HOPF'S LEMMA AND CONSTRAINED RADIAL SYMMETRY FOR THE FRACTIONAL LAPLACIAN

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Abstract. In this paper we prove Hopf's boundary point lemma for the fractional Laplacian. With respect to the classical formulation, in the non-local framework the normal derivative of the involved function u at $z \in \partial\Omega$ is replaced with the limit of the ratio $u(x)/(\delta_R(x))^s$, where $\delta_R(x) = \operatorname{dist}(x, \partial B_R)$ and $B_R \subset \Omega$ is a ball such that $z \in \partial B_R$. More precisely, we show that

$$\liminf_{B \ni x \to z} \frac{u(x)}{(\delta_R(x))^s} > 0\,.$$

Also we consider the *overdetermined* problem

$$\begin{cases} (-\Delta)^s\, u = 1 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ \lim\limits_{\Omega \ni x \to z} \frac{u(x)}{(\delta_\Omega(x))^s} = q(|z|) & \text{for every } z \in \partial\Omega\,. \end{cases}$$

Here Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 1$, containing the origin and satisfying the interior ball condition, $\delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, and $(-\Delta)^s$, $s \in (0,1)$, is the fractional Laplace operator defined, up to normalization factors, as

$$(-\Delta)^s\, u(x) = \text{ P.V.}\int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy\,.$$

We show that if the function $q(r)$ grows fast enough with respect to r, then the problem admits a solution only in a suitable ball *centered at the origin.* The proof is based on a comparison principle proved along the paper, and on the boundary point lemma mentioned before.

1. Introduction

Fractional non-local operators are subject to extensive investigation for both their theoretical interest and the multiplicity of their applications. Indeed, fractional and non-local operators appear in concrete applications in many fields such as, just to name a few, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, thin obstacle problem, optimal transport, image reconstruction, as well as diffusion problems (see, for instance, [2, 9, 28] and the references therein).

A special role is played by the fractional Laplace operator $(-\Delta)^s$, $s \in (0,1)$, which is the (linear) integro-differential operator whose symbol is $|\xi|^{2s}$ (see, for instance, [7, Section 3.1] and [12, p. 2]). The fractional Laplacian may also be defined equivalently as follows (see,

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e.g., [11]):

$$(-\Delta)^s u(x) = c_{N,s} \,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy$$

$$= c_{N,s} \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy\,, \tag{1.1}$$

where $u\colon \mathbb{R}^N \to \mathbb{R}$ is any semicontinuous function such that the last integral has a finite value for positive ε, as well as a finite limit as $\varepsilon \to 0^+$ for every fixed $x \in \mathbb{R}^N$. This is certainly the case if $u \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. In particular, the convergence for $\varepsilon \to 0^+$ follows by Taylor expansion of $u(y)$ around the point $y = x$. The abbreviation P.V. stands for *principal value*, and the constant $c_{N,s}$ (which is found, for instance, in [7, Remark 3.11]) is given by

$$c_{N,s} = \frac{4^s\, s\, \Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}}\,\Gamma(1-s)}.$$

Fractional Laplacian and problems driven by this operator and its generalizations have been widely studied in the literature. In the present paper we are mainly interested in the fractional counterpart of Hopf's boundary point lemma and on overdetermined problems in a fractional non-local setting.

1.1. **Hopf's lemma.** Hopf's boundary point lemma, firstly proved by Hopf in [23], provides a subtle analysis of the outer normal derivative of a subharmonic function at a boundary maximum point, under the assumption that an interior ball condition holds there.

Even though the maximum principle for elliptic equations was proved by Hopf independently of the boundary point lemma, the lemma may be used as a tool for proving the validity of the strong maximum principle for second order uniformly elliptic operators (see, for instance, [15, Lemma 3.4 and Theorem 3.5]).

Along this paper we prove that Hopf's boundary point lemma holds true also for the fractional Laplacian. The main difference between its classical formulation and the one given here is that the normal derivative of a function u at a boundary point $z \in \partial\Omega$ is replaced with the limit of the ratio $u(x)/(\delta_R(x))^s$, where δ_R is the function defined as follows

$$\delta_R(x) = \mathrm{dist}(x, \partial B_R)\,, \tag{1.2}$$

B_R being an interior ball at z. A precise statement is the following.

Definition 1.1 (Interior ball condition)**.** *A set $\Omega \subseteq \mathbb{R}^N$, $N \ge 1$, satisfies the interior ball condition at $z \in \partial\Omega$ if there exists a ball $B_R \subseteq \Omega$ such that $z \in \partial B_R$.*

With this definition we can state the following result.

Lemma 1.2 (Fractional Hopf's boundary point lemma)**.** *Let $\Omega \subseteq \mathbb{R}^N$, $N \ge 1$, be an open set satisfying the interior ball condition at $z \in \partial\Omega$, and let $c \in L^\infty(\Omega)$. Consider a lower semicontinuous function $u\colon \mathbb{R}^N \to \mathbb{R}$ satisfying*

$$(-\Delta)^s u(x) \ge c(x)\, u(x) \tag{1.3}$$

pointwise in Ω.

(i) In the case when Ω is bounded, if $u \ge 0$ in $\mathbb{R}^N \setminus \Omega$ and $c(x) \le 0$ in Ω, then either u vanishes identically in Ω, or

$$\liminf_{B_R \ni x \to z} \frac{u(x)}{(\delta_R(x))^s} > 0\,, \tag{1.4}$$

where B_R is the ball in Definition 1.1, and δ_R is as in (1.2)*.*

(ii) If $u \ge 0$ in all of $\mathbb{R}^N$, then either u vanishes identically in Ω, or (1.4) *holds true.*

1.2. **Fractional non-local overdetermined problems.** A typical question concerning free boundary problems is: *what is the shape of the free boundary?* The question often arises in connection with overdetermined problems, i.e., when some redundant condition is imposed on the free boundary.

For instance, consider a bounded domain $\Omega \subseteq \mathbb{R}^N$ whose boundary is a priori unknown, apart from some degree of regularity. Then, a celebrated result by Serrin and Weinberger (see [27, 29]) states that if the solution u of the torsion problem

$$\begin{cases} -\Delta u = 1 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.5}$$

satisfies the additional condition $-\frac{\partial u}{\partial \nu} = c$ (constant) along $\partial\Omega$ (here ν is the outer normal to $\partial\Omega$), then the domain Ω is a ball. Due to the translation-invariance of the problem, that ball can be centered at any point in $\mathbb{R}^N$.

Thus, a further question arises: what additional condition on the free boundary, in place of Serrin-Weinberger's condition $-\frac{\partial u}{\partial \nu} = c$, may force the domain to be a ball *centered at a prescribed point?* In [20] the author shows that if Ω contains the origin, and if the solution u to the torsion problem in Ω satisfies $-\frac{\partial u}{\partial \nu}(x) = c\,|x|$ on $\partial\Omega$, then Ω is a ball *centered at the origin.* This explains the meaning of *"constrained symmetry"*.

The result obtained in [20] is not at all obvious, since there exist non-spherical domains (for instance, ellipses) such that the solution u of the torsion problem satisfies $-\frac{\partial u}{\partial \nu}(x) = q(|x|)$ on the boundary, where q is a function depending only on the distance from x to the origin (see [17, Problem (6)] and [20, Section 5]). Therefore, some assumption on q has to be done in order to prove that Ω is a ball and that this ball is centered at a prescribed point.

One of the aims of this paper is to study fractional counterparts of the overdetermined problem described above. In order to do this, first of all we have to replace the classical notion of outer normal derivative with a fractional one.

At this purpose, let us consider the fractional counterpart of the torsion problem (1.5), that is the non-local linear problem

$$\begin{cases} (-\Delta)^s\, u = 1 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.6}$$

where Ω is a (possibly non-smooth and disconnected) bounded open set in $\mathbb{R}^N$, $N \geq 1$. A solution of (1.6) is a continuous function $u\colon \mathbb{R}^N \to \mathbb{R}$ satisfying pointwise the equation and the outer condition in there. Such kind of solutions are sometimes called *classical* (see, for instance, [13]).

A probabilistic interpretation of this problem is the following: consider the symmetric stable process in $\mathbb{R}^N$ of index $\alpha = 2s$, which is a Lévy process as the usual Brownian motion, but admitting jumps in the paths. Then, $u(x)$ expresses the average time needed for the particle starting at $x \in \Omega$ to exit from Ω (see [4–6, 14]).

In the case when Ω is the ball B_R of radius $R > 0$ centered at the origin, the solution $u = u_R$ of (1.6) is known explicitly (see [14, 26]) and it is given by

$$u_R(x) = \gamma_{N,s}\, ((R^2 - |x|^2)^+)^s\,, \quad x \in \mathbb{R}^N\,, \tag{1.7}$$

where the exponent $^+$ denotes the positive part, and $\gamma_{N,s}$ is the following constant:

$$\gamma_{N,s} = \frac{\Gamma(\frac{N}{2})}{4^s\,\Gamma(1+s)\,\Gamma(\frac{N}{2}+s)}. \tag{1.8}$$

Due to the exponent $s < 1$, the radial solution u_R in (1.7) does not belong to the class $C^1(\overline{B}_R)$. However, it turns out that the ratio $u_R(x)/(\delta_R(x))^s$ is continuous in the closed ball $\overline{B}_R$: indeed, taking into account that in this case $\delta_R(x) = R - |x|$, we easily compute

$$\lim_{B_R \ni x \to z} \frac{u_R(x)}{(\delta_R(x))^s} = 2^s\, \gamma_{N,s}\, R^s \text{ for every } z \in \partial B_R. \tag{1.9}$$

This boundary value may be thought of as a fractional replacement of the inner derivative $-\partial u_R/\partial\nu$.

More generally, if u is a solution of $(-\Delta)^s\, u(x) = g(x)$ in a sufficiently smooth bounded domain Ω, satisfying $u = 0$ in $\mathbb{R}^N \setminus \Omega$, the boundary regularity of the ratio $u(x)/(\delta_\Omega(x))^s$, where $\delta_\Omega = \operatorname{dist}(x, \partial\Omega)$, has been investigated in [26, Theorem 1.2]. In particular, the authors prove that the mentioned ratio is Hölder continuous up to the boundary of Ω.

Therefore, a non-local fractional counterpart of the overdetermined problem described before can be formulated as follows:

$$\begin{cases} (-\Delta)^s\, u = 1 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ \lim\limits_{\Omega \ni x \to z} \frac{u(x)}{(\delta_\Omega(x))^s} = q(|z|) & \text{for every } z \in \partial\Omega\,. \end{cases} \tag{1.10}$$

Suppose that problem (1.10) is solvable, i.e., there exists a continuous function $u \in C^0(\mathbb{R}^N)$ such that the ratio $u(x)/(\delta_\Omega(x))^s$ has a continuous extension to $\overline{\Omega}$ and the three conditions prescribed in there are satisfied. Can we infer that the domain Ω is a ball?

In the sequel we show that if the function $q(r)$ grows fast enough with respect to r, then problem (1.10) possesses a solution only in suitable balls *centered at the origin.* The radii r of such balls are the positive solutions to (1.13) below (if there are some). More precisely, our result is the following:

Theorem 1.3 (Constrained radial symmetry)**.** *Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$, containing the origin and satisfying the interior ball condition at any $z \in \partial\Omega$, and let $q(r)$ be a non-negative function of the variable $r > 0$. Assume that either*

$$\textit{the ratio } \frac{q(r)}{r^s} \textit{ is strictly increasing in } r > 0\,, \tag{1.11}$$

or the set Ω is connected and

$$\textit{the ratio } \frac{q(r)}{r^s} \textit{ is non-decreasing in } r > 0. \tag{1.12}$$

Then, problem (1.10) *admits a solution if and only if Ω is a ball B_r, centered at the origin, whose radius $r > 0$ solves the equation*

$$q(r) = 2^s\, \gamma_{N,s}\, r^s, \tag{1.13}$$

where $\gamma_{N,s}$ is the constant given in (1.8)*.*

Alternatively, if the boundary $\partial\Omega$ is regular enough to have an outer normal ν at each point, then we may consider the following variant of problem (1.10):

$$\begin{cases} (-\Delta)^s\, u = 1 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ -(\partial_\nu)^s\, u(z) = q(|z|) & \text{for every } z \in \partial\Omega, \end{cases} \tag{1.14}$$

where the fractional inner derivative $-(\partial_\nu)^s\, u$ is defined as

$$-(\partial_\nu)^s\, u(z) = \lim_{\varepsilon \to 0^+} \frac{u(z - \varepsilon\, \nu)}{\varepsilon^s}\,, \qquad z \in \partial\Omega. \tag{1.15}$$

A solution of problem (1.14) is a continuous function $u \in C^0(\mathbb{R}^N)$ such that the limit in (1.15) exists for every $z \in \partial\Omega$ and the three conditions prescribed in (1.14) are satisfied. Now it is not required that the ratio $u(x)/(\delta_\Omega(x))^s$ has a continuous extension to $\overline{\Omega}$.

The case when $q =$ const. is considered in [12]: assuming that Ω is a bounded open set with a C^2 boundary, in this paper the authors prove that such a problem is solvable *only* if the domain is a ball, thus extending to the fractional case the celebrated result by Serrin [27] and Weinberger [29] (see also [10] for the case $N = 2$ and $s = \frac{1}{2}$).

Under the conditions on q given in Theorem 1.3, problem (1.14) is solvable only if Ω is a ball centered at the origin, as stated in the following result:

Theorem 1.4 (Constrained radial symmetry with fractional inner derivative). *Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$, containing the origin and such that its boundary $\partial\Omega$ has an outer normal ν at each point. Then, the statement of Theorem 1.3 continues to hold if problem* (1.10) *is replaced with* (1.14).

Note that in Theorem 1.4 the interior ball condition is not required. Furthermore, Theorem 1.4 (as well as Theorem 1.3) does not apply to the case $q =$ constant, and this is not a surprise, because when q is constant one cannot expect the center of Ω to occur at a prescribed point.

Results similar to Theorem 1.3 and Theorem 1.4 for the case $s = 1$ (which corresponds to the classical Laplacian), with extensions to the p-Laplacian and other quasilinear operators can be found in [17, 18, 20]. There, a non-linear right-hand side is also taken into consideration.

Constrained radial symmetry for local operators in domains with cavities is also investigated in [16, 19, 21], while singular solutions of the Laplace equation are considered in [1].

The method of proof for this kind of results is outlined in [20], and it is based on the comparison with radial solutions. Without entering into details for the moment, we observe that when the radius R is let vary, formula (1.9) shows that the boundary value of the ratio $u_R(x)/(\delta_R(x))^s$ increases as fast as R^s. Assumption (1.11), instead, requires the function $q(r)$ to grow faster than r^s: using the comparison principle, this leads to a contradiction unless Ω is a ball centered at the origin. The argument is a refinement of the one used in [22] to investigate overdetermined problems associated to the Laplace operator in domains with cavities.

The present paper is organized as follows. In Section 2 we establish the strong minimum principle for inequality (1.3), and derive a monotonicity property of the solution of the nonlocal linear problem (1.6) with respect to the domain. Section 3 is devoted to Hopf's lemma for the fractional Laplacian, while Section 4 focuses on the overdetermined problems (1.10) and (1.14). Finally, in Appendix A we give some comments on boundary regularity.

2. Minimum principle and monotonicity

In this section we establish a minimum principle for solutions of inequality (1.3), and a monotonicity result for problem (1.6). Both are essential for the subsequent development.

Theorem 2.1 (Strong minimum principle). *Let $u\colon \mathbb{R}^N \to \mathbb{R}$ be a lower semicontinuous function satisfying* (1.3) *pointwise in an open set $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, and let $c\colon \Omega \to \mathbb{R}$ be any real-valued function.*

(*i*) *In the case when Ω is bounded, if $u \geq 0$ in $\mathbb{R}^N \setminus \Omega$ and $c(x) \leq 0$ in Ω, then either u vanishes identically in Ω, or $u > 0$ in Ω.*

(*ii*) *If $u \geq 0$ in all of $\mathbb{R}^N$, then either u vanishes identically in Ω, or $u > 0$ in Ω.*

Proof. First of all, we show that, under the assumptions of assertion (*i*), the function u is non-negative in $\mathbb{R}^N$.

At this purpose, let us assume that $u \geq 0$ in $\mathbb{R}^N \setminus \Omega$ and $c \leq 0$ in Ω bounded. Now, we argue by contradiction. If u were negative somewhere in Ω, then, by compactness of $\overline{\Omega}$ and using the fact that $u \geq 0$ in $\mathbb{R}^N \setminus \Omega$, u would reach its (negative) minimum at some $x_0 \in \Omega$. By (1.1) we may write

$$(-\Delta)^s\, u(x_0) \leq c_{N,s}\ \mathrm{P.V.}\int_\Omega \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}}\, dy + c_{N,s} \int_{\mathbb{R}^N\setminus\Omega} \frac{u(x_0)}{|x_0 - y|^{N+2s}}\, dy.$$

Since the first integral is non-positive and the second one is strictly negative (being $u(x_0)$ the negative minimum of u), we see that $(-\Delta)^s\, u(x_0) < 0$. However, $c(x_0) \le 0$ by assumption, hence $c(x_0)\, u(x_0) \ge 0$, thus contradicting (1.3). Hence, $u \ge 0$ in all of $\mathbb{R}^N$, as claimed.

To complete the proof of Theorem 2.1, we refine the argument above, taking into account that u is non-negative in $\mathbb{R}^N$. Here, the sign of $c(x)$ is not relevant.

Both for proving (i) and (ii), we have to show that either u vanishes identically in Ω, or $u > 0$ in Ω.

Suppose that $u \not\equiv 0$ in Ω. We claim that $u > 0$ in all of Ω. By lower semicontinuity, there exists $x_1 \in \Omega$ and $\varepsilon_1 > 0$ such that $u(y) \ge \varepsilon_1$ for all $y \in B_1 = B(x_1, \varepsilon_1) \subset \Omega$. If, contrary to the claim, $u(\bar{x}) = 0$ at some $\bar{x} \in \Omega \setminus B_1$, then we have

$$(-\Delta)^s\, u(\bar{x}) \le c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N \setminus B_1} \frac{-u(y)}{|\bar{x} - y|^{N+2s}}\, dy + c_{N,s} \int_{B_1} \frac{-\varepsilon_1}{|\bar{x} - y|^{N+2s}}\, dy.$$

Again, the first integral is non-positive and the second one is strictly negative, hence $(-\Delta)^s\, u(\bar{x}) < 0$. Since $c(\bar{x})\, u(\bar{x}) = 0$, a contradiction with (1.3) is reached. Thus, we must have $u > 0$ in Ω, and the proof is complete. □

Now we focus our attention on the linear problem (1.6). Firstly, the comparison principle is derived. To be more precise, we prove the monotonicity of the solution u with respect to the domain Ω. We also notice that uniqueness of the solution of problem (1.6) follows immediately.

Lemma 2.2 (Monotonicity)**.** *Let $\Omega_1 \subseteq \Omega_2$ be two bounded open sets in $\mathbb{R}^N$, $N \ge 1$ and let u_i be a (continuous) solution of* (1.6) *in $\Omega = \Omega_i$, $i = 1, 2$. Then, $u_1 \le u_2$ in $\mathbb{R}^N$.*

Proof. Since $(-\Delta)^s\, u_2 \ge 0$ in Ω_2, by using Claim (i) of Theorem 2.1 with $c(x) \equiv 0$ we get $u_2 \ge 0$ in $\mathbb{R}^N$. Furthermore, since $u_1 = 0$ in $\mathbb{R}^N \setminus \Omega_1$ by assumption, the difference $w = u_2 - u_1$ satisfies

$$w \ge 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega_1\,.$$

Also, since the equation in (1.6) is linear, we have

$$(-\Delta)^s\, w = 0 \quad \text{in} \quad \Omega_1\,.$$

A further application of Claim (i) of Theorem 2.1 yields $w \ge 0$ in all of $\mathbb{R}^N$, and the conclusion follows. □

Corollary 2.3 (Uniqueness)**.** *Let Ω be a bounded open set in $\mathbb{R}^N$, $N \ge 1$. Then, problem* (1.6) *admits at most one solution.*

Proof. The corollary follows by letting $\Omega_1 = \Omega_2 = \Omega$ in Lemma 2.2. □

3. Hopf's lemma

This section is devoted to the fractional counterpart of Hopf's boundary point lemma. As outlined in the Introduction, with respect to the classical result proved by Hopf in [23], in our formulation we replace the normal derivative at a boundary point with the inferior limit of the ratio $u(x)/(\delta_R(x))^s$, which estimates from below the fractional inner derivative in (1.15) whenever such a derivative exists.

For simplicity, first of all we prove Hopf's lemma for the operator $(-\Delta)^s$ in the case when Ω is a ball. The extension to any open set Ω satisfying the interior ball condition is considered afterwards. Throughout the present section, the function u is required to be just lower semicontinuous, the definition of the operator being as in (1.1).

3.1. **The case of the ball.** In this subsection we prove Hopf's lemma in the ball. The argument is based on a suitable barrier method.

Lemma 3.1 (Hopf's lemma in the ball)**.** *Let* $u\colon \mathbb{R}^N \to \mathbb{R}$ *be a lower semicontinuous function satisfying* (1.3) *in a given ball* B_R*,* $R>0$*, and let* $c \in L^\infty(B_R)$*.*

(i) If $u \geq 0$ *in* $\mathbb{R}^N \setminus B_R$ *and* $c(x) \leq 0$ *in* B_R*, then either* u *vanishes identically in* B_R*, or* (1.4) *holds true at every* $z \in \partial B_R$*.*

(ii) If $u \geq 0$ *in all of* $\mathbb{R}^N$*, then either* u *vanishes identically in* B_R*, or* (1.4) *holds true at every* $z \in \partial B_R$*.*

Proof. Assuming that u does not vanish identically in B_R, let us prove (1.4). Observe, firstly, that in both cases (i) and (ii) we have

$$u(x) > 0 \quad \text{for all } x \in B_R \tag{3.1}$$

by Theorem 2.1. More precisely, for every compact subset $K \subset B_R$ we have

$$\inf_{y \in K} u(y) > 0. \tag{3.2}$$

To simplify the computation assume, without loss of generality, that the ball B_R is centered at the origin. Recall that the radial function u_R in (1.7) satisfies

$$(-\Delta)^s\, u_R = 1 \quad \text{in} \quad B_R. \tag{3.3}$$

Letting $v_n(x) = \frac{1}{n}\, u_R(x)$ for $x \in \mathbb{R}^N$ and $n = 1, 2, \ldots$, we aim to show that there exists some $\bar{n} \in \mathbb{N}$ such that

$$u \geq v_{\bar{n}} \text{ in } \mathbb{R}^N, \tag{3.4}$$

which implies (1.4), thanks to (1.9).

The argument for proving (3.4) is by contradiction. Suppose that for every $n \in \mathbb{N}$ the difference

$$w_n = v_n - u$$

is positive in $\mathbb{R}^N$.

Since $v_n = 0 \leq u$ in $\mathbb{R}^N \setminus B_R$, the upper semicontinuous function w_n attains its positive maximum at some $x_n \in B_R$. Taking into account that $w_n(x_n) > 0$ and (3.1) holds true, we may write $0 < u(x_n) < v_n(x_n)$. As a consequence of this and of the fact that

$$v_n \to 0 \text{ uniformly in } \mathbb{R}^N, \tag{3.5}$$

we have

$$\lim_{n \to +\infty} u(x_n) = 0. \tag{3.6}$$

This and (3.2) imply $|x_n| \to R$ as $n \to +\infty$. Consequently, as long as y ranges in the ball $\overline{B}_{\frac{R}{2}} \subset B_R$, the difference $x_n - y$ keeps far from zero when n is large. Hence, there exist constants $c_1, c_2 > 0$, independent of n, such that

$$c_1 < \int_{B_{\frac{R}{2}}} \frac{dy}{|x_n - y|^{N+2s}} < c_2 \text{ for } n \text{ large}. \tag{3.7}$$

In view of this, let us rewrite the inequality $c(x_n)\, u(x_n) \leq (-\Delta)^s\, u(x_n)$, which holds true by assumption, as follows:

$$c(x_n)\, u(x_n) \leq c_{N,s} \int_{B_{\frac{R}{2}}} \frac{u(x_n) - u(y)}{|x_n - y|^{N+2s}}\, dy + c_{N,s}\ \text{P.V.}\!\int_{\mathbb{R}^N \setminus B_{\frac{R}{2}}} \frac{u(x_n) - u(y)}{|x_n - y|^{N+2s}}\, dy. \tag{3.8}$$

To estimate the first integral, observe that by (3.2) there exists a positive constant $c_3 > 0$ such that $u(y) \geq c_3$ for $y \in B_{\frac{R}{2}}$. This, (3.6) and (3.7) imply

$$\limsup_{n \to +\infty} \int_{B_{\frac{R}{2}}} \frac{u(x_n) - u(y)}{|x_n - y|^{N+2s}}\, dy \leq -c_1\, c_3 < 0. \tag{3.9}$$

To deal with the second integral in (3.8), we recall that $w_n(y) \le w_n(x_n)$ for all $y \in \mathbb{R}^N$ (being x_n the maximum of w_n in $\mathbb{R}^N$), hence

$$u(x_n) - u(y) \le v_n(x_n) - v_n(y)\,.$$

Therefore, we have

$$\text{P.V.}\int_{\mathbb{R}^N \setminus B_{\frac{R}{2}}} \frac{u(x_n) - u(y)}{|x_n - y|^{N+2s}}\,dy \le \text{P.V.}\int_{\mathbb{R}^N \setminus B_{\frac{R}{2}}} \frac{v_n(x_n) - v_n(y)}{|x_n - y|^{N+2s}}\,dy. \tag{3.10}$$

To study the behavior as $n \to +\infty$, it is preferable to write

$$\text{P.V.}\int_{\mathbb{R}^N \setminus B_{\frac{R}{2}}} \frac{v_n(x_n) - v_n(y)}{|x_n - y|^{N+2s}}\,dy = (-\Delta)^s\, v_n(x_n) - \int_{B_{\frac{R}{2}}} \frac{v_n(x_n) - v_n(y)}{|x_n - y|^{N+2s}}\,dy. \tag{3.11}$$

By (3.3) and the definition of v_n we get

$$(-\Delta)^s\, v_n(x_n) = \frac{1}{n} \to 0 \quad \text{as } n \to +\infty\,.$$

Hence, as a consequence of this and of (3.5) and (3.7), the right-hand side in the equality (3.11) vanishes as $n \to +\infty$, which yields

$$\lim_{n\to+\infty} \text{P.V.}\int_{\mathbb{R}^N \setminus B_{\frac{R}{2}}} \frac{v_n(x_n) - v_n(y)}{|x_n - y|^{N+2s}}\,dy = 0$$

and so, by (3.10)

$$\limsup_{n\to+\infty} \text{P.V.}\int_{\mathbb{R}^N \setminus B_{\frac{R}{2}}} \frac{u(x_n) - u(y)}{|x_n - y|^{N+2s}}\,dy \le 0. \tag{3.12}$$

Taking into account that $c(x)$ is bounded by assumption, and (3.6), we also have

$$\lim_{n\to+\infty} c(x_n)\, u(x_n) = 0. \tag{3.13}$$

Inserting (3.12) and (3.9) into (3.8), we get a contradiction with (3.13). Hence, (3.4) must hold true for some $\bar{n}$, as claimed. □

3.2. **The general case.** The fractional boundary point lemma holds in any (possibly unbounded and disconnected) open set Ω satisfying an interior ball condition, as stated in Lemma 1.2.

Proof of Lemma 1.2. In both cases (i) and (ii), by Theorem 2.1 we conclude that either u vanishes identically in Ω, or $u > 0$ in all of Ω.

If $u > 0$ in Ω, then, in particular, $u > 0$ in B_R and $u \ge 0$ in $\mathbb{R}^N \setminus B_R$. Thus, by Lemma 3.1 we immediately deduce that (1.4) holds. □

3.3. **Some remarks.** (i) The main ingredient in the proof of the fractional Hopf's boundary point lemma is the construction of the barrier $v_{\bar{n}}$, starting from the solution u_R of problem (1.6) in a ball. This construction differs from the original procedure used by Hopf in [23].

A boundary point lemma for the inequality (1.3) is also found in [12, Proposition 3.3]: the proof is based on the construction of a suitable barrier, still involving the radial function u_R. Comparison is achieved using energy estimates.

The same function u_R is used in [26] to estimate the solution φ_2 of $(-\Delta)^s\,\varphi_2 = 0$ in an annulus, taking two constant values in the two connected components of the complement (see the proof of [26, Lemma 3.2]).

(ii) If we let $s = 1$ in (1.4), then the limit becomes obviously infinite and this was observed in the statement of [3, Lemma 4.3], which is a boundary point lemma dealing with the case $c(x) \equiv 0$.

An inspection in the proof of [3, Lemma 4.3] shows that an estimate with a power function was also found by the authors. Lemma 1.2 generalizes the cited statement, also putting into evidence the importance of the boundedness of Ω when the sign of u in Ω is unknown. In fact, if u is negative in an unbounded domain Ω, then, in general, Lemma 1.2 does not hold, as the following counterexamples show.

Counterexamples. Let us check that Claim (i) of Lemma 1.2 fails if the set Ω is not bounded. Indeed, consider $c(x) \equiv 0$ in $\Omega = \mathbb{R}^N \setminus \{0\}$. Of course, in this case, $\partial\Omega = \{0\}$, and we must take $z = 0$.

(a) The claim fails even in the classical case. Indeed, the function $u(x) = -|x|^2$ satisfies $-\Delta u > 0$ in $\mathbb{R}^N$ and vanishes on $\mathbb{R}^N \setminus \Omega = \{0\}$. However, inequality (1.4) with $s = 1$ does not hold because $Du(0) = 0$.

(b) A counterexample in the fractional case is the following. Let $s \in (\frac{1}{2}, 1)$. We claim that the concave function $u(x) = -|x|^\alpha$, $\alpha \in [1, 2s)$, satisfies

$$(-\Delta)^s\, u(x) > 0 \quad \text{for all } x \in \Omega = \mathbb{R}^N \setminus \{0\} \tag{3.14}$$

and

$$\liminf_{B_R \ni x \to z} \frac{u(x)}{(\delta_R(x))^s} = 0\,, \tag{3.15}$$

where B_R is any ball with $z = 0 \in \partial B_R$. Let us show that (3.14) holds true. Since u is of class C^2 in a neighborhood of every fixed $x \neq 0$, and $\alpha < 2s$, the integral in (1.1) converges. The idea for estimating its value is that $u(x) < u(y)$ if and only if $|x| > 0$ and $y \in B = B(0, |x|)$. The contribution of the ball B is compensated by the reflected ball B' given by

$$B' = \{\, y' \in \mathbb{R}^N : y + y' = 2x,\ y \in B \,\}.$$

More precisely, the integral in (1.1) may be splitted as follows:

$$(-\Delta)^s u(x) \;=\; c_{N,s}\ \text{P.V.} \left(\int_{B \cup B'} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy + \int_{\mathbb{R}^N \setminus (B \cup B')} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy \right).$$

By concavity we have $u(y) + u(y') \leq 2u(x)$, hence $u(x) - u(y) + u(x) - u(y') \geq 0$. This shows that the first integral is non-negative. On the other hand, the second one is positive, since $u(x) < u(y)$ in $\mathbb{R}^N \setminus (B \cup B')$. As a consequence we have that

$$\begin{aligned} (-\Delta)^s u(x) \;&=\; c_{N,s}\ \text{P.V.}\!\int_{B \cup B'} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy \\ &+\; c_{N,s}\ \text{P.V.}\!\int_{\mathbb{R}^N \setminus (B \cup B')} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy > 0\,, \end{aligned}$$

as claimed.

However, if B_R is any ball as in Definition 1.1 with $z = 0 \in \partial B_R$, then (3.15) is satisfied, because $\alpha \geq 1 > s$.

4. Fractional nonlocal overdetermined problems

This section is devoted to the overdetermined problems (1.10) and (1.14). Here we prove Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.3. First of all, let us show that if Ω is a ball whose radius satisfies equation (1.13), then problem (1.10) is solvable. If equation (1.13) has a solution $r = R_0 > 0$, then we let $R = R_0$ in (1.7), thus getting a radial solution of the Dirichlet problem (1.6) with $\Omega = B_{R_0}$. Such a solution also satisfies the third condition in (1.10) because it behaves as in (1.9) near the boundary: hence, the overdetermined problem (1.10) is solvable.

To complete the proof, assume that problem (1.10) has a solution u in an unknown open set Ω, bounded, containing the origin and satisfying the interior ball condition. Define the radii $R_1 \le R_2$ as follows:

$$R_1 = \min_{z \in \partial\Omega} |z|, \qquad R_2 = \max_{z \in \partial\Omega} |z|.$$

By definition, the ball B_{R_1} (respectively, B_{R_2}) is the largest (smallest) ball centered at the origin and contained in (containing) Ω, and there exist boundary points $z_i \in \partial\Omega$, $i = 1, 2$, satisfying $|z_i| = R_i$.

Let us prove that $R_1 = R_2$. Denote by u_{R_i} the solution of the Dirichlet problem (1.6) when $\Omega = B_{R_i}$, $i = 1, 2$. By Lemma 2.2 we have

$$u_{R_1} \le u \le u_{R_2} \text{ in } \mathbb{R}^N . \tag{4.1}$$

Let us compare the limit

$$\lim_{B_{R_1} \ni x \to z_1} \frac{u_{R_1}(x)}{(\delta_{R_1}(x))^s} = 2^s \, \gamma_{N,s} \, R_1^s, \tag{4.2}$$

which follows from (1.9), and

$$\lim_{\Omega \ni x \to z_1} \frac{u(x)}{(\delta_\Omega(x))^s} = q(|z_1|), \tag{4.3}$$

which holds by the third condition in (1.10). This comparison is not immediate because both the numerator and the denominator in (4.2) are smaller than the corresponding terms in (4.3). Therefore, we argue as follows. Let ν_1 be the outer normal to ∂B_1 at z_1. When the point $x = z_1 - t\,\nu_1$ runs along the ray r of B_1 passing through z_1, i.e. for $t \in [0, R_1]$, we have

$$\delta_{R_1}(x) = t = |x - z_1| = \delta_\Omega(x).$$

In particular, the last equality holds because $B_{R_1} \subseteq \Omega$, and therefore $\operatorname{dist}(x, z_1) \le \operatorname{dist}(x, z)$ for all x as above and for all $z \in \partial\Omega$. Since the denominators in (4.2) and (4.3) coincide along r, and $u_{R_1}(x) \le u(x)$, we may write

$$2^s \, \gamma_{N,s} \, R_1^s = -(\partial_{\nu_1})^s \, u_{R_1}(z_1) \le -(\partial_{\nu_1})^s \, u(z_1) = q(|z_1|) = q(R_1). \tag{4.4}$$

To go further, we need to compare the limits

$$\lim_{B_{R_2} \ni x \to z_2} \frac{u_{R_2}(x)}{(\delta_{R_2}(x))^s} = 2^s \, \gamma_{N,s} \, R_2^s$$

and

$$\lim_{\Omega \ni x \to z_2} \frac{u(x)}{(\delta_\Omega(x))^s} = q(|z_2|).$$

Now we use the interior ball $B_R \subseteq \Omega$ with $z_2 \in \partial B_R$, which exists by assumption. Since $B_R \subseteq B_{R_2}$, the outer normal ν_2 to ∂B_{R_2} at z_2 is also normal to ∂B_R. Hence, letting $x = z_2 - t\,\nu_2$ for $t \in [0, R]$, we have

$$\delta_{R_2}(x) = t = |x - z_2| = \delta_\Omega(x)$$

and, arguing as above, we get that

$$q(R_2) = q(|z_2|) = -(\partial_{\nu_2})^s \, u(z_2) \le -(\partial_{\nu_2})^s \, u_{R_2}(z_2) = 2^s \, \gamma_{N,s} \, R_2^s. \tag{4.5}$$

Hence, from (4.4) and (4.5) we deduce

$$\frac{q(R_2)}{R_2^s} \le 2^s \, \gamma_{N,s} \le \frac{q(R_1)}{R_1^s}. \tag{4.6}$$

Now, let us consider separately the two cases in the statement of Theorem 1.3. If (1.11) holds true, that is the ratio $q(r)/r^s$ is strictly increasing, from (4.6) we deduce $R_1 = R_2$. Consequently $\Omega = B_{R_1} = B_{R_2}$, which concludes the proof of Theorem 1.3 in this case.

Finally, suppose that Ω is connected and (1.12) occurs. In this case, to complete the proof we apply Lemma 3.1 to the difference $w = u - u_{R_1}$, which is non-negative in $\mathbb{R}^N$

by (4.1), and satisfies $(-\Delta)^s w = 0$ in B_{R_1}. The argument is the following. Since the ratio $q(r)/r^s$ is non-decreasing, equalities must hold in (4.6), and consequently also in (4.4). Hence $(\partial_{\nu_1})^s w(z_1) = 0$. From this we get

$$\liminf_{B_{R_1} \ni x \to z_1} \frac{w(x)}{(\delta(x))^s} = 0. \tag{4.7}$$

By Lemma 3.1 (both claims apply) it follows that $w = 0$ in B_{R_1}, i.e. $u = u_{R_1}$ in B_{R_1}.

Consequently, $u(x) = 0$ when $|x| = R_1$. However, u is positive in Ω by Claim (i) of Theorem 2.1, hence $\partial B_{R_1} \not\subseteq \Omega$. Since Ω is connected, it must coincide with B_{R_1}, and the proof is complete. □

Proof of Theorem 1.4. The proof is similar to that of Theorem 1.3. In particular, after having established inequality (4.1), one derives (4.4) and (4.5) directly from the third condition in (1.14), without the need of an interior ball B_R. Afterwards, the argument proceeds as before. □

Remark 4.1. In Theorem 1.3 we assume that Ω satisfies the interior ball condition at any point of its boundary, just for simplicity in the statement of the result. In fact, this condition can be weakened. More precisely, an inspection in the proof shows that Theorem 1.3 still holds if the third condition in (1.10) is replaced with

$$\lim_{\Omega \ni x \to z_i} \frac{u(x)}{(\delta(x))^s} = q(|z_i|), \ \ i = 1, 2,$$

where z_1, z_2 are *extremal* points in $\partial\Omega$. The interior ball condition, which is automatically satisfied at z_1, is needed only at z_2.

Similarly, Theorem 1.4 continues to hold if the third condition in (1.14) is replaced with $-(\partial_\nu)^s u(z_i) = q(|z_i|)$, $i = 1, 2$. Thus, it is enough that the domain Ω has an outer normal ν at z_1 and z_2.

Appendix A. Some comments on Hölder continuity

Let Ω be a smooth domain in $\mathbb{R}^N$, $N \geq 1$, and let ν be its outer normal. In this section we put into evidence that when a function u belongs to the Hölder class $C^s(\overline{\Omega})$, $s \in (0, 1)$, it is not told that the fractional inner derivative in (1.15) is well defined for all $z \in \partial\Omega$. A one-dimensional counterexample is constructed below. Multidimensional examples are readily derived.

Consequently, assuming $u \in C^s(\overline{\Omega})$ in Theorem 1.4 would not ensure that the third condition in (1.14) makes sense. One must still suppose that the derivative in there is well defined.

We also would like to note that the fractional case differs from the borderline cases $s = 0$ and $s = 1$. Indeed, the functions in $C^s(\overline{\Omega})$ are continuous up to the boundary, and in case $s = 0$ the limit in (1.15) trivially exists (and equals zero). If, instead, $s = 1$, then the limit in (1.15) is the inner derivative $-\partial u/\partial\nu$, which exists for all $u \in C^1(\overline{\Omega})$.

The following example also shows that the limit

$$\lim_{\Omega \ni x \to z} \frac{u(x)}{(\delta(x))^s}, \tag{A.1}$$

which appears in the third condition of problem (1.10), need not exist even though $u \in C^s(\overline{\Omega})$.

Let us construct a function $u = u(x)$ for $x \in \overline{\Omega} = [0, 1]$ belonging to the class $C^s(\overline{\Omega})$, $s = \frac{1}{2}$, and such that the limit

$$(\partial_\nu)^{\frac{1}{2}} u(0) = \lim_{\varepsilon \to 0^+} \frac{u(0) - u(\varepsilon)}{\sqrt{\varepsilon}} \tag{A.2}$$

does not exist. The function u (which is not required to satisfy any equation) is bounded between v_1 and v_2 given by $v_i(x) = (-1)^i \sqrt{x}$, $i = 1, 2$, whose graphs are dotted in the picture.

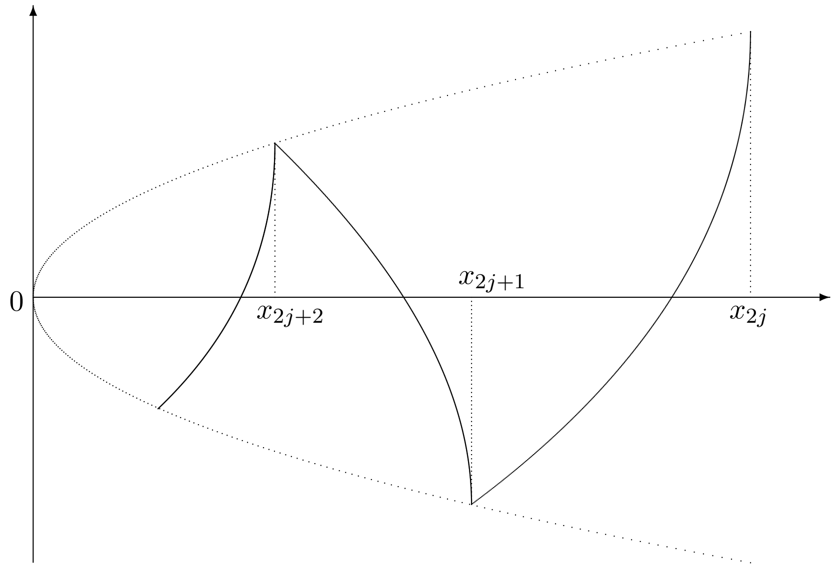

FIGURE 1. Construction of $u \in C^{\frac{1}{2}}([0,1])$ not possessing fractional derivative $(\partial_\nu)^{\frac{1}{2}} u(0)$.

The graph of u is made up of a countable sequence of suitable arcs joining the origin to the points defined as follows. For $k = 0, 1, 2, \dots$ let

$$u_k(x) = (-1)^k \left(\sqrt{x_k} - 2\sqrt{x_k - x} \right) \text{ for } x \le x_k \,. \tag{A.3}$$

Here $x_0 = 1$, and x_{k+1} is the unique positive solution $t < x_k$ of

$$u_k(t) + (-1)^k \sqrt{t} = 0, \tag{A.4}$$

which exists thanks to the coefficient 2 in (A.3). In fact, any coefficient $a > 1$ in place of 2 would suffice. Let u be defined as follows:

$$u(x) = \begin{cases} 0, & x = 0; \\ u_k(x), & x_{k+1} < x \le x_k. \end{cases}$$

Then, u belongs to the Hölder class $C^{\frac{1}{2}}([0,1])$ because

$$|u(x) - u(y)| \le 2\sqrt{|x-y|} \text{ for all } x, y \in [0,1].$$

Furthermore, we have

$$\liminf_{\varepsilon \to 0^+} \frac{u(\varepsilon)}{\sqrt{\varepsilon}} = -1; \qquad \limsup_{\varepsilon \to 0^+} \frac{u(\varepsilon)}{\sqrt{\varepsilon}} = 1.$$

Hence, the limit in (A.2) does not exist, as claimed, nor do the limit in (A.1) at $z = 0$.

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