\mathcal{B}^*$ is a ball centered at $\mathcal{R}_{t,1}$. Moreover, the successive iterates of $\mathcal{B}$ and $\mathcal{B}^*$ coincide due to the fact that Λ_t is symmetric with respect to the critical line $\mathcal{C}_0$. Therefore, if anyone of the two items of Theorem 4.2 is proved for $\mathcal{B}^*$, then the same item is proved for $\mathcal{B}$.*

We split the proof of Theorem 4.2 in several cases according in which Θ_{-k} the center of the ball q is situated. Let us begin with the simplest cases.

Lemma 4.7 *Let $t \in (t_0, 1]$ and $\mathcal{B} = B(q, r)$ be a ball in $\mathcal{R}_t$, then*

i) If $q \in \Theta_{-1}$, then $\mathcal{B} \cap \mathcal{C}_0 = \emptyset$.

ii) If $q \in \Theta_{-k}$ with $k = 4$ or $k \geq 6$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$, then there exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$.

Proof. If $\mathcal{B} = B(q, r) \subset \mathcal{R}_t$ and $q \in \Theta_{-1}$, then $\mathcal{B} \subset \Theta_{-1} \cup \mathbb{S}_1(\Theta_{-1})$, being as usual, $\mathbb{S}_1(x, y)$ the symmetric of a point $(x, y) \in \Theta_{-1}$ with respect to $\mathcal{C}_{-1}$. Now, see Figure 4, since $P_{-1,2} \in \overline{P_{0,2}P_{2,4}}$ one has that $\mathbb{S}_1(\Theta_{-1}) \cap \mathcal{C}_0 = \emptyset$ and the first statement follows. In order to prove the second statement we will firstly check that, if $q \in \Theta_{-k}$, then $P_k^\star \in \mathcal{B}$.

To this end, let us recall that we have denoted by D_k the maximum distance between $P_k^\star$ to the boundary of Θ_{-k}. Hence, let us now consider $d_k = dist(\Theta_{-k}, \mathcal{C}_0)$ and observe that it will be enough to check that $D_k < d_k$ for $k = 4$ and $k \geq 6$. Of course, if $\mathcal{B} = B(q, r)$ satisfies $q \in \Theta_{-k}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$ then it necessarily holds that $r \geq d_k > D_k$ and therefore $P_k^\star \in \mathcal{B}$. Recall that $D_k = (\sqrt{2}t)^{-k} D_0$ satisfies $D_{k+1} < D_k$ for every k. Let us suppose that we have proved $D_k < d_k$ for $k \in \{4, 6, 8\}$. Then, since $d_6 = dist(P_{-6,-3}, \mathcal{C}_0)$ and $d_7 = dist(P_{-7,-5}, \mathcal{C}_0)$ with $P_{-6,-3} \in \Theta_{-8}$ and $P_{-7,-5} \in \Theta_{-7}$, one has $d_7 > d_6 > D_6 > D_7$. Moreover, since $d_8 = dist(P_{-5,-3}, \mathcal{C}_0) < d_k$ for every $k > 8$ we also have $d_k > d_8 > D_8 > D_k$. Hence, it is enough to prove $D_k < d_k$ for $k \in \{4, 6, 8\}$.

A direct calculation gives

$$
\begin{aligned}
d_4 &= \frac{-1 + 2t - 4t^3 + 4t^4}{4t^4} \\
d_6 &= \frac{-1 + 2t - 8t^5 + 8t^6}{8t^6} \\
d_8 &= \frac{1 - 2t + 2t^2 - 8t^4 + 8t^5}{8t^5}
\end{aligned}
$$

while $D_k = (\sqrt{2}t)^{-k} D_0$, being (according to Lemma 3.3), $D_0 = D_{0,3} = dist(P_0^\star, P_{0,3})$ for $t \in (t_0, t_{0,3}]$, $D_0 = D_{3,5} = dist(P_0^\star, P_{3,5})$, for $t \in [t_{0,3}, 1]$, with $t_{0,3} \approx 0.9112$. The expression of $D_{0,3}$ and $D_{3,5}$ were given in (7). Now, a numerical analysis shows that, for every $t \in (t_0, 1]$, $D_k < d_k$ for $k = 4, 6$ and $k = 8$. Therefore we have proved that $P_k^\star \in \mathcal{B}$.

Hence, there exists a sufficiently small neighborhood $U \subset \mathcal{B}$ of $P_k^\star$ such that $\Lambda_t^k(U)$ is a neighborhood of the fixed point P. This easily follows taking into account that the orbit of $P_k^\star$ never visits the critical set. Now, let us recall the sets $\mathcal{A}_m = \mathcal{R}_t \setminus \bigcup_{j=0}^{m-1} \Theta_{-j}$, see again (4) and Figure 4. These sets are a basis of neighborhoods of P and moreover $\Lambda_t^m(\mathcal{A}_m) = \mathcal{R}_t$. Take m large enough so that $\mathcal{A}_m \subset \Lambda_t^k(U)$. Then, one has $\Lambda_t^{k+m}(U) = \mathcal{R}_t$. ■

Next, we will study the case in which $q \in \Theta_{-5}$. We will use the following result whose proof is given in the Section 5:

Lemma 4.8 *Let $t \in (t_0, 1]$ and $\mathcal{B} = B(q, r)$ be a ball in $\mathbb{R}^2$ with $q \in \Theta_0$.*

i) If $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 2, 3$ then there exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B} \cap \Theta_0) = \mathcal{R}_t$.

ii) If $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 5$ then there exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B} \cap \Theta_0) = \mathcal{R}_t$.

Proposition 4.9 *Let $t \in (t_0, 1]$ and $\mathcal{B} = B(q, r)$ be a ball in $\mathcal{R}_t$ with $q \in \Theta_{-5}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. Then the following statements holds:*

i) If $\mathcal{B} \cap \mathcal{C}_{-5} \neq \emptyset$, then there exists $j \in \mathbb{N}$ with $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$.

ii) If $\mathcal{B} \cap \mathcal{C}_{-5} = \emptyset$, then there exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \widetilde{r})$ contained in $\mathcal{B} \cap \Theta_{-5}$ such that

$$\Lambda_t^6(\widetilde{\mathcal{B}}) = B(\Lambda_t^6(\widetilde{q}), (\sqrt{2}t)^6 \widetilde{r})$$

with $(\sqrt{2}t)^6 \widetilde{r} > r$.

Proof. Let us take a ball $\mathcal{B} = B(q, r)$ with $q \in \Theta_{-5}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. Firstly, let us assume that $\mathcal{B} \cap \mathcal{C}_{-5} \neq \emptyset$. Let us consider the set $\mathcal{D} = \mathcal{B} \cap \Theta_{-5}$. Then, $\Lambda_t^5(\mathcal{D})$ is a subset of Θ_0 such that there exists a ball $\mathcal{B}^*$ in $\mathbb{R}^2$ centered at $\Lambda_t^5(q) \in \Theta_0$ with $\Lambda_t^5(\mathcal{D}) = \mathcal{B}^* \cap \Theta_0$. Moreover, $\mathcal{B}^* \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 5$. Therefore, applying the second statement of Lemma 4.8, there exists $j \in \mathbb{N}$ with $\Lambda_t^j(\mathcal{B}^* \cap \Theta_0) = \Lambda_t^{j+5}(\mathcal{D}) = \mathcal{R}_t$.

Let us assume that $\mathcal{B} \cap \mathcal{C}_{-5} = \emptyset$. Then, it is always possible to define a circular sector $CS(q, r, \alpha_1, \alpha_2)$ contained in $\mathcal{B} \cap \Theta_{-5}$, with $\alpha_2 - \alpha_1 = \frac{\pi}{4}$, see Figure 5. Applying Lemma 4.5 item iii) we may choose a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \delta_0 r)$, with $\delta_0 \approx 0.2767$, contained in the above circular sector, and therefore in $\mathcal{B} \cap \Theta_{-5}$. Hence, $\Lambda_t^j(\widetilde{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$ for $j = 0, 1, \ldots, 5$ and henceforth, from Lemma 4.1, $\Lambda_t^6(\widetilde{\mathcal{B}})$ is the ball $B(\Lambda_t^6(\widetilde{q}), (\sqrt{2}t)^6 \delta_0 r)$. Finally, it is easy to check that

$$(\sqrt{2}t)^6 \delta_0 > 1$$

for every $t \in (t_0, 1]$. ∎

Now, let us study the case in which $q \in \Theta_{-3}$. Hence, let us assume $\mathcal{B} = B(q, r)$ to be a ball in $\mathcal{R}_t$ satisfying $q \in \Theta_{-3}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$.

We will make use of the point $P_{-3,0} = \mathcal{C}_{-3} \cap \mathcal{C}_0$. In fact, we will distinguish between the case in which $P_{-3,0} \in \mathcal{B}$, see Proposition 4.11 and the case in which $P_{-3,0} \notin \mathcal{B}$, see

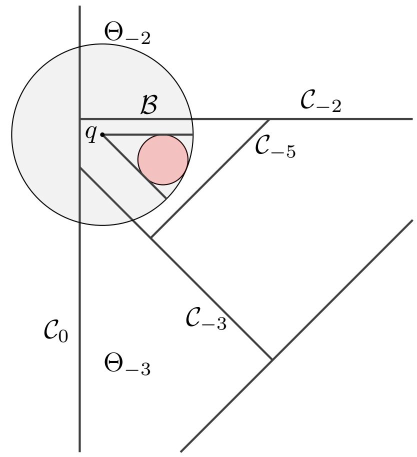

Figure 5: A circular sector for the case $q \in \Theta_{-5}$

Proposition 4.12. Let W be the intersection between $\mathcal{C}_0$ and $\mathbb{S}_3^{-1}(\mathcal{C}_{-5})$ (observe that this point always exists because, from statement i) of Lemma 3.2, $P \in \mathbb{S}_3(\Theta_{-3}) \subset \mathcal{A}_4$). Then, $W_3 = \Lambda_t^3(W) \in \mathcal{C}_3$ and we may construct an isosceles right triangle, denoted by Δ, contained in Θ_0 with vertices W_3, $P_{0,3}$ and $P_{-2,0}$, see Figure 6. Let us also denote by Δ_{-3} the triangle contained in Θ_{-3} with vertices W, $P_{-3,0}$ and $P_{-5,-3}$. Observe that $\Lambda_t^3(\Delta_{-3}) = \Delta$. We will use the following result whose proof is also given in Section 5:

Lemma 4.10 *Let $t \in (t_0, 1]$. If $\mathcal{B} = B(q, r)$ is a ball in $\mathbb{R}^2$ with $q \in \Delta$ such that $P_{0,3} \in \mathcal{B}$ and $\mathcal{B} \cap \overline{P_{-2,0}W_3} \neq \emptyset$, then there exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B} \cap \Delta) = \mathcal{R}_t$.*

Proposition 4.11 *Let $t \in (t_0, 1]$ and $\mathcal{B} = B(q, r)$ be a ball in $\mathcal{R}_t$ with $q \in \Theta_{-3}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. If $P_{-3,0} \in \mathcal{B}$, then the following statements hold:*

i) If $\overline{P_{-5,-3}W} \cap \mathcal{B} = \emptyset$, then there exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \widetilde{r})$ contained in $\mathcal{B} \cap \Delta_{-3}$ such that $\Lambda_t^i(\widetilde{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$, for $i = 0, \ldots, 5$, $\Lambda_t^6(\widetilde{\mathcal{B}}) = B(\Lambda_t^6(\widetilde{q}), (\sqrt{2}t)^6\widetilde{r})$ and $(\sqrt{2}t)^6\widetilde{r} > r$.

ii) If $\overline{P_{-5,-3}W} \cap \mathcal{B} \neq \emptyset$, then there exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$.

Proof. If $\overline{P_{-5,-3}W} \cap \mathcal{B} = \emptyset$, then we may always consider the circular sector $\Gamma = CS(q, r, \frac{3\pi}{2}, \frac{7\pi}{4})$ which is contained in $\mathcal{B} \cap \Delta_{-3}$.

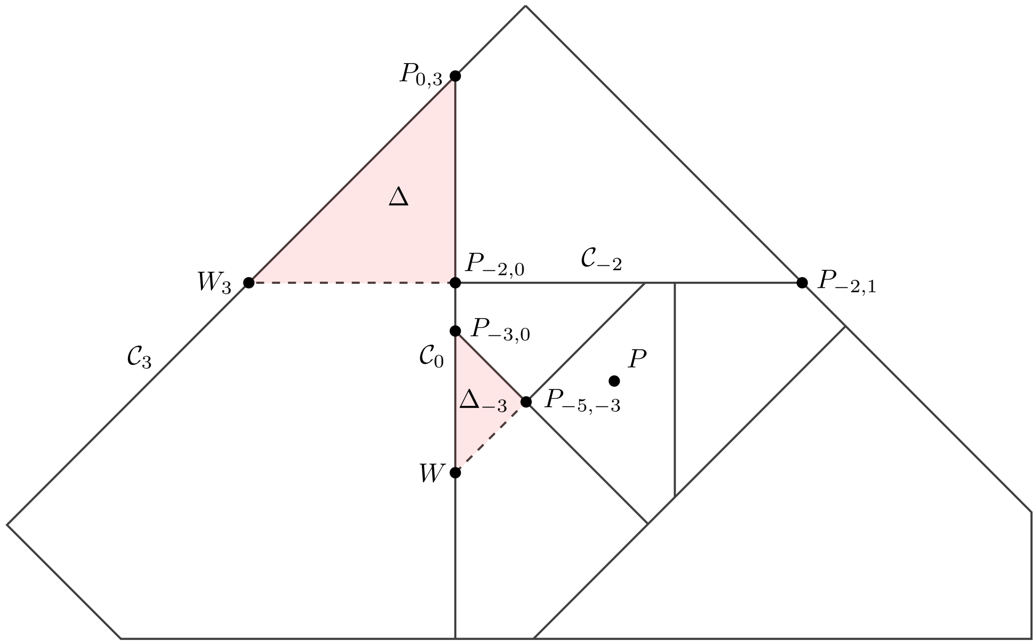

Figure 6: The triangles Δ and Δ_{-3}

Hence, $\mathbb{S}_3(\Gamma) = CS(\mathbb{S}_3(q), r, \frac{7\pi}{4}, 2\pi)$ is a circular sector contained in Θ_{-5}. So, we may pick a ball $\mathcal{B}^* = B(q^*, \delta_0 r) \subset \mathbb{S}_3(\Gamma)$ such that $\mathcal{B}^* \subset \Theta_{-5}$. Then, $\Lambda_t^j(\mathcal{B}^*) \cap \mathcal{C}_0 = \emptyset$, for $i = 0, \dots, 5$, $\Lambda_t^6(\mathcal{B}^*) = B(\Lambda_t^6(q^*), (\sqrt{2}t)^6 \delta_0 r)$, with $(\sqrt{2}t)^6\delta_0 > 1$ for every $t \in (t_0, 1]$. To conclude the proof of the first statement it is enough to take $\widetilde{\mathcal{B}} = \mathbb{S}_3^{-1}(\mathcal{B}^*) = B(\widetilde{q}, \delta_0 r)$, with $\widetilde{q} = \mathbb{S}_3^{-1}(q^*)$. From the second statement of Lemma 3.2, we have that $\Lambda_t^6(\widetilde{\mathcal{B}}) = \Lambda_t^6(\mathcal{B}^*) = B(\Lambda_t^6(\widetilde{q}), (\sqrt{2}t)^6\delta_0 r)$.

Now, let us assume that $\overline{P_{-5,-3}W} \cap \mathcal{B} \neq \emptyset$, with $\mathcal{B} = B(q, r)$. Let $\mathcal{D} = \mathcal{B} \cap \Delta_{-3}$. Then $\Lambda_t^3(\mathcal{D})$ is a subset of Δ such that there exists a ball $\mathcal{B}^*$ in $\mathbb{R}^2$ centered at $\Lambda_t^3(q) \in \Delta$ with $\Lambda_t^3(\mathcal{D}) = \mathcal{B}^* \cap \Delta$. Moreover, $P_{0,3} \in \mathcal{B}^*$ and $\mathcal{B}^* \cap \overline{P_{-2,0}W_3} \neq \emptyset$, then applying Lemma 4.10, there exists $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}^* \cap \Delta) = \Lambda_t^{j+3}(\mathcal{D}) = \mathcal{R}_t$. Hence $\Lambda_t^{j+3}(\mathcal{B}) = \mathcal{R}_t$. ■

Now, we deal with the case $P_{-3,0} \notin \mathcal{B}$.

Proposition 4.12 *Let $t \in (t_0, 1]$ and $\mathcal{B} = B(q, r)$ be a ball in $\mathcal{R}_t$ with $q \in \Theta_{-3}$ and $\mathcal{B} \cap \mathcal{C}_0 \neq \emptyset$. If $P_{-3,0} \notin \mathcal{B}$ then the following statement holds:*

i) If $\mathcal{B}$ does not intersects $\mathcal{C}_{-1}$ and $\mathcal{C}_{-3}$ at the same time, there is a ball $\widetilde{\mathcal{B}} = B(\widetilde{q}, \widetilde{r})$

contained in $\mathcal{B}\cap\Theta_{-3}$ such that

$$\Lambda_t^4(\widetilde{\mathcal{B}}) = B(\Lambda_t^4(\widetilde{q}),(\sqrt{2}t)^4\widetilde{r}),$$

and $(\sqrt{2}t)^4\widetilde{r} > r$.

ii) If $\mathcal{B}$ intersects $\mathcal{C}_{-1}$ and $\mathcal{C}_{-3}$, then there exists $j\in\mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$.

Proof. Let us first assume that $\mathcal{B}$ does not intersect $\mathcal{C}_{-1}$ and $\mathcal{C}_{-3}$. Then there is a circular sector $CS(q,r,\alpha_1,\alpha_2)$ contained in $\mathcal{B}\cap\Theta_{-3}$ with $\alpha_2-\alpha_1=\pi$. Then, from the first statement of Lemma 4.5 there is a ball $\widetilde{\mathcal{B}} = B(\widetilde{q},\frac{r}{2})\subset\mathcal{B}\cap\Theta_{-3}$. Hence $\Lambda_t^i(\widetilde{\mathcal{B}})\cap\mathcal{C}_0=\emptyset$, for $i=0,\ldots,3$, $\Lambda_t^4(\widetilde{\mathcal{B}}) = B(\Lambda_t^4(\widetilde{q}),(\sqrt{2}t)^4\frac{r}{2})$, with $2t^4>1$ for every $t\in(t_0,1]$.

Now, let us assume that $\mathcal{B}\cap\mathcal{C}_{-3}\neq\emptyset$ and $\mathcal{B}\cap\mathcal{C}_{-1}=\emptyset$ (the case $\mathcal{B}\cap\mathcal{C}_{-3}=\emptyset$ and $\mathcal{B}\cap\mathcal{C}_{-1}\neq\emptyset$ can be demonstrated in the same way). Let us denote by $q_1 = q+r\exp(i\varphi_1)$ and $\widetilde{q}_1 = q+r\exp(i\widetilde{\varphi}_1)$ the intersections between the boundary of $\mathcal{B}$ with $\mathcal{C}_0$ and by $q_2 = q+r\exp(i\varphi_2)$ and $\widetilde{q}_2 = q+r\exp(i\widetilde{\varphi}_2)$ the intersections between the boundary of $\mathcal{B}$ with $\mathcal{C}_{-3}$. See Figure 7. Since $\varphi_2-\varphi_1>0$, $\widetilde{\varphi}_1-\widetilde{\varphi}_2\geq 0$ and $\varphi_2-\varphi_1-(\widetilde{\varphi}_1-\widetilde{\varphi}_2)=\pi/2$ it holds that $\varphi_2-\varphi_1\geq\pi/2$. Then we may construct a circular sector $CS(q,r,\alpha_1,\alpha_2)\subset\mathcal{B}\cap\Theta_{-3}$ with $\alpha_2-\alpha_1=\pi/2$.

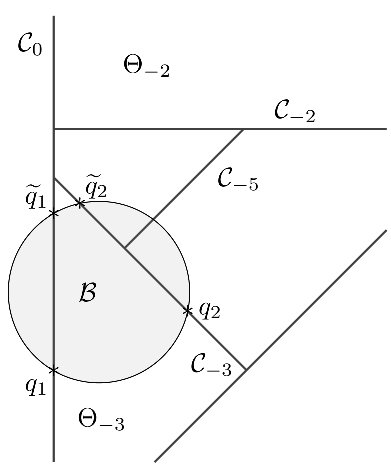

Figure 7: The case $\mathcal{B}\cap\mathcal{C}_{-3}\neq\emptyset$, $\mathcal{B}\cap\mathcal{C}_{-1}=\emptyset$

From the second statement of Lemma 4.5 there exists a ball $\widetilde{\mathcal{B}} = B(\widetilde{q},(\sqrt{2}-1)r)\subset\mathcal{B}\cap\Theta_{-3}$ such that $\Lambda_t^4(\widetilde{\mathcal{B}}) = B(\Lambda_t^4(q),(\sqrt{2}t)^4(\sqrt{2}-1)r)$, with $4t^4(\sqrt{2}-1)>1$, for every

$t \in (t_0, 1]$. Let us remark that this last inequality holds for every $t > t_0 = \frac{1}{\sqrt{2}}(\sqrt{2}+1)^{\frac{1}{4}}$. This fact gives us the value of t_0 in the Main Theorem. The first statement is proved.

Now, for proving the second statement let us assume that $\mathcal{B} = B(q,r)$ satisfies $q \in \Theta_{-3}$, $\mathcal{B} \cap \mathcal{C}_{-j} \neq \emptyset$ for $j = 0, 1, 3$. We take $\mathcal{D} = \mathcal{B} \cap \Theta_{-3}$. Then, the set $\Lambda_t^3(\mathcal{D})$ is a subset of Θ_0 such that there exists a ball $\mathcal{B}^*$ in $\mathbb{R}^2$ centered at $\Lambda_t^3(q) \in \Theta_0$ with $\Lambda_t^3(\mathcal{D}) = \mathcal{B}^* \cap \Theta_0$. Moreover, $\mathcal{B}^* \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 2, 3$. Therefore, applying the first statement of Lemma 4.8, we conclude that there is $j \in \mathbb{N}$ such that $\Lambda_t^j(\mathcal{B}^* \cap \Theta_0) = \Lambda_t^{j+3}(\mathcal{D}) = \mathcal{R}_t$. Hence $\Lambda_t^{j+3}(\mathcal{B}) = \mathcal{R}_t$.

■

In order to finish the proof of Theorem 4.2 we must assume that we have a ball $\mathcal{B} = B(q,r)$ under the assumptions of Theorem 4.2 with $q \in \Theta_{-2}$. Let us consider the set $\mathcal{C}^*_{-2} = \mathcal{C}_{-2} \cup \mathbb{S}_0^{-1}(\mathcal{C}_{-2}) = \overline{P_{-2,1}W_3}$, see Figure 6. Let us take $\mathcal{B}^* = B(q^*, r)$ the symmetric of $\mathcal{B}$ with respect to $\mathcal{C}^*_{-2}$. Then:

1) $q^* = \mathbb{S}_2(q) \in \mathcal{A}_3 = \mathcal{R}_t \setminus \bigcup_{j=0}^2 \Theta_{-j}$.

2) $\mathcal{B}^* \cap \mathcal{C}_0 \neq \emptyset$.

3) For every $j > 2$, $\Lambda_t^j(\mathcal{B}) = \Lambda_t^j(\mathcal{B}^*)$.

Let us distinguish between the following cases:

i) If q^* belongs to $\mathcal{A}_6 = \mathcal{R}_t \setminus \bigcup_{j=0}^5 \Theta_{-j}$ then we may apply Lemma 4.7 to conclude that, $\Lambda_t^j(\mathcal{B}^*) = \mathcal{R}_t$, for some $j \in \mathbb{N}$. Therefore it also holds that $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$.

ii) If q^* belongs to Θ_{-5} we apply Proposition 4.9 to get either some $j \in \mathbb{N}$ with $\Lambda_t^j(\mathcal{B}^*) = \mathcal{R}_t$ (in this case $\Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$ also holds) or a ball $\widehat{\mathcal{B}} \subset \mathcal{B}^* \cap \Theta_{-5}$ ($\Lambda_t^j(\widehat{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$ for $j = 0, \ldots, 5$) such that $\Lambda_t^6(\widehat{\mathcal{B}})$ is a ball with radius greater than r. So, if we define $\widetilde{\mathcal{B}} = \mathbb{S}_2^{-1}(\widehat{\mathcal{B}}) \subset \mathcal{B}$, then from Lemma 3.2, we have $\Lambda_t^j(\widehat{\mathcal{B}}) = \Lambda_t^j(\widetilde{\mathcal{B}})$, for every $j > 2$ and therefore Theorem 4.2 is also proved in this case.

iii) Finally, if q^* belongs to Θ_{-3}, then we apply either Proposition 4.11 (if $P_{-3,0} \in \mathcal{B}^*$) or Proposition 4.12 (if $P_{-3,0} \notin \mathcal{B}^*$). If $P_{-3,0} \in \mathcal{B}^*$, then we get either $j \in \mathbb{N}$ with

$\Lambda_t^j(\mathcal{B}^*) = \Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$ or a ball $\widehat{\mathcal{B}} \subset \mathcal{B}^* \cap \Delta_{-3}$ ($\Lambda_t^j(\widehat{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$ for $j = 0, \dots, 5$) such that $\Lambda_t^6(\widehat{\mathcal{B}})$ is a ball with radius greater than r. Defining $\widetilde{\mathcal{B}} = \mathbb{S}_2^{-1}(\widehat{\mathcal{B}})$, we have done.

If $P_{-3,0} \notin \mathcal{B}^*$ then we get either $j \in \mathbb{N}$ with $\Lambda_t^j(\mathcal{B}^*) = \Lambda_t^j(\mathcal{B}) = \mathcal{R}_t$ or a ball $\widehat{\mathcal{B}} \subset \mathcal{B}^* \cap \Theta_{-3}$ ($\Lambda_t^j(\widehat{\mathcal{B}}) \cap \mathcal{C}_0 = \emptyset$ for $j = 0, \dots, 3$) such that $\Lambda_t^4(\widehat{\mathcal{B}})$ is a ball with radius greater than r. Once again, if we define $\widetilde{\mathcal{B}} = \mathbb{S}_2^{-1}(\widehat{\mathcal{B}})$, then we have $\Lambda_t^j(\widehat{\mathcal{B}}) = \Lambda_t^j(\widetilde{\mathcal{B}})$, for every $j > 2$ and therefore Theorem 4.2 is also proved in this case.

Hence, we have ended the proof of Theorem 4.2.

5 Proofs of Lemma 4.8 and Lemma 4.10

5.1 Proof of Lemma 4.10

Let W be the intersection between $\mathcal{C}_0$ and $\mathbb{S}_3^{-1}(\mathcal{C}_{-5})$. Then, $W_3 = \Lambda_t^3(W) \in \mathcal{C}_3$ and recall that Δ is the isosceles right triangle contained in Θ_0 with vertices W_3, $P_{0,3}$ and $P_{-2,0}$, see Figure 6.

In order to avoid tedious notation, during this section, let us define $A = P_{0,3}$ and $Z = P_{-2,0}$. We consider the point H in $\mathcal{C}_0$ given by $H = \mathbb{S}_2^{-1}(P_{-3,0})$, see Figure 8. This point always exists for $t \in (t_1, 1]$ with $t_1 \approx 0.8326$ (for $t = t_1$ one has that $P_{2,5}$ and $P_{-1,2}$ are symmetric with respect to $\mathcal{C}_0$, see Figure 4). In fact, when $t = t_1$ we have $H = A$ (that is A and $P_{-3,0}$ are symmetric with respect to $\mathcal{C}_{-2}$), and when $t = 1$ it follows that $H = Z$ because the lines $\mathcal{C}_{-2}$ and $\mathcal{C}_{-3}$ intersect at the point $Z = P_{-2,0} = P_{-3,0}$.

Let us take the straight line L passing through H with slope -1 and define F the intersection between L and $\mathcal{C}_3$. The triangle whose vertices are A, H and F is denoted by Δ'. Observe that $\Delta' \subset \Delta$.

Lemma 5.1 *For every $t \in (t_1, 1]$ there exists a three-periodic point Q^3 of Λ_t in Δ'.*

Proof. Let us describe how Λ_t^3 acts on the vertices of Δ'. The second image of A is the point $P_{2,5}$. The coordinates of this point were obtained in Section 2 (in fact this point was earlier denoted by $K_2 = (2t - 2t^2 + 8t^4(1-t), 2t(1-t))$, see Figure 1). Therefore, applying the expression of Λ_t given at (2) one has that $A_3 = \Lambda_t^3(A) = \Lambda_t(K_2)$ is the point in $\mathcal{C}_3$ given by $A_3 = (4t^2(1-t+2t^3-2t^4), 8t^5(1-t))$. Now, since $H \in \mathbb{S}_2^{-1}(\mathcal{C}_{-3}) \cap \mathcal{C}_0$ one has $H_3 = \Lambda_t^3(H) \in \mathcal{C}_0 \cap \mathcal{C}_3$. Therefore, $H_3 = A$. Finally, since $F \in \mathbb{S}_0^{-1}(\mathbb{S}_2^{-1}(\mathcal{C}_{-3}))$ we conclude $F_3 = \Lambda_t^3(F) \in \mathcal{C}_0$. But, using that Λ_t^3 is not only linear on Δ' but also preserves angles, we know that $\Lambda_t^3(\Delta')$ must be an isosceles right triangle and therefore the ordinate of F_3 must coincide with the ordinate of A_3. So, $F_3 = \Lambda_t^3(F) = (1, 8t^5(1-t))$. Observe that, for $t = 1$, $\Lambda_t^3(\Delta')$ is the triangle $\mathcal{T}_0$ with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Moreover,

$$dist(A, F_3) = dist(H_3, F_3) = (\sqrt{2}t)^3 dist(H, F) = (\sqrt{2}t)^3 \frac{1}{\sqrt{2}} dist(A, H) > dist(A, H)$$

for every $t \in (\sqrt[3]{\frac{1}{2}}, 1]$. Hence $\Delta' \subset \Lambda_t^3(\Delta')$ and therefore there exists a three-periodic point, denoted by Q^3 of Λ_t in Δ'. ■

An easy numerical calculation shows that

$$Q^3 = (x_{Q^3}, y_{Q^3}) = \left(\frac{4t^2(1+2t^3)}{1+4t^3+8t^6}, \frac{8t^5}{1+4t^3+8t^6}\right). \tag{8}$$

Now, Lemma 4.10 follows from the following result:

Lemma 5.2 *For every $t \in (t_0, 1]$, there exists a set $U \subset \Delta$ satisfying the following properties:*

i) If $\mathcal{B} = B(q, r)$ is a ball with $q \in \Delta$ such that $A = P_{0,3} \in \mathcal{B}$ and $\mathcal{B} \cap \overline{ZW_3} = \mathcal{B} \cap \overline{P_{-2,0}W_3} \neq \emptyset$, then $U \subset \mathcal{B}$.

ii) There exists $m \in \mathbb{N}$ and a set Δ'_{-m} such that $Q^3 \in \Delta'_{-m} \subset U$ and $\Lambda_t^m(\Delta'_{-m}) = \Delta'$.

iii) There exists $j \in \mathbb{N}$ such that $\Lambda_t^j(U) = \mathcal{R}_t$.

Proof. Let us begin by constructing the set U. Let us take again the triangle Δ with vertices A, Z and W_3. For the sake of simplicity, we can make a linear change in coordinates $(X, Y) = L_t(x, y)$, depending on the parameter t, in order to transport the point W_3 into the origin, the point A into $(1, 1)$ and the point Z into $(1, 0)$. So, the triangle Δ becomes into the triangle $\mathcal{T}_0$ with vertices $(0, 0)$, $(1, 1)$ and $(1, 0)$. To this end, it is enough to denote by $l = l_t = dist(W_3, Z)$ and define

$$(X, Y) = L_t(x, y) = (l^{-1}(x - x_{W_3}), l^{-1}(y - y_{W_3})), \tag{9}$$

being $W_3 = (x_{W_3}, y_{W_3})$. In this way, any ball $\mathcal{B} = B(q, r)$ satisfying $q \in \Delta$, $A \in \mathcal{B}$ and $\mathcal{B} \cap \overline{ZW_3} \neq \emptyset$ is mapping by L_t into a ball $\mathcal{B}' = B(q', r')$, satisfying $q' \in \mathcal{T}_0$, $A' = (1, 1) \in \mathcal{B}'$ and $\mathcal{B}' \cap \widetilde{Y} \neq \emptyset$, being

$$\widetilde{Y} = \{(X, Y) \in \mathbb{R}^2 : Y = 0\}.$$

One may easily deduce that any such ball $\mathcal{B}'$ contains the polygonal domain U' whose vertices are $G' = (\frac{1}{2}, \frac{1}{2})$, $R' = (\frac{13}{16}, \frac{3}{16})$, $S' = (1, \frac{3}{8})$ and $A' = (1, 1)$. This last claim follows by bearing in mind that the set U' is convex and therefore it is enough to check that for any such ball $\mathcal{B}'$, $\Omega' \in \mathcal{B}'$ for every $\Omega' \in \{G', R', S', A'\}$. But this fact can be showed by using that the two sets

$$\{(X, Y) : dist((X, Y), (1, 1)) < dist((X, Y), \Omega')\}$$
$$\{(X, Y) : dist((X, Y), \widetilde{Y}) < dist((X, Y), \Omega')\}$$

are disjoint for every $\Omega' \in \{G', R', S', A'\}$. Hence, defining $U = L_t^{-1}(U') \subset \Delta$, the first statement of the lemma follows.

The vertices of the domain U are denoted by $G = L_t^{-1}(G')$ (this is a point in $\mathcal{C}_3$), $R = L_t^{-1}(R')$ (this is a point in the straight segment $\overline{GZ}$), $S = L_t^{-1}(S')$ (this is a point in $\overline{ZA}$) and $A = P_{0,3} = L_t^{-1}(A')$. The shape of this polygonal domain can be seen in Figure 8.

Now, let us show the second statement of the lemma. Recall that $Q^3 \in \Delta'$ and thus in order to prove that $Q^3 \in U$ it is enough to check that, denoting by $L_{RS} = \{(x, y) \in \mathbb{R}^2 : y - x = \alpha\}$ the straight line passing through R and S, then $y_{Q^3} - x_{Q^3} > \alpha$. But, taking into account equation (8) and the fact that

$$\alpha = \frac{-5 + 10t - 10t^2 - 24t^4 + 24t^5}{16t^2}$$

then, one may check that $y_{Q^3} - x_{Q^3} > \alpha$, for every $t > t_1 \approx 0.8326$ (the value of t for which the periodic point Q^3 arises). Observe that we have proved that $Q^3 \in int(U)$. From the

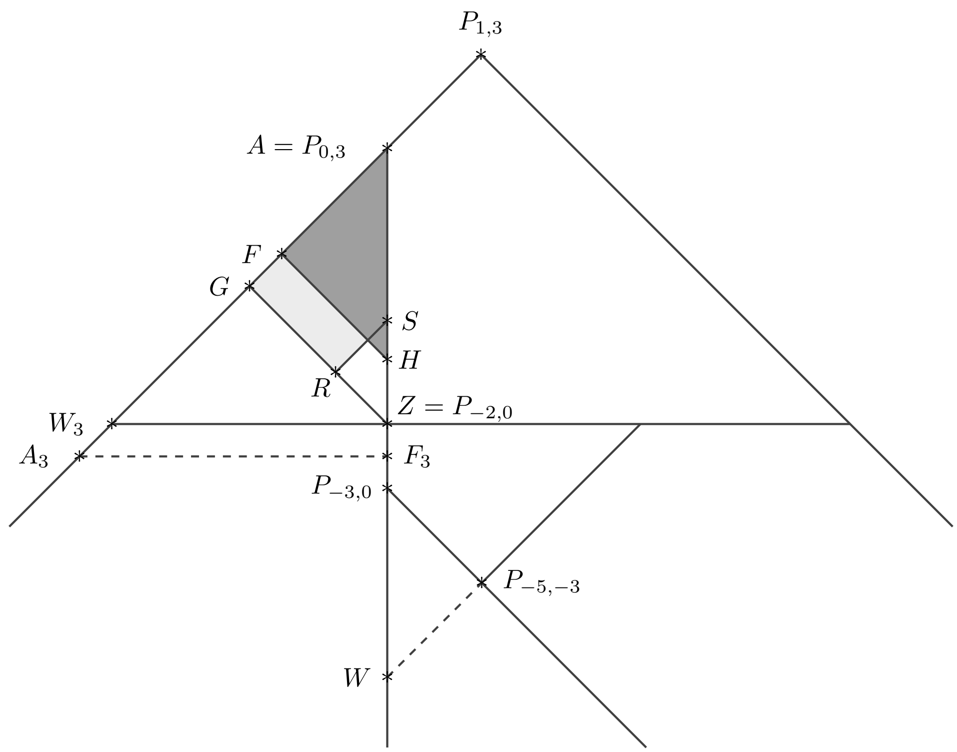

Figure 8: The case $F_3 \notin \Delta$

fact that Q^3 is a repelling periodic orbit and $\Lambda^3_{t|\Delta'}$ is linear, there is $m \in \mathbb{N}$ large enough and a neighborhood Δ'_{-m} of Q^3 such that $\Delta'_{-m} \subset U$ and $\Lambda_t^m(\Delta'_{-m}) = \Delta'$. Therefore, the second statement of the Lemma is proved.

To prove the third statement, let us recall that Δ is the triangle with vertices A, Z and W_3 and $\Lambda_t^3(\Delta')$ is the triangle with vertices $A = H_3$, $F_3 = (1, 8t^5(1-t))$ and A_3. Let us distinguish between the following cases:

i) Suppose that $F_3 \notin \Delta$ (or, equivalently, that $\Delta \subset \Lambda_t^3(\Delta')$) (this situation is showed in Figure 8). This fact takes place when the ordinate of the point F_3 is smaller than the ordinate of the point $Z = P_{-2,0}$. By using $\Lambda_t^2(P_{-2,0}) = P_{0,2} = (1, 2t(1-t))$ we have $P_{-2,0} = Z = (1, \frac{2t-1}{2t^2})$. Then, $F_3 \notin \Delta$ when

$$8t^5(1-t) < \frac{2t-1}{2t^2}.$$

An easy calculation shows that the above inequality holds if $t \in (t^*, 1]$ with $t^* \approx 0.8894$. Then, if $t \in (t^*, 1]$ third statement easily follows by taking into account that $\Theta_{-2} \subset \Lambda_t^6(\Delta')$.

This can be checked by observing that, in fact, $\Theta_{-2} \subset \Lambda_t^3(\Delta)$. This last inclusion can be obtained from the facts that $\Lambda_t^3(Z) = P_{1,3}$ and $\Lambda_t^3(\Delta)$ is an isosceles right triangle ($\Lambda^3_{t|\Delta}$ is linear and preserves angles) with one of its vertices coinciding with $P_{1,3}$ and the ordinate of $A_3 = \Lambda_t^3(A)$ smaller than the ordinate of Z. Since, from the third statement of Lemma 3.2, $P_2^\star \in \Theta_{-2}$ and $\Lambda_t^3(P_2^\star) = P$ we have $P \in \Lambda_t^6(\Delta) \subset \Lambda_t^9(\Delta') = \Lambda_t^{9+m}(\Delta'_{-m}) \subset \Lambda_t^{9+m}(U)$. Thus using the basin of neighborhoods of P given at (4), $\mathcal{A}_k = \mathcal{R}_t \setminus \bigcup_{j=0}^{k-1} \Theta_{-k}$, we get some $k \in \mathbb{N}$ with $\mathcal{A}_k \subset \Lambda_t^{9+m}(U)$. Hence, for $j = k+m+9$ we obtain $\Lambda_t^j(U) = \mathcal{R}_t$.

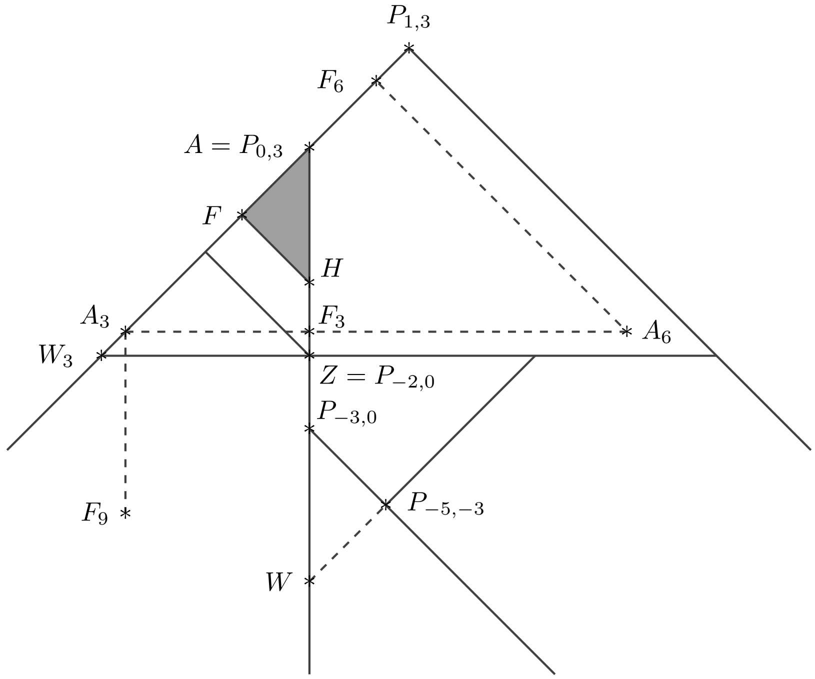

Figure 9: The case $F_3 \in \Delta$

ii) Suppose $F_3 \in \Delta$ or, equivalently, $t_0 < t < t^*$ (this case is showed in Figure 9). Let us consider $d = dist(H, Z)$. Let us observe that, by definition of Z the distance d coincides with $dist(Z, P_{-3,0})$. Recall that $P_{-2,0} = Z = (1, \frac{2t-1}{2t^2})$ and, moreover, since the image of $P_{-3,0}$ belongs to $\mathcal{C}_{-2}$, one easily has $P_{-3,0} = (1, \frac{2t^3-2t+1}{2t^3})$ and therefore

$$d = \frac{2t-1}{2t^2} - \frac{2t^3-2t+1}{2t^3} = \frac{-2t^3+2t^2+t-1}{2t^3}.$$

Let us also consider $d_1 = dist(H, F_3)$. In order to compute d_1, let us write

$$H = (1, \frac{2t-1}{2t^2} + d) = (1, \frac{-2t^3+4t^2-1}{2t^3}).$$

Since $F_3 = (1, 8t^5(1-t))$, we conclude that

$$d_1 = \frac{-2t^3 + 4t^2 - 1}{2t^2} - 8t^5(1-t) = \frac{16t^9 - 16t^8 - 2t^3 + 4t^2 - 1}{2t^3}.$$

Now, observe that F_6 is a point in $\mathcal{C}_3$ with $dist(F_6, A) = (\sqrt{2}t)^3 dist(F_3, H) = (\sqrt{2}t)^3 d_1$. Moreover, one may check that F_9 is a point in the same vertical line that of A_3 with $dist(F_9, A_3) = (\sqrt{2}t)^3 dist(F_6, A) = (\sqrt{2}t)^6 d_1 = 8t^6 d_1$. Then, the result follows if we prove that $d_1 + 8t^6 d_1 = d_1(1 + 8t^6) > d$. This holds for every $t \in (t_2, 1]$ with $t_2 \approx 0.85$. This means that for every $t \in (t_0, 1]$, $t_0 \approx 0.882$, the point $Z = P_{-2,0}$ belongs to $\Lambda_t^9(\Delta')$, so $P_{1,3}$ belongs to $\Lambda_t^{12}(\Delta')$ and hence $\Theta_{-2} \subset \Lambda_t^{12}(\Delta')$. Therefore, also using the second statement of this lemma, $P \in \Lambda_t^{15}(\Delta') = \Lambda_t^{15+m}(\Delta'_{-m}) \subset \Lambda_t^{15+m}(U)$ and we get again $\Lambda_t^j(U) = \mathcal{R}_t$, with $j = 15 + m + k$ (k large enough so that $\mathcal{A}_k \subset \Lambda_t^{15+m}(U)$). ■

5.2 Proof of Lemma 4.8

We will use the three-periodic point $Q^3 \in \Delta$ constructed during the proof of Lemma 4.10. Let us recall that (see (8)):

$$Q^3 = (x_{Q^3}, y_{Q^3}) = (\frac{4t^2(1+2t^3)}{1+4t^3+8t^6}, \frac{8t^5}{1+4t^3+8t^6}).$$

We also compute $Q = (x_Q, y_Q) = \Lambda_t^2(Q^3)$. It is easy to see that $Q \in \Theta_0$ and, since $\Lambda_t(Q) = Q^3$, from the expression of Λ_t given at (2) it follows that

$$Q = (x_Q, y_Q) = (\frac{x_{Q^3} + y_{Q^3}}{2t}, \frac{x_{Q^3} - y_{Q^3}}{2t}) = (\frac{2t + 8t^4}{1+4t^3+8t^6}, \frac{2t}{1+4t^3+8t^6}). \qquad (10)$$

First statement of Lemma 4.8 will be proved if we check that for every $t \in (t_0, 1]$, $t_0 \approx 0.882$, any ball $\mathcal{B} = B(q, r) \subset \mathbb{R}^2$ with $q \in \Theta_0$, $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 2, 3$, contains either $P_0^\star$ or Q. This claim follows by recalling that any sufficiently small neighborhood of $P_0^\star$ (respectively, Q) is sent by Λ_t (respectively, a convenient power of Λ_t) into some neighborhood of the fixed point P containing, for large enough k, some $\mathcal{A}_k = \mathcal{R}_t \setminus \bigcup_{j=0}^{k-1} \Theta_{-k}$ with $\Lambda_t^k(\mathcal{A}_k) = \mathcal{R}_t$ (the sets $\mathcal{A}_k$ where defined in (4)).

Let us recall that, see (5),

$$P_0^\star = (x_{P_0^\star}, y_{P_0^\star}) = (\frac{2t+2}{2t^2+2t+1}, \frac{2t}{2t^2+2t+1}). \qquad (11)$$

Let us divide $\Theta_0 = \Theta_0^+ \cup \Theta_0^-$ with

$$\Theta_0^+ = \{(x,y) \in \Theta_0 : y \geq y_{P_0^\star}\} \tag{12}$$

and $\Theta_0^- = \Theta_0 \setminus \Theta_0^+$.

Lemma 5.3 *Let $t \in (t_0, 1]$ and $\mathcal{B} = B(q,r)$ be a ball in $\mathbb{R}^2$ with $q \in \Theta_0^+$ and $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$, then $P_0^\star \in \mathcal{B}$.*

Proof. It will be enough to check that, if $q = (x_q, y_q) \in \Theta_0^+$ then $dist(q, P_0^\star) \leq dist(q, \mathcal{C}_2)$. But this last claim follows if we prove that $dist(q, P_0^\star) \leq dist(q, \mathcal{C}_2)$ whenever $y_q = y_{P_0^\star}$. Hence, denote by $Q_1 = (x_{Q_1}, y_{Q_1}) = (1, y_{P_0^\star})$ and by $Q_2 = (x_{Q_2}, y_{Q_2})$ the intersections between the line $y = y_{P_0^\star}$ and the sets $\mathcal{C}_0$ and $\mathcal{C}_3$, respectively, see Figure 10. It suffices to demonstrate that $dist(Q_1, P_0^\star) \leq dist(Q_1, \mathcal{C}_2)$ and $dist(Q_2, P_0^\star) \leq dist(Q_2, \mathcal{C}_2)$. Recall that the segment $\mathcal{C}_2$ is contained in the line

$$L_2 = \{(x,y) \in \mathbb{R}^2 : y = 2t(1-t)\}, \tag{13}$$

then

$$dist(Q_1, P_0^\star) = 1 - x_{P_0^\star} \leq y_{P_0^\star} - 2t(1-t) = dist(Q_1, \mathcal{C}_2)$$

for every $t \in [0,1]$. Moreover, since $\mathcal{C}_3$ is contained in the line

$$L_3 = \{(x,y) \in \mathbb{R}^2 : y = x - 4t^2 + 4t^3\} \tag{14}$$

the point Q_2 is given by $Q_2 = (x_{Q_2}, y_{Q_2}) = (y_{P_0^\star} + 4t^2 - 4t^3, y_{P_0^\star})$. Hence, one has

$$dist(Q_2, P_0^\star) = x_{P_0^\star} - y_{P_0^\star} - 4t^2 + 4t^3 = \frac{2}{2t^2 + 2t + 1} - 4t^2 + 4t^3 \leq y_{P_0^\star} - 2t(1-t) = dist(Q_2, \mathcal{C}_2)$$

for every $t \in (\frac{\sqrt{2}}{2}, 1]$. ■

Let us consider the straight segment $\mathcal{I}_1 = \overline{P_{0,3}A}$ formed by all the points (x,y) in Θ_0 satisfying $dist((x,y), \mathcal{C}_0) = dist((x,y), \mathcal{C}_3)$, see Figure 10. Then,

Lemma 5.4 *For any $t \in (t_0, 1]$, $A \in \mathcal{C}_2$.*

Proof. Denoting by $b = \sqrt{2}+1$, $\mathcal{I}_1$ is contained in the straight line

$$L_{\mathcal{I}_1} = \{(x,y) \in \mathbb{R}^2 : y = bx - \sqrt{2} - 4t^2 + 4t^3\}. \tag{15}$$

Now, since $\mathcal{C}_2$ is contained in the line L_2 (see(13)) and $\mathcal{C}_5$ is contained in the line

$$L_5 = \{(x,y) \in \mathbb{R}^2 : y + x = 4t - 4t^2 + 8t^4 - 8t^5\} \tag{16}$$

it holds that $L_{\mathcal{I}_1}$ intersects L_2 at the point $(\frac{1}{b}(\sqrt{2}+2t+2t^2-4t^3), 2t(1-t))$ and $L_{\mathcal{I}_1}$ intersects L_5 at the point $(\frac{\sqrt{2}+4t-4t^3+8t^4-8t^5}{1+b}, y_0)$ (the value of y_0 is not interesting). Since, for every $t \in (0.8576, 1]$, one may check that

$$\frac{\sqrt{2}+4t-4t^3+8t^4-8t^5}{1+b} < \frac{1}{b}(\sqrt{2}+2t+2t^2-4t^3)$$

we conclude that $A = (x_A, y_A) = (\frac{1}{b}(\sqrt{2}+2t+2t^2-4t^3), 2t(1-t)) \in \mathcal{C}_2$, for every $t \in (t_0, 1]$.

■

Now, let us consider the parabolas

$$V_{0,P_0^\star} = \{(x,y) \in \mathbb{R}^2 : dist((x,y), P_0^\star) = dist((x,y), \mathcal{C}_0)\}$$

$$V_{3,P_0^\star} = \{(x,y) \in \mathbb{R}^2 : dist((x,y), P_0^\star) = dist((x,y), \mathcal{C}_3)\}$$

(see Figure 10). Then, $V_{0,P_0^\star}$ and $V_{3,P_0^\star}$ intersect in two points. Let us denote by $H = (x_H, y_H)$ the one closer to $\mathcal{C}_2$ (the ordinate of H is smaller than the ordinate of $P_0^\star$). In fact, one may check that, since

$$dist(A, \mathcal{C}_0) = 1 - x_A < dist(A, P_0^\star)$$

for every $t \in (\frac{\sqrt{2}}{2}, 1]$, then for every $t \in (t_0, 1]$ this point H unfortunately always belongs to Θ_0 and henceforth it also follows that $H \in \mathcal{I}_1$. Moreover, if we define $A_1 = (x_{A_1}, y_{A_1})$ the intersection between $V_{3,P_0^\star}$ and $\mathcal{C}_2$ (the fact that this point belongs to $\mathcal{C}_2$ follows from Lemma 5.4) and A_2 the point where $V_{0,P_0^\star}$ intersects with the boundary of Θ_0 (it does not matter if this point belongs to $\mathcal{C}_2$ or $\mathcal{C}_5$) then the points H, A_1 and A_2 delimit a region Z in Θ_0 (lower shading region in Figure 10) formed by those points (x,y) satisfying

$$dist((x,y), P_0^\star) > dist((x,y), \mathcal{C}_0)$$

$$dist((x,y), P_0^\star) > dist((x,y), \mathcal{C}_3)$$

Of course, there exists another region, Z', in Θ_0 (containing $P_{0,3}$) for which points (x, y) in Z' also satisfy $dist((x,y), P_0^\star) > dist((x,y), \mathcal{C}_0)$ and $dist((x,y), P_0^\star) > dist((x,y), \mathcal{C}_3)$. Nevertheless, the first statement of Lemma 4.8 is proved when the center q of the ball $\mathcal{B}$ belong to this region Z'. This claim follows from Lemma 5.3 and the following result:

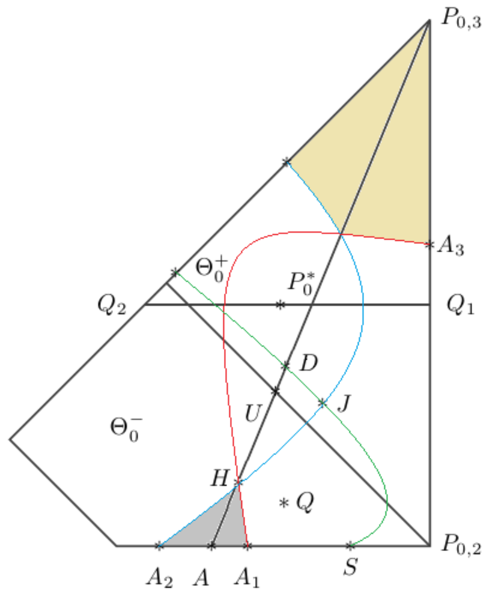

Figure 10: Regions Z and Z'

Lemma 5.5 *For every $t \in (t_0, 1]$ it holds that $Z' \subset \Theta_0^+$.*

Proof. If we denote by $A_3 = (x_{A_3}, y_{A_3})$ the intersection between $\mathcal{C}_0$ and $V_{3,P_0^\star}$. Then in order to prove that Z' is contained in the set Θ_0^+ (see (12)) it is enough to prove that $y_{A_3} > y_{P_0^\star}$. This fact follows if we prove that

$$dist(Q_1, P_0^\star) < dist(Q_1, \mathcal{C}_3),$$

being $Q_1 = (1, y_{P_0^\star})$. Using that $\mathcal{C}_3$ is contained in L_3 (see (14)) one has

$$dist^2((x,y), \mathcal{C}_3) = \frac{1}{2}(y - x + 4t^2 - 4t^3)^2 \tag{17}$$

for every $(x, y) \in \mathbb{R}^2$.

Then, since

$$dist^2(Q_1, P_0^\star) = (x_{P_0^\star} - 1)^2 < \frac{1}{2}(y_{Q_1} - x_{Q_1} + 4t^2 - 4t^3)^2 = dist^2(Q_1, \mathcal{C}_3),$$

holds for every $t \in (0.845, 1]$ the lemma follows. ■

Therefore, the first statement of Lemma 4.8 is now consequence of the following result:

Lemma 5.6 *Let $\mathcal{B} = B(q, r)$ be a ball in $\mathbb{R}^2$ with $q \in \Theta_0^-$, $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 2, 3$ then if $q \notin Z$ one has $P_0^\star \in \mathcal{B}$ and if $q \in Z$ then $Q \in \mathcal{B}$.*

Proof. Let us first assume that $q \notin Z$. This assumption directly implies that either $dist(q, P_0^\star) \leq dist(q, \mathcal{C}_0)$ or $dist(q, P_0^\star) \leq dist(q, \mathcal{C}_3)$. Then $P_0^\star \in \mathcal{B}$.

Let us now assume that $q \in Z$. Let us consider the new parabola:

$$V_{0,Q} = \{(x, y) \in \mathbb{R}^2 : dist((x, y), Q) = dist((x, y), \mathcal{C}_0)\}.$$

We remark that the lemma will be proved if we demonstrate that $Z \subset \{(x, y) \in \mathbb{R}^2 : dist((x, y), Q) < dist((x, y), \mathcal{C}_0)\}$ because in this case any ball in the hypotheses of the lemma (with $q \in Z$) must contains Q. Let us therefore consider $S = (x_S, y_S)$ the intersection between $V_{0,Q}$ and $\mathcal{C}_2$ and $D = (x_D, y_D)$ the intersection between $V_{0,Q}$ and $\mathcal{I}_1$. Then the lemma follows if we prove the following claims (see again Figure 10):

i) For every $t \in (t_0, 1]$ one has $x_S > x_{A_1}$.

ii) For every $t \in (t_0, 1]$ one has $y_D > y_H$.

To prove the first claim it is enough to check that, for every $t \in (t_0, 1]$:

$$dist(S, P_0^\star) < dist(S, \mathcal{C}_3), \tag{18}$$

and to prove the second claim it is enough to get, for every $t \in (t_0, 1]$, a point $U = (x_U, y_U) \in \mathcal{I}_1$ satisfying

$$\max\{dist(U, P_0^\star), dist(U, Q)\} < dist(U, \mathcal{C}_0) = dist(U, \mathcal{C}_3). \tag{19}$$

Remark 5.7 *Denoting by J the intersection in Θ_0 between $V_{0,P_0^\star}$ and $V_{0,Q}$ then, if (19) holds then J necessarily satisfies $dist(J, \mathcal{C}_0) < dist(J, \mathcal{C}_3)$ and therefore one easily obtains $y_D > y_J > y_H$.*

Let us start by proving (18). To this end, we compute the equation of $V_{0,Q}$ which is given by

$$2(1-x_Q)x + (y-y_Q)^2 + x_Q^2 = 1.$$

Then, also using that any point in $\mathcal{C}_2$ is contained in the line L_2 (see (13)), one gets

$$S = (x_S, y_S) = (\frac{1 - x_Q^2 - (2t(1-t) - y_Q)^2}{2(1-x_Q)}, 2t(1-t)). \tag{20}$$

Now, from (17), the inequality given at (18) follows by observing that for every $t \in (t_1, 1]$ ($t_1 \approx 0.8326$ the value of t for which the periodic point Q arises), it holds that

$$dist^2(S, P_0^\star) = (x_S - 1)^2 < \frac{1}{2}(y_S - x_S + 4t^2 - 4t^3)^2 = dist^2(S, \mathcal{C}_3).$$

To prove (19), let us consider the straight segment $\mathcal{I}_2$ formed by those points $(x, y) \in \Theta_0$ such that $dist((x, y), \mathcal{C}_0) = dist((x, y), \mathcal{C}_2)$. Then, $\mathcal{I}_2$ is contained in the straight line

$$L_{\mathcal{I}_2} = \{(x, y) \in \mathbb{R}^2 : y + x = 1 + 2t - 2t^2\}. \tag{21}$$

Let $U = (x_U, y_U)$ be the intersection between $L_{\mathcal{I}_2}$ and $L_{\mathcal{I}_1}$ (see equation (15)). We obtain (recall that $b = \sqrt{2} + 1$)

$$U = (x_U, y_U) = (\frac{1}{1+b}(b + 2t + 2t^2 - 4t^3), 1 + 2t - 2t^2 - \frac{1}{1+b}(b + 2t + 2t^2 - 4t^3)).$$

Then the inequality given at (19) holds for $t \in (0.833, 1]$.

Therefore the result is proved ■

To demonstrate the second statement of Lemma 4.8, we will need the next result:

Lemma 5.8 *Let $t \in (t_0, 1]$ and $\mathcal{B} = B(q, r)$ be a ball in $\mathbb{R}^2$ with $q \in \Theta_0$ and $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 5$ then:*

i) $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$, for every $t \in (\widetilde{t}, 1]$ with $\widetilde{t} \approx 0.928$.

ii) If $t \in (t_0, \widetilde{t}\,]$, then either $\mathcal{B} \cap \mathcal{C}_3 \neq \emptyset$ or $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$.

Proof. Along the proof it will be very useful Figure 11. Let us again consider $\mathcal{I}_1 = \overline{P_{0,3}A}$ formed by all the points (x, y) in Θ_0 satisfying $dist((x, y), \mathcal{C}_0) = dist((x, y), \mathcal{C}_3)$. Recall that the segment $\mathcal{I}_1$ is contained in $L_{\mathcal{I}_1}$, see (15). We also use again $\mathcal{I}_2 = \overline{P_{0,2}R_1}$ the straight segment formed by all the points (x, y) in Θ_0 satisfying $dist((x, y), \mathcal{C}_0) = dist((x, y), \mathcal{C}_2)$. This segment is contained in the straight line $L_{\mathcal{I}_2}$, see (21). The point R_1 belongs to $\mathcal{C}_3$ and since $\mathcal{C}_3$ is contained in the straight line L_3, see (14), one gets

$$R_1 = (x_{R_1}, y_{R_2}) = (\frac{1 + 2t + 2t^2 - 4t^3}{2}, \frac{1 + 2t - 6t^2 + 4t^3}{2}) \tag{22}$$

Let us consider $\mathcal{I}_3 = \overline{P_{2,5}R_2}$ the straight segment formed by all the points (x, y) in Θ_0 satisfying $dist((x, y), \mathcal{C}_2) = dist((x, y), \mathcal{C}_5)$. This segment is contained in the line

$$L_{\mathcal{I}_3} = \{(x, y) \in \mathbb{R}^2 : y - 2t + 2t^2 = b(x - h)\}, \tag{23}$$

being as usual $b = \sqrt{2} + 1$ and where we have introduced

$$h = h(t) = 2t - 2t^2 + 8t^4 - 8t^5.$$

Since $L_{\mathcal{I}_3}$ and $L_{\mathcal{I}_1}$ (see (15)) have the same slope (equal to $b = \sqrt{2}+1$), Lemma 5.4 implies that $R_2 \in \mathcal{C}_3$. From the fact that $\mathcal{C}_3$ is contained in the straight line L_3 (see (14)) we conclude that

$$R_2 = (x_{R_2}, y_{R_2}) = (\frac{1}{b-1}(4t^3 - 2t^2 - 2t + bh), \frac{1}{b-1}(4t^3 - 2t^2 - 2t + bh) + 4t^3 - 4t^2). \tag{24}$$

Now from the equation of R_1 given at (22), we have that $x_{R_1} > x_{R_2}$ (and also $y_{R_1} > y_{R_2}$) for every $t \in (\widetilde{t}, 1]$ with $\widetilde{t} \approx 0.928$. This means that, if $t \in (\widetilde{t}, 1]$ there is no ball $\mathcal{B} = B(q, r)$ with $q \in \Theta_0$, $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$ for $i = 0, 5$ and $\mathcal{B} \cap \mathcal{C}_2 = \emptyset$. The first statement if proved.

To prove the second statement, let us assume that $t \in (t_0, \widetilde{t}\,]$. In this case $L_{\mathcal{I}_2}$ and $L_{\mathcal{I}_3}$ ((21) and (23)) intersect in one point $R_3 \in \Theta_0$ with

$$R_3 = (x_{R_3}, y_{R_3}) = (\frac{1 + bh}{b + 1}, 1 + 2t - 2t^2 - \frac{1 + bh}{b + 1}). \tag{25}$$

Observe that, since $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 5$, if $\mathcal{B} \cap \mathcal{C}_2 = \emptyset$ then q must belong to the triangle Z_1 with vertices R_1, R_2 and R_3, see Figure 11.

Let us take $\mathcal{I}_4 = \overline{P_{3,5}R_4}$ the straight segment formed by all the points (x, y) in Θ_0 satisfying $dist((x, y), \mathcal{C}_3) = dist((x, y), \mathcal{C}_5)$. Of course, $\mathcal{I}_4$ is horizontal and (see Section 2) since $P_{3,5} = M_3 = (2t - 2t^3 + 4t^4(1 - t), 2t - 4t^2 + 2t^3 + 4t^4(1 - t))$, $\mathcal{I}_4$ is contained in the line

$$L_{\mathcal{I}_4} = \{(x, y) \in \mathbb{R}^2 : y = 2t - 4t^2 + 2t^3 + 4t^4(1 - t)\}. \tag{26}$$

Hence, R_4 is the point in $\mathcal{C}_0$ given by $R_4 = (1, 2t - 4t^2 + 2t^3 + 4t^4(1 - t))$. The segment $\mathcal{I}_4$ intersects $\mathcal{I}_1$ in a point $R_5 = (x_{R_5}, y_{R_5})$ with

$$y_{R_5} = 2t - 4t^2 + 2t^3 + 4t^4(1 - t).$$

Then, if some ball $\mathcal{B} = B(q, r)$ satisfies $q \in \Theta_0$, $\mathcal{B} \cap \mathcal{C}_i \neq \emptyset$, for $i = 0, 5$, if $\mathcal{B} \cap \mathcal{C}_3 = \emptyset$ then q must belong to the polygonal region Z_2 with vertices A, R_4, R_5 and $P_{0,2}$ (see Figure 11).

To conclude the proof of the lemma it is enough to check that $Z_1 \cap Z_2 = \emptyset$. This follows from the fact that the ordinate of the point R_3 is greater than the ordinate of the point R_5 for every $t \in (0.857, 1]$. $\blacksquare$

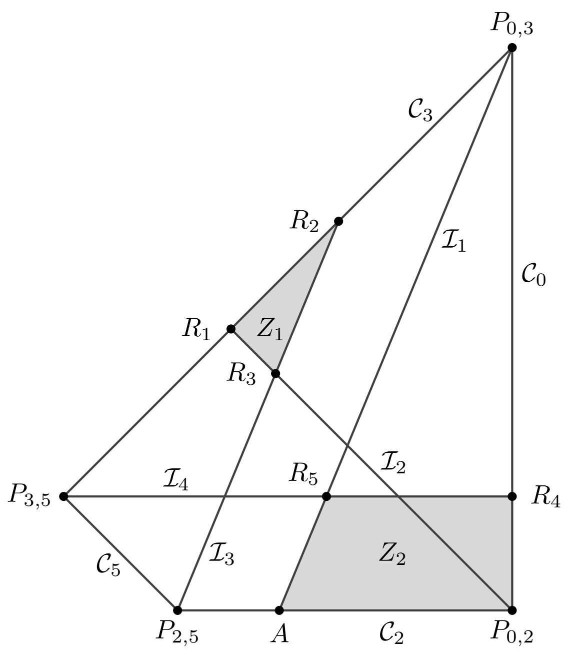

Figure 11: The case $t \in (t_0, \widetilde{t}\,]$

Now, we may prove the second statement of Lemma 4.8. We will distinguish between two cases:

A) Let us assume that $t \in (\,\widetilde{t}, 1]$. Then from Lemma 5.8 we have that $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$. Therefore, we may suppose that $\mathcal{B} \cap \mathcal{C}_3 = \emptyset$, because if not we may apply the first statement of Lemma 4.8 to conclude. Hence, q must belong to Z_2 (recall that this set exists for every $t \in (t_0, 1]$). We will demonstrate that the three-periodic orbit Q given at (10) must belong to $\mathcal{B}$. To this end, let us again use the parabola

$$V_{0,Q} = \{(x, y) \in \mathbb{R}^2 : dist((x, y), Q) = dist((x, y), \mathcal{C}_0)\}.$$

This parabola intersects $\mathcal{C}_2$ in the point S, see (20) and Figures 10 and 12. We also need to compute $\widetilde{S}$ the intersection between $V_{0,Q}$ and $\mathcal{I}_4 \subset L_{\mathcal{I}_4}$ (see (26)):

$$\widetilde{S} = (x_{\widetilde{S}}, y_{\widetilde{S}}) = (\frac{1 - x_Q^2 - (c - y_Q)^2}{2(1 - x_Q)}, c)$$

with $c = c(t) = 2t - 4t^2 + 2t^3 + 4t^4(1 - t)$.

Now, it is sufficient to demonstrate that if $(x, y) \in Z_2$ with $dist((x, y), Q) > dist((x, y), \mathcal{C}_0)$ then $dist((x, y), Q) < dist((x, y), L_5)$, being L_5 (see (16)) the straight line containing $\mathcal{C}_5$ (the region $\{(x, y) \in Z_2 : dist((x, y), Q) > dist((x, y), \mathcal{C}_0)\}$ is the shading region in Figure 12). To this end, it suffices to check that $dist((x, y), Q) < dist((x, y), L_5)$ for $(x, y) \in \{S, \widetilde{S}, P_{0,2}, R_4\}$. This is because the set $\{(x, y) \in \mathbb{R}^2 : \ dist((x, y), Q) < dist((x, y), L_5)\}$ is convex. Hence, if $dist((x, y), Q) < dist((x, y), L_5)$ for $(x, y) \in \{S, \widetilde{S}, P_{0,2}, R_4\}$ then $dist((x, y), Q) < dist((x, y), L_5)$ for every (x, y) in the polygonal domain with vertices $S, \widetilde{S}, P_{0,2}$ and R_4. Since one easily gets

$$dist^2((x, y), L_5) = \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x - y)^2, \tag{27}$$

for every $(x, y) \in \mathbb{R}^2$, we conclude

$$dist^2(S, Q) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x_S - y_S)^2 = dist^2(S, L_5),$$

for every $t \in [0.846, 1]$,

$$dist^2(\widetilde{S}, Q) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x_{\widetilde{S}} - y_{\widetilde{S}})^2 = dist^2(\widetilde{S}, L_5),$$

for every $t \in [0.85, 1]$. Moreover, recalling that $P_{0,2} = (1, y_{P_{0,2}}) = (1, 2t(1 - t))$,

$$dist^2(P_{0,2}, Q) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - 1 - y_{P_{0,2}})^2 = dist^2(P_{0,2}, L_5),$$

for every $t \in [0.858, 1]$ and finally

$$dist^2(R_4, Q) < \frac{1}{2}(4t - 4t^2 + 8t^4 - 8t^5 - x_{R_4} - y_{R_4})^2 = dist^2(R_4, L_5),$$

for every $t \in [0.833, 1]$.

Remark 5.9 *Observe that all the arguments used in the case $t \in (\widetilde{t}, 1]$ still remain valid for $t \in (t_0, 1]$ under the assumption $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$.*

B) Let us assume that $t \in (t_0, \widetilde{t}\,]$. In this case applying Lemma 5.8 we have that either $\mathcal{B} \cap \mathcal{C}_3 \neq \emptyset$ or $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$. If $\mathcal{B} \cap \mathcal{C}_2 \neq \emptyset$, then we may assume that $\mathcal{B} \cap \mathcal{C}_3 = \emptyset$ (in other case, the result will follow from the first statement of Lemma 4.8). Hence, we may repeat the arguments of the previous case (see Remark 5.9), because they do not depend on the value of $t \in (t_0, 1]$.

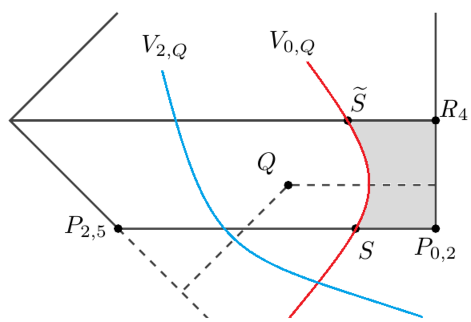

Figure 12: The region in Z_2 with $d((x,y), Q) > d((x,y), \mathcal{C}_0)$

Hence let us assume that $t \in (t_0, \widetilde{t}\,]$ and $\mathcal{B} \cap \mathcal{C}_3 \neq \emptyset$. Therefore we may assume that $\mathcal{B} \cap \mathcal{C}_2 = \emptyset$. Then q must belong to the triangle Z_1 with vertices R_1, R_2 and R_3 (see Figure 11). To conclude the proof of the lemma we will prove that any such ball must contain $P_0^\star$, see (5). To this end, let us consider the parabola

$$V_{5,P_0^\star} = \{(x,y) \in \mathbb{R}^2 : dist((x,y), P_0^\star) = dist((x,y), \mathcal{C}_5)\}.$$

The proof of the lemma ends if we check that the set Z_1 is contained in the set of points satisfying $dist((x,y), P_0^\star) < dist((x,y), L_5)$. However, since both sets are convex it suffices to obtain $dist((x,y),P_0^\star) < dist((x,y),L_5)$ for $(x,y) \in \{R_1,R_2,R_3\}$. Using the formula (27) and the expressions given for $P_0^\star$, R_1, R_2 and R_3 in (11), (22), (24) and (25), one may deduce that

$$dist^2(R_1,P_0^\star) < \frac{1}{2}(4t-4t^2+8t^4-8t^5-x_{R_1}-y_{R_1})^2 = dist^2(R_1,L_5)$$

for every $t \in (1/\sqrt[3]{2},1]$,

$$dist^2(R_2,P_0^\star) < \frac{1}{2}(4t-4t^2+8t^4-8t^5-x_{R_2}-y_{R_2})^2 = dist^2(R_2,L_5)$$

for every $t \in [0.759, 0.944]$ (recall that if suffices to suppose $t < \widetilde{t} \approx 0.928$), and

$$dist^2(R_3,P_0^\star) < \frac{1}{2}(4t-4t^2+8t^4-8t^5-x_{R_3}-y_{R_3})^2 = dist^2(R_3,L_5)$$

for every $t \in [0.847, 1]$.

Lemma 4.8 is proved.

6 Conclusions and final remarks

Along this paper we have proved the existence of a pentagonal domain $\mathcal{R}_t$ which is the maximal attracting set for the $2-D$ tent map Λ_t for every $t \in (\frac{\sqrt{2}}{2},1]$. Moreover, for every $t \in (t_0,1]$, $t_0 \approx 0.882$ we have proved that $\mathcal{R}_t$ is, in fact, a two-dimensional strange attractor for Λ_t and there is a unique ergodic a.c.i.m. μ_t with $supp(\mu_t) = \mathcal{R}_t$.

In order to extend Theorem 1.1 to every $t \in [\sqrt[3]{\frac{1}{2}},1]$ it seems enough to use a large number of pre-images of P (or more periodic orbits) in all the arguments of this paper.

When $t \in (\frac{\sqrt{2}}{2}, \sqrt[3]{\frac{1}{2}})$ the pentagonal domain $\mathcal{R}_t$ is no longer transitive as the numerical evidences show to us, see Figure 13. We will consider these dynamics in a forthcoming paper.

Finally, it seems possible to use similar results as the ones given at the main result in [1] in order to prove that the map

$$t \in (t_0,1] \to \frac{d\mu_t}{dLeb}$$

is continuous.

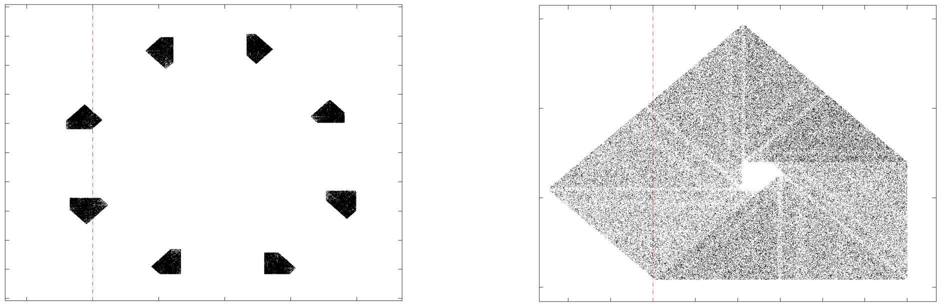

Figure 13: The attractor for $t = 0.73$ and $t = 0.77$ respectively

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