ON THE EXISTENCE OF STATIONARY SOLUTIONS FOR SOME NONLINEAR HEAT EQUATIONS

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Abstract: We establish the existence in $H^3(\mathbb{R}^5)$ of stationary solutions of certain nonlinear heat equations using the Fixed Point Technique. The equation for the perturbed solution involves the second order differential operator without Fredholm property.

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1. Introduction

Let us consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ either vanishes or converges to 0 at infinity. In the case of $a \geq 0$, the essential spectrum of the operator $A : E \to F$ correspondent to the left-hand side of equation (1.1) contains the origin. Consequently, this operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimensions of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of certain properties of such operators. Let us note that elliptic problems involving operators without Fredholm property were studied extensively in recent years (see [12], [14], [15], [16], [17], [18], [19], [20], [21], [22], also [6]) along with their potential applications to the theory of reaction-diffusion equations (see [9], [10]). Non Fredholm operators arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). In the particular case where $a = 0$ the operator A satisfies the Fredholm property in certain

properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is essentially different and the approach developed in these works cannot be applied.

One of the important questions about problems with non-Fredholm operators concerns their solvability. We will consider the nonlinear heat equation

$$\frac{\partial u}{\partial t} = \Delta u + \varepsilon g(u) + f(x), \quad x \in \mathbb{R}^5 \tag{1.2}$$

with the parameter $\varepsilon \geq 0$. Seeking the stationary solutions of problem (1.2) yields the nonlinear Poisson equation

$$-\Delta u = f(x) + \varepsilon g(u). \tag{1.3}$$

Let us make the following technical assumption about the source term of problem (1.3).

Assumption 1. *Let $f(x) : \mathbb{R}^5 \to \mathbb{R}$ be nontrivial, $f(x) \in L^1(\mathbb{R}^5)$ and $\nabla f(x) \in L^2(\mathbb{R}^5)$.*

Note that we do not need to assume additionally the square integrability of $f(x)$. Indeed, by means of the Sobolev inequality (see e.g. p.183 of [11]) under the assumption above $f(x) \in L^{\frac{10}{3}}(\mathbb{R}^5)$. Evidently, $f(x) \in L^1(\mathbb{R}^5) \cap L^{\frac{10}{3}}(\mathbb{R}^5)$ yields

$$f(x) \in L^2(\mathbb{R}^5).$$

We will be using the Sobolev space

$$H^3(\mathbb{R}^5) = \{u(x) : \mathbb{R}^5 \to \mathbb{C} \mid u(x) \in L^2(\mathbb{R}^5),\ (-\Delta)^{\frac{3}{2}} u \in L^2(\mathbb{R}^5)\}$$

equipped with the norm

$$\|u\|^2_{H^3(\mathbb{R}^5)} = \|u\|^2_{L^2(\mathbb{R}^5)} + \|(-\Delta)^{\frac{3}{2}} u\|^2_{L^2(\mathbb{R}^5)}. \tag{1.4}$$

The operator $(-\Delta)^{\frac{3}{2}}$ is defined via the spectral calculus. By means of the Sobolev embedding we have

$$\|u\|_{L^\infty(\mathbb{R}^5)} \leq c_e \|u\|_{H^3(\mathbb{R}^5)}, \tag{1.5}$$

where $c_e > 0$ is the constant of the embedding. The hat symbol will stand for the standard Fourier transform, such that

$$\widehat{u}(p) = \frac{1}{(2\pi)^{\frac{5}{2}}} \int_{\mathbb{R}^5} u(x) e^{-ipx} dx. \tag{1.6}$$

This enables us to express the Sobolev norm as

$$\|u\|^2_{H^3(\mathbb{R}^5)} = \int_{\mathbb{R}^5} (1 + |p|^6) |\widehat{u}(p)|^2 dp. \tag{1.7}$$

When the parameter ε vanishes, we arrive at the standard Poisson equation

$$-\Delta u = f(x). \tag{1.8}$$

Under Assumption 1 by means of Lemma 7 of [22] problem (1.8) admits a unique solution $u_0(x) \in H^2(\mathbb{R}^5)$ and no orthogonality relations are required. As discussed in Lemmas 5 and 6 of [22], in dimensions $d < 5$ we need certain orthogonality conditions for the solvability of equation (1.8) in $H^2(\mathbb{R}^d)$. We do not discuss the problem in dimensions $d > 5$ to avoid extra technicalities since the argument will rely on similar ideas (see Lemma 7 of [22]). Due to our Assumption 1

$$\nabla(-\Delta u_0) = \nabla f(x) \in L^2(\mathbb{R}^5).$$

Therefore, for the unique solution of the linear Poisson equation (1.8) we have $u_0(x) \in H^3(\mathbb{R}^5)$. By seeking the resulting solution of the nonlinear Poisson equation (1.3) as

$$u(x) = u_0(x) + u_p(x) \tag{1.9}$$

we clearly arrive at the perturbative equation

$$-\Delta u_p = \varepsilon g(u_0 + u_p). \tag{1.10}$$

Let us introduce a closed ball in our Sobolev space

$$B_\rho := \{u(x) \in H^3(\mathbb{R}^5) \mid \|u\|_{H^3(\mathbb{R}^5)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.11}$$

We will seek the solution of (1.10) as the fixed point of the auxiliary nonlinear problem

$$-\Delta u = \varepsilon g(u_0 + v). \tag{1.12}$$

in the ball (1.11). Note that the left side of (1.12) involves the operator $-\Delta : H^2(\mathbb{R}^5) \to L^2(\mathbb{R}^5)$, which has no Fredholm property, since its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$ and therefore, a bounded inverse of this operator does not exist. The similar situation arised in [20] and [22] but as distinct from the present work, the problems treated there were nonlocal. The fixed point technique was used in [13] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger type operator involved in such nonlinear problem possessed the Fredholm property (see Assumption 1 of [13], also [7]). Let us define the interval on the real line as

$$I := [-c_e\|u_0\|_{H^3(\mathbb{R}^5)} - c_e,\ c_e\|u_0\|_{H^3(\mathbb{R}^5)} + c_e]. \tag{1.13}$$

We make the following assumption about the nonlinear part of problem (1.3).

Assumption 2. *Let $g(s) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$ and $g'(0) = 0$. We also assume that $g(s) \in C_2(\mathbb{R})$, such that*

$$a_2 := sup_{s\in I}|g''(s)| > 0.$$

Note that $a_1 := sup_{s\in I}|g'(s)| > 0$ as well, otherwise the function $g(s)$ will be constant on the interval I and a_2 will vanish. For instance, $g(s) = s^2$ clearly satisfies the assumption above. Our main statement is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then equation (1.12) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon*$ for a certain $\varepsilon* > 0$. The unique fixed point $u_p(x)$ of the map T_g is the only solution of problem (1.10) in B_ρ.*

Note that the resulting solution of problem (1.3) given by (1.9) will be nontrivial since the source term $f(x)$ is nontrivial and $g(0)$ vanishes according to our assumptions. We will make use of the following elementary technical lemma.

Lemma 4. *Consider the function $\varphi(R) := \alpha R + \dfrac{\beta}{R^4}$ on the positive semi- axis $(0, +\infty)$ with the constants $\alpha, \beta > 0$. It attains the minimal value at $R^* = \left(\dfrac{4\beta}{\alpha}\right)^{\frac{1}{5}}$, which is given by $\varphi(R^*) = \dfrac{5}{4^{\frac{4}{5}}}\alpha^{\frac{4}{5}}\beta^{\frac{1}{5}}$.*

Let us proceed to the proof of our main result.

2. The existence of the perturbed solution

Proof of Theorem 3. Let us choose arbitrarily $v(x) \in B_\rho$ and denote the right side of equation (1.12) as $G(x) := g(u_0 + v)$. By applying the standard Fourier transform (1.6) to both sides of problem (1.12), we arrive at

$$\widehat{u}(p) = \varepsilon\frac{\widehat{G}(p)}{p^2},$$

such that for the norm we have

$$\|u\|^2_{L^2(\mathbb{R}^5)} = \varepsilon^2 \int_{\mathbb{R}^5} \frac{|\widehat{G}(p)|^2}{|p|^4} dp. \tag{2.14}$$

Clearly,

$$\|\widehat{G}(p)\|_{L^\infty(\mathbb{R}^5)} \le \frac{1}{(2\pi)^{\frac{5}{2}}}\|G(x)\|_{L^1(\mathbb{R}^5)}. \tag{2.15}$$

Let us estimate the right side of (2.14) using (2.15) with $R > 0$ as

$$\varepsilon^2 \int_{|p|\le R} \frac{|\widehat{G}(p)|^2}{|p|^4} dp + \varepsilon^2 \int_{|p|>R} \frac{|\widehat{G}(p)|^2}{|p|^4} dp \le$$

$$\le \varepsilon^2 \frac{1}{(2\pi)^5}\|G(x)\|^2_{L^1(\mathbb{R}^5)}|S_5|R + \varepsilon^2\frac{1}{R^4}\|G(x)\|^2_{L^2(\mathbb{R}^5)}. \tag{2.16}$$

Here and below S_5 stands for the unit sphere in the space of five dimensions centered at the origin and $|S_5|$ for its Lebesgue measure (see e.g. p.6 of [11]). Since $v(x) \in B_\rho$, we have

$$\|u_0 + v\|_{L^2(\mathbb{R}^5)} \leq \|u_0\|_{H^3(\mathbb{R}^5)} + 1.$$

Also, the Sobolev embedding (1.5) yields

$$|u_0 + v| \leq c_e \|u_0\|_{H^3(\mathbb{R}^5)} + c_e.$$

Using the representation $G(x) = \int_0^{u_0+v} g'(s)ds$, with the interval I given by (1.13), we easily obtain

$$|G(x)| \leq sup_{s\in I}|g'(s)||u_0 + v| = a_1|u_0 + v|,$$

such that

$$\|G(x)\|_{L^2(\mathbb{R}^5)} \leq a_1\|u_0 + v\|_{L^2(\mathbb{R}^5)} \leq a_1(\|u_0\|_{H^3(\mathbb{R}^5)} + 1).$$

Similarly, $G(x) = \int_0^{u_0+v} ds\Big[\int_0^s g''(t)dt\Big]$. Therefore, we estimate

$$|G(x)| \leq \frac{1}{2} sup_{t\in I}|g''(t)||u_0 + v|^2 = \frac{a_2}{2}|u_0 + v|^2,$$

$$\|G(x)\|_{L^1(\mathbb{R}^5)} \leq \frac{a_2}{2}\|u_0 + v\|^2_{L^2(\mathbb{R}^5)} \leq \frac{a_2}{2}(\|u_0\|_{H^3(\mathbb{R}^5)} + 1)^2.$$

Thus we arrive at the upper bound for the right side of (2.16) as

$$\frac{\varepsilon^2}{(2\pi)^5}|S_5|\frac{a_2^2}{4}(\|u_0\|_{H^3(\mathbb{R}^5)} + 1)^4 R + \varepsilon^2 a_1^2(\|u_0\|_{H^3(\mathbb{R}^5)} + 1)^2\frac{1}{R^4}$$

with $R \in (0, +\infty)$. By means of Lemma 4 we obtain the minimal value of the expression above. Hence

$$\|u\|^2_{L^2(\mathbb{R}^5)} \leq \varepsilon^2\frac{|S_5|^{\frac{4}{5}}}{(2\pi)^4}a_2^{\frac{8}{5}}(\|u_0\|_{H^3(\mathbb{R}^5)} + 1)^{3\frac{3}{5}}a_1^{\frac{2}{5}}\frac{5}{4^{\frac{8}{5}}}. \tag{2.17}$$

Clearly, (1.12) implies that

$$\nabla(-\Delta u) = \varepsilon g'(u_0 + v)(\nabla u_0 + \nabla v).$$

We will make use of the identity

$$g'(u_0 + v) = \int_0^{u_0+v} g''(s)ds$$

along with the Sobolev embedding (1.5), such that

$$|g'(u_0+v)| \le sup_{s\in I}|g''(s)||u_0+v| \le a_2c_e(\|u_0\|_{H^3(\mathbb{R}^5)}+1)$$

and

$$|\nabla(-\Delta u)| \le \varepsilon a_2c_e(\|u_0\|_{H^3(\mathbb{R}^5)}+1)|\nabla u_0+\nabla v|.$$

Using the inequality, which can be trivially derived via the standard Fourier transform, namely

$$\|\nabla u\|_{L^2(\mathbb{R}^5)} \le \|u\|_{H^3(\mathbb{R}^5)}, \tag{2.18}$$

we easily arrive at

$$\|(-\Delta)^{\frac{3}{2}}u\|^2_{L^2(\mathbb{R}^5)} \le \varepsilon^2a_2^2c_e^2(\|u_0\|_{H^3(\mathbb{R}^5)}+1)^4. \tag{2.19}$$

By virtue of the definition of the norm (1.4) along with estimates (2.17) and (2.19) we derive

$$\|u\|_{H^3(\mathbb{R}^5)} \le \varepsilon(\|u_0\|_{H^3(\mathbb{R}^5)}+1)^2a_2^{\frac{4}{5}}\sqrt{\frac{|S_5|^{\frac{4}{5}}}{(2\pi)^4}a_1^{\frac{2}{5}}\frac{5}{4^{\frac{8}{5}}}+a_2^{\frac{2}{5}}c_e^2} \le \rho$$

for all positive values of the parameter ε small enough, such that $u(x) \in B_\rho$ as well. Suppose for some $v(x) \in B_\rho$ there are two solutions $u_{1,2}(x) \in B_\rho$ of problem (1.12). Then their difference $u(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^5)$ satisfies the Laplace equation. Since there are no nontrivial square integrable harmonic functions, $u(x) = 0$ a.e. in $\mathbb{R}^5$. Therefore, equation (1.12) defines a map $T_g : B_\rho \to B_\rho$ when $\varepsilon > 0$ is sufficiently small.

Let us show that this map is a strict contraction. We choose arbitrarily $v_{1,2}(x) \in B_\rho$, such that by virtue of the argument above $u_{1,2} = T_gv_{1,2} \in B_\rho$ as well. Explicitly, via (1.12) we have

$$-\Delta u_1 = \varepsilon g(u_0+v_1), \quad -\Delta u_2 = \varepsilon g(u_0+v_2). \tag{2.20}$$

Let us introduce

$$G_1(x) := g(u_0+v_1), \quad G_2(x) := g(u_0+v_2).$$

Then by applying the standard Fourier transform (1.6) to both sides of each of the equations (2.20), we obtain

$$\widehat{u_1}(p) = \varepsilon\frac{\widehat{G_1}(p)}{p^2}, \quad \widehat{u_2}(p) = \varepsilon\frac{\widehat{G_2}(p)}{p^2}.$$

Therefore, we express the norm

$$\|u_1-u_2\|^2_{L^2(\mathbb{R}^5)} = \varepsilon^2\int_{\mathbb{R}^5}\frac{|\widehat{G_1}(p)-\widehat{G_2}(p)|^2}{|p|^4}dp,$$

which can be estimated via (2.15) as

$$\varepsilon^2 \int_{|p|\leq R} \frac{|\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^4} dp + \varepsilon^2 \int_{|p|>R} \frac{|\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^4} dp \leq$$

$$\leq \frac{\varepsilon^2}{(2\pi)^5} \|G_1(x) - G_2(x)\|^2_{L^1(\mathbb{R}^5)} |S_5| R + \frac{\varepsilon^2}{R^4} \|G_1(x) - G_2(x)\|^2_{L^2(\mathbb{R}^5)}$$

with $R \in (0, +\infty)$. Let us make use of the representation

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} g'(s) ds,$$

such that

$$|G_1(x) - G_2(x)| \leq sup_{s\in I} |g'(s)||v_1 - v_2| = a_1 |v_1 - v_2|$$

and therefore

$$\|G_1(x) - G_2(x)\|_{L^2(\mathbb{R}^5)} \leq a_1 \|v_1 - v_2\|_{L^2(\mathbb{R}^5)} \leq a_1 \|v_1 - v_2\|_{H^3(\mathbb{R}^5)}.$$

We can also express

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} ds \Big[\int_0^s g''(t) dt \Big].$$

This enables us to estimate $G_1(x) - G_2(x)$ in the absolute value from above by

$$\frac{1}{2} sup_{t\in I} |g''(t)||(v_1 - v_2)(2u_0 + v_1 + v_2)| = \frac{a_2}{2} |(v_1 - v_2)(2u_0 + v_1 + v_2)|.$$

Via the Schwarz inequality we derive the upper bound for the norm $\|G_1(x) - G_2(x)\|_{L^1(\mathbb{R}^5)}$ as

$$\frac{a_2}{2} \|v_1 - v_2\|_{L^2(\mathbb{R}^5)} \|2u_0 + v_1 + v_2\|_{L^2(\mathbb{R}^5)} \leq a_2 \|v_1 - v_2\|_{H^3(\mathbb{R}^5)} (\|u_0\|_{H^3(\mathbb{R}^5)} + 1).$$

Thus we arrive at

$$\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^5)} \leq \varepsilon^2 \|v_1 - v_2\|^2_{H^3(\mathbb{R}^5)} \Big\{ \frac{{a_2}^2}{(2\pi)^5} (\|u_0\|_{H^3(\mathbb{R}^5)} + 1)^2 |S_5| R + \frac{{a_1}^2}{R^4} \Big\}.$$

Lemma 4 enables us to minimize the right side of the inequality above over $R > 0$, such that we obtain

$$\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^5)} \leq \varepsilon^2 \|v_1 - v_2\|^2_{H^3(\mathbb{R}^5)} \frac{5}{4^{\frac{4}{5}}} \frac{a_2^{\frac{8}{5}}}{(2\pi)^4} (\|u_0\|_{H^3(\mathbb{R}^5)} + 1)^2 |S_5|^{\frac{4}{5}} a_1^{\frac{2}{5}}. \tag{2.21}$$

Using (2.20) we express $\nabla(-\Delta)(u_1-u_2)$ as

$$\varepsilon[g'(u_0+v_1)(\nabla u_0+\nabla v_1)-g'(u_0+v_2)(\nabla u_0+\nabla v_2)]=$$

$$=\varepsilon\Big[(\nabla u_0+\nabla v_1)\int_{u_0+v_2}^{u_0+v_1}g''(s)ds+(\nabla v_1-\nabla v_2)\int_0^{u_0+v_2}g''(s)ds\Big].$$

This yields the upper bound for $|\nabla(-\Delta)(u_1-u_2)|$ as

$$\varepsilon sup_{s\in I}|g''(s)||v_1-v_2||\nabla u_0+\nabla v_1|+\varepsilon sup_{s\in I}|g''(s)||u_0+v_2||\nabla v_1-\nabla v_2|,$$

which can be easily estimated from above by virtue of the Sobolev embedding (1.5) by

$$\varepsilon a_2c_e\|v_1-v_2\|_{H^3(\mathbb{R}^5)}|\nabla u_0+\nabla v_1|+\varepsilon a_2c_e\|u_0+v_2\|_{H^3(\mathbb{R}^5)}|\nabla v_1-\nabla v_2|,$$

such that

$$\|\nabla(-\Delta)(u_1-u_2)\|_{L^2(\mathbb{R}^5)}\le\varepsilon a_2c_e\|v_1-v_2\|_{H^3(\mathbb{R}^5)}\|\nabla u_0+\nabla v_1\|_{L^2(\mathbb{R}^5)}+$$

$$+\varepsilon a_2c_e(\|u_0\|_{H^3(\mathbb{R}^5)}+1)\|\nabla v_1-\nabla v_2\|_{L^2(\mathbb{R}^5)}.$$

By virtue of (2.18) using that $v_1\in B_\rho$ we arrive at

$$\|\nabla(-\Delta)(u_1-u_2)\|^2_{L^2(\mathbb{R}^5)}\le 4\varepsilon^2a_2^2c_e^2(\|u_0\|_{H^3(\mathbb{R}^5)}+1)^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^5)}. \quad (2.22)$$

Inequalities (2.21) and (2.22) imply that

$$\|u_1-u_2\|_{H^3(\mathbb{R}^5)}\le\varepsilon(\|u_0\|_{H^3(\mathbb{R}^5)}+1)a_2^{\frac{4}{5}}\Big[\frac{5}{4^{\frac{4}{5}}}\frac{a_1^{\frac{2}{5}}}{(2\pi)^4}|S_5|^{\frac{4}{5}}+4a_2^{\frac{2}{5}}c_e^2\Big]^{\frac{1}{2}}\|v_1-v_2\|_{H^3(\mathbb{R}^5)}.$$

Therefore, the map $T_g: B_\rho\to B_\rho$ defined by equation (1.12) is a strict contraction for all values of $\varepsilon>0$ sufficiently small. Its unique fixed point $u_p(x)$ is the only solution of problem (1.10) in B_ρ, such that the resulting $u(x)\in H^3(\mathbb{R}^5)$ given by (1.9) is the stationary solution of our nonlinear heat equation (1.2). ■

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