Microscopic Conductivity of Lattice Fermions at Equilibrium – Part I: Non–Interacting Particles

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Abstract

We consider free lattice fermions subjected to a static bounded potential and a time– and space–dependent electric field. For any bounded convex region $\mathcal{R} \subset \mathbb{R}^d$ ($d \geq 1$) of space, electric fields $\mathcal{E}$ within $\mathcal{R}$ drive currents. At leading order, uniformly with respect to the volume $|\mathcal{R}|$ of $\mathcal{R}$ and the particular choice of the static potential, the dependency on $\mathcal{E}$ of the current is linear and described by a conductivity (tempered, operator–valued) distribution. Because of the positivity of the heat production, the real part of its Fourier transform is a positive measure, named here (microscopic) conductivity measure of $\mathcal{R}$, in accordance with Ohm's law in Fourier space. This finite measure is the Fourier transform of a time–correlation function of current fluctuations, i.e., the conductivity distribution satisfies Green–Kubo relations. We additionally show that this measure can also be seen as the boundary value of the Laplace–Fourier transform of a so–called quantum current viscosity. The real and imaginary parts of conductivity distributions are related to each other via the Hilbert transform, i.e., they satisfy Kramers–Kronig relations. At leading order, uniformly with respect to parameters, the heat production is the classical work performed by electric fields on the system in presence of currents. The conductivity measure is uniformly bounded with respect to parameters of the system and it is never the trivial measure $0\,\mathrm{d}\nu$. Therefore, electric fields generally produce heat in such systems. In fact, the conductivity measure defines a quadratic form in the space of Schwartz functions, the Legendre–Fenchel transform of which describes the resistivity of the system. This leads to Joule's law, i.e., the

heat produced by currents is proportional to the resistivity and the square of currents.

Contents

1 Introduction

The present paper belongs to a succession of works on Ohm and Joule's laws starting with [BPK1], where heat production of free lattice fermions subjected to a static bounded potential and a time– and space–dependent electric field has been analyzed in detail.

Ohm's law is not only valid at macroscopic scales. Indeed, in a recent work [W] the authors experimentally verified the validity of Ohm's law at the atomic scale for a purely quantum system. Such a behavior was unexpected [F]:

...In the 1920s and 1930s, it was expected that classical behavior would operate at macroscopic scales but would break down at the microscopic scale, where it would be replaced by the new quantum mechanics. The pointlike electron motion of the classical world would be replaced by the spread out quantum waves. These quantum waves would lead to very different behavior. ... Ohm's law remains valid, even at very low temperatures, a surprising result that reveals classical behavior in the quantum regime.

[D. K. Ferry, 2012]

One aim of the present paper is to establish a form of Ohm and Joule's laws at *microscopic* scales, by introducing the concept of microscopic *conductivity distri-*

butions for bounded regions $\mathcal{R} \subset \mathbb{R}^d$ of space, whose existence and basic properties follow from rather general properties of fermion systems at equilibrium.

More precisely, consider any arbitrary smooth compactly supported function $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ which yields a space–homogeneous electric field $\mathbf{1}[x \in \mathcal{R}]\, \mathcal{E}_t\, \vec{w}$ at time $t \in \mathbb{R}$ oriented along the normalized vector $\vec{w} := (w_1, \ldots, w_d) \in \mathbb{R}^d$ in some open convex domain $\mathcal{R} \subset \mathbb{R}^d$. For free lattice fermions at thermal equilibrium subjected to a static bounded potential, we show the existence of finite symmetric measures $\{\mu_{\mathcal{R}}\}_{\mathcal{R}\subset\mathbb{R}^d}$ on $\mathbb{R}$ taking values in the set $\mathcal{B}_+(\mathbb{R}^d)$ of positive linear operators on $\mathbb{R}^d$ such that, uniformly with respect to (w.r.t.) the volume $|\mathcal{R}|$ and the choice of the static potential, the induced mean current response $\mathbb{J}_{\mathcal{R}}^{(\mathcal{E})}(t)$ at time t within $\mathcal{R}$ obeys:

$$\mathbb{J}_{\mathcal{R}}^{(\mathcal{E})}(t) = \frac{1}{2}\int_{\mathbb{R}} \hat{\mathcal{E}}_\nu^{(t)} \mu_{\mathcal{R}}(\mathrm{d}\nu)\, \vec{w} + \frac{i}{2}\int_{\mathbb{R}} \mathbb{H}(\hat{\mathcal{E}}^{(t)})(\nu)\, \mu_{\mathcal{R}}(\mathrm{d}\nu)\, \vec{w} + \mathcal{O}\left(\|\mathcal{E}\|_\infty^2\right) ,$$

with $\hat{\mathcal{E}}$ being the Fourier transform of $\mathcal{E}$, $\hat{\mathcal{E}}_\nu^{(t)} := \mathrm{e}^{i\nu t}\hat{\mathcal{E}}_\nu$, and where $\mathbb{H}$ is the Hilbert transform. This expression allows us to define $\mathcal{B}(\mathbb{R}^d)$–valued tempered distributions $\mu_{\mathcal{R}}^{\|}, \mu_{\mathcal{R}}^{\perp}$ satisfying Kramers–Kronig relations and such that

$$\mathbb{J}_{\mathcal{R}}^{(\mathcal{E})}(t) = \left(\mu_{\mathcal{R}}^{\|}(\hat{\mathcal{E}}^{(t)}) + i\mu_{\mathcal{R}}^{\perp}(\hat{\mathcal{E}}^{(t)})\right)\vec{w} + \mathcal{O}\left(\|\mathcal{E}\|_\infty^2\right) ,$$

see Equations (54)–(55). By $\mathcal{B}(\mathbb{R}^d)$–valued tempered distributions, we mean a map from the space $\mathcal{S}(\mathbb{R};\mathbb{C})$ of Schwartz functions to the space $\mathcal{B}(\mathbb{R}^d)$ of linear operators on $\mathbb{R}^d$ where each entry w.r.t. the canonical orthonormal basis of $\mathbb{R}^d$ is a (tempered) distribution. $\mu_{\mathcal{R}}^{\|}$ is the linear response in–phase component of the total conductivity in Fourier space and $\mu_{\mathcal{R}}^{\|} + i\mu_{\mathcal{R}}^{\perp}$ is named the (microscopic, $\mathcal{B}(\mathbb{R}^d)$–valued) *conductivity distribution* of the region $\mathcal{R}$, while $\mu_{\mathcal{R}}$ is the (in–phase) conductivity measure, similar to [KLM].

We show four important properties of $\mu_{\mathcal{R}}$:

- It is the Fourier transform of a time–correlation function of current fluctuations, i.e., the microscopic conductivity measures satisfy *Green–Kubo relations*. See Theorem 3.1 and Equation (46).

- $\|\mu_{\mathcal{R}}(\mathbb{R})\|_{\mathrm{op}}$ is uniformly bounded w.r.t. $\mathcal{R}$ and $\mu_{\mathcal{R}}(\mathbb{R}\backslash\{0\}) > 0$. See Theorem 3.1.

- If the GNS representation of the equilibrium state of the system is denoted by $(\mathcal{H}, \pi, \Psi)$, then $\mu_{\mathcal{R}}$ is the spectral measure of the Liouvillean $\mathcal{L}$ of the system w.r.t. a vector $\Psi_{\mathcal{R}} \in \mathcal{H}$. We show that $\mu_{\mathcal{R}}(\mathbb{R}\backslash\{0\}) = 0$ if and only if $\Psi_{\mathcal{R}} \in \ker \mathcal{L}$. Thus, $\mu_{\mathcal{R}}(\mathbb{R}\backslash\{0\}) > 0$ is equivalent to the geometric condition $\Psi_{\mathcal{R}} \notin \ker \mathcal{L}$ which is easily verified in the present case. See Equation (109), Theorem 5.6 and Corollary 5.7.

- $\mu_{\mathcal{R}}$ can also be constructed on $\mathbb{R}\backslash\{0\}$ as the boundary value of the Laplace–Fourier transform of a so–called quantum current viscosity. See Equations (32) and (40) as well as Theorem 5.9.

If the first law of thermodynamics holds true for the system under consideration, then the existence and basic properties of the microscopic conductivity measures are, roughly speaking, consequences of very general properties of KMS states and decay bounds of space–time correlation functions of the equilibrium state.

Indeed, the existence of the (in–phase) conductivity measure is related to the positivity of the heat production induced by the electric field on the fermion system at thermal equilibrium. When the so–called AC–condition

$$\int_{\mathbb{R}} \mathcal{E}_t \mathrm{d}t = 0 \tag{1}$$

holds, the total heat production per unit of volume (of $\mathcal{R}$) as the electric field is switched off turns out to be equal to

$$\int_{\mathbb{R}} \hat{\mathcal{E}}_\nu \left\langle \vec{w}, \mu_{\mathcal{R}}(\mathrm{d}\nu)\, \vec{w} \right\rangle + \mathcal{O}\left(\|\mathcal{E}\|_\infty^3\right) = \int_{\mathbb{R}} \left\langle \mathcal{E}_t \vec{w}, \mu_{\mathcal{R}}^{\parallel}(\hat{\mathcal{E}}^{(t)}) \vec{w} \right\rangle \mathrm{d}t + \mathcal{O}\left(\|\mathcal{E}\|_\infty^3\right) ,$$

uniformly w.r.t. $|\mathcal{R}|$ and the choice of the static potential. Since

$$\int_{\mathbb{R}} \left\langle \mathcal{E}_t \vec{w}, \mu_{\mathcal{R}}^{\perp}(\hat{\mathcal{E}}^{(t)}) \vec{w} \right\rangle \mathrm{d}t = 0 ,$$

this expression is the classical work performed by the electric field on the fermion system in the presence of currents $\mathbb{J}_{\mathcal{R}}^{(\mathcal{E})}$:

$$\int_{\mathbb{R}} \left\langle \mathcal{E}_t \vec{w}, \mathbb{J}_{\mathcal{R}}^{(\mathcal{E})}(t) \right\rangle \mathrm{d}t + \mathcal{O}\left(\|\mathcal{E}\|_\infty^3\right) . \tag{2}$$

As $\mu_{\mathcal{R}}(\mathbb{R}\backslash\{0\}) > 0$, this implies that electric fields generally produce heat in such systems and heat production is directly related to the electric conductivity.

Note that the elements of the dual $\mathcal{S}_0^*$ of the space $\mathcal{S}_0$ of Schwartz functions $\mathbb{R} \to \mathbb{R}$ satisfying the AC–condition (1) are restrictions to $\mathcal{S}_0$ of tempered distributions. $\mathcal{S}_0^*$ is interpreted here as a space of AC–currents and $(\mathcal{S}_0, \mathcal{S}_0^*)$ is a dual pair. To obtain Joule's law in its original formulation, which relates the heat production with currents rather than with electric fields, we consider the Legendre–Fenchel transform $\mathbf{Q}_{\mathcal{R}}^*$ of the positive quadratic form

$$\mathbf{Q}_{\mathcal{R}}(\mathcal{E}) := \int_{\mathbb{R}} \left\langle \mathcal{E}_t \vec{w}, \mu_{\mathcal{R}}^{\parallel}(\hat{\mathcal{E}}^{(t)}) \right\rangle \mathrm{d}t \, .$$

Let $\partial \mathbf{Q}_{\mathcal{R}}(\mathcal{E}) \subset \mathcal{S}_0^*$ be the subdifferential of $\mathbf{Q}_{\mathcal{R}}$ at the point $\mathcal{E} \in \mathcal{S}_0$.The multifunction

$$\mathcal{E} \mapsto \boldsymbol{\sigma}_{\mathcal{R}}(\mathcal{E}) = \frac{1}{2} \partial \mathbf{Q}_{\mathcal{R}}(\mathcal{E})$$

from $\mathcal{S}_0$ to $\mathcal{S}_0^*$ (i.e., the set–valued map from $\mathcal{S}_0$ to $2^{\mathcal{S}_0^*}$) is single–valued with domain $\mathrm{Dom}(\boldsymbol{\sigma}_{\mathcal{R}}) = \mathcal{S}_0$. It is interpreted as the *AC–conductivity* of the region $\mathcal{R}$. Similarly, the multifunction

$$\mathcal{J} \mapsto \boldsymbol{\rho}_{\mathcal{R}}(\mathcal{J}) = \frac{1}{2} \partial \mathbf{Q}_{\mathcal{R}}^*(\mathcal{J})$$

from $\mathcal{S}_0^*$ to $\mathcal{S}_0$ (i.e., the set–valued map from $\mathcal{S}_0^*$ to $2^{\mathcal{S}_0}$) is the *AC–resistivity* of the region $\mathcal{R}$. Indeed, for all $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho}_{\mathcal{R}}) \neq \emptyset$ and $\mathcal{E} \in \mathrm{Dom}(\boldsymbol{\sigma}_{\mathcal{R}}) = \mathcal{S}_0$,

$$\boldsymbol{\sigma}_{\mathcal{R}}(\boldsymbol{\rho}_{\mathcal{R}}(\mathcal{J})) = \{\mathcal{J}\} \qquad \text{and} \qquad \boldsymbol{\rho}_{\mathcal{R}}(\boldsymbol{\sigma}_{\mathcal{R}}(\mathcal{E})) \supset \{\mathcal{E}\} \, .$$

Moreover, the multifunction $\boldsymbol{\rho}_{\mathcal{R}}$ is *linear*, in the sense described in Section 4.5, and, for any $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho}_{\mathcal{R}})$,

$$\{\mathbf{Q}_{\mathcal{R}}^*(\mathcal{J})\} = \langle \mathcal{J}, \boldsymbol{\rho}_{\mathcal{R}}(\mathcal{J}) \rangle = \mathbf{Q}_{\mathcal{R}}(\boldsymbol{\rho}_{\mathcal{R}}(\mathcal{J})) \, . \tag{3}$$

Thus, $\langle \mathcal{J}, \boldsymbol{\rho}_{\mathcal{R}}(\mathcal{J}) \rangle$ is the heat production (per unit of volume) in presence of the current $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho}_{\mathcal{R}})$. In other words, (3) is an expression of Joule's law in its original formulation, that is, the heat produced by currents is proportional to the resistivity and the square of currents.

Remark that we use the Weyl gauge for which $\mathcal{E}$ is minus the time derivative of the potential $\mathcal{A}$. Thus, the quantity $\int_{\mathbb{R}} \mathcal{E}_t \mathrm{d}t$ is the total shift of the electromagnetic potential $\mathcal{A}$ between the times where the field $\mathcal{E}$ is turned on and off. For this reason, we impose the AC–condition (1) to identify the total electromagnetic work with the total *internal* energy change of the system, which turns out to be the heat production, by [BPK1, Theorem 3.2]. This condition is however not used in our proofs and a general expression of the heat production as a function of the applied electric field at any time is obtained.

Indeed, based on Araki's notion of relative entropy, [BPK1] proves for the fermion system under consideration that the first law of thermodynamics holds at any time: We identify the heat production with an *internal* energy increment and define an electromagnetic *potential* energy as being the difference between the total and the internal energy increments. Both energies are studied in detail here to get the heat production at microscopic scales for all times.

Besides the internal energy increment we introduce the *paramagnetic* and *diamagnetic* energy increments. The first one is the part of electromagnetic work implying a change of the internal state of the system, whereas the diamagnetic energy is the raw electromagnetic energy given to the system at thermal equilibrium. The paramagnetic energy increment is associated to the presence of paramagnetic currents, whereas the second one is caused by thermal and diamagnetic currents. We show that these currents have different physical origins:

- *Thermal* currents are currents coming from the space inhomogeneity of the system. They exist, in general, even at thermal equilibrium.
- *Diamagnetic* currents correspond to the raw ballistic flow of charged particles due to the electric field, starting at thermal equilibrium.
- Diamagnetic currents produced by the electric field create a kind of "propagating wave front" that destabilizes the whole system by changing its internal state. In presence of inhomogeneities the system opposes itself to the propagation of that front by progressively creating so–called *paramagnetic* currents. Such induced currents act as a sort of friction (cf. current viscosity) to the diamagnetic current and produce heat as well as a modification of the electromagnetic potential energy.

We thus analyze the linear response in terms of diamagnetic and paramagnetic currents, which form altogether the total current of the system and yield the conductivity distribution. For more details on the features of such currents, see Sections 3.5 and 4.4.

For the sake of technical simplicity and without loss of generality, note that we only consider in the sequel an increasing sequence $\{\Lambda_l\}_{l=1}^{\infty}$ of boxes instead of general convex regions $\mathcal{R}$ where the electric field is non–vanishing. We obtain *uniform* bounds permitting to control the behavior of μ_{Λ_l} at large size $l \gg 1$ of the boxes $\{\Lambda_l\}_{l=1}^{\infty}$. The uniformity of our results w.r.t. l and the choice of the static potential is a consequence of tree–decay bounds of the n–point, $n \in 2\mathbb{N}$, correlations of the many–fermion system [BPK1, Section 4]. Such uniform bounds are crucial in our next paper [BPK2] on Ohm's law to construct the macroscopic conductivity distribution in the case of free fermions subjected to random static potentials (i.e., in the presence of disorder).

The validity of Ohm's law at atomic scales mentioned in [W, F] suggests a fast convergence of μ_{Λ_l}, as $l \to \infty$. Hence, we expect that the family $\{\mu_{\Lambda_l}\}_{l=1}^{\infty}$ of measures on $\mathbb{R}$ obeys a large deviation principle, for some relevant class of interactions between lattice fermions. This question is, however, not addressed here.

To conclude, our main assertions are Theorems 3.1 (existence of the conductivity measure), 3.3 (cf. Ohm's law) and 4.1, 4.7 (cf. Joule's law). This paper is organized as follows:

- In Section 2 we briefly describe the non–autonomous C^*–dynamical system for (free) fermions associated to a discrete Schrödinger operator with bounded static potential in presence of an electric field that is time– and space–dependent. For more details, see also [BPK1, Section 2].

- Section 3 introduces Ohm's law at microscopic scales via paramagnetic and diamagnetic currents. Mathematical properties of the corresponding conductivities are explained in detail and a notion of current viscosity is discussed.

- Section 4 is devoted to the derivation of Joule's law at microscopic scales. In particular, we introduce there four kinds of energy increments: the internal energy increment or heat production, the electromagnetic potential energy, the paramagnetic energy increment and the diamagnetic energy. The AC–resistivity is also described.

- All technical proofs are postponed to Section 5. Additional properties on the conductivity measure are also proven, see Section 5.1.2.

- Finally, Section A is an appendix on the Duhamel two–point function. It is indeed an important mathematical tool used here which frequently appears in the context of linear response theory.

Notation 1.1 (Generic constants)
To simplify notation, we denote by D any generic positive and finite constant. These constants do not need to be the same from one statement to another.

2 Setup of the Problem

The aim of this section is to describe the non–autonomous C^*–dynamical system under consideration. Since almost everything is already described in detail in [BPK1, Section 2], we only focus on the specific concepts or definitions that are important in the sequel.

2.1 Free Fermion Systems on Lattices

2.1.1 Algebraic Formulation of Fermion Systems on Lattices

The d–dimensional lattice $\mathfrak{L} := \mathbb{Z}^d$ ($d \in \mathbb{N}$) represents the (cubic) crystal and we define $\mathcal{P}_f(\mathfrak{L}) \subset 2^{\mathfrak{L}}$ to be the set of all *finite* subsets of $\mathfrak{L}$. We denote by $\mathcal{U}$ the CAR C^*–algebra of the infinite system and define annihilation and creation operators of (spinless) fermions with wave functions $\psi \in \ell^2(\mathfrak{L})$ by

$$a(\psi) := \sum_{x\in\mathfrak{L}} \overline{\psi(x)} a_x \in \mathcal{U} \ , \quad a^*(\psi) := \sum_{x\in\mathfrak{L}} \psi(x) a_x^* \in \mathcal{U} \ .$$

Here, a_x, a_x^*, $x \in \mathfrak{L}$, and the identity $\mathbf{1}$ are generators of $\mathcal{U}$ and satisfy the canonical anti–commutation relations: For any $x, y \in \mathfrak{L}$,

$$a_x a_y + a_y a_x = 0 \ , \qquad a_x a_y^* + a_y^* a_x = \delta_{x,y} \mathbf{1} \ . \tag{4}$$

2.1.2 Static External Potentials

Let $\Omega := [-1,1]^{\mathfrak{L}}$. For any $\omega \in \Omega$, $V_\omega \in \mathcal{B}(\ell^2(\mathfrak{L}))$ is defined to be the self–adjoint multiplication operator with the function $\omega : \mathfrak{L} \to [-1,1]$. The static external potential V_ω is of order $\mathcal{O}(1)$ and we rescale below its strength by an additional parameter $\lambda \in \mathbb{R}_0^+$ (i.e., $\lambda \geq 0$).

2.1.3 Dynamics on the One–Particle Hilbert Space

Let $\Delta_{\mathrm{d}} \in \mathcal{B}(\ell^2(\mathfrak{L}))$ be (up to a minus sign) the usual d–dimensional discrete Laplacian defined by

$$[\Delta_{\mathrm{d}}(\psi)](x) := 2d\psi(x) - \sum_{z\in\mathfrak{L},\,|z|=1} \psi(x+z) \ , \qquad x \in \mathfrak{L},\ \psi \in \ell^2(\mathfrak{L}) \ . \tag{5}$$

Then, for $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, the dynamics in the one–particle Hilbert space $\ell^2(\mathfrak{L})$ is implemented by the unitary group $\{\mathrm{U}_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ generated by the (anti–self–adjoint) operator $-i(\Delta_{\mathrm{d}} + \lambda V_\omega)$:

$$\mathrm{U}_t^{(\omega,\lambda)} := \exp(-it(\Delta_{\mathrm{d}} + \lambda V_\omega)) \in \mathcal{B}(\ell^2(\mathfrak{L})) \ , \qquad t \in \mathbb{R} \ . \tag{6}$$

2.1.4 Dynamics on the CAR C^*–Algebra

For all $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, the condition

$$\tau_t^{(\omega,\lambda)}(a(\psi)) = a((\mathrm{U}_t^{(\omega,\lambda)})^*(\psi)) \ , \qquad t \in \mathbb{R} \ , \ \psi \in \ell^2(\mathfrak{L}) \ , \tag{7}$$

uniquely defines a family $\tau^{(\omega,\lambda)} := \{\tau_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ of (Bogoliubov) $*$–automorphisms of $\mathcal{U}$, see [BR2, Theorem 5.2.5]. The one–parameter group $\tau^{(\omega,\lambda)}$ is strongly continuous and we denote its generator by $\delta^{(\omega,\lambda)}$. Clearly,

$$\tau_t^{(\omega,\lambda)}(B_1B_2) = \tau_t^{(\omega,\lambda)}(B_1)\tau_t^{(\omega,\lambda)}(B_2) \ , \qquad B_1, B_2 \in \mathcal{U} \ , \ t \in \mathbb{R} \ . \tag{8}$$

In the following, we will need the *time–reversal* operation Θ. It is the unique map $\Theta : \mathcal{U} \to \mathcal{U}$ satisfying the following properties:

- Θ is antilinear and continuous.
- $\Theta(\mathbf{1}) = \mathbf{1}$ and $\Theta(a_x) = a_x$ for all $x \in \mathfrak{L}$.
- $\Theta(B_1B_2) = \Theta(B_1)\,\Theta(B_2)$ for all $B_1, B_2 \in \mathcal{U}$.
- $\Theta(B^*) = \Theta(B)^*$ for all $B \in \mathcal{U}$.

In particular, Θ is involutive, i.e., $\Theta \circ \Theta = \mathrm{Id}_{\mathcal{U}}$. This operation can be explicitly defined by using the Fock representation of $\mathcal{U}$. It is called *time–reversal* of the dynamics $\tau_t^{(\omega,\lambda)}$ because of the following identity

$$\Theta \circ \tau_t^{(\omega,\lambda)} = \tau_{-t}^{(\omega,\lambda)} \circ \Theta \ ,$$

which is valid for all $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \in \mathbb{R}$, see Lemma 5.1. This feature is important to obtain a symmetric conductivity measure.

2.1.5 Thermal Equilibrium State

For any realization $\omega \in \Omega$ and strength $\lambda \in \mathbb{R}_0^+$ of the static external potential, the thermal equilibrium state of the system at inverse temperature $\beta \in \mathbb{R}^+$ (i.e., $\beta > 0$) is by definition the unique $(\tau^{(\omega,\lambda)}, \beta)$–KMS state $\varrho^{(\beta,\omega,\lambda)}$, see [BR2, Example 5.3.2.] or [P, Theorem 5.9]. It is well–known that such a state is stationary with respect to (w.r.t.) the dynamics, that is,

$$\varrho^{(\beta,\omega,\lambda)} \circ \tau_t^{(\omega,\lambda)} = \varrho^{(\beta,\omega,\lambda)} , \qquad \beta \in \mathbb{R}^+, \ \omega \in \Omega, \ \lambda \in \mathbb{R}_0^+, \ t \in \mathbb{R} . \tag{9}$$

The state $\varrho^{(\beta,\omega,\lambda)}$ is *gauge–invariant and quasi–free*. Such states are uniquely characterized by bounded positive operators $\mathbf{d} \in \mathcal{B}(\ell^2(\mathfrak{L}))$ obeying $0 \leq \mathbf{d} \leq \mathbf{1}$. These operators are named *symbols* of the corresponding states. The symbol of $\varrho^{(\beta,\omega,\lambda)}$ is given by

$$\mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)} := \frac{1}{1+\mathrm{e}^{\beta(\Delta_{\mathrm{d}}+\lambda V_\omega)}} \in \mathcal{B}(\ell^2(\mathfrak{L})) . \tag{10}$$

Let us remark here that $\varrho^{(\beta,\omega,\lambda)}$ is time–reversal invariant, i.e., for all parameters $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$,

$$\varrho^{(\beta,\omega,\lambda)} \circ \Theta\,(B) = \overline{\varrho^{(\beta,\omega,\lambda)}\,(B)} , \qquad B \in \mathcal{U} .$$

See Lemma 5.1.

2.2 Fermion Systems in Presence of Electromagnetic Fields

2.2.1 Electric Fields

Using the Weyl gauge (also named temporal gauge), the electric field is defined from a compactly supported potential

$$\mathbf{A} \in \mathbf{C}_0^\infty = \bigcup_{l \in \mathbb{R}^+} C_0^\infty(\mathbb{R} \times [-l,l]^d \,; (\mathbb{R}^d)^*)$$

by

$$E_{\mathbf{A}}(t,x) := -\partial_t \mathbf{A}(t,x) , \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d . \tag{11}$$

Here, $(\mathbb{R}^d)^*$ is the set of one–forms[1] on $\mathbb{R}^d$ that take values in $\mathbb{R}$ and $\mathbf{A}(t,x) \equiv 0$ whenever $x \notin [-l,l]^d$ and $\mathbf{A} \in C_0^\infty(\mathbb{R} \times [-l,l]^d \,; (\mathbb{R}^d)^*)$. Since $\mathbf{A} \in \mathbf{C}_0^\infty$,

[1] In a strict sense, one should take the dual space of the tangent spaces $T(\mathbb{R}^d)_x$, $x \in \mathbb{R}^d$.

$\mathbf{A}(t,x) = 0$ for all $t \leq t_0$, where $t_0 \in \mathbb{R}$ is some initial time. We also define the integrated electric field between $x^{(2)} \in \mathfrak{L}$ and $x^{(1)} \in \mathfrak{L}$ at time $t \in \mathbb{R}$ by

$$\mathbf{E}_t^{\mathbf{A}}(\mathbf{x}) := \int_0^1 \left[E_{\mathbf{A}}(t, \alpha x^{(2)} + (1-\alpha)x^{(1)}) \right] (x^{(2)} - x^{(1)}) \mathrm{d}\alpha \ , \qquad (12)$$

where $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$.

2.2.2 Discrete Magnetic Laplacian

We consider without loss of generality *negatively* charged fermions. Thus, using the (minimal) coupling of $\mathbf{A} \in \mathbf{C}_0^\infty$ to the discrete Laplacian $-\Delta_{\mathrm{d}}$, the discrete *time–dependent* magnetic Laplacian is (up to a minus sign) the self–adjoint operator

$$\Delta_{\mathrm{d}}^{(\mathbf{A})} \equiv \Delta_{\mathrm{d}}^{(\mathbf{A}(t,\cdot))} \in \mathcal{B}(\ell^2(\mathfrak{L})) \ , \qquad t \in \mathbb{R} \ ,$$

defined by

$$\langle \mathfrak{e}_x, \Delta_{\mathrm{d}}^{(\mathbf{A})} \mathfrak{e}_y \rangle = \exp\left(-i \int_0^1 \left[\mathbf{A}(t, \alpha y + (1-\alpha)x) \right] (y-x) \mathrm{d}\alpha \right) \langle \mathfrak{e}_x, \Delta_{\mathrm{d}} \mathfrak{e}_y \rangle \quad (13)$$

for all $t \in \mathbb{R}$ and $x, y \in \mathfrak{L}$. Here, $\langle \cdot, \cdot \rangle$ is the scalar product in $\ell^2(\mathfrak{L})$ and $\{\mathfrak{e}_x\}_{x \in \mathfrak{L}}$ is the canonical orthonormal basis $\mathfrak{e}_x(y) \equiv \delta_{x,y}$ of $\ell^2(\mathfrak{L})$. In (13), $\alpha y + (1-\alpha)x$ and $y - x$ are seen as vectors in $\mathbb{R}^d$.

2.2.3 Perturbed Dynamics on the One–Particle Hilbert Space

The dynamics of the system under the influence of an electromagnetic potential is defined via the two–parameter group $\{\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t \geq s}$ of unitary operators on $\ell^2(\mathfrak{L})$ generated by the (time–dependent anti–self–adjoint) operator $-i(\Delta_{\mathrm{d}}^{(\mathbf{A})} + \lambda V_\omega)$ for any $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$:

$$\forall s,t \in \mathbb{R},\ t \geq s: \quad \partial_t \mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})} = -i(\Delta_{\mathrm{d}}^{(\mathbf{A}(t,\cdot))} + \lambda V_\omega) \mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})} \ , \quad \mathrm{U}_{s,s}^{(\omega,\lambda)} := \mathbf{1} \ . \qquad (14)$$

The dynamics is well–defined because the map

$$t \mapsto (\Delta_{\mathrm{d}}^{(\mathbf{A}(t,\cdot))} + \lambda V_\omega) \in \mathcal{B}(\ell^2(\mathfrak{L}))$$

from $\mathbb{R}$ to the set $\mathcal{B}(\ell^2(\mathfrak{L}))$ of bounded operators acting on $\ell^2(\mathfrak{L})$ is continuously differentiable for every $\mathbf{A} \in \mathbf{C}_0^\infty$.

Note that, as explained in [BPK1, Section 2.3], the interaction between magnetic fields and electron spins is here neglected because such a term will become negligible for electromagnetic potentials slowly varying in space, see Section 2.3.1. This justifies the assumption of fermions with zero–spin.

2.2.4 Perturbed Dynamics on the CAR C^*–Algebra

For all $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$, the condition

$$\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}(a(\psi)) = a((\mathrm{U}_{t,s}^{(\omega,\lambda,\mathbf{A})})^*(\psi)) \ , \qquad t \geq s, \ \psi \in \ell^2(\mathfrak{L}) \ , \tag{15}$$

uniquely defines a family of Bogoliubov automorphisms of the C^*–algebra $\mathcal{U}$, see [BR2, Theorem 5.2.5]. The family $\{\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}\}_{t\geq s}$ is itself the solution of a non–autonomous evolution equation, see [BPK1, Sections 5.2-5.3].

2.2.5 Time–Dependent State

Since $\varrho^{(\beta,\omega,\lambda)}$ is stationary (cf. (9)) and $\mathbf{A}(t,x) = 0$ for all $t \leq t_0$, the time evolution of the state of the system equals

$$\rho_t^{(\beta,\omega,\lambda,\mathbf{A})} := \begin{cases} \varrho^{(\beta,\omega,\lambda)} & , \qquad t \leq t_0 \ , \\ \varrho^{(\beta,\omega,\lambda)} \circ \tau_{t,t_0}^{(\omega,\lambda,\mathbf{A})} & , \qquad t \geq t_0 \ . \end{cases} \tag{16}$$

This state is gauge–invariant and quasi–free for all times, by construction.

2.3 Space–Scale of Fields, Linear Response Theory and SGM

2.3.1 From Microscopic to Macroscopic Electromagnetic Fields

For space scales large compared to 10^{-14} m, electron and nuclei are usually treated as point systems and electromagnetic phenomena are governed by *microscopic* Maxwell equations. However, the electromagnetic fields produced by these point charges fluctuate very much in space and time and macroscopic devices generally measure averages over intervals in space and time much larger than the scale of these fluctuations. This implies relatively smooth and slowly varying macroscopic quantities. As explained in [Ja, Section 6.6], *"only a spatial averaging is necessary."* The *macroscopic* electromagnetic fields are thus coarse–grainings of microscopic ones and satisfy the so–called macroscopic Maxwell equations. In particular, their spacial variations become negligible on the atomic scale.

Similarly, we consider that the infinite bulk containing conducting fermions only experiences mesoscopic electromagnetic fields, which are produced by mesoscopic devices. In other words, the heat production or the conductivity is measured in a local region which is very small w.r.t. the size of the bulk, but very large w.r.t. the lattice spacing of the crystal. We implement this hierarchy of space scales by rescaling vector potentials. That means, for any $l \in \mathbb{R}^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$, we consider the space–rescaled vector potential $\mathbf{A}_l$ defined by

$$\mathbf{A}_l(t,x) := \mathbf{A}(t, l^{-1}x) , \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d . \tag{17}$$

Then, to ensure that an infinite number of lattice sites is involved, we eventually perform the limit $l \to \infty$. See [BPK2] for more details.

Indeed, the scaling factor l^{-1} used in (17) means, at fixed l, that the space scale of the electric field (11) is infinitesimal w.r.t. the macroscopic bulk (which is the whole space), whereas the lattice spacing gets infinitesimal w.r.t. the space scale of the electric field when $l \to \infty$.

2.3.2 Linear Response Theory

Linear response theory refers here to linearized non–equilibrium statistical mechanics and has been initiated by Kubo [K] and Mori [M]. Ohm's law is one of the first and certainly one of the most important examples thereof. It is indeed a linear response to electric fields. Therefore, we also rescale the strength of the electromagnetic potential $\mathbf{A}_l$ by a real parameter $\eta \in \mathbb{R}$ and eventually take the limit $\eta \to 0$.

When $|\eta| \ll 1$ and $l \gg 1$, it turns out that, uniformly w.r.t. l, the mean currents $\mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}$ and $\mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}$, defined below by (42)–(43), are of order $\mathcal{O}(\eta)$. Similarly, the energy increments $\mathbf{S}^{(\omega,\eta\mathbf{A}_l)}$, $\mathbf{P}^{(\omega,\eta\mathbf{A}_l)}$, $\mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}$ and $\mathfrak{I}_{\mathrm{d}}^{(\omega,\eta\mathbf{A}_l)}$, respectively defined by (58), (59), (62) and (63), are all of order $\mathcal{O}\left(\eta^2 l^d\right)$. Such results are derived in the next sections by using tree–decay bounds of the n–point, $n \in 2\mathbb{N}$, correlations of the many–fermion system [BPK1, Section 4].

2.3.3 Experimental Setting of Scanning Gate Microscopy

Our setting is reminiscent of the so–called scanning gate microscopy used to perform imaging of electron transport in two–dimensional semiconductor quantum structures. See, e.g., [S]. In this experimental situation, the two–dimensional electron system on a lattice experiences a time–periodic space–homogeneous electromagnetic potential perturbed by a mesoscopic or microscopic *time–independent* electric potential. Physically speaking, this situation is, mutatis mutandis, analogous to the one considered here. Therefore, we expect that our setting can also be implemented in experiments by similar technics combined with calorimetry to measure the heat production.

3 Microscopic Ohm's Law

In his original work [O] G.S. Ohm states that the current in the steady regime is proportional to the voltage applied to the conducting material. The proportionality coefficient is the conductivity of the physical system. Ohm's laws is among the most resilient laws of (classical) electricity theory and is usually justified from a microscopic point of view by the Drude model or some of its improvements that take into account quantum corrections. [Cf. the Landau theory of fermi liquids.] As in the Drude model we do not consider here interactions between charge carriers, but our approach will be also applied to interacting fermions in subsequent papers.

In this section, we study, among other things, (microscopic) Ohm's law in Fourier space for the system of free fermions described in Section 2. Without loss of generality, we only consider space–homogeneous (though time–dependent) electric fields in the box

$$\Lambda_l := \{(x_1, \ldots, x_d) \in \mathfrak{L} \, : \, |x_1|, \ldots, |x_d| \leq l\} \in \mathcal{P}_f(\mathfrak{L}) \tag{18}$$

with $l \in \mathbb{R}^+$. More precisely, let $\vec{w} := (w_1, \ldots, w_d) \in \mathbb{R}^d$ be any (normalized) vector, $\mathcal{A} \in C_0^\infty(\mathbb{R}; \mathbb{R})$ and set $\mathcal{E}_t := -\partial_t \mathcal{A}_t$ for all $t \in \mathbb{R}$. Then, $\bar{\mathbf{A}} \in \mathbf{C}_0^\infty$ is defined to be the electromagnetic potential such that the value of the electric field equals $\mathcal{E}_t \vec{w}$ at time $t \in \mathbb{R}$ for all $x \in [-1,1]^d$ and $(0,0,\ldots,0)$ for $t \in \mathbb{R}$ and $x \notin [-1,1]^d$. This choice yields rescaled electromagnetic potentials $\eta \bar{\mathbf{A}}_l$ as defined by (17) for $l \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$.

Before stating Ohm's law for the system under consideration we first need some definitions.

3.1 Current Observables

For any pair $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$, we define the *paramagnetic* and *diamagnetic* current observables $I_{\mathbf{x}} = I_{\mathbf{x}}^*$ and $\mathrm{I}_{\mathbf{x}}^{\mathbf{A}} = (\mathrm{I}_{\mathbf{x}}^{\mathbf{A}})^*$ for $\mathbf{A} \in \mathbf{C}_0^\infty$ at time $t \in \mathbb{R}$ by

$$I_{\mathbf{x}} := -2 \operatorname{Im}(a_{x^{(2)}}^* a_{x^{(1)}}) = i(a_{x^{(2)}}^* a_{x^{(1)}} - a_{x^{(1)}}^* a_{x^{(2)}}) \tag{19}$$

and

$$\mathrm{I}_{\mathbf{x}}^{\mathbf{A}} := -2\,\mathrm{Im}\left(\left(\mathrm{e}^{-i\int_0^1[\mathbf{A}(t,\alpha x^{(2)}+(1-\alpha)x^{(1)})](x^{(2)}-x^{(1)})\mathrm{d}\alpha}-1\right)a_{x^{(2)}}^{*}a_{x^{(1)}}\right) . \tag{20}$$

These are seen as currents because, by (14)–(15), they satisfy the discrete continuity equation

$$\partial_t n_x(t) = -\tau_{t,t_0}^{(\omega,\lambda,\mathbf{A})}\left(\sum_{z\in\mathfrak{L}}\mathbf{1}\left[|z|=1\right]\left(I_{(x,x+z)}+\mathrm{I}_{(x,x+z)}^{\mathbf{A}}\right)\right) \tag{21}$$

for $x\in\mathfrak{L}$ and $t\geq t_0$, where

$$n_x(t) := \tau_{t,t_0}^{(\omega,\lambda,\mathbf{A})}(a_x^{*}a_x) \tag{22}$$

is the density observable at lattice site $x\in\mathfrak{L}$ and time $t\geq t_0$. The notions of paramagnetic and diamagnetic current observables come from the physics literature, see, e.g., [GV, Eq. (A2.14)]. The paramagnetic current observable $\mathbf{1}\left[|z|=1\right]I_{(x,x+z)}$ is intrinsic to the system whereas the diamagnetic one $\mathrm{I}_{\mathbf{x}}^{\mathbf{A}}$ is only non–vanishing in presence of electromagnetic potentials.

Observe that the minus sign in the right hand side of (21) comes from the fact that the particles are negatively charged, $I_{(x,y)}$ being the observable related to the flow of particles from the lattice site x to the lattice site y or the current from y to x without external electromagnetic potential. [Positively charged particles can of course be treated in the same way.] As one can see from (21), current observables on bonds of nearest neighbors are especially important. Thus, we define the subset

$$\mathfrak{K} := \left\{\mathbf{x}:=(x^{(1)},x^{(2)})\in\mathfrak{L}^2\ :\ |x^{(1)}-x^{(2)}|=1\right\} \tag{23}$$

of bonds of nearest neighbors.

In fact, by using the canonical orthonormal basis $\{e_k\}_{k=1}^{d}$ of the Euclidian space $\mathbb{R}^d$, we define the current sums in the box Λ_l (18) for any $l\in\mathbb{R}^+$, $\mathbf{A}\in\mathbf{C}_0^\infty$, $t\in\mathbb{R}$ and $k\in\{1,\ldots,d\}$ by

$$\mathbb{I}_{k,l} := \sum_{x\in\Lambda_l}I_{(x+e_k,x)}-\varrho^{(\beta,\omega,\lambda)}\left(I_{(x+e_k,x)}\right)\mathbf{1}\qquad\text{and}\qquad\mathbf{I}_{k,l}^{\mathbf{A}} := \sum_{x\in\Lambda_l}\mathrm{I}_{(x+e_k,x)}^{\mathbf{A}} . \tag{24}$$

In particular, $\varrho^{(\beta,\omega,\lambda)}\left(\mathbb{I}_{k,l}\right)=0$, while $\mathbf{I}_{k,l}^{\mathbf{A}}=0$ when $\mathbf{A}(t,\cdot)=0$.

3.2 Adjacency Observables

Let $P_{\mathbf{x}}$, $\mathbf{x} = (x^{(1)}, x^{(2)})$, be the second–quantization of the *adjacency matrix* of the oriented graph containing exactly the pairs $(x^{(2)}, x^{(1)})$ and $(x^{(1)}, x^{(2)})$, i.e.,

$$P_{\mathbf{x}} := a_{x^{(2)}}^{*} a_{x^{(1)}} + a_{x^{(1)}}^{*} a_{x^{(2)}} \ , \qquad \mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^{2} \ . \tag{25}$$

The observable $P_{\mathbf{x}}$ is related to the current observable $I_{\mathbf{x}}$ in the following way: For any $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^{2}$,

$$2a_{x^{(1)}}^{*} a_{x^{(2)}} = P_{\mathbf{x}} + iI_{\mathbf{x}} \ , \qquad [P_{\mathbf{x}}, I_{\mathbf{x}}] = 2i \left(a_{x^{(1)}}^{*} a_{x^{(1)}} - a_{x^{(2)}}^{*} a_{x^{(2)}} \right) \ . \tag{26}$$

The importance of the adjacency observable $P_{\mathbf{x}}$ in the linear response regime results from the fact that

$$\mathrm{I}_{\mathbf{x}}^{\eta \mathbf{A}} = \eta P_{\mathbf{x}} \int_{0}^{1} [\mathbf{A}(t, \alpha x^{(2)} + (1-\alpha) x^{(1)})](x^{(2)} - x^{(1)}) \mathrm{d}\alpha + \mathcal{O}\left(\eta^{2}\right) \ . \tag{27}$$

Then, similar to the *diamagnetic* current sum $\mathbf{I}_{k,l}^{\mathbf{A}}$ (24), we define the observables

$$\mathbb{P}_{k,l} := \sum_{x \in \Lambda_{l}} P_{(x+e_{k}, x)} \in \mathcal{U} \ , \qquad l \in \mathbb{R}^{+} \ , \ k \in \{1, \ldots, d\} \ . \tag{28}$$

3.3 Microscopic Transport Coefficients

Now, for any $\beta \in \mathbb{R}^{+}$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_{0}^{+}$ we define two important functions associated with the observables $I_{\mathbf{x}}$ and $P_{\mathbf{x}}$:

(p) The *paramagnetic* transport coefficient $\sigma_{\mathrm{p}}^{(\omega)} \equiv \sigma_{\mathrm{p}}^{(\beta,\omega,\lambda)}$ is defined by

$$\sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, t) := \int_{0}^{t} \varrho^{(\beta,\omega,\lambda)} \left(i[I_{\mathbf{y}}, \tau_{s}^{(\omega,\lambda)}(I_{\mathbf{x}})] \right) \mathrm{d}s \ , \quad \mathbf{x}, \mathbf{y} \in \mathfrak{L}^{2} \ , \ t \in \mathbb{R} \ . \tag{29}$$

(d) The *diamagnetic* transport coefficient $\sigma_{\mathrm{d}}^{(\omega)} \equiv \sigma_{\mathrm{d}}^{(\beta,\omega,\lambda)}$ is defined by

$$\sigma_{\mathrm{d}}^{(\omega)}(\mathbf{x}) := \varrho^{(\beta,\omega,\lambda)}(P_{\mathbf{x}}) \ , \qquad \mathbf{x} \in \mathfrak{L}^{2} \ . \tag{30}$$

At $\mathbf{x} \in \mathfrak{L}^{2}$, $\sigma_{\mathrm{d}}^{(\omega)}(\mathbf{x})$ is obviously the expectation value of the adjacency observable $P_{\mathbf{x}}$ in the thermal state $\varrho^{(\beta,\omega,\lambda)}$ of the fermion system. This coefficient is diamagnetic because of (27). For any bond $\mathbf{x} \in \mathfrak{K}$, it can be interpreted as being minus the kinetic energy in $\mathbf{x}$: The total kinetic energy observable in the box Λ_l equals

$$2d\sum_{x\in\Lambda_l}a_x^*a_x-\sum_{\mathbf{x}=(x^{(1)},x^{(2)})\in\mathfrak{K}\cap\Lambda_l^2}a_{x^{(2)}}^*a_{x^{(1)}}=2d\sum_{x\in\Lambda_l}a_x^*a_x-\frac{1}{2}\sum_{\mathbf{x}\in\mathfrak{K}\cap\Lambda_l^2}P_{\mathbf{x}}\ .$$

The particle number observables $a_x^*a_x$, $x\in\Lambda_l$, are rather related to the (kinetic) energy in the lattice sites.

The physical meaning of $\sigma_{\mathrm{p}}^{(\omega)}$ is less obvious. We motivate in the following that it is a linear coupling between the diamagnetic current in the bond $\mathbf{y}$ and the paramagnetic current in the bond $\mathbf{x}$: Indeed, define by $\delta^{(\omega,\lambda)}$ the generator of the group $\tau^{(\omega,\lambda)}$, see (7). Then, for any fixed $\beta\in\mathbb{R}^+$, $\omega\in\Omega$, $\lambda\in\mathbb{R}_0^+$, $\eta\in\mathbb{R}$ and $\mathbf{y}\in\mathfrak{K}$, let the symmetric derivation

$$\tilde{\delta}^{(\eta,\mathbf{y})}:=\delta^{(\omega,\lambda)}+i\eta\,[I_{\mathbf{y}},\ \cdot\ ] \tag{31}$$

be the generator of the (perturbed) group $\{\tilde{\tau}_t^{(\eta,\mathbf{y})}\}_{t\in\mathbb{R}}$ of automorphisms of the C^*–algebra $\mathcal{U}$. Note that this perturbation corresponds at leading order in η to an electromagnetic potential $\eta\mathbf{A}^{(\mathbf{y})}$ of order η along the bond $\mathbf{y}$. See, e.g., Lemma 5.11. This small electromagnetic potential yields a diamagnetic current observable of the order $\eta P_{\mathbf{y}}$ on the same bond $\mathbf{y}$, cf. (27). Since $I_{\mathbf{y}}\in\mathcal{U}$ (cf. (19)), we may use a Dyson–Phillips series to obtain for small $|\eta|\ll 1$ that

$$\tilde{\tau}_t^{(\eta,\mathbf{y})}(B)=\tau_t^{(\omega,\lambda)}(B)+\eta\int_0^t\tau_{t-s}^{(\omega,\lambda)}\left(i[I_{\mathbf{y}},\tau_s^{(\omega,\lambda)}(B)]\right)\mathrm{d}s+\mathcal{O}\left(\eta^2\right)$$

for any $B\in\mathcal{U}$. If $|\eta|\ll 1$, then the diamagnetic current behaves as

$$\mathbb{J}_{\mathrm{d}}^{(\eta,\mathbf{y})}:=\varrho^{(\beta,\omega,\lambda)}(\tilde{\tau}_t^{(\eta,\mathbf{y})}(\mathrm{I}_{\mathbf{y}}^{\eta\mathbf{A}^{(\mathbf{y})}}))=\eta\varrho^{(\beta,\omega,\lambda)}\left(P_{\mathbf{y}}\right)+\mathcal{O}\left(\eta^2\left|t\right|\right)$$

with $\varrho^{(\beta,\omega,\lambda)}\left(P_{\mathbf{y}}\right)=\mathcal{O}\left(1\right)$, see (25) and (27). On the other hand, by (9) and (29), the so–called paramagnetic current

$$\mathbb{J}_{\mathrm{p}}^{(\eta,\mathbf{y})}\left(\mathbf{x},t\right):=\varrho^{(\beta,\omega,\lambda)}(\tilde{\tau}_t^{(\eta,\mathbf{y})}(I_{\mathbf{x}}))-\varrho^{(\beta,\omega,\lambda)}\left(I_{\mathbf{x}}\right)$$

satisfies

$$\partial_t\mathbb{J}_{\mathrm{p}}^{(\eta,\mathbf{y})}\left(\mathbf{x},t\right)=\mathbb{J}_{\mathrm{d}}^{(\eta,\mathbf{y})}\mathfrak{v}^{(\mathbf{y})}\left(\mathbf{x},t\right)+\mathcal{O}\left(|\mathbb{J}_{\mathrm{d}}^{(\eta,\mathbf{y})}|^2\left|t\right|\right)$$

for any $\mathbf{x},\mathbf{y}\in\mathfrak{K}$ and $t\in\mathbb{R}$, where

$$\mathfrak{v}^{(\mathbf{y})}\left(\mathbf{x},t\right):=\frac{1}{\varrho^{(\beta,\omega,\lambda)}\left(P_{\mathbf{y}}\right)}\varrho^{(\beta,\omega,\lambda)}\left(i[I_{\mathbf{y}},\tau_t^{(\omega,\lambda)}(I_{\mathbf{x}})]\right)=\frac{\partial_t\sigma_{\mathrm{p}}^{(\omega)}\left(\mathbf{x},\mathbf{y},t\right)}{\sigma_{\mathrm{d}}^{(\omega)}\left(\mathbf{y}\right)}\ . \tag{32}$$

In other words, $\mathfrak{v}$ can be interpreted as a (time–dependent) *quantum current viscosity*.

For any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$ we define two further important functions, the analogues of $\sigma_{\mathrm{p}}^{(\omega)}$ and $\sigma_{\mathrm{d}}^{(\omega)}$, associated with the observables $\mathbb{I}_{k,l}$ and $\mathbb{P}_{k,l}$:

(p) The space–averaged *paramagnetic* transport coefficient

$$t \mapsto \Xi_{\mathrm{p},l}^{(\omega)}(t) \equiv \Xi_{\mathrm{p},l}^{(\beta,\omega,\lambda)}(t) \in \mathcal{B}(\mathbb{R}^d)$$

is defined, w.r.t. the canonical orthonormal basis of $\mathbb{R}^d$, by

$$\left\{ \Xi_{\mathrm{p},l}^{(\omega)}(t) \right\}_{k,q} := \frac{1}{|\Lambda_l|} \int_0^t \varrho^{(\beta,\omega,\lambda)} \left(i[\mathbb{I}_{k,l}, \tau_s^{(\omega,\lambda)}(\mathbb{I}_{q,l})] \right) \mathrm{d}s \tag{33}$$

for any $k, q \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$.

(d) The space–averaged *diamagnetic* transport coefficient

$$\Xi_{\mathrm{d},l}^{(\omega)} \equiv \Xi_{\mathrm{d},l}^{(\beta,\omega,\lambda)} \in \mathcal{B}(\mathbb{R}^d)$$

corresponds to the diagonal matrix defined by

$$\left\{ \Xi_{\mathrm{d},l}^{(\omega)} \right\}_{k,q} := \frac{\delta_{k,q}}{|\Lambda_l|} \varrho^{(\beta,\omega,\lambda)} (\mathbb{P}_{k,l}) \ , \quad k, q \in \{1, \ldots, d\} \ . \tag{34}$$

Of course, by (24) and (29)–(30),

$$\left\{ \Xi_{\mathrm{p},l}^{(\omega)}(t) \right\}_{k,q} = \frac{1}{|\Lambda_l|} \sum_{x,y \in \Lambda_l} \sigma_{\mathrm{p}}^{(\omega)} (x + e_q, x, y + e_k, y, t) \tag{35}$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $k, q \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$, while

$$\left\{ \Xi_{\mathrm{d},l}^{(\omega)} \right\}_{k,k} = \frac{1}{|\Lambda_l|} \sum_{x \in \Lambda_l} \sigma_{\mathrm{d}}^{(\omega)} (x + e_k, x) \ . \tag{36}$$

Both coefficients are typically the paramagnetic and diamagnetic conductivity one experimentally measures for large samples, i.e., large enough boxes Λ_l. Indeed, we show in [BPK2] that the limits $l \to \infty$ of $\Xi_{\mathrm{p},l}^{(\omega)}$ and $\Xi_{\mathrm{d},l}^{(\omega)}$ generally exist and define so–called macroscopic paramagnetic and diamagnetic conductivities. Before going further, we first discuss some important mathematical properties of $\Xi_{\mathrm{p},l}^{(\omega)}$ and $\Xi_{\mathrm{d},l}^{(\omega)}$.

By using the scalar product $\langle \cdot, \cdot \rangle$ in $\ell^2(\mathfrak{L})$, the canonical orthonormal basis $\{\mathfrak{e}_x\}_{x\in\mathfrak{L}}$ of $\ell^2(\mathfrak{L})$ and the symbol $\mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)}$ defined by (10), we observe from (36) that

$$\left\{\Xi_{\mathrm{d},l}^{(\omega)}\right\}_{k,k} = \frac{2}{|\Lambda_l|}\sum_{x\in\Lambda_l} \mathrm{Re}\left\{\langle \mathfrak{e}_{x+e_k}, \mathbf{d}_{\mathrm{fermi}}^{(\beta,\omega,\lambda)}\mathfrak{e}_x\rangle\right\} \in [-2,2] \tag{37}$$

for any $l,\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $k \in \{1,\ldots,d\}$.

The main property of the paramagnetic transport coefficient $\Xi_{\mathrm{p},l}^{(\omega)}$ is proven in Section 5.1.2 and given in the next theorem. To present it, we introduce the notation $\mathcal{B}_+(\mathbb{R}^d) \subset \mathcal{B}(\mathbb{R}^d)$ for the set of positive linear operators on $\mathbb{R}^d$. For any $\mathcal{B}(\mathbb{R}^d)$–valued measure μ on $\mathbb{R}$, we additionally denote by $\|\mu\|_{\mathrm{op}}$ the measure on $\mathbb{R}$ taking values in $\mathbb{R}_0^+$ that is defined, for any Borel set $\mathcal{X}$, by

$$\|\mu\|_{\mathrm{op}}(\mathcal{X}) := \sup\left\{\sum_{i\in I}\|\mu(\mathcal{X}_i)\|_{\mathrm{op}} : \{\mathcal{X}_i\}_{i\in I} \text{ is a finite Borel partition of } \mathcal{X}\right\} . \tag{38}$$

We, moreover, say that μ is symmetric if $\mu(\mathcal{X}) = \mu(-\mathcal{X})$ for any Borel set $\mathcal{X} \subset \mathbb{R}$. With these definitions we have the following assertion:

Theorem 3.1 (Microscopic paramagnetic conductivity measures)
For any $l,\beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, there exists a non–zero symmetric $\mathcal{B}_+(\mathbb{R}^d)$–valued measure $\mu_{\mathrm{p},l}^{(\omega)} \equiv \mu_{\mathrm{p},l}^{(\beta,\omega,\lambda)}$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} (1+|\nu|)\,\|\mu_{\mathrm{p},l}^{(\omega)}\|_{\mathrm{op}}(\mathrm{d}\nu) < \infty , \tag{39}$$

uniformly w.r.t. $l,\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and

$$\Xi_{\mathrm{p},l}^{(\omega)}(t) = \int_{\mathbb{R}} (\cos(t\nu) - 1)\,\mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu) , \qquad t \in \mathbb{R} .$$

Proof: The assertions follow from Theorems 5.4 and 5.5 combined with Corollary 5.7 and Lemma 5.10. ■

Corollary 3.2 (Properties of the microscopic paramagnetic conductivity)
For $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$ *and* $\lambda \in \mathbb{R}_0^+$, $\Xi_{\mathrm{p},l}^{(\omega)}$ *has the following properties:*
(i) *Time–reversal symmetry:* $\Xi_{\mathrm{p},l}^{(\omega)}(0) = 0$ *and*

$$\Xi_{\mathrm{p},l}^{(\omega)}(-t) = \Xi_{\mathrm{p},l}^{(\omega)}(t) \ , \qquad t \in \mathbb{R} \ .$$

(ii) *Negativity of* $\Xi_{\mathrm{p},l}^{(\omega)}$*:*

$$\Xi_{\mathrm{p},l}^{(\omega)}(t) \leq 0 \ , \qquad t \in \mathbb{R} \ .$$

(iii) *Cesàro mean of* $\Xi_{\mathrm{p},l}^{(\omega)}$*:*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \Xi_{\mathrm{p},l}^{(\omega)}(s) \, \mathrm{d}s = -\mu_{\mathrm{p},l}^{(\omega)}(\mathbb{R} \backslash \{0\}) \leq 0 \ .$$

(iv) *Equicontinuity: The family* $\{\Xi_{\mathrm{p},l}^{(\beta,\omega,\lambda)}\}_{l,\beta \in \mathbb{R}^+, \omega \in \Omega, \lambda \in \mathbb{R}_0^+}$ *of maps from* $\mathbb{R}$ *to* $\mathcal{B}(\mathbb{R}^d)$ *is equicontinuous.*
(v) *Macroscopic paramagnetic conductivity measures: The family* $\{\mu_{\mathrm{p},l}^{(\omega)}\}_{l \in \mathbb{R}^+}$ *has weak***–accumulation points.*

Proof: (i)–(iii) are direct consequences of Theorem 3.1 and Lebesgue's dominated convergence theorem. To prove (iv), observe that the uniform bound (39) implies that, for any $\nu_0 \in \mathbb{R}_0^+$,

$$\mu_{\mathrm{p},l}^{(\omega)}(\mathbb{R} \backslash [-\nu_0, \nu_0]) = \mathcal{O}\left(\nu_0^{-1}\right)$$

uniformly w.r.t. $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$. (v) follows from Theorem 3.1 and the weak*–compactness of the unit ball in the set of measures on $\mathbb{R}$ taking values in the set of positive elements of $\mathcal{B}(\mathbb{R}^d)$. ■

The $\mathcal{B}_+(\mathbb{R}^d)$–valued measures $\mu_{\mathrm{p},l}^{(\omega)}$ can be represented in terms of the spectral measure of an explicit self–adjoint operator w.r.t. explicitly given vectors, see Equation (109). From this representation, one concludes for instance that, if the operator $(\Delta_{\mathrm{d}} + \lambda V_\omega)$ has purely (absolutely) continuous spectrum (as for $\lambda = 0$) then, for any $k, q \in \{1, \ldots, d\}$,

$$\left\{\mu_{\mathrm{p},l}^{(\omega)}(\mathbb{R} \backslash \{0\})\right\}_{k,q} = \frac{1}{|\Lambda_l|} (\mathbb{I}_{k,l}, \mathbb{I}_{q,l})_{\sim}^{(\omega)} \ .$$

Here, $(\cdot,\cdot)_{\sim}^{(\omega)}$ is the Duhamel two–point function $(\cdot,\cdot)_{\sim}^{(\omega)}$, which is studied in detail in Section A. In fact, the constant $\mu_{\mathrm{p},l}^{(\omega)}(\mathbb{R} \backslash \{0\})$ is the so–called static admittance of linear response theory, see Theorem 5.8. Moreover, Theorem 5.9 explains how $\mu_{\mathrm{p},l}^{(\omega)}$ can also be constructed from the *space–averaged* quantum current viscosity

$$\mathbf{V}_{l}^{(\omega)}(t) := \left(\Xi_{\mathrm{d},l}^{(\omega)}\right)^{-1}\partial_{t}\Xi_{\mathrm{p},l}^{(\omega)}(t) \in \mathcal{B}(\mathbb{R}^{d}) \tag{40}$$

for any $l,\beta \in \mathbb{R}^{+}$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_{0}^{+}$ and $t \in \mathbb{R}$. Compare with (32). More precisely, it is the boundary value of the (imaginary part of the) Laplace–Fourier transform of $\Xi_{\mathrm{d},l}^{(\omega)}\mathbf{V}_{l}^{(\omega)}$.

Recall that, as asserted in Theorem 3.1, the measure $\mu_{\mathrm{p},l}^{(\omega)}$ is never the zero–measure. Nevertheless, it is a priori not clear whether the weak*–accumulation points of the family $\{\mu_{\mathrm{p},l}^{(\omega)}\}_{l\in\mathbb{R}^{+}}$ also have this property. We show in a companion paper that, as $l \to \infty$, the measure $\mu_{\mathrm{p},l}^{(\omega)}$ converges to the zero–measure if $\lambda = 0$ but, for $\lambda \in \mathbb{R}^{+}$, there is generally a unique weak*–accumulation point of $\{\mu_{\mathrm{p},l}^{(\omega)}\}_{l\in\mathbb{R}^{+}}$, which is not the zero–measure.

3.4 Paramagnetic and Diamagnetic Currents

Recall that we assume in this section that the current results from a space–homogeneous electric field $\eta\mathcal{E}_{t}\vec{w}$ at time $t \in \mathbb{R}$ in the box Λ_{l}, where $\vec{w} := (w_{1},\ldots,w_{d}) \in \mathbb{R}^{d}$, $\mathcal{E}_{t} := -\partial_{t}\mathcal{A}_{t}$ for all $t \in \mathbb{R}$, and $\mathcal{A} \in C_{0}^{\infty}(\mathbb{R};\mathbb{R})$. This electric field corresponds to the (rescaled) electromagnetic potential $\eta\bar{\mathbf{A}}_{l}$. We also remind that $\{e_{k}\}_{k=1}^{d}$ is the canonical orthonormal basis of the Euclidian space $\mathbb{R}^{d}$.

Generally, even in the absence of electromagnetic fields, i.e., if $\eta = 0$, there exist (thermal) currents coming from the inhomogeneity of the fermion system for $\lambda \in \mathbb{R}^{+}$. For any $l,\beta \in \mathbb{R}^{+}$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_{0}^{+}$ and $k \in \{1,\ldots,d\}$,

$$\mathbb{J}_{k,l}^{(\omega)} \equiv \mathbb{J}_{k,l}^{(\beta,\omega,\lambda)} := |\Lambda_{l}|^{-1}\sum_{x\in\Lambda_{l}}\varrho^{(\beta,\omega,\lambda)}(I_{(x+e_{k},x)}) \tag{41}$$

is the density of current along the direction e_{k} in the box Λ_{l}. In the space–homogeneous case, by symmetry, $\mathbb{J}_{k,l}^{(\omega)} = 0$ but in general, $\mathbb{J}_{k,l}^{(\omega)} \neq 0$. We prove in [BPK2] that

$$\lim_{l\to\infty}\mathbb{J}_{k,l}^{(\omega)} = 0$$

almost surely if $\omega \in \Omega$ is the realization of some ergodic random potential.

Then, for any $l,\beta \in \mathbb{R}^{+}$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_{0}^{+}$, $\eta \in \mathbb{R}$, $\vec{w} \in \mathbb{R}^{d}$, $\mathcal{A} \in C_{0}^{\infty}(\mathbb{R};\mathbb{R})$ and $t \geq t_{0}$, the (increment of) current density resulting from the space–homogeneous electric perturbation $\mathcal{E}$ in the box Λ_{l} is the sum of two current densities defined from (24):

(p) The paramagnetic current density

$$\mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) \equiv \mathbb{J}_{\mathrm{p}}^{(\beta,\omega,\lambda,\eta\bar{\mathbf{A}}_l)}(t) \in \mathbb{R}^d$$

is defined by the space average of the current increment vector inside the box Λ_l, that is for any $k \in \{1,\ldots,d\}$,

$$\left\{\mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t)\right\}_k := |\Lambda_l|^{-1}\,\rho_t^{(\beta,\omega,\lambda,\eta\bar{\mathbf{A}}_l)}(\mathbb{I}_{k,l})\ . \tag{42}$$

(d) The diamagnetic (or ballistic) current density

$$\mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) \equiv \mathbb{J}_{\mathrm{d}}^{(\beta,\omega,\lambda,\eta\bar{\mathbf{A}}_l)}(t) \in \mathbb{R}^d$$

is defined analogously, for any $k \in \{1,\ldots,d\}$, by

$$\left\{\mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t)\right\}_k := |\Lambda_l|^{-1}\,\rho_t^{(\beta,\omega,\lambda,\eta\bar{\mathbf{A}}_l)}(\mathbf{I}_{k,l}^{\eta\bar{\mathbf{A}}_l})\ . \tag{43}$$

The paramagnetic current density is only related to the *change of internal state* $\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}$ produced by the electromagnetic field. We will show below that these currents carry the paramagnetic energy increment defined in Section 4.3. The diamagnetic current density corresponds to a raw ballistic flow of charged particles caused by the electric field, at thermal equilibrium. It directly comes from the change of the electromagnetic potential expressed in terms of the observable (57) defined below. We will show that it yields the diamagnetic energy defined in Section 4.3. With this, diamagnetic and paramagnetic currents are respectively "first order" and "second order" with respect to changes of the electromagnetic potentials and thus have different physical properties. See for instance Theorems 3.3 and 4.1.

3.5 Current Linear Response

We are now in position to derive a microscopic version of Ohm's law. We use the space–averaged paramagnetic and diamagnetic transport coefficients $\Xi_{\mathrm{p},l}^{(\omega)}$ (33) and $\Xi_{\mathrm{d},l}^{(\omega)}$ (34) to define the $\mathbb{R}^d$–valued functions

$$J_{\mathrm{p},l}^{(\omega,\mathcal{A})} \equiv J_{\mathrm{p},l}^{(\beta,\omega,\lambda,\vec{w},\mathcal{A})} \qquad \text{and} \qquad J_{\mathrm{d},l}^{(\omega,\mathcal{A})} \equiv J_{\mathrm{d},l}^{(\beta,\omega,\lambda,\vec{w},\mathcal{A})}$$

by

$$J_{\mathrm{p},l}^{(\omega,\mathcal{A})}(t) := \int_{t_0}^{t} \left(\Xi_{\mathrm{p},l}^{(\omega)}(t-s)\,\vec{w}\right)\mathcal{E}_s \mathrm{d}s\ , \qquad t \geq t_0\ , \tag{44}$$

$$J_{\mathrm{d},l}^{(\omega,\mathcal{A})}(t) := \left(\Xi_{\mathrm{d},l}^{(\omega)}\vec{w}\right)\int_{t_0}^{t} \mathcal{E}_s \mathrm{d}s\ , \qquad t \geq t_0\ , \tag{45}$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\vec{w} \in \mathbb{R}^d$ and $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$. They are the linear responses of the paramagnetic and diamagnetic current densities, respectively:

Theorem 3.3 (Microscopic Ohm's law)
For any $\vec{w} \in \mathbb{R}^d$ and $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$, there is $\eta_0 \in \mathbb{R}^+$ such that, for $|\eta| \in [0,\eta_0]$,

$$\mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) = \eta J_{\mathrm{p},l}^{(\omega,\mathcal{A})}(t) + \mathcal{O}\left(\eta^2\right) \qquad \text{and} \qquad \mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) = \eta J_{\mathrm{d},l}^{(\omega,\mathcal{A})}(t) + \mathcal{O}\left(\eta^2\right)\ ,$$

uniformly for $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$.

Proof: See Lemmata 5.14–5.15. ■

The fact that the asymptotics obtained are uniform w.r.t. $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$ is a crucial property to get macroscopic Ohm's law in [BPK2]. Note also that Theorem 3.3 can easily be extended to macroscopically space–inhomogeneous electromagnetic fields, that is, for all space–rescaled vector potentials $\mathbf{A}_l$ (17) with $\mathbf{A} \in \mathbf{C}_0^\infty$, by exactly the same methods as in the proof of Theorem 4.1. We refrain from doing it at this point, for technical simplicity. The result above can indeed be deduced from Theorem 4.1, see Equations (65)–(66).

As a consequence, $\Xi_{\mathrm{p},l}^{(\omega)}$ and $\Xi_{\mathrm{d},l}^{(\omega)}$ can be interpreted as *charge* transport coefficients. Observe that $\Xi_{\mathrm{p},l}^{(\omega)}(0) = 0$, by Corollary 3.2 (i). Therefore, when the electric field is switched on, it accelerates the charged particles and first induces diamagnetic currents, cf. (45). This creates a kind of "wave front" that destabilizes the whole system by changing its internal state. By the phenomenon of current viscosity discussed in Section 3.3, the presence of such diamagnetic currents leads to the progressive appearance of paramagnetic currents. We prove in Section 4 that these paramagnetic currents are responsible for heat production and modify as well the electromagnetic potential energy of charge carriers. Indeed, the positive measures of Theorem 3.1 are directly related to heat production (cf. Section 4.4) and are the boundary values of the (imaginary part of the) Laplace–Fourier transforms of the current viscosities as discussed in the previous section.

Note that Theorem 3.3 also leads to (finite–volume) *Green–Kubo relations*, by (33) and (44). Indeed, by (24), $|\Lambda_l|^{-\frac{1}{2}}\,\mathbb{I}_{k,l}$ is a *current fluctuation* and (33) gives:

$$\left\{\Xi_{\mathrm{p},l}^{(\omega)}\,(t)\right\}_{k,q}=\int_0^t \varrho^{(\beta,\omega,\lambda)}\left(i\left[|\Lambda_l|^{-\frac{1}{2}}\,\mathbb{I}_{k,l},|\Lambda_l|^{-\frac{1}{2}}\,\tau_s^{(\omega,\lambda)}(\mathbb{I}_{q,l})\right]\right)\mathrm{d}s \tag{46}$$

for any $l,\beta\in\mathbb{R}^+$, $\omega\in\Omega$, $\lambda\in\mathbb{R}_0^+$, $t\in\mathbb{R}$ and $k,q\in\{1,\ldots,d\}$. In the limit $l\to\infty$ we show in [BPK2] that $\Xi_{\mathrm{p},l}^{(\omega)}$ is related to a quasi–free dynamics on the CCR algebra of (current) fluctuations.

Theorem 3.3 together with (44)–(45) gives a natural notion of linear conductivity of the fermion system in the box Λ_l: It is the map

$$t\mapsto\mathbf{\Sigma}_l^{(\omega)}\equiv\mathbf{\Sigma}_l^{(\beta,\omega,\lambda)}\,(t)\in\mathcal{B}(\mathbb{R}^d)$$

defined by

$$\mathbf{\Sigma}_l^{(\omega)}(t):=\begin{cases}0 & ,\qquad t\leq 0\,,\\ \Xi_{\mathrm{d},l}^{(\omega)}+\Xi_{\mathrm{p},l}^{(\omega)}\,(t) & ,\qquad t\geq 0\,,\end{cases} \tag{47}$$

for $l,\beta\in\mathbb{R}^+$, $\omega\in\Omega$, $\lambda\in\mathbb{R}_0^+$. The *total* current

$$J_l^{(\omega,\mathcal{A})}(t):=J_{\mathrm{p},l}^{(\omega,\mathcal{A})}(t)+J_{\mathrm{d},l}^{(\omega,\mathcal{A})}(t)\,,\qquad t\geq t_0\,,$$

which as in [GV, Eq. (A2.14)] is the sum of paramagnetic and diamagnetic current densities, has the following linear response:

$$J_l^{(\omega,\mathcal{A})}(t)=\int_{\mathbb{R}}\left(\mathbf{\Sigma}_l^{(\omega)}\,(t-s)\,\vec{w}\right)\mathcal{E}_s\mathrm{d}s=\begin{pmatrix}\{\mathbf{\Sigma}_l^{(\omega)}\vec{w}\}_1{*}\mathcal{E}\\ \vdots\\ \{\mathbf{\Sigma}_l^{(\omega)}\vec{w}\}_d{*}\mathcal{E}\end{pmatrix}\,. \tag{48}$$

In particular, if the electric field stays constant for sufficiently large times, i.e., $\mathcal{E}_t=D$ for arbitrary large times $t\in[T,\infty)$ with $T>t_0$, then in the situation where $t\gg T$, i.e., in the DC–regime, we deduce from Corollary 3.2 (iii) and (47)–(48) that

$$|t|^{-1}\,J_l^{(\omega,\mathcal{A})}(t)=D(\Xi_{\mathrm{d},l}^{(\omega)}-\mu_{\mathrm{p},l}^{(\omega)}\,(\mathbb{R}\backslash\,\{0\}))+o\,(1)\,. \tag{49}$$

It is not a priori clear whether $\mu_{\mathrm{p},l}^{(\omega)}\,(\mathbb{R}\backslash\,\{0\})=\Xi_{\mathrm{d},l}^{(\omega)}$ or not. We prove in [BPK2] that this last equality actually holds in the limit $l\to\infty$. [Recall that $\mathbf{A}\in\mathbf{C}_0^\infty$ is compactly supported in space and time, but it can be switched off at arbitrary large times.]

In order to express the *in–phase current* from (48), we define by $\mathbf{\Sigma}_{l,+}^{(\omega)}$ the symmetrization of $\mathbf{\Sigma}_{l}^{(\omega)}$, that is,

$$\mathbf{\Sigma}_{l,+}^{(\omega)}(t) := \mathbf{\Sigma}_{l}^{(\omega)}(|t|) = \Xi_{\mathrm{d},l}^{(\omega)} + \Xi_{\mathrm{p},l}^{(\omega)}(t) \ , \qquad t \in \mathbb{R} \ , \tag{50}$$

see Corollary 3.2 (i). Similarly, the anti–symmetrization $\mathbf{\Sigma}_{l,-}^{(\omega)}$ of $\mathbf{\Sigma}_{l}^{(\omega)}$ is given by

$$\mathbf{\Sigma}_{l,-}^{(\omega)} := \mathrm{sign}(t)\mathbf{\Sigma}_{l}^{(\omega)}(|t|) \ , \qquad t \in \mathbb{R} \ . \tag{51}$$

With these definitions the current linear response (48) equals

$$J_{l}^{(\omega,\mathcal{A})}(t) = \frac{1}{2}\int_{\mathbb{R}} \left(\mathbf{\Sigma}_{l,+}^{(\omega)}(t-s)\,\vec{w}\right)\mathcal{E}_{s}\mathrm{d}s + \frac{1}{2}\int_{\mathbb{R}} \left(\mathbf{\Sigma}_{l,-}^{(\omega)}(t-s)\,\vec{w}\right)\mathcal{E}_{s}\mathrm{d}s \ . \tag{52}$$

The first part in the right hand side of this equality is by definition the in–phase current.

This last equation is directly related to Ohm's law in Fourier space: Similar to [KLM], it is indeed natural to define the *conductivity measure* $\mu_{\Lambda_l}^{(\omega)} \equiv \mu_{\Lambda_l}^{(\beta,\omega,\lambda)}$ as being the Fourier transform of $\mathbf{\Sigma}_{l,+}^{(\omega)}(t)$. By Theorem 3.1 and (50),

$$\mu_{\Lambda_l}^{(\omega)}(\mathcal{X}) = \mu_{\mathrm{p},l}^{(\omega)}(\mathcal{X}) + (\Xi_{\mathrm{d},l}^{(\omega)} - \mu_{\mathrm{p},l}^{(\omega)}(\mathbb{R}))\mathbf{1}\left[0 \in \mathcal{X}\right]$$

with $\mathcal{X} \subset \mathbb{R}$ being any Borel set. Therefore, we can rewrite the current linear response (52) as

$$J_{l}^{(\omega,\mathcal{A})}(t) = \frac{1}{2}\int_{\mathbb{R}} \hat{\mathcal{E}}_{\nu}^{(t)}\mu_{\Lambda_l}^{(\omega)}(\mathrm{d}\nu)\,\vec{w} + \frac{i}{2}\int_{\mathbb{R}} \mathbb{H}(\hat{\mathcal{E}}^{(t)})(\nu)\,\mu_{\Lambda_l}^{(\omega)}(\mathrm{d}\nu)\,\vec{w} \tag{53}$$

with $\hat{\mathcal{E}}$ being the Fourier transform of $\mathcal{E}$, $\hat{\mathcal{E}}_{\nu}^{(t)} := \mathrm{e}^{i\nu t}\hat{\mathcal{E}}_{\nu}$, and where $\mathbb{H}$ is the Hilbert transform, i.e.,

$$\mathbb{H}(f)(\nu) := -\frac{1}{\pi}\lim_{\varepsilon\to 0^{+}}\int_{[-\varepsilon^{-1},-\varepsilon]\cup[\varepsilon,\varepsilon^{-1}]} \frac{f(\nu - x)}{x}\mathrm{d}x \ , \qquad \nu \in \mathbb{R} \ .$$

Here, $f : \mathbb{R} \to \mathbb{C}$ belongs to the space Υ of functions which are the Fourier transforms of compactly supported and piece–wise smooth functions $\mathbb{R} \to \mathbb{R}$. Equation (53) corresponds to Ohm's law in Fourier space at microscopic scales, in accordance with experimental results of [F, W].

Moreover, by Corollary 3.2 (v) together with Equation (37), Theorem 3.1 and the Bolzano–Weierstrass theorem, the family $\{\mu_{\Lambda_l}^{(\omega)}\}_{l\in\mathbb{R}^{+}}$ has weak*–accumulation points. As a consequence, the current linear response converges pointwise along a subsequence to

$$J_{\infty}^{(\omega,\mathcal{A})}(t)=\frac{1}{2}\int_{\mathbb{R}}\hat{\mathcal{E}}_{\nu}^{(t)}\mu_{\mathbb{R}^d}^{(\omega)}(\mathrm{d}\nu)\,\vec{w}+\frac{i}{2}\int_{\mathbb{R}}\mathbb{H}(\hat{\mathcal{E}}^{(t)})\,(\nu)\,\mu_{\mathbb{R}^d}^{(\omega)}(\mathrm{d}\nu)\,\vec{w}$$

with $\mu_{\mathbb{R}^d}^{(\omega)}$ being some weak*–accumulation point of $\{\mu_{\Lambda_l}^{(\omega)}\}_{l\in\mathbb{R}^+}$. $\mu_{\mathbb{R}^d}^{(\omega)}$ can be interpreted as a *macroscopic conductivity measure* and is under reasonable circumstances unique. In fact, we give in [BPK2] a detailed analysis of such limits by considering random static external potentials.

Observe that $i\mathbb{H}(\Upsilon)\subset\Upsilon$ and $\mathbb{H}\circ\mathbb{H}=-1$ on Υ. In particular, the two functionals

$$\begin{aligned}\mu_{\Lambda_l}^{\|} &: \Upsilon\to\mathbb{R}\,,\ \mu_{\Lambda_l}^{\|}(f):=\frac{1}{2}\int_{\mathbb{R}}f(\nu)\mu_{\Lambda_l}^{(\omega)}(\mathrm{d}\nu)\ ,\\ \mu_{\Lambda_l}^{\perp} &: \Upsilon\to\mathbb{R}\,,\ \mu_{\Lambda_l}^{\perp}(f):=\frac{1}{2}\int_{\mathbb{R}}\mathbb{H}(f)(\nu)\mu_{\Lambda_l}^{(\omega)}(\mathrm{d}\nu)\ ,\end{aligned}$$

satisfy Kramers–Kronig relations:

$$\mu_{\Lambda_l}^{\|}\circ\mathbb{H}=\mu_{\Lambda_l}^{\perp}\qquad\text{and}\qquad\mu_{\Lambda_l}^{\perp}\circ\mathbb{H}=-\mu_{\Lambda_l}^{\|}\,. \tag{54}$$

Note that, w.r.t. the usual topology of the space $\mathcal{S}(\mathbb{R};\mathbb{C})$ of Schwartz functions, $\Upsilon\cap\mathcal{S}(\mathbb{R};\mathbb{C})$ is dense in $\mathcal{S}(\mathbb{R};\mathbb{C})$ and $\mu_{\Lambda_l}^{\|},\mu_{\Lambda_l}^{\perp}$ are continuous on $\Upsilon\cap\mathcal{S}(\mathbb{R};\mathbb{C})$. Hence, each entry of $\mu_{\Lambda_l}^{\|},\mu_{\Lambda_l}^{\perp}$ w.r.t. the canonical orthonormal basis of $\mathbb{R}^d$ can be seen as a tempered distribution. Moreover, (53) yields

$$J_{l}^{(\omega,\mathcal{A})}(t)=\left(\mu_{\Lambda_l}^{\|}(\hat{\mathcal{E}}^{(t)})+i\mu_{\Lambda_l}^{\perp}(\hat{\mathcal{E}}^{(t)})\right)\vec{w}\,. \tag{55}$$

Therefore, the $\mathcal{B}(\mathbb{R}^d)$–valued distribution $\mu_{\Lambda_l}^{\|}$ is the linear response in–phase component of the total conductivity in Fourier space. For this reason, $\mu_{\Lambda_l}^{\|}+i\mu_{\Lambda_l}^{\perp}$ is named here the (microscopic, $\mathcal{B}(\mathbb{R}^d)$–valued) *conductivity distribution* of the box Λ_l. Similarly, the limit $J_{\infty}^{(\omega,\mathcal{A})}$ obeys (55) with $\mu_{\mathbb{R}^d}^{(\omega)}$ replacing $\mu_{\Lambda_l}^{(\omega)}$.

4 Microscopic Joule's Law

...the calorific effects of equal quantities of transmitted electricity are proportional to the resistances opposed to its passage, whatever may be the length, thickness,

shape, or kind of metal which closes the circuit : and also that, coeteris paribus, these effects are in the duplicate ratio of the quantities of transmitted electricity ; and consequently also in the duplicate ratio of the velocity of transmission.

[Joule, 1840]

In other words, as originally observed [J] by the physicist J. P. Joule, the heat (per second) produced within an electric circuit is proportional to the electric resistance and the square of the current.

The aim of this section is to prove such a phenomenology for the fermion system under consideration. Before studying Joule's effect we need to define energy observables and increments:

4.1 Energy Observables

For any $L \in \mathbb{R}^+$, the *internal* energy observable in the box Λ_L (18) is defined by

$$H_L^{(\omega,\lambda)} := \sum_{x,y\in\Lambda_L} \langle \mathfrak{e}_x, (\Delta_{\mathrm{d}} + \lambda V_\omega)\mathfrak{e}_y\rangle a_x^* a_y \in \mathcal{U}\ . \tag{56}$$

It is the second quantization of the one–particle operator $\Delta_{\mathrm{d}} + \lambda V_\omega$ restricted to the subspace $\ell^2(\Lambda_L) \subset \ell^2(\mathfrak{L})$. When the electromagnetic field is switched on, i.e., for $t \geq t_0$, the (time–dependent) *total* energy observable in the box Λ_L is then equal to $H_L^{(\omega,\lambda)} + W_t^{\mathbf{A}}$, where, for any $\mathbf{A} \in \mathbf{C}_0^\infty$ and $t \in \mathbb{R}$,

$$W_t^{\mathbf{A}} := \sum_{x,y\in\Lambda_L} \langle \mathfrak{e}_x, (\Delta_{\mathrm{d}}^{(\mathbf{A})} - \Delta_{\mathrm{d}})\mathfrak{e}_y\rangle a_x^* a_y \in \mathcal{U} \tag{57}$$

is the electromagnetic *potential* energy observable.

We define below four types of energies because we have the two above energy observables as well as two relevant states, the thermal equilibrium state $\varrho^{(\beta,\omega,\lambda)}$ and its time evolution $\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}$.

4.2 Time–dependent Thermodynamic View Point

In [BPK1], we investigate the *heat* production of the (non–autonomous) C^*–dynamical system $(\mathcal{U}, \tau_{t,s}^{(\omega,\lambda,\mathbf{A})})$ for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$. We show in [BPK1, Theorem 3.2] that the fermion system under consideration obeys the first law of thermodynamics. It means that the heat production due to the electromagnetic field is equal to an *internal* energy increment. The latter is directly related to the family $\{H_L^{(\omega,\lambda)}\}_{L\in\mathbb{R}^+}$ of internal energy observables. We also consider an electromagnetic potential energy defined from the observable $W_t^{\mathbf{A}}$. Hence, we define the following energy increments:

(Q) The *internal* energy increment $\mathbf{S}^{(\omega,\mathbf{A})} \equiv \mathbf{S}^{(\beta,\omega,\lambda,\mathbf{A})}$ is a map from $\mathbb{R}$ to $\mathbb{R}_0^+$ defined by

$$\mathbf{S}^{(\omega,\mathbf{A})}(t) := \lim_{L\to\infty} \left\{ \rho_t^{(\beta,\omega,\lambda,\mathbf{A})}(H_L^{(\omega,\lambda)}) - \varrho^{(\beta,\omega,\lambda)}(H_L^{(\omega,\lambda)}) \right\} . \tag{58}$$

It takes positive finite values because of [BPK1, Theorem 3.2].

(P) The electromagnetic *potential* energy (increment) $\mathbf{P}^{(\omega,\mathbf{A})} \equiv \mathbf{P}^{(\beta,\omega,\lambda,\mathbf{A})}$ is a map from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathbf{P}^{(\omega,\mathbf{A})}(t) := \rho_t^{(\beta,\omega,\lambda,\mathbf{A})}(W_t^{\mathbf{A}}) = \rho_t^{(\beta,\omega,\lambda,\mathbf{A})}(W_t^{\mathbf{A}}) - \varrho^{(\beta,\omega,\lambda)}(W_{t_0}^{\mathbf{A}}) . \tag{59}$$

In other words, $\mathbf{S}^{(\omega,\mathbf{A})}$ is the increase of internal energy of the fermion system due to the change of its internal state, whereas $\mathbf{P}^{(\omega,\mathbf{A})}$ is the electromagnetic potential energy of the fermion system in the state $\rho_t^{(\beta,\omega,\lambda,\mathbf{A})}$. By [BPK1, Theorem 3.2], $\mathbf{S}^{(\omega,\mathbf{A})}$ equals the heat production of the fermion system. Moreover, by [BPK1, Eq. (24)], the increase of *total* energy of the *infinite* system

$$\lim_{L\to\infty} \left\{ \rho_t^{(\beta,\omega,\lambda,\mathbf{A})}(H_L^{(\omega,\lambda)} + W_t^{\mathbf{A}}) - \varrho^{(\beta,\omega,\lambda)}(H_L^{(\omega,\lambda)}) \right\} = \mathbf{S}^{(\omega,\mathbf{A})}(t) + \mathbf{P}^{(\omega,\mathbf{A})}(t) \tag{60}$$

is exactly the work performed by the electromagnetic field at time $t \geq t_0$:

$$\mathbf{S}^{(\omega,\mathbf{A})}(t) + \mathbf{P}^{(\omega,\mathbf{A})}(t) = \int_{t_0}^{t} \rho_s^{(\beta,\omega,\lambda,\mathbf{A})}\left(\partial_s W_s^{\mathbf{A}}\right) \mathrm{d}s . \tag{61}$$

4.3 Electromagnetic View Point

In the previous subsection the *total* energy increment is decomposed into two components (60) that can be identified with heat production and potential energy. This total energy increment can also be decomposed in two other components which have interesting features in terms of currents. Indeed, for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $\mathbf{A} \in \mathbf{C}_0^\infty$, we define:

(p) The *paramagnetic* energy increment $\mathfrak{J}_{\mathrm{p}}^{(\omega,\mathbf{A})} \equiv \mathfrak{J}_{\mathrm{p}}^{(\beta,\omega,\lambda,\mathbf{A})}$ is the map from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathfrak{J}_{\mathrm{p}}^{(\omega,\mathbf{A})}(t) := \lim_{L\rightarrow\infty} \left\{ \rho_t^{(\beta,\omega,\lambda,\mathbf{A})}(H_L^{(\omega,\lambda)} + W_t^{\mathbf{A}}) - \varrho^{(\beta,\omega,\lambda)}(H_L^{(\omega,\lambda)} + W_t^{\mathbf{A}}) \right\} . \tag{62}$$

(d) The *diamagnetic* energy (increment) $\mathfrak{J}_{\mathrm{d}}^{(\omega,\mathbf{A})} \equiv \mathfrak{J}_{\mathrm{d}}^{(\beta,\omega,\lambda,\mathbf{A})}$ is the map from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$\mathfrak{J}_{\mathrm{d}}^{(\omega,\mathbf{A})}(t) := \varrho^{(\beta,\omega,\lambda)}(W_t^{\mathbf{A}}) = \varrho^{(\beta,\omega,\lambda)}(W_t^{\mathbf{A}}) - \varrho^{(\beta,\omega,\lambda)}(W_{t_0}^{\mathbf{A}}) . \tag{63}$$

Note that the limit (62) exists at all times because of (60)–(61). In particular,

$$\mathfrak{J}_{\mathrm{p}}^{(\omega,\mathbf{A})}(t) + \mathfrak{J}_{\mathrm{d}}^{(\omega,\mathbf{A})}(t) = \int_{t_0}^{t} \rho_s^{(\beta,\omega,\lambda,\mathbf{A})}\left(\partial_s W_s^{\mathbf{A}}\right) \mathrm{d}s \tag{64}$$

for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$ and times $t \geq t_0$.

The term $\mathfrak{J}_{\mathrm{p}}^{(\omega,\mathbf{A})}$ is the part of electromagnetic work implying a change of the internal state of the system, whereas the diamagnetic energy is the raw electromagnetic energy given to the system at thermal equilibrium. Indeed, because of the second law of thermodynamics, in presence of non–zero electromagnetic fields the system constantly tends to minimize the (instantaneous) free–energy associated with $H_L^{(\omega,\lambda)} + W_t^{\mathbf{A}}$ and it is thus forced to change its state as time evolves.

We show below that $\mathfrak{J}_{\mathrm{p}}^{(\omega,\mathbf{A})}$ and $\mathfrak{J}_{\mathrm{d}}^{(\omega,\mathbf{A})}$ *cannot* be identified with either $\mathbf{P}^{(\omega,\mathbf{A})}$ or $\mathbf{S}^{(\omega,\mathbf{A})}$ but are directly related to paramagnetic and diamagnetic currents, respectively.

4.4 Joule's Effect and Energy Increments

By Theorem 3.3, for each $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and any electromagnetic potential $\mathbf{A} \in \mathbf{C}_0^\infty$, the electric field in its integrated form $\mathbf{E}_t^{\eta\mathbf{A}_l}$ (cf. (11)–(12) and (17)) implies paramagnetic and diamagnetic currents with linear coefficients being respectively equal to

$$J_{\mathrm{p},l}^{(\omega,\mathbf{A})}(t,\mathbf{x}) := \frac{1}{2}\int_{t_0}^{t} \sum_{\mathbf{y}\in\mathfrak{K}} \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x},\mathbf{y},t-s)\, \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{y})\mathrm{d}s , \tag{65}$$

$$J_{\mathrm{d},l}^{(\omega,\mathbf{A})}(t,\mathbf{x}) := \int_{t_0}^{t} \sigma_{\mathrm{d}}^{(\omega)}(\mathbf{x})\, \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x})\mathrm{d}s , \tag{66}$$

at any bond $\mathbf{x} \in \mathfrak{K}$ (see (23)) and time $t \geq t_0$. Recall that $\sigma_{\mathrm{p}}^{(\omega)}$ and $\sigma_{\mathrm{d}}^{(\omega)}$ are the microscopic charge transport coefficients defined by (29)–(30).

Provided $|\eta| \ll 1$, the electric work produced at any time $t \geq t_0$ by paramagnetic currents is then equal to

$$\frac{\eta^2}{2} \int_{t_0}^{t} \sum_{\mathbf{x} \in \mathfrak{K}} J_{\mathrm{p},l}^{(\omega,\mathbf{A})}(s,\mathbf{x}) \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x}) \mathrm{d}s \ , \tag{67}$$

whereas the diamagnetic work equals

$$\frac{\eta^2}{2} \int_{t_0}^{t} \sum_{\mathbf{x} \in \mathfrak{K}} J_{\mathrm{d},l}^{(\omega,\mathbf{A})}(s,\mathbf{x}) \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x}) \mathrm{d}s = \frac{\eta^2}{4} \sum_{\mathbf{x} \in \mathfrak{K}} J_{\mathrm{d},l}^{(\omega,\mathbf{A})}(t,\mathbf{x}) \int_{t_0}^{t} \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x}) \mathrm{d}s \ . \tag{68}$$

Remark that the factor $\eta^2/2$ (instead of η^2) in (67)–(68) is due to the fact that $\mathfrak{K}$ is a set of *oriented* bonds and thus each bond is counted twice.

As explained in Section 3.4, there exist also *thermal* currents

$$\varrho^{(\beta,\omega,\lambda)}(I_{\mathbf{x}}) \ , \qquad \mathbf{x} \in \mathfrak{K} \ , \tag{69}$$

coming from the inhomogeneity of the fermion system for $\lambda \in \mathbb{R}^+$. Thermal currents imply an additional raw electromagnetic work

$$\frac{\eta}{2} \sum_{\mathbf{x} \in \mathfrak{K}} \varrho^{(\beta,\omega,\lambda)}(I_{\mathbf{x}}) \int_{t_0}^{t} \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x}) \mathrm{d}s \tag{70}$$

at any time $t \geq t_0$.

Since $\mathbf{A}$ is by assumption compactly supported in time, the corresponding electric field satisfies the *AC–condition*

$$\int_{t_0}^{t} E_{\mathbf{A}}(s,x) \mathrm{d}s = 0 \ , \quad x \in \mathbb{R}^d \ , \tag{71}$$

for times $t \geq t_1 \geq t_0$. Here,

$$t_1 := \min \left\{ t \geq t_0 : \quad \int_{t_0}^{t'} E_{\mathbf{A}}(s,x) \mathrm{d}s = 0 \quad \text{for all } x \in \mathbb{R}^d \text{ and } t' \geq t \right\}$$

is the time at which the electric field is definitively turned off. In this case, the electric works (68) and (70) vanish for $t \geq t_1$ and (67) stays constant. Following Joule's effect, for $t \geq t_1$, this energy should correspond to a *heat production* as defined in [BPK1, Definition 3.1]. The latter equals the energy increment $\mathbf{S}^{(\omega,\eta\mathbf{A}_l)}$, by [BPK1, Theorem 3.2].

We prove this heuristics in Section 5.2.1 and obtain the following theorem:

Theorem 4.1 (Microscopic Joule's law – I)
For any $\mathbf{A} \in \mathbf{C}_0^\infty$*, there is* $\eta_0 \in \mathbb{R}^+$ *such that, for all* $|\eta| \in (0, \eta_0]$*,* $l, \beta \in \mathbb{R}^+$*,* $\omega \in \Omega$*,* $\lambda \in \mathbb{R}_0^+$ *and* $t \geq t_0$*, the following assertions hold true:*
(p) *Paramagnetic energy increment:*

$$\mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}(t) = \frac{\eta^2}{2}\int_{t_0}^{t}\sum_{\mathbf{x}\in\mathfrak{K}} J_{\mathrm{p},l}^{(\omega,\mathbf{A})}(s,\mathbf{x})\mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x})\mathrm{d}s + \mathcal{O}(\eta^3 l^d)\ .$$

(d) *Diamagnetic energy:*

$$\begin{aligned}\mathfrak{I}_{\mathrm{d}}^{(\omega,\eta\mathbf{A}_l)}(t) &= \frac{\eta}{2}\sum_{\mathbf{x}\in\mathfrak{K}}\varrho^{(\beta,\omega,\lambda)}(I_{\mathbf{x}})\left(\int_{t_0}^{t}\mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x})\mathrm{d}s\right) \\ &\quad +\frac{\eta^2}{4}\sum_{\mathbf{x}\in\mathfrak{K}} J_{\mathrm{d},l}^{(\omega,\mathbf{A})}(t,\mathbf{x})\int_{t_0}^{t}\mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x})\mathrm{d}s + \mathcal{O}(\eta^3 l^d)\ .\end{aligned}$$

(**Q**) *Heat production – Internal energy increment:*

$$\begin{aligned}\mathbf{S}^{(\omega,\eta\mathbf{A}_l)}(t) &= -\frac{\eta^2}{2}\sum_{\mathbf{x}\in\mathfrak{K}} J_{\mathrm{p},l}^{(\omega,\mathbf{A})}(t,\mathbf{x})\left(\int_{t_0}^{t}\mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x})\mathrm{d}s\right) \\ &\quad +\mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}(t) + \mathcal{O}(\eta^3 l^d)\end{aligned}$$

(**P**) *Electromagnetic potential energy:*

$$\begin{aligned}\mathbf{P}^{(\omega,\eta\mathbf{A}_l)}(t) &= \frac{\eta^2}{2}\sum_{\mathbf{x}\in\mathfrak{K}} J_{\mathrm{p},l}^{(\omega,\mathbf{A})}(t,\mathbf{x})\left(\int_{t_0}^{t}\mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x})\mathrm{d}s\right) \\ &\quad +\mathfrak{I}_{\mathrm{d}}^{(\omega,\eta\mathbf{A}_l)}(t) + \mathcal{O}(\eta^3 l^d)\ .\end{aligned}$$

The correction terms of order $\mathcal{O}(\eta^3 l^d)$ *in assertions (p), (d), (**Q**) and (**P**) are uniformly bounded in* $\beta \in \mathbb{R}^+$*,* $\omega \in \Omega$*,* $\lambda \in \mathbb{R}_0^+$ *and* $t \geq t_0$*.*

Proof: The first two assertions are Theorem 5.12, whereas (**Q**) and (**P**) are direct consequences of (58)–(59), (62)–(63), Theorem 5.12 and Lemma 5.13. ■

We emphasize the fact that the asymptotics obtained are uniform w.r.t. $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$. This is a crucial property to get macroscopic Joule's law when $l \to \infty$. See [BPK2].

Remark 4.2 (Total energy)
One can easily deduce from Lemma 5.11 the asymptotics of the total work performed by the electric field, which is equal to

$$\int_{t_0}^{t} \rho_s^{(\beta,\omega,\lambda,\mathbf{A})} \left(\partial_s W_s^{\mathbf{A}}\right) \mathrm{d}s \ ,$$

similar to what is done in Theorem 4.1.

Theorem 4.1 describes, among other things, how resistance in the fermion system converts electric energy into heat. Indeed, by [BPK1, Theorem 3.2], for any $\mathbf{A} \in \mathbf{C}_0^\infty$, there is $\eta_0 \in \mathbb{R}^+$ such that, for all $|\eta| \in (0,\eta_0]$, $l,\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$,

$$\frac{\eta^2}{2}\int_{t_0}^{t} \sum_{\mathbf{x}\in\mathfrak{K}} J_{\mathrm{p},l}^{(\omega,\mathbf{A})}(s,\mathbf{x})\mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x})\mathrm{d}s - \frac{\eta^2}{2}\sum_{\mathbf{x}\in\mathfrak{K}} J_{\mathrm{p},l}^{(\omega,\mathbf{A})}(t,\mathbf{x}) \left(\int_{t_0}^{t} \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x})\mathrm{d}s\right) \geq \mathcal{O}(\eta^3 l^d) \ .$$

The latter is the positivity of the heat production, i.e., $\mathbf{S}^{(\omega,\eta\mathbf{A}_l)}(t) \in \mathbb{R}_0^+$, which for times $t \geq t_1 \geq t_0$ equals, at leading order, the work of paramagnetic currents (67), that is,

$$\frac{\eta^2}{4}\int_{t_0}^{t} \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2 \sum_{\mathbf{x},\mathbf{y}\in\mathfrak{K}} \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x},\mathbf{y},s_1-s_2)\, \mathbf{E}_{s_2}^{\mathbf{A}_l}(\mathbf{x})\mathbf{E}_{s_1}^{\mathbf{A}_l}(\mathbf{y}) \geq \mathcal{O}(\eta^3 l^d) \ . \quad (72)$$

This is nothing but Joule's law expressed w.r.t. electric fields and conductivity (instead of currents and resistance).

In fact, for any space–homogeneous electric field $\mathcal{E} \in C_0^\infty(\mathbb{R};\mathbb{R})$ in the box Λ_l for $l \in \mathbb{R}^+$ (as described at the beginning of Section 3), the left hand side of Equation (72) can be rewritten by using (35) and Theorem 3.1 as

$$\begin{aligned} &\eta^2 |\Lambda_l| \int_{t_0}^{t} \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2 \langle \vec{w}, \Xi_{\mathrm{p},l}^{(\omega)}(s_1-s_2)\vec{w}\rangle \mathcal{E}_{s_2}\mathcal{E}_{s_1} \\ &= \frac{\eta^2 |\Lambda_l|}{2} \int_{\mathbb{R}} |\hat{\mathcal{E}}_\nu|^2 \langle \vec{w}, \mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\vec{w}\rangle \geq 0 \end{aligned} \quad (73)$$

for all $t \geq t_1$, with $\hat{\mathcal{E}}_\nu$ being the Fourier transform of $\mathcal{E}_t$. In particular,

$$\frac{\eta^2}{2}|\hat{\mathcal{E}}_\nu|^2 \langle \vec{w}, \mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\vec{w}\rangle$$

is, at leading order, the heat production per unit volume due to the component of frequency ν of the electric field, in accordance with Joule's law in the AC–regime.

In presence of electromagnetic fields, i.e., at times $t \in [t_0, t_1]$ for which the AC–condition (71) does not hold, the situation is more complex. Indeed, at these times, $\mathfrak{J}_{\mathrm{p}}^{(\omega,\mathbf{A})}$ and $\mathfrak{J}_{\mathrm{d}}^{(\omega,\mathbf{A})}$ cannot be identified with either $\mathbf{P}^{(\omega,\mathbf{A})}$ or $\mathbf{S}^{(\omega,\mathbf{A})}$. From Theorem 4.1 (p), the energy $\mathfrak{J}_{\mathrm{p}}^{(\omega,\mathbf{A})}$ is generated by paramagnetic currents, see (65). By contrast, the raw electromagnetic energy $\mathfrak{J}_{\mathrm{d}}^{(\omega,\mathbf{A})}$ is carried by diamagnetic and thermal currents, see (66) and (69) and compare Theorem 4.1 (d) with (68) and (70). These currents are physically different: Diamagnetic currents correspond to the raw ballistic flow of charged particles due to the electric field, whereas only paramagnetic currents *partially* participates to the heat production $\mathbf{S}^{(\omega,\mathbf{A})}$, a portion of paramagnetic currents being also responsible for the modification of the electromagnetic potential energy:

- Part of the electric work performed by paramagnetic currents participates to the electromagnetic potential energy as explained in Theorem 4.1 (**P**). The same phenomenon appears for thermal currents defined by (69). Indeed, observe that any current $J(t, \mathbf{x})$ on the bound $\mathbf{x}$ at time t yields a contribution

$$J(t, \mathbf{x}) \left(\int_{t_0}^{t} \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x}) \mathrm{d}s \right)$$

  to the electromagnetic potential energy. Compare (70) and $\mathbf{P}^{(\omega,\eta\mathbf{A}_l)} - \mathfrak{J}_{\mathrm{d}}^{(\omega,\eta\mathbf{A}_l)}$ via Theorem 4.1 (**P**) . This potential energy disappears as soon as the electromagnetic potential is switched off.

- Then, the remaining energy coming from the whole paramagnetic energy $\mathfrak{J}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}$ is a heat energy or quantity of heat, by Theorem 4.1 (**Q**) and [BPK1, Theorem 3.2]. It survives even after turning off the electromagnetic potential.

4.5 Resistivity and Joule's Law

Joule's observation in [J] associates heat production in electric circuits with currents and resistance, rather than electric fields and conductivity. We thus explain in this subsection how to get such a relation between heat production and currents from (72)–(73), which express the total heat production as a function of electric fields and conductivity. For the sake of simplicity, we restrict our analysis to space–homogeneous electric fields $\mathcal{E}_t \vec{w}$ in the box Λ_l for $l \in \mathbb{R}^+$, as described at the beginning of Section 3. Here, $\mathcal{E} \in C_0^\infty(\mathbb{R};\mathbb{R})$ and $\vec{w} := (w_1, \ldots, w_d) \in \mathbb{R}^d$. In this subsection, we fix $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$.

By Corollary 3.2 (i), observe that, for times $t \geq t_1 \geq t_0$,

$$\begin{aligned} &\int_{t_0}^{t} \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2 \langle \vec{w}, \Xi_{\mathrm{p},l}^{(\omega)}(s_1 - s_2)\vec{w}\rangle \mathcal{E}_{s_2}\mathcal{E}_{s_1} \\ = \ & \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}s_1 \int_{\mathbb{R}} \mathrm{d}s_2 \langle \vec{w}, \Xi_{\mathrm{p},l}^{(\omega)}(s_1 - s_2)\vec{w}\rangle \mathcal{E}_{s_2}\mathcal{E}_{s_1} \mathrm{d}s_2 \mathrm{d}s_1 \ . \end{aligned}$$

Therefore, we define the subspace

$$\mathcal{S}_0 := \left\{ \mathcal{E} \in \mathcal{S}(\mathbb{R};\mathbb{R}) : \int_{\mathbb{R}} \mathcal{E}_s \mathrm{d}s = 0 \right\}$$

of $\mathbb{R}$–valued Schwartz functions satisfying the AC–condition as well as the functional $\mathbf{Q}_{\Lambda_l} \equiv \mathbf{Q}_{\Lambda_l}^{(\beta,\omega,\lambda)}$ on $\mathcal{S}_0$, the *total* heat production per unit of volume, by

$$\mathbf{Q}_{\Lambda_l}(\mathcal{E}) := \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}s_1 \int_{\mathbb{R}} \mathrm{d}s_2 \langle \vec{w}, \Xi_{\mathrm{p},l}^{(\omega)}(s_1 - s_2)\vec{w}\rangle \mathcal{E}_{s_2}\mathcal{E}_{s_1} \mathrm{d}s_2 \mathrm{d}s_1 \ , \quad \mathcal{E} \in \mathcal{S}_0 \ . \quad (74)$$

It is a finite, positive quadratic form on $\mathcal{S}_0$. Indeed, by Theorem 3.1,

$$\mathbf{Q}_{\Lambda_l}(\mathcal{E}) = \frac{1}{2}\int_{\mathbb{R}} |\hat{\mathcal{E}}_\nu|^2 \, \langle \vec{w}, \mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\vec{w}\rangle \in \mathbb{R}_0^+ \ , \quad \mathcal{E} \in \mathcal{S}_0 \ , \quad (75)$$

and $\langle \vec{w}, \mu_{\mathrm{p},l}^{(\omega)}\vec{w}\rangle$ is a positive measure. It thus defines a semi–norm $\|\cdot\|_{\Lambda_l} \equiv \|\cdot\|_{\Lambda_l}^{(\beta,\omega,\lambda)}$ on $\mathcal{S}_0$ by

$$\|\mathcal{E}\|_{\Lambda_l} := \sqrt{\mathbf{Q}_{\Lambda_l}(\mathcal{E})} \ , \qquad \mathcal{E} \in \mathcal{S}_0 \ . \quad (76)$$

Note that $\mathcal{S}_0$ is a closed subspace of the locally convex (Fréchet) space $\mathcal{S}(\mathbb{R};\mathbb{R})$. Let $\mathcal{S}_0^*$ be the dual space of $\mathcal{S}_0$, i.e., the set of all continuous linear functionals on $\mathcal{S}_0$. $\mathcal{S}_0^*$ is equipped with the weak*–topology. By the Hahn–Banach theorem, the elements of the dual $\mathcal{S}_0^*$ are restrictions to $\mathcal{S}_0$ of tempered distributions. $\mathcal{S}_0^*$ is in fact a space of in–phase AC–currents.

Let $\partial \mathbf{Q}_{\Lambda_l}(\mathcal{E}) \subset \mathcal{S}_0^*$ be the subdifferential of $\mathbf{Q}_{\Lambda_l}$ at the point $\mathcal{E} \in \mathcal{S}_0$. The multifunction $\boldsymbol{\sigma}_{\Lambda_l} \equiv \boldsymbol{\sigma}_{\Lambda_l}^{(\beta,\omega,\lambda)}$ from $\mathcal{S}_0$ to $\mathcal{S}_0^*$ (i.e., the set–valued map from $\mathcal{S}_0$ to $2^{\mathcal{S}_0^*}$) is defined by

$$\mathcal{E} \mapsto \boldsymbol{\sigma}_{\Lambda_l}(\mathcal{E}) = \frac{1}{2}\partial \mathbf{Q}_{\Lambda_l}(\mathcal{E}) \ .$$

It is single–valued with domain $\mathrm{Dom}(\boldsymbol{\sigma}_{\Lambda_l}) = \mathcal{S}_0$:

Lemma 4.3 (Properties of the AC–conductivity)
The multifunction $\boldsymbol{\sigma}_{\Lambda_l}$ has domain

$$\mathrm{Dom}(\boldsymbol{\sigma}_{\Lambda_l}) := \{\mathcal{E} \in \mathcal{S}_0 : \partial \mathbf{Q}_{\Lambda_l}(\mathcal{E}) \neq \emptyset\} = \mathcal{S}_0$$

and, for all $\mathcal{E} \in \mathcal{S}_0$, $\boldsymbol{\sigma}_{\Lambda_l}(\mathcal{E}) = \{\mathcal{J}_{\mathcal{E}}\}$ with

$$\left\langle \mathcal{J}_{\mathcal{E}}, \tilde{\mathcal{E}} \right\rangle = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle \vec{w}, \Xi_{\mathrm{p},l}^{(\omega)}(s_1 - s_2)\vec{w}\rangle \tilde{\mathcal{E}}_{s_1} \mathcal{E}_{s_2} \mathrm{d}s_2 \mathrm{d}s_1 \ , \quad \tilde{\mathcal{E}} \in \mathcal{S}_0 \ . \tag{77}$$

[We use the standard notation for distributions: $\langle \mathcal{J}_{\mathcal{E}}, \tilde{\mathcal{E}} \rangle \equiv \mathcal{J}_{\mathcal{E}}(\tilde{\mathcal{E}})$.]

Proof: We prove that, for all $\mathcal{E} \in \mathcal{S}_0$, $2\mathcal{J}_{\mathcal{E}}$ is the unique tangent functional of $\mathbf{Q}_{\Lambda_l}$ at the point $\mathcal{E}$. Indeed,

$$\mathbf{Q}_{\Lambda_l}(\mathcal{E} + \mathcal{E}_1) - \mathbf{Q}_{\Lambda_l}(\mathcal{E}) = 2\langle \mathcal{J}_{\mathcal{E}}, \mathcal{E}_1 \rangle + \mathbf{Q}_{\Lambda_l}(\mathcal{E}_1) \tag{78}$$

for all $\mathcal{E}_1 \in \mathcal{S}_0$. Since $\mathbf{Q}_{\Lambda_l}(\mathcal{E}_1) \geq 0$, the functional $2\mathcal{J}_{\mathcal{E}}$ is tangent to $\mathbf{Q}_{\Lambda_l}$ at $\mathcal{E} \in \mathcal{S}_0$. In particular, $\mathrm{Dom}(\boldsymbol{\sigma}_{\Lambda_l}) = \mathcal{S}_0$. The uniqueness of the tangent functional follows from the fact that $2\mathcal{J}_{\mathcal{E}}$ is the Gâteaux derivative of $\mathbf{Q}_{\Lambda_l}$ at $\mathcal{E} \in \mathcal{S}_0$. To see this, replace $\mathcal{E}_1$ with $\epsilon\mathcal{E}_1$ in (78) and take the limit $\epsilon \to 0$. ■

Equation (77) is directly related to Ohm's law in Fourier space. For this reason, $\boldsymbol{\sigma}_{\Lambda_l}$ is named here the *AC–conductivity* of the region Λ_l.

By Ohm and Joule's laws, a more resistive system produces less heat at fixed electric field. We thus define a *AC–resistivity order* from the total heat production $\mathbf{Q}_{\Lambda_l} \equiv \mathbf{Q}_{\Lambda_l}^{(\beta,\omega,\lambda)}$ (per unit of volume) on the space $\mathcal{S}_0$ of electric fields:

Definition 4.4 (AC–Resistivity order)
For all $l \in \mathbb{R}^+$, we define the partial order relation $\prec$ for the system parameters $(\beta, \omega, \lambda) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}_0^+$ by

$$(\beta_1, \omega_1, \lambda_1) \prec (\beta_2, \omega_2, \lambda_2) \qquad \textit{iff} \qquad \mathbf{Q}_{\Lambda_l}^{(\beta_1,\omega_1,\lambda_1)} \geq \mathbf{Q}_{\Lambda_l}^{(\beta_2,\omega_2,\lambda_2)} \ .$$

This definition is reminiscent of the approach of [LY] to the entropy. Observe also that

$$(\beta_1, \omega_1, \lambda_1) \prec (\beta_2, \omega_2, \lambda_2) \qquad \text{iff} \qquad \mu_{\mathrm{p},l}^{(\beta_1,\omega_1,\lambda_1)}|_{\mathbb{R}\backslash\{0\}} \geq \mu_{\mathrm{p},l}^{(\beta_2,\omega_2,\lambda_2)}|_{\mathbb{R}\backslash\{0\}} \ .$$

Furthermore, this partial order can be rewritten in terms of a quadratic function of currents, in accordance with Joule's law in its original form.

To see this, observe that $(\mathcal{S}_0, \mathcal{S}_0^*)$ is a dual pair, by [R, Theorem 3.10]. Therefore, $\mathbf{Q}_{\Lambda_l} : \mathcal{S}_0 \rightarrow [0, \infty)$ has a well–defined Legendre–Fenchel transform $\mathbf{Q}_{\Lambda_l}^* \equiv (\mathbf{Q}_{\Lambda_l}^{(\beta,\omega,\lambda)})^*$ which is the convex lower semi–continuous functional from $\mathcal{S}_0^*$ to $(-\infty, \infty]$ defined in our setting by

$$\mathbf{Q}_{\Lambda_l}^* (\mathcal{J}) := 2 \sup_{\mathcal{E} \in \mathcal{S}_0} \left\{ \langle \mathcal{J}, \mathcal{E} \rangle - \frac{1}{2} \mathbf{Q}_{\Lambda_l} (\mathcal{E}) \right\} , \qquad \mathcal{J} \in \mathcal{S}_0^* . \tag{79}$$

The square root of $\mathbf{Q}_{\Lambda_l}^* (\mathcal{J})$ can be seen as the norm of the linear map $\mathcal{J} : (\mathcal{S}_0, \|\cdot\|_{\Lambda_l}) \rightarrow \mathbb{R}$:

Lemma 4.5 ($\mathbf{Q}_{\Lambda_l}^*$ as a semi–norm on $\mathcal{S}_0^*$)
Assume that $\mathbf{Q}_{\Lambda_l}$ is not identically zero. Then,

$$\mathbf{Q}_{\Lambda_l}^* (\mathcal{J}) = \left(\sup \left\{ |\langle \mathcal{J}, \mathcal{E} \rangle| : \mathcal{E} \in \mathcal{S}_0, \ \|\mathcal{E}\|_{\Lambda_l} = 1 \right\} \right)^2 .$$

If $\mathbf{Q}_{\Lambda_l}$ is identically zero, $\mathbf{Q}_{\Lambda_l}^* (\mathcal{J}) = \infty$ for all $\mathcal{J} \in \mathcal{S}_0^* \backslash \{0\}$ and $\mathbf{Q}_{\Lambda_l}^* (0) = 0$.

Proof: The assertion for $\mathbf{Q}_{\Lambda_l} \equiv 0$ is a direct consequence of (79). Assume that $\mathbf{Q}_{\Lambda_l}$ is not identically zero. For any $\mathcal{J} \in \mathcal{S}_0^*$, define the map

$$x \mapsto f_{\mathcal{J}} (x) := \sup_{\mathcal{E} \in \mathcal{S}_0 : \|\mathcal{E}\|_{\Lambda_l} = x} \left\{ |\langle \mathcal{J}, \mathcal{E} \rangle| - \frac{x^2}{2} \right\}$$

from $\mathbb{R}_0^+$ to $\mathbb{R}$. By rescaling, observe that, for any $x \in \mathbb{R}^+$,

$$f_{\mathcal{J}} (x) = \sup_{\mathcal{E} \in \mathcal{S}_0 : \|\mathcal{E}\|_{\Lambda_l} = 1} \left\{ x \, |\langle \mathcal{J}, \mathcal{E} \rangle| - \frac{x^2}{2} \right\} . \tag{80}$$

In particular, for any $\mathcal{J} \in \mathcal{S}_0^*$, $f_{\mathcal{J}}$ is clearly continuous. Therefore, we infer from (79) that

$$\mathbf{Q}_{\Lambda_l}^* (\mathcal{J}) = 2 \sup_{x \in \mathbb{R}_0^+} f_{\mathcal{J}} (x) = 2 \sup_{x \in \mathbb{R}^+} f_{\mathcal{J}} (x) \ , \tag{81}$$

which, combined with (80) and straightforward computations, leads to the assertion. ■

The above lemma implies that the domain

$$\mathrm{Dom} \left(\mathbf{Q}_{\Lambda_l}^* \right) = \left\{ \mathcal{J} \in \mathcal{S}_0^* : \mathbf{Q}_{\Lambda_l}^* (\mathcal{J}) < \infty \right\}$$

of the functional $\mathbf{Q}_{\Lambda_l}^*$ is a subspace of $\mathcal{S}_0^*$. Similar to (76), we define the semi–norm $\|\cdot\|_{\Lambda_l}^{(*)} \equiv \|\cdot\|_{\Lambda_l}^{(*,\beta,\omega,\lambda)}$ by

$$\|\mathcal{J}\|_{\Lambda_l}^{(*)} := \sqrt{\mathbf{Q}_{\Lambda_l}^*(\mathcal{J})} = \sup\left\{|\langle \mathcal{J}, \mathcal{E}\rangle| : \mathcal{E} \in \mathcal{S}_0,\ \|\mathcal{E}\|_{\Lambda_l} = 1\right\} \tag{82}$$

for any $\mathcal{J} \in \mathcal{S}_0^*$.

Let $\partial\mathbf{Q}_{\Lambda_l}^*(\mathcal{J}) \subset \mathcal{S}_0$ be the subdifferential of $\mathbf{Q}_{\Lambda_l}^*$ at the point $\mathcal{J} \in \mathcal{S}_0^*$. The multifunction $\boldsymbol{\rho}_{\Lambda_l} \equiv \boldsymbol{\rho}_{\Lambda_l}^{(\beta,\omega,\lambda)}$ from $\mathcal{S}_0^*$ to $\mathcal{S}_0$ (i.e., the set–valued map from $\mathcal{S}_0$ to $2^{\mathcal{S}_0^*}$) is defined by

$$\mathcal{J} \mapsto \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}) = \frac{1}{2}\partial\mathbf{Q}_{\Lambda_l}^*(\mathcal{J}) \ .$$

It is named here the *AC–resistivity* of the region Λ_l because it is a sort of inverse of the AC–conductivity:

Lemma 4.6 (Properties of the AC–resistivity)
The multifunction $\boldsymbol{\rho}_{\Lambda_l}$ has non–empty domain equal to

$$\mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l}) := \left\{\mathcal{J} \in \mathcal{S}_0^* : \partial\mathbf{Q}_{\Lambda_l}^*(\mathcal{J}) \neq \emptyset\right\} = \bigcup_{\mathcal{E}\in\mathcal{S}_0} \boldsymbol{\sigma}_{\Lambda_l}(\mathcal{E}) \ .$$

Furthermore, for all $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$ and $\mathcal{E} \in \mathrm{Dom}(\boldsymbol{\sigma}_{\Lambda_l}) = \mathcal{S}_0$,

$$\boldsymbol{\sigma}_{\Lambda_l}\left(\boldsymbol{\rho}_{\Lambda_l}(\mathcal{J})\right) = \{\mathcal{J}\} \qquad \textit{and} \qquad \boldsymbol{\rho}_{\Lambda_l}(\boldsymbol{\sigma}_{\Lambda_l}(\mathcal{E})) \supset \{\mathcal{E}\} \ . \tag{83}$$

Proof: Young's inequality asserts that

$$\frac{1}{2}\mathbf{Q}_{\Lambda_l}^*(\mathcal{J}) + \frac{1}{2}\mathbf{Q}_{\Lambda_l}(\mathcal{E}) \geq \langle \mathcal{J}, \mathcal{E}\rangle$$

with equality iff $2\mathcal{J} \in \partial\mathbf{Q}_{\Lambda_l}(\mathcal{E})$. As $\mathbf{Q}_{\Lambda_l} = \mathbf{Q}_{\Lambda_l}^{**}$,

$$\frac{1}{2}\mathbf{Q}_{\Lambda_l}^*(\mathcal{J}) + \frac{1}{2}\mathbf{Q}_{\Lambda_l}(\mathcal{E}) = \langle \mathcal{J}, \mathcal{E}\rangle$$

iff $2\mathcal{E} \in \partial\mathbf{Q}_{\Lambda_l}^*(\mathcal{J})$. In other words,

$$\mathcal{E} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}) \Longleftrightarrow \mathcal{J} \in \boldsymbol{\sigma}_{\Lambda_l}(\mathcal{E}) \ . \tag{84}$$

As a consequence, $\mathcal{J}_{\mathcal{E}} \in \boldsymbol{\sigma}_{\Lambda_l}(\mathcal{E})$ (cf. Lemma 4.3) yields $\mathcal{E} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_{\mathcal{E}})$. It follows that

$$\bigcup_{\mathcal{E}\in\mathcal{S}_0} \boldsymbol{\sigma}_{\Lambda_l}(\mathcal{E}) \subset \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$$

and

$$\boldsymbol{\rho}_{\Lambda_l}\left(\boldsymbol{\sigma}_{\Lambda_l}\left(\mathcal{E}\right)\right) \supset \{\mathcal{E}\} \ .$$

Now, let $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$ and $\mathcal{E} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J})$. Then, by (84), $\mathcal{J} \in \boldsymbol{\sigma}_{\Lambda_l}(\mathcal{E})$ and we infer from the uniqueness of the tangent functional (Lemma 4.3) that $\mathcal{J} = \mathcal{J}_{\mathcal{E}}$. Therefore,

$$\boldsymbol{\sigma}_{\Lambda_l}\left(\boldsymbol{\rho}_{\Lambda_l}\left(\mathcal{J}\right)\right) = \{\mathcal{J}\}$$

and

$$\mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l}) \subset \bigcup_{\mathcal{E}\in\mathcal{S}_0} \boldsymbol{\sigma}_{\Lambda_l}\left(\mathcal{E}\right) \ .$$

■

Note that $\mathbf{Q}_{\Lambda_l} : \mathcal{S}_0 \to [0,\infty)$ is a convex continuous functional, by positivity of the conductivity measure, see Theorem 3.1 and (75). In particular,

$$\mathbf{Q}_{\Lambda_l}\left(\mathcal{E}\right) := 2 \sup_{\mathcal{J}\in\mathcal{S}_0^*} \left\{ \langle \mathcal{J},\mathcal{E}\rangle - \frac{1}{2}\mathbf{Q}_{\Lambda_l}^*\left(\mathcal{J}\right) \right\} \ . \tag{85}$$

Therefore, we deduce from (79) and (85) that

$$(\beta_1,\omega_1,\lambda_1) \prec (\beta_2,\omega_2,\lambda_2) \qquad \text{iff} \qquad (\mathbf{Q}_{\Lambda_l}^{(\beta_1,\omega_1,\lambda_1)})^* \leq (\mathbf{Q}_{\Lambda_l}^{(\beta_2,\omega_2,\lambda_2)})^* \ .$$

Furthermore, by using (76) and similar arguments as in Lemma 4.5, if $\mathbf{Q}_{\Lambda_l}$ is not identically zero, then:

$$\|\mathcal{E}\|_{\Lambda_l} = \sup\left\{ |\langle \mathcal{J},\mathcal{E}\rangle| : \mathcal{J}\in\mathcal{S}_0^*,\ \|\mathcal{J}\|_{\Lambda_l}^{(*)} = 1 \right\} \ .$$

We are now in position to obtain Joule's law in its original form. To this end, we say that a multifunction $\boldsymbol{\rho}$ from $\mathcal{S}_0^*$ to $\mathcal{S}_0$ is *linear* if:

(a) Its domain $\mathrm{Dom}(\boldsymbol{\rho})$ is a subspace of $\mathcal{S}_0^*$.

(b) For $\alpha \in \mathbb{R}\backslash\{0\}$ and $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho})$, $\boldsymbol{\rho}(\alpha\mathcal{J}) = \alpha\boldsymbol{\rho}(\mathcal{J})$ and $0 \in \boldsymbol{\rho}(0)$.

(c) For $\mathcal{J}_1, \mathcal{J}_2 \in \mathrm{Dom}(\boldsymbol{\rho})$, $\boldsymbol{\rho}(\mathcal{J}_1 + \mathcal{J}_2) = \boldsymbol{\rho}(\mathcal{J}_1) + \boldsymbol{\rho}(\mathcal{J}_2)$.

Then, one gets that the heat produced by currents is proportional to the resistivity and the square of currents:

Theorem 4.7 (Microscopic Joule's law – II)
(i) $\boldsymbol{\rho}_{\Lambda_l}$ *is a linear multifunction and* $\boldsymbol{\sigma}_{\Lambda_l}\left(\boldsymbol{\rho}_{\Lambda_l}(\mathcal{J})\right) = \{\mathcal{J}\}$ *for all* $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$.
(ii) *For any* $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$,

$$\{\mathbf{Q}^*_{\Lambda_l}(\mathcal{J})\} = \left\langle \mathcal{J}, \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J})\right\rangle = \mathbf{Q}_{\Lambda_l}\left(\boldsymbol{\rho}_{\Lambda_l}(\mathcal{J})\right) .$$

(iii) *There is a bilinear symmetric positive map* $(\cdot,\cdot)^{(*)}_{\Lambda_l}$ *on* $\mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$ *such that*

$$\mathbf{Q}^*_{\Lambda_l}(\mathcal{J}_1) = (\mathcal{J}_1,\mathcal{J}_1)^{(*)}_{\Lambda_l} \qquad \text{and} \qquad \left\langle \mathcal{J}_1, \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_2)\right\rangle = \{(\mathcal{J}_1,\mathcal{J}_2)^{(*)}_{\Lambda_l}\}$$

for all $\mathcal{J}_1,\mathcal{J}_2 \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$.

Proof: (i.a) The fact that $\mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$ is a subspace of $\mathcal{S}_0^*$ is a direct consequence of Lemmata 4.3 and 4.6.
(i.b) Let $\alpha \in \mathbb{R}$ and $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$. Take any $\mathcal{E}_{\mathcal{J}} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J})$ and observe that $\mathcal{J} = \mathcal{J}_{\mathcal{E}_{\mathcal{J}}}$, by using Lemmata 4.3 and 4.6. Then,

$$\alpha\mathcal{J} = \mathcal{J}_{\alpha\mathcal{E}_{\mathcal{J}}} \in \boldsymbol{\sigma}_{\Lambda_l}(\alpha\mathcal{E}_{\mathcal{J}}) .$$

From (84) it follows that $\alpha\boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}) \subset \boldsymbol{\rho}_{\Lambda_l}(\alpha\mathcal{J})$. If $\alpha \neq 0$ then, by replacing $(\mathcal{J},\alpha)$ with $(\alpha\mathcal{J},\alpha^{-1})$, one gets that $\boldsymbol{\rho}_{\Lambda_l}(\alpha\mathcal{J}) \subset \alpha\boldsymbol{\rho}_{\Lambda_l}(\mathcal{J})$.
(i.c) Let $\mathcal{J}_1,\mathcal{J}_2 \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$ and take any $\mathcal{E}_{\mathcal{J}_1} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1)$ and $\mathcal{E}_{\mathcal{J}_2} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_2)$. As above, $\mathcal{J}_1 = \mathcal{J}_{\mathcal{E}_{\mathcal{J}_1}}$ and $\mathcal{J}_2 = \mathcal{J}_{\mathcal{E}_{\mathcal{J}_2}}$. Then,

$$\mathcal{J}_1 + \mathcal{J}_2 = \mathcal{J}_{\mathcal{E}_{\mathcal{J}_1}+\mathcal{E}_{\mathcal{J}_2}} \in \boldsymbol{\sigma}_{\Lambda_l}(\mathcal{E}_{\mathcal{J}_1}+\mathcal{E}_{\mathcal{J}_2}) .$$

Hence, using again (84), we arrive at

$$\boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1) + \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_2) \subset \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1+\mathcal{J}_2) .$$

Now, let $\mathcal{J}_1,\mathcal{J}_2 \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$ and take any $\mathcal{E}_{\mathcal{J}_1+\mathcal{J}_2} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1+\mathcal{J}_2)$. Then, $\mathcal{J}_{\mathcal{E}_{\mathcal{J}_1+\mathcal{J}_2}} = \mathcal{J}_1 + \mathcal{J}_2$. Similarly, choose also $\mathcal{E}_{\mathcal{J}_1} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1)$ and $\mathcal{E}_{\mathcal{J}_2} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_2)$ with $\mathcal{J}_1 = \mathcal{J}_{\mathcal{E}_{\mathcal{J}_1}}$ and $\mathcal{J}_2 = \mathcal{J}_{\mathcal{E}_{\mathcal{J}_2}}$. Obviously, by Equation (77),

$$\mathcal{J}_2 = \mathcal{J}_{\mathcal{E}_{\mathcal{J}_2}} = \mathcal{J}_{\mathcal{E}_{\mathcal{J}_1+\mathcal{J}_2}} - \mathcal{J}_{\mathcal{E}_{\mathcal{J}_1}} = \mathcal{J}_{\mathcal{E}_{\mathcal{J}_1+\mathcal{J}_2}-\mathcal{E}_{\mathcal{J}_1}} ,$$

which together with (84) yields the converse inclusion

$$\boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1+\mathcal{J}_2) \subset \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1) + \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_2) .$$

(ii) Take any $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$ and $\mathcal{E}_{\mathcal{J}} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J})$. We infer from (74) and Lemma 4.3 that

$$\langle \mathcal{J}, \mathcal{E}_{\mathcal{J}} \rangle = \langle \mathcal{J}_{\mathcal{E}_{\mathcal{J}}}, \mathcal{E}_{\mathcal{J}} \rangle = \mathbf{Q}_{\Lambda_l}(\mathcal{E}_{\mathcal{J}}) \ .$$

Since

$$\frac{1}{2}\mathbf{Q}^*_{\Lambda_l}(\mathcal{J}) + \frac{1}{2}\mathbf{Q}_{\Lambda_l}(\mathcal{E}_{\mathcal{J}}) = \langle \mathcal{J}, \mathcal{E}_{\mathcal{J}} \rangle \ ,$$

we also deduce that $\mathbf{Q}^*_{\Lambda_l}(\mathcal{J}) = \mathbf{Q}_{\Lambda_l}(\mathcal{E}_{\mathcal{J}})$.

(iii) For all $\mathcal{J}_1, \mathcal{J}_2 \in \mathrm{Dom}(\mathbf{Q}^*_{\Lambda_l})$, define

$$(\mathcal{J}_1, \mathcal{J}_2)^{(*)}_{\Lambda_l} := \frac{1}{4}\left(\mathbf{Q}^*_{\Lambda_l}(\mathcal{J}_1 + \mathcal{J}_2) - \mathbf{Q}^*_{\Lambda_l}(\mathcal{J}_1 - \mathcal{J}_2)\right) \ . \tag{86}$$

This quantity is clearly symmetric w.r.t. $\mathcal{J}_1, \mathcal{J}_2$ and

$$(\mathcal{J}, \mathcal{J})^{(*)}_{\Lambda_l} = \mathbf{Q}^*_{\Lambda_l}(\mathcal{J}) \geq 0 \ , \qquad \mathcal{J} \in \mathrm{Dom}(\mathbf{Q}^*_{\Lambda_l}) \ ,$$

by Lemma 4.5. Using the linearity of $\boldsymbol{\rho}_{\Lambda_l}$ and the fact that $\langle \mathcal{J}, \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}) \rangle \subset \mathbb{R}^+_0$ contains exactly one element for all $\mathcal{J} \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$, we compute that, for any $\mathcal{J}_1, \mathcal{J}_2 \in \mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$,

$$\frac{1}{2}\{\mathbf{Q}^*_{\Lambda_l}(\mathcal{J}_1 + \mathcal{J}_2) - \mathbf{Q}^*_{\Lambda_l}(\mathcal{J}_1 - \mathcal{J}_2)\} = \langle \mathcal{J}_2, \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1) \rangle + \langle \mathcal{J}_1, \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_2) \rangle \ .$$

Again by linearity of $\boldsymbol{\rho}_{\Lambda_l}$, this implies that (86) defines a bilinear form on $\mathrm{Dom}(\boldsymbol{\rho}_{\Lambda_l})$. We also infer from the above equation that the set $\langle \mathcal{J}_2, \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1) \rangle \subset \mathbb{R}$ contains exactly one element. Let $\mathcal{E}_{\mathcal{J}_1} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1)$ and $\mathcal{E}_{\mathcal{J}_2} \in \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_2)$ with $\mathcal{J}_1 = \mathcal{J}_{\mathcal{E}_{\mathcal{J}_1}}$ and $\mathcal{J}_2 = \mathcal{J}_{\mathcal{E}_{\mathcal{J}_2}}$. Then, by Lemma 4.3,

$$\langle \mathcal{J}_2, \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_1) \rangle = \{\langle \mathcal{J}_{\mathcal{E}_{\mathcal{J}_2}}, \mathcal{E}_{\mathcal{J}_1} \rangle\} = \{\langle \mathcal{J}_{\mathcal{E}_{\mathcal{J}_1}}, \mathcal{E}_{\mathcal{J}_2} \rangle\} = \langle \mathcal{J}_1, \boldsymbol{\rho}_{\Lambda_l}(\mathcal{J}_2) \rangle \ .$$

■

5 Technical Proofs

This section is divided in two parts: Section 5.1 gives a detailed proof of Theorem 3.1 as well as additional properties of paramagnetic transport coefficients defined in Section 3.3. In Section 5.2 we prove Theorems 3.3 and 4.1. Note that we start in this second subsection with the proof of Theorem 4.1 because the other one is simpler and uses similar arguments.

5.1 Paramagnetic Transport Coefficients

5.1.1 Microscopic Paramagnetic Transport Coefficients

We study in this subsection the microscopic paramagnetic transport coefficient $\sigma_{\mathrm{p}}^{(\omega)}$ which is defined by (29), that is,

$$\sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x},\mathbf{y},t) := \int_0^t \varrho^{(\beta,\omega,\lambda)}\left(i[I_{\mathbf{y}},\tau_s^{(\omega,\lambda)}(I_{\mathbf{x}})]\right)\mathrm{d}s\ , \quad \mathbf{x},\mathbf{y}\in\mathfrak{L}^2\ ,\ t\in\mathbb{R}\ .$$

Recall that $I_{\mathbf{x}}$ is the paramagnetic current observable defined by (19), that is,

$$I_{\mathbf{x}} := i(a_{x^{(2)}}^* a_{x^{(1)}} - a_{x^{(1)}}^* a_{x^{(2)}})\ , \qquad \mathbf{x} := (x^{(1)},x^{(2)})\in\mathfrak{L}^2\ . \tag{87}$$

The coefficient $\sigma_{\mathrm{p}}^{(\omega)}$ can explicitly be written in terms of a scalar product involving current observables. To show this, we introduce the Duhamel two–point function $(\cdot,\cdot)_{\sim}^{(\omega)}$ defined by

$$(B_1,B_2)_\sim \equiv (B_1,B_2)_\sim^{(\beta,\omega,\lambda)} := \int_0^\beta \varrho^{(\beta,\omega,\lambda)}\left(B_1^*\tau_{i\alpha}^{(\omega,\lambda)}(B_2)\right)\mathrm{d}\alpha \tag{88}$$

for any $B_1,B_2\in\mathcal{U}$. The properties of this sesquilinear form are described in detail in Appendix A. In particular, by Theorem A.1 for $\mathcal{X}=\mathcal{U}$, $\tau=\tau^{(\omega,\lambda)}$ and $\varrho=\varrho^{(\beta,\omega,\lambda)}$, $(B_1,B_2)\mapsto(B_1,B_2)_\sim$ is a positive sesquilinear form on $\mathcal{U}$. We then infer from Lemma A.14 that

$$\sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x},\mathbf{y},t) = (I_{\mathbf{y}},\tau_t^{(\omega,\lambda)}(I_{\mathbf{x}}))_\sim - (I_{\mathbf{y}},I_{\mathbf{x}})_\sim\ , \tag{89}$$

for all $\beta\in\mathbb{R}^+$, $\omega\in\Omega$, $\lambda\in\mathbb{R}_0^+$, $\mathbf{x},\mathbf{y}\in\mathfrak{L}^2$ and $t\in\mathbb{R}$. By Theorem A.16, it follows that $\sigma_{\mathrm{p}}^{(\omega)}$ is symmetric w.r.t. time–reversal and permutation of bonds.

Indeed, by using the time–reversal operation $\Theta:\mathcal{U}\to\mathcal{U}$ defined in Section 2.1.4, one proves:

Lemma 5.1 (Time–reversal symmetry of the fermion system)
Let $\beta\in\mathbb{R}^+$, $\omega\in\Omega$ and $\lambda\in\mathbb{R}_0^+$. Then,

$$\Theta\circ\tau_t^{(\omega,\lambda)} = \tau_{-t}^{(\omega,\lambda)}\circ\Theta\ , \qquad t\in\mathbb{R}\ , \tag{90}$$

and

$$\varrho^{(\beta,\omega,\lambda)}(B) = \overline{\varrho^{(\beta,\omega,\lambda)}\circ\Theta(B)}\ , \qquad B\in\mathcal{U}\ . \tag{91}$$

Proof: By continuity of the maps Θ and $\tau_t^{(\omega,\lambda)}$ as well as the density of polynomials in the creation and annihilation operators in $\mathcal{U}$, it suffices to prove the first assertion for monomials in a_x, a_x^*, $x \in \mathfrak{L}$. Now, since $\Theta(H_L^{(\omega,\lambda)}) = H_L^{(\omega,\lambda)}$ (see (56)), by [BPK1, Theorem A.3 (i)],

$$\Theta \circ \tau_t^{(\omega,\lambda)}(B) = \tau_{-t}^{(\omega,\lambda)} \circ \Theta(B) \ , \qquad B \in \mathcal{U}_0, \ t \in \mathbb{R} \ ,$$

which implies (90). The second assertion is a consequence of the uniqueness of the $(\tau^{(\omega,\lambda)}, \beta)$–KMS state $\varrho^{(\beta,\omega,\lambda)}$ together with Lemma A.12. ■

Since $\Theta(I_{\mathbf{x}}) = -I_{\mathbf{x}}$ for any $\mathbf{x} \in \mathfrak{L}^2$, we deduce from Lemma 5.1 and Theorem A.16 for $\mathcal{X} = \mathcal{U}$, $\tau = \tau^{(\omega,\lambda)}$ and $\varrho = \varrho^{(\beta,\omega,\lambda)}$ that the function $\sigma_{\mathrm{p}}^{(\omega)}$ from $\mathfrak{L}^4 \times \mathbb{R}$ to $\mathbb{R}$ is symmetric w.r.t. time–reversal and permutation of bonds:

$$\sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, t) = \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, -t) = \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{y}, \mathbf{x}, t) \ , \qquad \mathbf{x}, \mathbf{y} \in \mathfrak{L}^2 \ , \ t \in \mathbb{R} \ .$$

Thermal equilibrium states $\varrho^{(\beta,\omega,\lambda)}$ are by construction quasi–free and gauge–invariant. This fact implies that $\sigma_{\mathrm{p}}^{(\omega)}$ can be expressed in terms of complex–time two–point correlation functions $C_{t+i\alpha}^{(\omega)} \equiv C_{t+i\alpha}^{(\beta,\omega,\lambda)}$ defined by

$$C_{t+i\alpha}^{(\omega)}(\mathbf{x}) := \varrho^{(\beta,\omega,\lambda)}(a_{x^{(1)}}^* \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(2)}})) \ , \qquad \mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2 \ , \qquad (92)$$

for all $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$ and $\alpha \in [0, \beta]$. This is shown in the following assertion:

Lemma 5.2 ($\sigma_{\mathrm{p}}^{(\omega)}$ in terms of two–point correlation functions)
Let $\beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$. Then, for all $\mathbf{x}, \mathbf{y} \in \mathfrak{L}^2$ and $t \in \mathbb{R}$,

$$\sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, t) = \int_0^\beta \left(\mathfrak{C}_{t+i\alpha}^{(\omega)}(\mathbf{x}, \mathbf{y}) - \mathfrak{C}_{i\alpha}^{(\omega)}(\mathbf{x}, \mathbf{y}) \right) \mathrm{d}\alpha \in \mathbb{R} \ ,$$

where $\mathfrak{C}_{t+i\alpha}^{(\omega)} \equiv \mathfrak{C}_{t+i\alpha}^{(\beta,\omega,\lambda)}$ is the map from $\mathfrak{L}^4$ to $\mathbb{C}$ defined by

$$\mathfrak{C}_{t+i\alpha}^{(\omega)}(\mathbf{x}, \mathbf{y}) := \sum_{\pi,\pi' \in S_2} \varepsilon_\pi \varepsilon_{\pi'} C_{t+i\alpha}^{(\omega)}(y^{\pi'(1)}, x^{\pi(1)}) C_{-t+i(\beta-\alpha)}^{(\omega)}(x^{\pi(2)}, y^{\pi'(2)}) \qquad (93)$$

for any $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$ and $\mathbf{y} := (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$. Here, $\pi, \pi' \in S_2$ are by definition permutations of $\{1, 2\}$ with signatures $\varepsilon_\pi, \varepsilon_{\pi'} \in \{-1, 1\}$.

Proof: Fix $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$, $\alpha \in [0, \beta]$, $\mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2$ and $\mathbf{y} := (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$. From Equation (89) together with (164),

$$\sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, t) = \int_0^\beta \left(\varrho^{(\beta,\omega,\lambda)} \left(I_{\mathbf{y}} \tau_{t+i\alpha}^{(\omega,\lambda)}(I_{\mathbf{x}}) \right) - \varrho^{(\beta,\omega,\lambda)} \left(I_{\mathbf{y}} \tau_{i\alpha}^{(\omega,\lambda)}(I_{\mathbf{x}}) \right) \right) \mathrm{d}\alpha \ . \tag{94}$$

Direct computations using (8) and (19) yield

$$\begin{aligned} I_{\mathbf{y}} \tau_{t+i\alpha}^{(\omega,\lambda)}(I_{\mathbf{x}}) \ &= \ -\left(a_{y^{(1)}}^* a_{y^{(2)}} - a_{y^{(2)}}^* a_{y^{(1)}} \right) \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}^*) \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(2)}}) \\ &\quad + \left(a_{y^{(1)}}^* a_{y^{(2)}} - a_{y^{(2)}}^* a_{y^{(1)}} \right) \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(2)}}^*) \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}) \ . \end{aligned} \tag{95}$$

Note that, for all $\mathbf{x} \in \mathfrak{L}^2$ and $x \in \mathfrak{L}$, the maps

$$z \mapsto \tau_z^{(\omega,\lambda)}(I_{\mathbf{x}}) \ , \quad z \mapsto \tau_z^{(\omega,\lambda)}(a_x^*) \ , \quad z \mapsto \tau_z^{(\omega,\lambda)}(a_x) \ , \tag{96}$$

defined on $\mathbb{R}$ have unique analytic continuations for $z \in \mathbb{C}$ and (95) makes sense.

Recall that $\mathfrak{e}_x(y) \equiv \delta_{x,y}$ is the canonical orthonormal basis of $\ell^2(\mathfrak{L})$ and, as usual,

$$\{B_1, B_2\} := B_1 B_2 + B_2 B_1 \ , \qquad B_1, B_2 \in \mathcal{U} \ .$$

Therefore, using the anti–commutator relation

$$\{a_{y^{(2)}}, \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}^*)\} = \langle \mathfrak{e}_{y^{(2)}}, (\mathrm{U}_{t+i\alpha}^{(\omega,\lambda)})^* \mathfrak{e}_{x^{(1)}} \rangle \mathbf{1} \ ,$$

see (4) and (7), we get the equality

$$\begin{aligned} & \varrho^{(\beta,\omega,\lambda)} \left(a_{y^{(1)}}^* a_{y^{(2)}} \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}^*) \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(2)}}) \right) \\ = \ & -\varrho^{(\beta,\omega,\lambda)} \left(a_{y^{(1)}}^* \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}^*) a_{y^{(2)}} \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(2)}}) \right) \\ & +\varrho^{(\beta,\omega,\lambda)} \left(\{a_{y^{(2)}}, \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}^*)\} \right) \varrho^{(\beta,\omega,\lambda)} \left(a_{y^{(1)}}^* \tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(2)}}) \right) \ . \end{aligned} \tag{97}$$

Since $\varrho^{(\beta,\omega,\lambda)}$ is by construction a quasi–free state, we use [BR2, p. 48], that is here,

$$\begin{aligned} & \varrho^{(\beta,\omega,\lambda)}(a^*(f_1)\, a^*(f_2)\, a(g_1)\, a(g_2)) \\ = \ & \varrho^{(\beta,\omega,\lambda)}(a^*(f_1)\, a(g_2)) \varrho^{(\beta,\omega,\lambda)}(a^*(f_2)\, a(g_1)) \\ & -\varrho^{(\beta,\omega,\lambda)}(a^*(f_1)\, a(g_1)) \varrho^{(\beta,\omega,\lambda)}(a^*(f_2)\, a(g_2)) \ , \end{aligned}$$

to infer from Equation (97) that

$$\begin{aligned}
&\varrho^{(\beta,\omega,\lambda)}\left(a_{y^{(1)}}^{*}a_{y^{(2)}}\tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}^{*})\tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(2)}})\right)\\
=\ &\varrho^{(\beta,\omega,\lambda)}(a_{y^{(1)}}^{*}a_{y^{(2)}})\varrho^{(\beta,\omega,\lambda)}(\tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}^{*})\tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(2)}}))\\
&+\varrho^{(\beta,\omega,\lambda)}\left(a_{y^{(1)}}^{*}\tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(2)}})\right)\varrho^{(\beta,\omega,\lambda)}\left(a_{y^{(2)}}\tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}^{*})\right)\ . \qquad (98)
\end{aligned}$$

Remark that the KMS property (162) together with (9) and the Phragmén–Lindelöf theorem [BR2, Proposition 5.3.5] yields

$$\varrho^{(\beta,\omega,\lambda)}(\tau_{t+i\alpha}^{(\omega,\lambda)}(B))=\varrho^{(\beta,\omega,\lambda)}(B)\ ,\qquad B\in\mathcal{U}\ . \qquad (99)$$

See also [BR2, Proposition 5.3.7]. We thus combine (99) and (162) with Equation (8) and the analyticity of the maps (96) to deduce from (92) that

$$C_{-t+i(\beta-\alpha)}^{(\omega)}(\mathbf{x})=\varrho^{(\beta,\omega,\lambda)}(a_{x^{(2)}}\tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}^{*}))\ .$$

Using this together with (92), (99) and again the analyticity of the maps (96), we get from Equation (98) that

$$\begin{aligned}
&\varrho^{(\beta,\omega,\lambda)}\left(a_{y^{(1)}}^{*}a_{y^{(2)}}\tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(1)}}^{*})\tau_{t+i\alpha}^{(\omega,\lambda)}(a_{x^{(2)}})\right)\\
=\ &C_{0}^{(\omega)}(y^{(1)},y^{(2)})C_{0}^{(\omega)}(x^{(1)},x^{(2)})+C_{t+i\alpha}^{(\omega)}(y^{(1)},x^{(2)})C_{-t+i(\beta-\alpha)}^{(\omega)}(x^{(1)},y^{(2)})\ .
\end{aligned}$$

Then we use this last equality together with (95) to get

$$\begin{aligned}
&\varrho^{(\beta,\omega,\lambda)}\left(I_{\mathbf{y}}\tau_{t+i\alpha}^{(\omega,\lambda)}(I_{\mathbf{x}})\right)\\
=\ &-\sum_{\pi,\pi'\in S_2}\varepsilon_{\pi}\varepsilon_{\pi'}\Big(C_{t+i\alpha}^{(\omega)}(y^{\pi'(1)},x^{\pi(2)})C_{-t+i(\beta-\alpha)}^{(\omega)}(x^{\pi(1)},y^{\pi'(2)})\\
&+C_{0}^{(\omega)}(y^{\pi'(1)},y^{\pi'(2)})C_{0}^{(\omega)}(x^{\pi(1)},x^{\pi(2)})\Big)\ . \qquad (100)
\end{aligned}$$

Therefore, the assertion follows by combining (94) with (100) for any $\beta\in\mathbb{R}^{+}$, $\omega\in\Omega$, $\lambda\in\mathbb{R}_{0}^{+}$, $t\in\mathbb{R}$, $\alpha\in[0,\beta]$, $\mathbf{x}:=(x^{(1)},x^{(2)})\in\mathfrak{L}^{2}$ and $\mathbf{y}:=(y^{(1)},y^{(2)})\in\mathfrak{L}^{2}$. ■

Lemma 5.2 is a useful technical result because the complex–time two–point correlation functions $C_{t+i\alpha}^{(\omega)}$ can be expressed in terms of the one–particle bounded self–adjoint operator $(\Delta_{\mathrm{d}}+\lambda V_{\omega})\in\mathcal{B}(\ell^{2}(\mathfrak{L}))$ to which the spectral theorem can be applied. Indeed, for all $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$ and $\alpha \in [0, \beta]$, one gets from (7), (10) and (92) that

$$C_{t+i\alpha}^{(\omega)}(\mathbf{x}) = \langle \mathfrak{e}_{x^{(2)}}, \mathrm{e}^{-it(\Delta_\mathrm{d}+\lambda V_\omega)} F_\alpha^\beta \left(\Delta_\mathrm{d} + \lambda V_\omega\right) \mathfrak{e}_{x^{(1)}} \rangle \ , \tag{101}$$

where F_α^β is the real function defined, for every $\beta \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$, by

$$F_\alpha^\beta\left(\varkappa\right) := \frac{\mathrm{e}^{\alpha\varkappa}}{1+\mathrm{e}^{\beta\varkappa}} \ , \qquad \varkappa \in \mathbb{R} \ .$$

Equation (101) provides useful estimates like space–decay properties of complex–time two–point correlation functions $C_{t+i\alpha}^{(\omega)}$, see [BPK2]. An important consequence of (101) is the fact that the coefficient $\mathfrak{C}_{t+i\alpha}^{(\omega)}$ defined by (93) can be seen as the kernel (w.r.t. the canonical basis $\{\mathfrak{e}_x \otimes \mathfrak{e}_{x'}\}_{x,x'\in\mathfrak{L}}$) of a bounded operator on $\ell^2(\mathfrak{L}) \otimes \ell^2(\mathfrak{L})$. This operator is again denoted by $\mathfrak{C}_{t+i\alpha}^{(\omega)}$:

Lemma 5.3 ($\mathfrak{C}_{t+i\alpha}^{(\omega)}$ as a bounded operator)
Let $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$ and $\alpha \in [0, \beta]$. Then, there is a unique bounded operator $\mathfrak{C}_{t+i\alpha}^{(\omega)}$ on $\ell^2(\mathfrak{L}) \otimes \ell^2(\mathfrak{L})$ with

$$\langle \mathfrak{e}_{x^{(1)}} \otimes \mathfrak{e}_{x^{(2)}}, \mathfrak{C}_{t+i\alpha}^{(\omega)}(\mathfrak{e}_{y^{(1)}} \otimes \mathfrak{e}_{y^{(2)}})\rangle_{\ell^2(\mathfrak{L})\otimes\ell^2(\mathfrak{L})} = \mathfrak{C}_{t+i\alpha}^{(\omega)}((x^{(1)}, x^{(2)}), (y^{(1)}, y^{(2)}))$$

for all $(x^{(1)}, x^{(2)}), (y^{(1)}, y^{(2)}) \in \mathfrak{L}^2$, and

$$\|\mathfrak{C}_{t+i\alpha}^{(\omega)}\|_\mathrm{op} \leq 4 \qquad \textit{and} \qquad \lim_{\alpha\to 0^+} \|\mathfrak{C}_{i\alpha}^{(\omega)} - \mathfrak{C}_0^{(\omega)}\|_\mathrm{op} = 0 \ ,$$

where $\|\cdot\|_\mathrm{op}$ is the operator norm.

Proof: By (93) and (101), the bounded operator $\mathfrak{C}_{t+i\alpha}^{(\omega)}$ exists, is unique, and one directly gets

$$\frac{1}{4}\|\mathfrak{C}_{t+i\alpha}^{(\omega)}\|_\mathrm{op} \leq \left\| \frac{\mathrm{e}^{(-it+\alpha)(\Delta_\mathrm{d}+\lambda V_\omega)}}{1+\mathrm{e}^{\beta(\Delta_\mathrm{d}+\lambda V_\omega)}} \right\|_\mathrm{op} \left\| \frac{\mathrm{e}^{(it+\beta-\alpha)(\Delta_\mathrm{d}+\lambda V_\omega)}}{1+\mathrm{e}^{\beta(\Delta_\mathrm{d}+\lambda V_\omega)}} \right\|_\mathrm{op} \leq 1$$

for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $t \in \mathbb{R}$ and $\alpha \in [0, \beta]$. Moreover, in the same way, (93) and (101) also lead to

$$\frac{1}{4}\|\mathfrak{C}_{i\alpha}^{(\omega)} - \mathfrak{C}_0^{(\omega)}\|_\mathrm{op} \leq \left\|\mathrm{e}^{\alpha(\Delta_\mathrm{d}+\lambda V_\omega)} - \mathbf{1}\right\|_\mathrm{op} + \left\|\mathrm{e}^{-\alpha(\Delta_\mathrm{d}+\lambda V_\omega)} - \mathbf{1}\right\|_\mathrm{op} \tag{102}$$

for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $\alpha \in [0, \beta]$. Recall that the self–adjoint operator $\Delta_\mathrm{d} + \lambda V_\omega$ is bounded, i.e., $\Delta_\mathrm{d} + \lambda V_\omega \in \mathcal{B}(\ell^2(\mathfrak{L}))$. It follows that the one–parameter group $\{\mathrm{e}^{\alpha(\Delta_\mathrm{d}+\lambda V_\omega)}\}_{\alpha\in\mathbb{R}}$ is uniformly continuous (norm continuous). Therefore, the second assertion is deduced from (102) in the limit $\alpha \to 0^+$. ■

5.1.2 Space–Averaged Paramagnetic Transport Coefficients

Equation (33) and Lemma A.14 for $\mathcal{X} = \mathcal{U}$, $\tau = \tau^{(\omega,\lambda)}$ and $\varrho = \varrho^{(\beta,\omega,\lambda)}$ yield

$$\left\{ \Xi_{\mathrm{p},l}^{(\omega)}(t) \right\}_{k,q} = \frac{1}{|\Lambda_l|} \left[(\mathbb{I}_{k,l}, \tau_t^{(\omega,\lambda)}(\mathbb{I}_{q,l}))_\sim - (\mathbb{I}_{k,l}, \mathbb{I}_{q,l})_\sim \right] \qquad (103)$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $k, q \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$. Since $\Theta(I_{\mathbf{x}}) = -I_{\mathbf{x}}$ for any $\mathbf{x} \in \mathfrak{L}^2$, by Theorem A.16, the operator $\Xi_{\mathrm{p},l}^{(\omega)}(t)$ is symmetric at any fixed time $t \in \mathbb{R}$ while the $\mathcal{B}(\mathbb{R}^d)$–valued function $\Xi_{\mathrm{p},l}^{(\omega)}$ is symmetric w.r.t. time–reversal. In other words,

$$\left\{ \Xi_{\mathrm{p},l}^{(\omega)}(t) \right\}_{k,q} = \left\{ \Xi_{\mathrm{p},l}^{(\omega)}(-t) \right\}_{k,q} = \left\{ \Xi_{\mathrm{p},l}^{(\omega)}(t) \right\}_{q,k} \in \mathbb{R} \qquad (104)$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $k, q \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$.

Because of (103) it is convenient to use the *Duhamel* GNS representation

$$(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{\Psi}) \equiv (\tilde{\mathcal{H}}^{(\beta,\omega,\lambda)}, \tilde{\pi}^{(\beta,\omega,\lambda)}, \tilde{\Psi}^{(\beta,\omega,\lambda)})$$

of the $(\tau^{(\omega,\lambda)}, \beta)$–KMS state $\varrho^{(\beta,\omega,\lambda)}$ for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$. See Definition A.8 with $\mathcal{X} = \mathcal{U}$ and $\varrho = \varrho^{(\beta,\omega,\lambda)}$. Note that we identify here the Duhamel two–point function defined by (88) on the CAR algebra $\mathcal{U}$ with the scalar product $(\cdot,\cdot)_\sim$ of the Hilbert space $\tilde{\mathcal{H}}$, see Remark A.11.

The CAR C^*–algebra $\mathcal{U}$ is the inductive limit of (finite dimensional) simple C^*–algebras $\{\mathcal{U}_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathfrak{L})}$, see [Si, Lemma IV.1.2]. By [BR1, Corollary 2.6.19.], $\mathcal{U}$ is thus *simple*. This property has some important consequences: The $(\tau^{(\omega,\lambda)}, \beta)$–KMS state $\varrho^{(\beta,\omega,\lambda)}$ is faithful. In particular, $\tilde{\pi}$ is injective. Remark that $\tilde{\Psi} \equiv \mathbf{1} \in \mathcal{U}$ and $\mathcal{U}$ is a dense set of $\tilde{\mathcal{H}}$, but $\tilde{\pi}(B)\tilde{\Psi}$ is generally not equal to $B \in \mathcal{U}$, in contrast to the usual GNS representation. For this reason, we do not identify $\tilde{\pi}(\mathcal{U})$ with $\mathcal{U}$. Moreover, by Theorem A.9 for $\mathcal{X} = \mathcal{U}$ and $\varrho = \varrho^{(\beta,\omega,\lambda)}$, the $*$–automorphism group $\tau = \tau^{(\omega,\lambda)}$ can be extended to a unitary group on the whole Hilbert space $\tilde{\mathcal{H}}$:

$$\tau_t^{(\omega,\lambda)}(B) = \mathrm{e}^{it\tilde{\mathcal{L}}} B \, , \qquad t \in \mathbb{R} \, , \ B \in \mathcal{U} \subset \tilde{\mathcal{H}} \, , \qquad (105)$$

with $\tilde{\mathcal{L}} \equiv \tilde{\mathcal{L}}^{(\beta,\omega,\lambda)}$ being a self–adjoint operator acting on $\tilde{\mathcal{H}}$. The domain of $\tilde{\mathcal{L}}$ includes the domain of the generator $\delta^{(\omega,\lambda)}$ of the one–parameter group $\tau^{(\omega,\lambda)}$, i.e., $\mathrm{Dom}(\tilde{\mathcal{L}}) \supset \mathrm{Dom}(\delta^{(\omega,\lambda)})$, while

$$\tilde{\mathcal{L}}(B) = -i\delta^{(\omega,\lambda)}(B) \, , \qquad B \in \mathrm{Dom}(\delta^{(\omega,\lambda)}) \subset \mathcal{U} \subset \tilde{\mathcal{H}} \, . \qquad (106)$$

Equation (105) is an important representation of the dynamics because we can deduce from (103) the existence of the paramagnetic conductivity measure from the spectral theorem.

To present this result, recall that $\mathcal{B}_+(\mathbb{R}^d) \subset \mathcal{B}(\mathbb{R}^d)$ denotes the set of positive linear operators on $\mathbb{R}^d$ and any $\mathcal{B}(\mathbb{R}^d)$–valued measure μ on $\mathbb{R}$ is symmetric iff $\mu(\mathcal{X}) = \mu(-\mathcal{X})$ for any Borel set $\mathcal{X} \subset \mathbb{R}$. Then, we derive the paramagnetic conductivity measure:

Theorem 5.4 (Conductivity measures as spectral measures)
For any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, there exists a finite symmetric $\mathcal{B}_+(\mathbb{R}^d)$–valued measure $\mu_{\mathrm{p},l}^{(\omega)} \equiv \mu_{\mathrm{p},l}^{(\beta,\omega,\lambda)}$ on $\mathbb{R}$ such that

$$\Xi_{\mathrm{p},l}^{(\omega)}(t) = \int_{\mathbb{R}} (\cos(t\nu) - 1)\, \mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\ , \qquad t \in \mathbb{R}\ . \tag{107}$$

Proof: Fix $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$. Let $\tilde{E} \equiv \tilde{E}^{(\beta,\omega,\lambda)}$ be the (projection–valued) spectral measure of the self–adjoint operator $\tilde{\mathcal{L}}$. Then, by combining (103)–(104) with (105), we directly arrive at the equality

$$\begin{aligned} \left\{ \Xi_{\mathrm{p},l}^{(\omega)}(t) \right\}_{k,q} &= \frac{1}{4|\Lambda_l|} \int_{\mathbb{R}} \left(\mathrm{e}^{it\nu} - 1 \right) (\mathbb{I}_{k,l}, \tilde{E}(\mathrm{d}\nu)\mathbb{I}_{q,l})_\sim \\ &\quad + \frac{1}{4|\Lambda_l|} \int_{\mathbb{R}} \left(\mathrm{e}^{it\nu} - 1 \right) (\mathbb{I}_{q,l}, \tilde{E}(\mathrm{d}\nu)\mathbb{I}_{k,l})_\sim \\ &\quad + \frac{1}{4|\Lambda_l|} \int_{\mathbb{R}} \left(\mathrm{e}^{-it\nu} - 1 \right) (\mathbb{I}_{k,l}, \tilde{E}(\mathrm{d}\nu)\mathbb{I}_{q,l})_\sim \\ &\quad + \frac{1}{4|\Lambda_l|} \int_{\mathbb{R}} \left(\mathrm{e}^{-it\nu} - 1 \right) (\mathbb{I}_{q,l}, \tilde{E}(\mathrm{d}\nu)\mathbb{I}_{k,l})_\sim \end{aligned} \tag{108}$$

for any $k, q \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$. Note that, for any Borel set $\mathcal{X} \subset \mathbb{R}$ and all $k, q \in \{1, \ldots, d\}$,

$$(\mathbb{I}_{k,l}, \tilde{E}(\mathcal{X})\, \mathbb{I}_{q,l})_\sim + (\mathbb{I}_{q,l}, \tilde{E}(\mathcal{X})\, \mathbb{I}_{k,l})_\sim \in \mathbb{R}\ .$$

Thus, define the $\mathcal{B}(\mathbb{R}^d)$–valued measure $\mu_{\mathrm{p},l}^{(\omega)}$ by

$$\begin{aligned}\left\langle \vec{u},\mu_{\mathrm{p},l}^{(\omega)}\left(\mathcal{X}\right)\vec{w}\right\rangle &= \frac{1}{4\left|\Lambda_l\right|}\sum_{k,q\in\{1,\ldots,d\}}u_k w_q(\mathbb{I}_{k,l},\tilde{E}\left(\mathcal{X}\right)\mathbb{I}_{q,l})_{\sim} \\ &\quad+\frac{1}{4\left|\Lambda_l\right|}\sum_{k,q\in\{1,\ldots,d\}}u_k w_q(\mathbb{I}_{q,l},\tilde{E}\left(\mathcal{X}\right)\mathbb{I}_{k,l})_{\sim} \\ &\quad+\frac{1}{4\left|\Lambda_l\right|}\sum_{k,q\in\{1,\ldots,d\}}u_k w_q(\mathbb{I}_{k,l},\tilde{E}\left(-\mathcal{X}\right)\mathbb{I}_{q,l})_{\sim} \\ &\quad+\frac{1}{4\left|\Lambda_l\right|}\sum_{k,q\in\{1,\ldots,d\}}u_k w_q(\mathbb{I}_{q,l},\tilde{E}\left(-\mathcal{X}\right)\mathbb{I}_{k,l})_{\sim} \qquad (109)\end{aligned}$$

for any $\vec{u}:=(u_1,\ldots,u_d)\in\mathbb{R}^d$, $\vec{w}:=(w_1,\ldots,w_d)\in\mathbb{R}^d$ and all Borel sets $\mathcal{X}\subset\mathbb{R}$. Here, $\langle\cdot,\cdot\rangle$ denotes the usual scalar product of $\mathbb{R}^d$. Obviously, by construction,

$$\left\langle \vec{u},\mu_{\mathrm{p},l}^{(\omega)}\left(\mathcal{X}\right)\vec{w}\right\rangle = \left\langle \vec{w},\mu_{\mathrm{p},l}^{(\omega)}\left(\mathcal{X}\right)\vec{u}\right\rangle \qquad \text{and} \qquad \left\langle \vec{w},\mu_{\mathrm{p},l}^{(\omega)}\left(\mathcal{X}\right)\vec{w}\right\rangle \geq 0\ ,$$

for any $\vec{u},\vec{w}\in\mathbb{R}^d$ and all Borel sets $\mathcal{X}\subset\mathbb{R}$. Moreover, $\mu_{\mathrm{p},l}^{(\omega)}$ is a symmetric measure and, by (108), we obtain Equation (107). ∎

For any $\beta\in\mathbb{R}^+$, $\omega\in\Omega$ and $\lambda\in\mathbb{R}_0^+$, it is useful at this point to also consider any GNS representation

$$(\mathcal{H},\pi,\Psi)\equiv(\mathcal{H}^{(\beta,\omega,\lambda)},\pi^{(\beta,\omega,\lambda)},\Psi^{(\beta,\omega,\lambda)})$$

of the $(\tau^{(\omega,\lambda)},\beta)$–KMS state $\varrho^{(\beta,\omega,\lambda)}$ and to describe its relation to the Duhamel GNS representation. To this end, we denote by $\mathcal{L}\equiv\mathcal{L}^{(\beta,\omega,\lambda)}$ the standard Liouvillean of the system under consideration, i.e., the self–adjoint operator acting on $\mathcal{H}$ which implements the dynamics as

$$\pi\left(\tau_t\left(B\right)\right)=\mathrm{e}^{it\mathcal{L}}\pi\left(B\right)\mathrm{e}^{-it\mathcal{L}}\ ,\qquad t\in\mathbb{R},\ B\in\mathcal{U}\ , \qquad (110)$$

with $\mathcal{L}\Psi=\Psi$. Let $E\equiv E^{(\beta,\omega,\lambda)}$ be the (projection–valued) spectral measure of $\mathcal{L}$. We also use the (Tomita–Takesaki) modular objects

$$\Delta\equiv\Delta^{(\beta,\omega,\lambda)}:=\mathrm{e}^{-\beta\mathcal{L}}\ ,\qquad J\equiv J^{(\beta,\omega,\lambda)}\ ,$$

of the pair $(\pi\left(\mathcal{U}\right)'',\Psi)$.

Theorem A.1 says that

$$(B_1, B_2)_\sim = \langle \mathfrak{T}\pi(B_1)\Psi, \mathfrak{T}\pi(B_2)\Psi\rangle_{\mathcal{H}} , \qquad B_1, B_2 \in \mathcal{U} , \tag{111}$$

where $\mathfrak{T} \equiv \mathfrak{T}^{(\beta,\omega,\lambda)}$ is the operator defined by (158) for $\tau = \tau^{(\omega,\lambda)}$ and $\varrho = \varrho^{(\beta,\omega,\lambda)}$, that is,

$$\mathfrak{T} := \beta^{1/2}\left(\frac{1-\mathrm{e}^{-\beta\mathcal{L}}}{\beta\mathcal{L}}\right)^{1/2} . \tag{112}$$

Note that $\mathfrak{T}$ is unbounded, but

$$\pi(\mathcal{U})\Psi \subset \mathrm{Dom}(\Delta^{1/2}) \subset \mathrm{Dom}(\mathfrak{T}) . \tag{113}$$

The $\mathcal{B}_+(\mathbb{R}^d)$–valued measure $\mu_{\mathrm{p},l}^{(\omega)}$ of Theorem 5.4, which is defined by (109), can also be studied via (111). Indeed, (111) and (113) together with Theorem A.7 and (161) imply that

$$(\mathbb{I}_{k,l}, \tilde{E}(\mathcal{X})\mathbb{I}_{q,l})_\sim = \langle \mathfrak{T}E(\mathcal{X})\pi(\mathbb{I}_{k,l})\Psi, \mathfrak{T}E(\mathcal{X})\pi(\mathbb{I}_{q,l})\Psi\rangle_{\mathcal{H}} \tag{114}$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $k, q \in \{1, \ldots, d\}$ and any Borel set $\mathcal{X} \subset \mathbb{R}$. The existence of the first moment of $\mu_{\mathrm{p},l}^{(\omega)}$ is a direct consequence of the above equation.

To see this, recall that $\|\mu_{\mathrm{p},l}^{(\omega)}\|_{\mathrm{op}}$ is the measure on $\mathbb{R}$ taking values in $\mathbb{R}_0^+$ that is defined, for any Borel set $\mathcal{X} \subset \mathbb{R}$ and $\mu = \mu_{\mathrm{p},l}^{(\omega)}$, by (38). Then, one gets the following assertions:

Theorem 5.5 (Existence of the first moment of $\mu_{\mathrm{p},l}^{(\omega)}$)
For any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, the $\mathcal{B}_+(\mathbb{R}^d)$–valued measure $\mu_{\mathrm{p},l}^{(\omega)}$ of Theorem 5.4 satisfies the following bounds:

$$\begin{aligned}
\int_{\mathbb{R}} \|\mu_{\mathrm{p},l}^{(\omega)}\|_{\mathrm{op}}(\mathrm{d}\nu) &\leq \frac{1}{|\Lambda_l|}\sum_{k=1}^{d} \varrho^{(\beta,\omega,\lambda)}\left(\mathbb{I}_{k,l}^2\right) , \\
\int_{\mathbb{R}} |\nu|\, \|\mu_{\mathrm{p},l}^{(\omega)}\|_{\mathrm{op}}(\mathrm{d}\nu) &\leq \frac{2}{|\Lambda_l|}\sum_{k=1}^{d} \varrho^{(\beta,\omega,\lambda)}\left(\mathbb{I}_{k,l}^2\right) , \\
\int_{\mathbb{R}} |\nu|\, \|\mu_{\mathrm{p},l}^{(\omega)}\|_{\mathrm{op}}(\mathrm{d}\nu) &\leq \frac{2}{|\Lambda_l|}\sum_{k=1}^{d} \sqrt{\varrho^{(\beta,\omega,\lambda)}\left(\mathbb{I}_{k,l}^2\right)}\sqrt{\varrho^{(\beta,\omega,\lambda)}\left(\left(\delta^{(\omega,\lambda)}\left(\mathbb{I}_{k,l}\right)\right)^2\right)} .
\end{aligned}$$

Proof: Fix $l,\beta\in\mathbb{R}^{+}$, $\omega\in\Omega$ and $\lambda\in\mathbb{R}_{0}^{+}$. By positivity of the measure $\mu_{\mathrm{p},l}^{(\omega)}$ and linearity of the trace,

$$\|\mu_{\mathrm{p},l}^{(\omega)}\|_{\mathrm{op}}(\mathcal{X})\leq\mathrm{Trace}_{\mathcal{B}(\mathbb{R}^{d})}\left(\mu_{\mathrm{p},l}^{(\omega)}(\mathcal{X})\right)$$

for any Borel set $\mathcal{X}\subset\mathbb{R}$. This implies that

$$\int_{\mathbb{R}}\|\mu_{\mathrm{p},l}^{(\omega)}\|_{\mathrm{op}}(\mathrm{d}\nu)\leq\mathrm{Trace}_{\mathcal{B}(\mathbb{R}^{d})}\left(\int_{\mathbb{R}}\mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\right)$$

and

$$\int_{\mathbb{R}}|\nu|\,\|\mu_{\mathrm{p},l}^{(\omega)}\|_{\mathrm{op}}(\mathrm{d}\nu)\leq\mathrm{Trace}_{\mathcal{B}(\mathbb{R}^{d})}\left(\int_{\mathbb{R}}|\nu|\,\mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\right)\ .$$

Hence, by (109), it suffices to prove that

$$\int_{\mathbb{R}}(\mathbb{I}_{k,l},\tilde{E}(\mathrm{d}\nu)\mathbb{I}_{k,l})_{\sim}\ \leq\ \varrho^{(\beta,\omega,\lambda)}\left(\mathbb{I}_{k,l}^{2}\right)\ ,\tag{115}$$

$$\int_{\mathbb{R}}|\nu|\,(\mathbb{I}_{k,l},\tilde{E}(\mathrm{d}\nu)\mathbb{I}_{k,l})_{\sim}\ \leq\ 2\varrho^{(\beta,\omega,\lambda)}\left(\mathbb{I}_{k,l}^{2}\right)\ ,\tag{116}$$

$$\int_{\mathbb{R}}|\nu|\,(\mathbb{I}_{k,l},\tilde{E}(\mathrm{d}\nu)\mathbb{I}_{k,l})_{\sim}\ \leq\ 2\sqrt{\varrho^{(\beta,\omega,\lambda)}\left(\mathbb{I}_{k,l}^{2}\right)\varrho^{(\beta,\omega,\lambda)}\left(\left(\delta^{(\omega,\lambda)}\left(\mathbb{I}_{k,l}\right)\right)^{2}\right)}\ ,\tag{117}$$

for any $k\in\{1,\ldots,d\}$.

Inequality (115) is a direct consequence of Theorem A.4. The second upper bound is derived as follows: Fix $k\in\{1,\ldots,d\}$. We infer from (112) and (114) that

$$\int_{\mathbb{R}}|\nu|\,(\mathbb{I}_{k,l},\tilde{E}(\mathrm{d}\nu)\mathbb{I}_{k,l})_{\sim}\ =\ \left\|\left(1-\mathrm{e}^{-\beta\mathcal{L}}\right)^{1/2}E\left(\mathbb{R}_{0}^{+}\right)\pi\left(\mathbb{I}_{k,l}\right)\Psi\right\|_{\mathcal{H}}^{2}+\left\|\left(\mathrm{e}^{-\beta\mathcal{L}}-1\right)^{1/2}E\left(\mathbb{R}^{-}\right)\pi\left(\mathbb{I}_{k,l}\right)\Psi\right\|_{\mathcal{H}}^{2}\ .\tag{118}$$

Clearly, one has the upper bound

$$\left\|\left(1-\mathrm{e}^{-\beta\mathcal{L}}\right)^{1/2}E\left(\mathbb{R}_{0}^{+}\right)\pi\left(\mathbb{I}_{k,l}\right)\Psi\right\|_{\mathcal{H}}^{2}\leq\varrho^{(\beta,\omega,\lambda)}\left(\mathbb{I}_{k,l}^{2}\right)\ ,\tag{119}$$

while

$$\left\|\left(\mathrm{e}^{-\beta\mathcal{L}}-1\right)^{1/2}E\left(\mathbb{R}^{-}\right)\pi\left(\mathbb{I}_{k,l}\right)\Psi\right\|_{\mathcal{H}}^{2}\leq\left\|\Delta^{1/2}\pi\left(\mathbb{I}_{k,l}\right)\Psi\right\|_{\mathcal{H}}^{2}\ ,\tag{120}$$

with $\Delta := \mathrm{e}^{-\beta\mathcal{L}}$ being the modular operator. Using now the anti–unitarity of J, $J^2 = \mathbf{1}$ and

$$J\Delta^{1/2}\pi\left(\mathbb{I}_{k,l}\right)\Psi = \pi\left(\mathbb{I}_{k,l}\right)^{*}\Psi = \pi\left(\mathbb{I}_{k,l}\right)\Psi\ ,$$

one gets that

$$\left\|\Delta^{1/2}\pi\left(\mathbb{I}_{k,l}\right)\Psi\right\|_{\mathcal{H}}^{2} = \left\|\pi\left(\mathbb{I}_{k,l}\right)\Psi\right\|_{\mathcal{H}}^{2} = \varrho^{(\beta,\omega,\lambda)}\left(\mathbb{I}_{k,l}^{2}\right)\ . \tag{121}$$

Therefore, by combining Equation (118) with (119)–(121) we arrive at Inequality (116).

Finally, to prove (117), observe that

$$\begin{aligned}\int_{\mathbb{R}}|\nu|\left(\mathbb{I}_{k,l},\tilde{E}(\mathrm{d}\nu)\mathbb{I}_{k,l}\right)_{\sim} &= \left\langle\mathfrak{T}\pi\left(\mathbb{I}_{k,l}\right)\Psi, E\left(\mathbb{R}_0^{+}\right)\mathfrak{T}\mathcal{L}\pi\left(\mathbb{I}_{k,l}\right)\Psi\right\rangle_{\mathcal{H}} \\ &\quad -\left\langle\mathfrak{T}\pi\left(\mathbb{I}_{k,l}\right)\Psi, E\left(\mathbb{R}^{-}\right)\mathfrak{T}\mathcal{L}\pi\left(\mathbb{I}_{k,l}\right)\Psi\right\rangle_{\mathcal{H}}\ .\end{aligned} \tag{122}$$

Since $\mathbb{I}_{k,l} \in \mathcal{U}_0 \subset \mathrm{Dom}(\delta^{(\omega,\lambda)})$,

$$\mathcal{L}\pi\left(\mathbb{I}_{k,l}\right)\Psi = -i\pi\left(\delta^{(\omega,\lambda)}\left(\mathbb{I}_{k,l}\right)\right)\Psi\ , \tag{123}$$

see (110). Therefore, by additionally using the Cauchy–Schwarz inequality of $(\cdot,\cdot)_{\sim}$ and Theorem A.4, one gets (117) similarly as above. ■

Equation (114) also leads to a characterization of the non–triviality of the conductivity measure at non–zero frequencies via a geometric condition:

Theorem 5.6 (Geometric interpretation of the AC–conductivity measure) *Let $l,\beta \in \mathbb{R}^{+}$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^{+}$. Then,*

$$\mathrm{lin}\left\{\pi\left(\mathbb{I}_{k,l}\right)\Psi : k \in \{1,\ldots,d\}\right\} \subset \ker\left(\mathcal{L}\right) \qquad \textit{iff} \qquad \mu_{\mathrm{p},l}^{(\omega)}\left(\mathbb{R}\backslash\{0\}\right) = 0\ .$$

Here, lin *stands for the linear hull of some set.*

Proof: Fix $l,\beta \in \mathbb{R}^{+}$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^{+}$. If

$$\mathrm{lin}\left\{\pi\left(\mathbb{I}_{k,l}\right)\Psi : k \in \{1,\ldots,d\}\right\} \subset \ker\left(\mathcal{L}\right)\ ,$$

then we infer from (109) and (114) that $\mu_{\mathrm{p},l}^{(\omega)}\left(\mathbb{R}\backslash\{0\}\right) = 0$. Observe that $\mathfrak{T}$ acts as the identity on the kernel of $\mathcal{L}$. Assume now that $\mu_{\mathrm{p},l}^{(\omega)}\left(\mathbb{R}\backslash\{0\}\right) = 0$. Then,

$$\mu_{\mathrm{p},l}^{(\omega)}\left(\mathbb{R}\backslash\{0\}\right) = 0\ ,$$

which, by (109) for $\mathcal{X} = \mathbb{R}\backslash\{0\}$, implies that

$$(\mathbb{I}_{k,l}, \tilde{E}\,(\mathbb{R}\backslash\{0\})\,\mathbb{I}_{k,l})_{\sim} = 0 \;, \qquad k \in \{1, \ldots, d\} \;.$$

As a consequence, any linear combination of elements $\{\mathbb{I}_{k,l}\}_{k\in\{1,\ldots,d\}} \in \mathcal{U} \subset \tilde{\mathcal{H}}$ belongs to the kernel of $\tilde{\mathcal{L}}$, i.e.,

$$\mathrm{lin}\,\{\mathbb{I}_{k,l} : k \in \{1, \ldots, d\}\} \subset \ker(\tilde{\mathcal{L}}) \;.$$

By Theorem A.7 and (161), this property in turn yields

$$\mathrm{lin}\,\{\pi\,(\mathbb{I}_{k,l})\,\Psi : k \in \{1, \ldots, d\}\} \subset \ker\,(\mathcal{L}) \;.$$

■

Corollary 5.7 (Non–triviality of the measure $\mu_{\mathrm{p},l}^{(\omega)}$)
For any $l, \beta \in \mathbb{R}^{+}$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_{0}^{+}$, the $\mathcal{B}_{+}(\mathbb{R}^{d})$–valued measure $\mu_{\mathrm{p},l}^{(\omega)}$ of Theorem 5.4 satisfies $\mu_{\mathrm{p},l}^{(\omega)}\,(\mathbb{R}\backslash\{0\}) > 0$.

Proof: By explicit computations, for any $k \in \{1, \ldots, d\}$,

$$\delta^{(\omega,\lambda)}\,(\mathbb{I}_{k,l}) = \lambda \mathbb{A}_{k,l}^{(\omega)} + \mathbb{B}_{k,l} \;, \tag{124}$$

where $\mathbb{A}_{k,l}^{(\omega)}, \mathbb{B}_{k,l} \in \mathcal{U}$ are defined, for $\omega \in \Omega$ and $l \in \mathbb{R}^{+}$, by

$$\mathbb{A}_{k,l}^{(\omega)} := \sum_{x\in\Lambda_l} \left(V_\omega\,(x + e_k) - V_\omega\,(x)\right) P_{(x,x+e_k)}$$

and

$$\mathbb{B}_{k,l} := \sum_{x,z\in\mathfrak{L},|z|=1,z\neq\pm e_k} \left(\mathbf{1}\left[x \in (\Lambda_l + z)\backslash\Lambda_l\right] - \mathbf{1}\left[x \in \Lambda_l\backslash\,(\Lambda_l + z)\right]\right) P_{(x,x+e_k+z)}$$
$$+ \sum_{x\in\mathfrak{L}} \left(\mathbf{1}\left[x \in (\Lambda_l + e_k)\backslash\Lambda_l\right] - \mathbf{1}\left[x \in \Lambda_l\backslash\,(\Lambda_l + e_k)\right]\right)\left(2a_x^{*}a_x - P_{(x+e_k,x-e_k)}\right)$$

with $P_{(x,y)}$ being defined by (25) for any $x, y \in \mathfrak{L}$. In particular, $\delta^{(\omega,\lambda)}\,(\mathbb{I}_{k,l})$ is not zero and hence $\pi\,(\mathbb{I}_{k,l})\,\Psi \notin \ker\,(\mathcal{L})$, because π is injective and the cyclic vector Ψ is separating for $\pi\,(\mathcal{U})''$, see [BR2, Corollary 5.3.9.]. Therefore, the assertion is a direct consequence of Theorem 5.6. ■

We now give another construction of the (AC–conductivity) measure $\mu_{\mathrm{p},l}^{(\omega)}$ on $\mathbb{R}\backslash\{0\}$ from the diamagnetic transport coefficient $\Xi_{\mathrm{d},l}^{(\omega)}$ (34) and the space–averaged quantum current viscosity

$$t \mapsto \mathbf{V}_l^{(\omega)}(t) \equiv \mathbf{V}_l^{(\beta,\omega,\lambda)}(t) \in \mathcal{B}(\mathbb{R}^d)\ ,$$

see (40). W.r.t. the canonical orthonormal basis of $\mathbb{R}^d$,

$$\left\{\mathbf{V}_l^{(\omega)}(t)\right\}_{k,q} = \frac{1}{\varrho^{(\beta,\omega,\lambda)}(\mathbb{P}_{k,l})}\varrho^{(\beta,\omega,\lambda)}\left(i[\mathbb{I}_{k,l},\tau_t^{(\omega,\lambda)}(\mathbb{I}_{q,l})]\right) \tag{125}$$

for any $k,q \in \{1,\ldots,d\}$ and $t \in \mathbb{R}$. Compare (125) with (32). Its Laplace transform

$$\mathbf{L}[\mathbf{V}_l^{(\omega)}](\epsilon) := \int_0^\infty \mathrm{e}^{-\epsilon s}\mathbf{V}_l^{(\omega)}(s)\,\mathrm{d}s$$

exists for all $\epsilon \in \mathbb{R}^+$, by the boundedness of $\mathbf{V}_l^{(\omega)}$. In fact, one has:

Theorem 5.8 (Static admittance)
Let $l,\beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$. Then the limit of $\mathbf{L}[\mathbf{V}_l^{(\omega)}](\epsilon)$ exists as $\epsilon \downarrow 0$ and satisfies:

$$\Xi_{\mathrm{d},l}^{(\omega)}\lim_{\epsilon\downarrow 0}\mathbf{L}[\mathbf{V}_l^{(\omega)}](\epsilon) = \mu_{\mathrm{p},l}^{(\omega)}(\mathbb{R}\backslash\{0\}) = \frac{1}{|\Lambda_l|}\left\{(\mathbb{I}_{k,l},\tilde{E}(\mathbb{R}\backslash\{0\})\,\mathbb{I}_{q,l})_\sim^{(\omega)}\right\}_{k,q\in\{1,\ldots,d\}}$$

Note that $\tilde{E}(\mathbb{R}\backslash\{0\})$ is not the identity because $\tilde{\mathcal{L}}\mathbf{1} = 0$.

Proof: Fix $l,\beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$. By [NVW, Theorems III.3-III.4], observe that

$$\Xi_{\mathrm{d},l}^{(\omega)}\lim_{\epsilon\downarrow 0}\mathbf{L}[\mathbf{V}_l^{(\omega)}](\epsilon) = \frac{1}{|\Lambda_l|}\left\{(\mathbb{I}_{k,l},\tilde{E}(\mathbb{R}\backslash\{0\})\,\mathbb{I}_{q,l})_\sim\right\}_{k,q\in\{1,\ldots,d\}}.$$

On the other hand, by (103) and (105),

$$\frac{1}{t}\int_0^t \left\{\Xi_{\mathrm{p},l}^{(\omega)}(s)\right\}_{k,q}\mathrm{d}s = \frac{1}{t\,|\Lambda_l|}\int_0^t (\mathbb{I}_{k,l},\mathrm{e}^{it\tilde{\mathcal{L}}}\mathbb{I}_{q,l})_\sim \mathrm{d}s - \frac{1}{|\Lambda_l|}(\mathbb{I}_{k,l},\mathbb{I}_{q,l})_\sim \tag{126}$$

for any $t \in \mathbb{R}^+$ and $k,q \in \{1,\ldots,d\}$. The von Neumann or mean ergodic theorem (see, e.g., [P, Theorem 3.13]) implies that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (\mathbb{I}_{k,l},\mathrm{e}^{it\tilde{\mathcal{L}}}\mathbb{I}_{q,l})_\sim \mathrm{d}s = (\mathbb{I}_{k,l},\tilde{E}(\{0\})\,\mathbb{I}_{q,l})_\sim\ , \tag{127}$$

where $\tilde{E}(\{0\})$ is the orthogonal projection on the kernel of $\tilde{\mathcal{L}}$. By combining (126)–(127) we obviously get

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \left\{ \Xi_{\mathrm{p},l}^{(\omega)}(s) \right\}_{k,q} \mathrm{d}s = -\frac{1}{|\Lambda_l|} (\mathbb{I}_{k,l}, \tilde{E}(\mathbb{R}\backslash\{0\}) \mathbb{I}_{q,l})_\sim ,$$

which, by Corollary 3.2 (iii), implies that

$$\mu_{\mathrm{p},l}^{(\omega)}(\mathbb{R}\backslash\{0\}) = \frac{1}{|\Lambda_l|} \left\{ (\mathbb{I}_{k,l}, \tilde{E}(\mathbb{R}\backslash\{0\}) \mathbb{I}_{q,l})_\sim \right\}_{k,q\in\{1,\ldots,d\}} .$$

■

Note that the quantity

$$\Xi_{\mathrm{d},l}^{(\omega)} \lim_{\epsilon\downarrow 0} \mathbf{L}[\mathbf{V}_l^{(\omega)}](\epsilon) \in \mathcal{B}(\mathbb{R}^d)$$

is the so–called *static admittance* of linear response theory, which equals, in our case, the measure of $\mathbb{R}\backslash\{0\}$ w.r.t. the AC–conductivity measure. In fact, the quantum current viscosity uniquely defines the AC–conductivity measure:

Theorem 5.9 (Reconstruction of $\mu_{\mathrm{p},l}^{(\omega)}$ from the quantum current viscosity)
Let $l,\beta \in \mathbb{R}^+$, $\omega\in\Omega$ and $\lambda\in\mathbb{R}_0^+$. Then, for all $\vec{w} := (w_1,\ldots,w_d) \in \mathbb{R}^d$ and any continuous and compactly supported real–valued function $\hat{\mathcal{E}}$ with $\hat{\mathcal{E}}_0 = 0$,

$$\int_{\mathbb{R}} \hat{\mathcal{E}}_\nu \left\langle \vec{w}, \mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\vec{w} \right\rangle = \lim_{\epsilon\downarrow 0} \frac{1}{\pi} \int_{\mathbb{R}} \mathrm{d}\nu \int_0^\infty \mathrm{d}s \frac{(\epsilon\cos(\nu s) - \nu \sin(\nu s))\,\mathrm{e}^{-\epsilon s}}{\nu^2+\epsilon^2} \times \hat{\mathcal{E}}_\nu \left\langle \vec{w}, \Xi_{\mathrm{d},l}^{(\omega)} \mathbf{V}_l^{(\omega)}(s)\,\vec{w} \right\rangle .$$

Proof: Fix $l,\beta\in\mathbb{R}^+$, $\omega\in\Omega$ and $\lambda\in\mathbb{R}_0^+$. For any $\vec{w}\in\mathbb{R}^d$, define the complex–valued function

$$F_{\vec{w}}(z) := \int_{\mathbb{R}} \frac{1}{\nu - z} \left\langle \vec{w}, \mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\vec{w} \right\rangle , \qquad z\in\mathbb{C}^+ ,$$

where $\mathbb{C}^+$ is the set of complex numbers with strictly positive imaginary part. $F_{\mathrm{p},l}^{(\omega)}$ is the so–called Borel transform of the positive measure

$$\left\langle \vec{w}, \mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\vec{w} \right\rangle . \tag{128}$$

By (109), observe that

$$F_{\vec{w}}(z) = \frac{1}{4|\Lambda_l|}\sum_{k,q\in\{1,\ldots,d\}} w_k w_q \left(\mathbb{I}_{k,l},((\tilde{\mathcal{L}}-z)^{-1}+(-\tilde{\mathcal{L}}-z)^{-1})\mathbb{I}_{q,l}\right)_\sim + \frac{1}{4|\Lambda_l|}\sum_{k,q\in\{1,\ldots,d\}} w_k w_q \left(\mathbb{I}_{q,l},((\tilde{\mathcal{L}}-z)^{-1}+(-\tilde{\mathcal{L}}-z)^{-1})\mathbb{I}_{k,l}\right)_\sim$$

for any $z\in\mathbb{C}^+$ and $\vec{w}:=(w_1,\ldots,w_d)\in\mathbb{R}^d$. Using

$$(\pm\tilde{\mathcal{L}}-z)^{-1} = i\int_0^\infty \mathrm{e}^{izs}\mathrm{e}^{\mp is\tilde{\mathcal{L}}}\mathrm{d}s\ , \qquad z\in\mathbb{C}^+\ ,$$

as well as Theorem A.16 for $\mathcal{X}=\mathcal{U}$, $\tau=\tau^{(\omega,\lambda)}$ and $\varrho=\varrho^{(\beta,\omega,\lambda)}$, we obtain

$$F_{\vec{w}}(z) = \frac{i}{|\Lambda_l|}\sum_{k,q\in\{1,\ldots,d\}} w_k w_q \int_0^\infty \mathrm{e}^{izs}(\mathbb{I}_{k,l},\tau_t^{(\omega,\lambda)}(\mathbb{I}_{q,l}))_\sim \mathrm{d}s$$

for every $z\in\mathbb{C}^+$ and $\vec{w}:=(w_1,\ldots,w_d)\in\mathbb{R}^d$. Using (33) and (103), we now integrate by parts the r.h.s of the above equation to get

$$F_{\vec{w}}(z) = -\frac{1}{|\Lambda_l|}\sum_{k,q\in\{1,\ldots,d\}} w_k w_q z^{-1}\int_0^\infty \mathrm{e}^{izs}\varrho^{(\beta,\omega,\lambda)}\left(i[\mathbb{I}_{k,l},\tau_s^{(\omega,\lambda)}(\mathbb{I}_{q,l})]\right)\mathrm{d}s - \frac{1}{|\Lambda_l|}\sum_{k,q\in\{1,\ldots,d\}} w_k w_q z^{-1}(\mathbb{I}_{k,l},\mathbb{I}_{q,l})_\sim \qquad (129)$$

for any $z\in\mathbb{C}^+$ and $\vec{w}:=(w_1,\ldots,w_d)\in\mathbb{R}^d$. The function $\mathrm{Im}\,F_{\vec{w}}$ is the Poisson transform of the positive measure (128). Hence, we invoke [Jak, Theorem 3.7] to conclude that, for any real–valued continuous compactly supported function $\hat{\mathcal{E}}:\mathbb{R}\to\mathbb{R}$,

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}}\hat{\mathcal{E}}_\nu\,\mathrm{Im}\,F_{\vec{w}}(\nu+i\epsilon)\,\mathrm{d}\nu = \int_{\mathbb{R}}\hat{\mathcal{E}}_\nu\left\langle \vec{w},\mu_{\mathrm{p},l}^{(\omega)}(\mathrm{d}\nu)\vec{w}\right\rangle\ .$$

In particular, by (129) and under the condition that $\hat{\mathcal{E}}_0=0$, we arrive at the assertion. ■

To conclude, we show the uniformity of the upper bounds of Theorem 5.5 w.r.t. to the parameters $l,\beta\in\mathbb{R}^+$, $\omega\in\Omega$ and $\lambda\in\mathbb{R}_0^+$. These upper bounds all depend on the observable $|\Lambda_l|^{-\frac{1}{2}}\,\mathbb{I}_{k,l}$, which is a *current fluctuation*, by (24).

With this aim we define the linear subspace

$$\mathcal{I} := \mathrm{lin}\left\{\mathrm{Im}(a^*(\psi_1)\, a(\psi_2)) : \psi_1, \psi_2 \in \ell^1(\mathfrak{L}) \subset \ell^2(\mathfrak{L})\right\} \subset \mathcal{U}\ , \tag{130}$$

which is the linear hull (lin) of short range bond currents. It is an invariant subspace of the one–parameter group $\tau^{(\omega,\lambda)} = \{\tau_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ for any $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$. Indeed, the unitary group $\{(\mathrm{U}_t^{(\omega,\lambda)})^*\}_{t\in\mathbb{R}}$ (see (6) and (7)) defines a strongly continuous group on $(\ell^1(\mathfrak{L}) \subset \ell^2(\mathfrak{L}), \|\cdot\|_1)$.

Let the positive sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{I},l}^{(\omega)} \equiv \langle\cdot,\cdot\rangle_{\mathcal{I},l}^{(\beta,\omega,\lambda)}$ in $\mathcal{I}$ be defined by

$$\langle I, I'\rangle_{\mathcal{I},l}^{(\omega)} := \varrho^{(\beta,\omega,\lambda)}\left(\mathbb{F}^{(l)}(I)^*\, \mathbb{F}^{(l)}(I')\right)\ , \qquad I, I' \in \mathcal{I}\ , \tag{131}$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$. Here, $\mathbb{F}^{(l)}$ is the fluctuation observable defined by

$$\mathbb{F}^{(l)}(I) = \frac{1}{|\Lambda_l|^{1/2}} \sum_{x\in\Lambda_l} \left\{\chi_x(I) - \varrho^{(\beta,\omega,\lambda)}(I)\,\mathbf{1}\right\}\ , \qquad I \in \mathcal{I}\ , \tag{132}$$

for each $l \in \mathbb{R}^+$, where χ_x, $x \in \mathfrak{L}$, are the (space) translation automorphisms. Compare (24) with (132). For instance, the first upper bound of Theorem 5.5 can be rewritten as

$$\int_{\mathbb{R}} \|\mu_{\mathrm{p},l}^{(\omega)}\|_{\mathrm{op}}(\mathrm{d}\nu) \leq \sum_{k=1}^{d} \langle I_{(e_k,0)}, I_{(e_k,0)}\rangle_{\mathcal{I},l}^{(\omega)}\ .$$

Therefore, we show that the fermion system has uniformly bounded fluctuations, i.e., the quantity $\langle I, I'\rangle_{\mathcal{I},l}^{(\omega)}$, $I, I' \in \mathcal{I}$, is uniformly bounded w.r.t. $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$:

Lemma 5.10 (Uniform boundedness of $\langle\cdot,\cdot\rangle_{\mathcal{I},l}^{(\omega)}$)
There is a constant $D \in \mathbb{R}^+$ such that, for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and all $\psi_1, \psi_2, \psi_1', \psi_2' \in \ell^1(\mathfrak{L})$,

$$\left|\langle \mathrm{Im}(a^*(\psi_1)\, a(\psi_2)), \mathrm{Im}(a^*(\psi_1')\, a(\psi_2'))\rangle_{\mathcal{I},l}^{(\omega)}\right| \leq D\, \|\psi_1\|_1\, \|\psi_2\|_1\, \|\psi_1'\|_1\, \|\psi_2'\|_1\ .$$

Proof: Let $\psi_1, \psi_2, \psi_1', \psi_2' \in \ell^1(\mathfrak{L}) \subset \ell^2(\mathfrak{L})$ and without loss of generality assume that the functions $\psi_1, \psi_2, \psi'_1, \psi'_2$ are real–valued. Then, by definition,

$$\begin{aligned}
&\langle \mathrm{Im}(a^*(\psi_1)\, a(\psi_2)), \mathrm{Im}(a^*(\psi'_1)\, a(\psi'_2))\rangle_{\mathcal{I},l}^{(\omega)} \\
= &\sum_{\mathbf{x}:=(x^{(1)},x^{(2)}),\mathbf{y}:=(y^{(1)},y^{(2)})\in\mathfrak{L}^2} \psi_1(y^{(1)})\psi_2(y^{(2)})\psi'_1(x^{(1)})\psi'_2(x^{(2)}) \\
&\times \left[\frac{1}{4|\Lambda_l|}\sum_{z_1,z_2\in\Lambda_l} \varrho^{(\beta,\omega,\lambda)}\left(I^{\mathrm{fl}}_{\mathbf{y}+(z_2,z_2)} I^{\mathrm{fl}}_{\mathbf{x}+(z_1,z_1)}\right)\right] ,
\end{aligned}$$

where

$$I^{\mathrm{fl}}_{\mathbf{x}} := I_{\mathbf{x}} - \varrho^{(\beta,\omega,\lambda)}(I_{\mathbf{x}})\,\mathbf{1} , \qquad \mathbf{x}\in\mathfrak{L}^2 .$$

Recall that $I_{\mathbf{x}}$ is the paramagnetic current observable defined by (19). Hence, it suffices to prove the existence of a finite constant $D\in\mathbb{R}^+$ such that, for any $l,\beta\in\mathbb{R}^+$, $\omega\in\Omega$, $\lambda\in\mathbb{R}_0^+$ and all $\mathbf{x},\mathbf{y}\in\mathfrak{L}^2$,

$$\left|\frac{1}{4|\Lambda_l|}\sum_{z_1,z_2\in\Lambda_l} \varrho^{(\beta,\omega,\lambda)}\left(I^{\mathrm{fl}}_{\mathbf{y}+(z_2,z_2)} I^{\mathrm{fl}}_{\mathbf{x}+(z_1,z_1)}\right)\right| \leq D . \tag{133}$$

This can be shown by using Lemma 5.3.

Indeed, we infer from (100) at $t=\alpha=0$ that, for any $l,\beta\in\mathbb{R}^+$, $\omega\in\Omega$, $\lambda\in\mathbb{R}_0^+$, $\mathbf{x},\mathbf{y}\in\mathfrak{L}^2$ and all $z_1,z_2\in\Lambda_l$,

$$\begin{aligned}
\varrho^{(\beta,\omega,\lambda)}\left(I^{\mathrm{fl}}_{\mathbf{y}+(z_2,z_2)} I^{\mathrm{fl}}_{\mathbf{x}+(z_1,z_1)}\right) &= \varrho^{(\beta,\omega,\lambda)}\left(I_{\mathbf{y}+(z_2,z_2)} I_{\mathbf{x}+(z_1,z_1)}\right) \\
&\quad -\varrho^{(\beta,\omega,\lambda)}\left(I_{\mathbf{y}+(z_2,z_2)}\right)\varrho^{(\beta,\omega,\lambda)}\left(I_{\mathbf{x}+(z_1,z_1)}\right) \\
&= \mathfrak{C}_0^{(\omega)}\left(\mathbf{x}+(z_1,z_1), \mathbf{y}+(z_2,z_2)\right) ,
\end{aligned} \tag{134}$$

where $\mathfrak{C}_{t+i\alpha}^{(\omega)}$ is the map from $\mathfrak{L}^4$ to $\mathbb{C}$ defined at $t\in\mathbb{R}$ and $\alpha\in[0,\beta]$ by (93). Now, take the canonical orthonormal basis $\{\mathbf{e}_{\mathbf{x}}\}_{\mathbf{x}\in\mathfrak{L}^2}$ of $\ell^2(\mathfrak{L})\otimes\ell^2(\mathfrak{L})$ defined by

$$\mathbf{e}_{\mathbf{x}} := \mathfrak{e}_{x^{(1)}}\otimes\mathfrak{e}_{x^{(2)}} , \qquad \mathbf{x} := (x^{(1)},x^{(2)})\in\mathfrak{L}^2 .$$

Recall that $\mathfrak{e}_x(y)\equiv\delta_{x,y}\in\ell^2(\mathfrak{L})$. Then, the coefficient $\mathfrak{C}_{t+i\alpha}^{(\omega)}$ can be seen as a kernel – w.r.t. the canonical basis $\{\mathbf{e}_{\mathbf{x}}\}_{\mathbf{x}\in\mathfrak{L}^2}$ – of an operator on $\ell^2(\mathfrak{L})\otimes\ell^2(\mathfrak{L})$, again denoted by $\mathfrak{C}_{t+i\alpha}^{(\omega)}$. Then, we observe from (134) that

$$\begin{aligned}
&\frac{1}{4|\Lambda_l|}\sum_{z_1,z_2\in\Lambda_l} \varrho^{(\beta,\omega,\lambda)}\left(I^{\mathrm{fl}}_{\mathbf{y}+(z_2,z_2)} I^{\mathrm{fl}}_{\mathbf{x}+(z_1,z_1)}\right) \\
&\quad = \frac{1}{4|\Lambda_l|}\sum_{z_1,z_2\in\Lambda_l}\left\langle \mathbf{e}_{\mathbf{x}+(z_1,z_1)}, \mathfrak{C}_0^{(\omega)}(\mathbf{e}_{\mathbf{y}+(z_2,z_2)})\right\rangle
\end{aligned} \tag{135}$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $\mathbf{x}, \mathbf{y} \in \mathfrak{L}^2$.

By Lemma 5.3, the operator $\mathfrak{C}_{t+i\alpha}^{(\omega)}$ always satisfies $\|\mathfrak{C}_{t+i\alpha}^{(\omega)}\|_{\mathrm{op}} \leq 4$ and hence,

$$\left| \frac{1}{4|\Lambda_l|} \sum_{z_1,z_2 \in \Lambda_l} \left\langle \mathbf{e}_{\mathbf{x}+(z_1,z_1)}, \mathfrak{C}_0^{(\omega)} \mathbf{e}_{\mathbf{y}+(z_2,z_2)} \right\rangle \right| \leq 1 \tag{136}$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $\mathbf{x}, \mathbf{y} \in \mathfrak{L}^2$. By (135), it follows that

$$\left| \frac{1}{4|\Lambda_l|} \sum_{z_1,z_2 \in \Lambda_l} \varrho^{(\beta,\omega,\lambda)} \left(I_{\mathbf{y}+(z_2,z_2)}^{\mathrm{fl}} I_{\mathbf{x}+(z_1,z_1)}^{\mathrm{fl}} \right) \right| \leq 1$$

for any $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and all $\mathbf{x}, \mathbf{y} \in \mathfrak{L}^2$. ■

5.2 Tree–Decay Bounds and Uniformity of Responses

5.2.1 Uniformity of Energy Increment Responses

For the reader's convenience we start by reminding a few definitions and some standard mathematical facts used in our proofs. First of all, we recall that in [BPK1, Section 5.2] we give an explicit expression of the automorphisms $\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}$ of $\mathcal{U}$ in terms of series involving *multi–commutators*, see [BPK1, Eqs. (3.14)-(3.15)]. Indeed, in [BPK1, Eq. (5.15)] we represent the automorphisms $\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}$ as the following Dyson–Phillips series

$$\begin{aligned} &\tau_{t,s}^{(\omega,\lambda,\mathbf{A})}(B) - \tau_{t-s}^{(\omega,\lambda)}(B) \\ = &\sum_{k \in \mathbb{N}} i^k \int_s^t \mathrm{d}s_1 \cdots \int_s^{s_{k-1}} \mathrm{d}s_k [W_{s_k-s,s_k}^{\mathbf{A}}, \ldots, W_{s_1-s,s_1}^{\mathbf{A}}, \tau_{t-s}^{(\omega,\lambda)}(B)]^{(k+1)} \end{aligned} \tag{137}$$

for any $B \in \mathcal{U}$ and $t \geq s$. Here, for any $t, s \in \mathbb{R}$,

$$W_{t,s}^{\mathbf{A}} := \tau_t^{(\omega,\lambda)}(W_s^{\mathbf{A}}) \in \mathcal{U} \tag{138}$$

is the time–evolution of the electromagnetic potential energy observable $W_s^{\mathbf{A}}$ defined by (57), that is,

$$\begin{aligned} W_s^{\mathbf{A}} := &\sum_{x,y \in \mathfrak{L}} \left[\exp\left(-i \int_0^1 [\mathbf{A}(s, \alpha y + (1-\alpha)x)] (y-x) \mathrm{d}\alpha \right) - 1 \right] \\ &\times \langle \mathfrak{e}_x, \Delta_{\mathrm{d}} \mathfrak{e}_y \rangle a_x^* a_y , \end{aligned} \tag{139}$$

for any $\mathbf{A} \in \mathbf{C}_0^\infty$ and $s \in \mathbb{R}$.

The expression (137) is useful because we can apply *tree–decay bounds* on multi–commutators. These bounds, derived in [BPK1, Section 4], are useful to analyze multi–commutators of products of annihilation and creation operators. Using them, we show for instance in [BPK1, Lemma 5.10] that, for any $\mathbf{A} \in \mathbf{C}_0^\infty$, there is $\eta_0 \in \mathbb{R}^+$ such that, for $l, \varepsilon \in \mathbb{R}^+$, there is a ball

$$B(0,R) := \{x \in \mathfrak{L} : |x| \leq R\}$$

of radius $R \in \mathbb{R}^+$ centered at 0 such that, for all $|\eta| \in (0, \eta_0]$, $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, and $t_0 \leq s_1, \ldots, s_k \leq t$,

$$\sum_{x \in \Lambda_L \backslash B_R} \sum_{z \in \mathfrak{L}, |z| \leq 1} \sum_{k \in \mathbb{N}} \frac{(t-t_0)^k}{k!} \left| \varrho^{(\beta,\omega,\lambda)} \left([W_{s_k - t_0, s_k}^{\eta \mathbf{A}_l}, \ldots, W_{s_1 - t_0, s_1}^{\eta \mathbf{A}_l}, \tau_{t-t_0}^{(\omega,\lambda)}(a_x^* a_{x+z})]^{(k+1)} \right) \right| \leq \varepsilon \, .$$

This property together with (58) and (137) implies that, for all $|\eta| \in (0, \eta_0]$, $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$,

$$\begin{aligned} \mathbf{S}^{(\omega, \eta \mathbf{A}_l)}(t) \;&=\; \sum_{k \in \mathbb{N}} \sum_{x, z \in \mathfrak{L}, |z| \leq 1} \langle \mathfrak{e}_x, (\Delta_{\mathrm{d}} + \lambda V_\omega) \, \mathfrak{e}_{x+z} \rangle i^k \int_{t_0}^{t} \mathrm{d}s_1 \cdots \int_{t_0}^{s_{k-1}} \mathrm{d}s_k \\ &\quad \varrho^{(\beta,\omega,\lambda)} \left([W_{s_k - t_0, s_k}^{\eta \mathbf{A}_l}, \ldots, W_{s_1 - t_0, s_1}^{\eta \mathbf{A}_l}, \tau_{t-t_0}^{(\omega,\lambda)}(a_x^* a_{x+z})]^{(k+1)} \right) \, . \end{aligned} \tag{140}$$

See [BPK1, Section 5.5] for more details.

These assertions are important to get uniform bounds as explained in Theorems 3.3 and 4.1. Indeed, it is relatively straightforward to get the asymptotics of the elements $W_t^{\eta \mathbf{A}_l}$ and $\partial_t W_t^{\mathbf{A}}$ when $(\eta, l^{-1}) \to (0,0)$ by using the integrated electric field

$$\mathbf{E}_t^{\mathbf{A}}(\mathbf{x}) := \int_0^1 \left[E_{\mathbf{A}}(t, \alpha x^{(2)} + (1-\alpha) x^{(1)}) \right] (x^{(2)} - x^{(1)}) \mathrm{d}\alpha \tag{141}$$

between $x^{(2)} \in \mathfrak{L}$ and $x^{(1)} \in \mathfrak{L}$ at time $t \in \mathbb{R}$ (cf. (12)) and the subset

$$\mathfrak{K} := \left\{ \mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2 \ : \ |x^{(1)} - x^{(2)}| = 1 \right\} \tag{142}$$

of bonds of nearest neighbors (cf. (23)).

Lemma 5.11 (Asymptotics of the potential energy observable)
For any $\eta, l \in \mathbb{R}^{+}$, $\mathbf{A} \in \mathbf{C}_0^\infty$ *and* $t \geq t_0$, *there are complex numbers*

$$\left\{ \tilde{D}_{x,y}^{\eta \mathbf{A}_l}(t) \right\}_{x,y\in\mathfrak{L}} \subset \mathbb{C}$$

and a (η, t)*–independent subset* $\widetilde{\Lambda}_l \in \mathcal{P}_f(\mathfrak{L})$ *of diameter of order* $\mathcal{O}(l)$ *such that*

$$\begin{aligned} W_t^{\eta \mathbf{A}_l} &= \frac{1}{2}\sum_{\mathbf{x}\in\mathfrak{K}} \left\{ \eta \left(\int_{t_0}^{t} \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x}) \mathrm{d}s \right) I_{\mathbf{x}} + \frac{\eta^2}{2} \left(\int_{t_0}^{t} \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x}) \mathrm{d}s \right)^2 P_{\mathbf{x}} \right\} \\ &\quad + \eta^3 \sum_{x\in\widetilde{\Lambda}_l} \sum_{z\in\mathfrak{L},|z|\leq 1} \tilde{D}_{x,x+z}^{\eta \mathbf{A}_l}(t) a_x^* a_{x+z} \end{aligned}$$

with $\tilde{D}_{x,x+z}^{\eta \mathbf{A}_l}(t)$ *and* $\partial_t \tilde{D}_{x,x+z}^{\eta \mathbf{A}_l}(t)$ *being uniformly bounded for all* η *in compact sets, all* $x, z \in \mathfrak{L}$ *such that* $|z| \leq 1$, *and all* $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ *and* $l \in \mathbb{R}^+$.

Proof: Note that (141) yields

$$\mathbf{E}_t^{\mathbf{A}}(\mathbf{x}) \equiv \mathbf{E}_t^{\mathbf{A}}(x^{(1)}, x^{(2)}) = -\mathbf{E}_t^{\mathbf{A}}(x^{(2)}, x^{(1)}) , \quad \mathbf{x} := (x^{(1)}, x^{(2)}) \in \mathfrak{L}^2 , \ t \in \mathbb{R} .$$

Therefore, the statement is a straightforward consequence of Equations (5), (139) and (141) together with [BPK1, Eqs. (5.37)–(5.39), (5.41)]. ∎

By combining this lemma with (140) one can obtain Theorem 4.1 (**S**). However, by using (64), it is easier to start with the paramagnetic and diamagnetic energies $\mathfrak{I}_{\mathrm{p}}^{(\omega,\mathbf{A})}$ and $\mathfrak{I}_{\mathrm{d}}^{(\omega,\mathbf{A})}$ respectively defined by (62) and (63):

Theorem 5.12 (Microscopic paramagnetic and diamagnetic energies)
For any $\mathbf{A} \in \mathbf{C}_0^\infty$, *there is* $\eta_0 \in \mathbb{R}^+$ *such that, for all* $|\eta| \in (0, \eta_0]$, $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ *and* $t \geq t_0$, *one has:*
(p) *Paramagnetic energy increment:*

$$\mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}(t) = \frac{\eta^2}{4} \int_{t_0}^{t} \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2 \sum_{\mathbf{x},\mathbf{y}\in\mathfrak{K}} \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x},\mathbf{y},s_1 - s_2) \, \mathbf{E}_{s_2}^{\mathbf{A}_l}(\mathbf{y}) \mathbf{E}_{s_1}^{\mathbf{A}_l}(\mathbf{x}) + \mathcal{O}(\eta^3 l^d) .$$

(d) *Diamagnetic energy:*

$$\begin{aligned} \mathfrak{I}_{\mathrm{d}}^{(\omega,\eta\mathbf{A}_l)}(t) &= \frac{\eta}{2}\sum_{\mathbf{x}\in\mathfrak{K}} \varrho^{(\beta,\omega,\lambda)}(I_{\mathbf{x}}) \int_{t_0}^{t} \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x}) \mathrm{d}s \\ &\quad + \frac{\eta^2}{2} \int_{t_0}^{t} \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2 \sum_{\mathbf{x}\in\mathfrak{K}} \sigma_{\mathrm{d}}^{(\omega)}(\mathbf{x}) \, \mathbf{E}_{s_2}^{\mathbf{A}_l}(\mathbf{x}) \mathbf{E}_{s_1}^{\mathbf{A}_l}(\mathbf{x}) + \mathcal{O}(\eta^3 l^d) . \end{aligned}$$

The correction terms of order $\mathcal{O}(l^d\eta^3)$ in assertions (p) and (d) are uniformly bounded in $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$.

Proof: (p) Using $W_t^{\mathbf{A}} = 0$ for any $t \leq t_0$ and (9) we note that, for any $t \geq t_0$,

$$\varrho^{(\beta,\omega,\lambda)}(W_t^{\eta\mathbf{A}_l}) = \int_{t_0}^t \varrho^{(\beta,\omega,\lambda)}\left(\partial_s W_s^{\eta\mathbf{A}_l}\right) \mathrm{d}s = \int_{t_0}^t \varrho^{(\beta,\omega,\lambda)} \circ \tau_{s-t_0}^{(\omega,\lambda)}\left(\partial_s W_s^{\eta\mathbf{A}_l}\right) \mathrm{d}s \ .$$

For all $s \in \mathbb{R}$,

$$W_s^{\eta\mathbf{A}_l}, \partial_s W_s^{\eta\mathbf{A}_l} \in \mathcal{U}_{\widetilde{\Lambda}_l}$$

for some finite subset $\widetilde{\Lambda}_l \in \mathcal{P}_f(\mathfrak{L})$ of diameter of order $\mathcal{O}(l)$, see, e.g., [BPK1, Eqs. (5.41)]. As a consequence, by (62)–(64), the paramagnetic energy increment equals

$$\mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}(t) = \int_{t_0}^t \varrho^{(\beta,\omega,\lambda)} \circ \left(\tau_{s,t_0}^{(\omega,\lambda,\eta\mathbf{A}_l)} - \tau_{s-t_0}^{(\omega,\lambda)}\right)\left(\partial_s W_s^{\eta\mathbf{A}_l}\right) \mathrm{d}s \qquad (143)$$

for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{A} \in \mathbf{C}_0^\infty$ and $t \geq t_0$.

Similar to the proof of [BPK1, Lemma 5.10], one uses Dyson–Phillips expansions (137) and tree–decay bounds on multi–commutators [BPK1, Corollary 4.3] to infer from Lemma 5.11 and Equation (143) that, for any $\mathbf{A} \in \mathbf{C}_0^\infty$, there is $\eta_0 \in \mathbb{R}^+$ such that, for all $|\eta| \in (0, \eta_0]$, $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$,

$$\begin{aligned}\mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}(t) &= \int_{t_0}^t \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2\, \varrho^{(\beta,\omega,\lambda)}\left(i\left[\tau_{s_2-t_0}^{(\omega,\lambda)}\left(W_{s_2}^{\eta\mathbf{A}_l}\right), \tau_{s_1-t_0}^{(\omega,\lambda)}\left(\partial_{s_1} W_{s_1}^{\eta\mathbf{A}_l}\right)\right]\right) \\ &\quad + \mathcal{O}(\eta^3 l^d) \ . \end{aligned} \qquad (144)$$

This last correction term of order $\mathcal{O}(l^d\eta^3)$ is *uniformly bounded* in $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$.

Note that (8)–(9) combined with the group property of the family $\{\tau_t^{(\omega,\lambda)}\}_{t\in\mathbb{R}}$ imply that

$$\varrho^{(\beta,\omega,\lambda)}\left([\tau_{s_2-t_0}^{(\omega,\lambda)}(B_2), \tau_{s_1-t_0}^{(\omega,\lambda)}(B_1)]\right) = \varrho^{(\beta,\omega,\lambda)}\left([\tau_{s_2}^{(\omega,\lambda)}(B_2), \tau_{s_1}^{(\omega,\lambda)}(B_1)]\right)$$

for any $B_1, B_2 \in \mathcal{U}$ and all $s_1, s_1 \in \mathbb{R}$. Therefore, we insert this equality and the asymptotics given by Lemma 5.11 in Equation (144) to arrive at the equality

$$\begin{aligned}\mathfrak{I}_{\mathrm{p}}^{(\omega,\eta\mathbf{A}_l)}(t) &= \frac{\eta^2}{4} \sum_{\mathbf{x},\mathbf{y}\in\mathfrak{K}} \int_{t_0}^t \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2 \int_{t_0}^{s_2} \mathrm{d}s_3 \\ &\quad \times \mathbf{E}_{s_1}^{\mathbf{A}_l}(\mathbf{x}) \mathbf{E}_{s_3}^{\mathbf{A}_l}(\mathbf{y}) \varrho^{(\beta,\omega,\lambda)}\left(i[\tau_{s_2}^{(\omega,\lambda)}(I_{\mathbf{y}}), \tau_{s_1}^{(\omega,\lambda)}(I_{\mathbf{x}})]\right) \\ &\quad + \mathcal{O}(\eta^3 l^d) \ , \end{aligned} \qquad (145)$$

uniformly for $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$.

For any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{x}, \mathbf{y} \in \mathfrak{L}^2$ and $s_1, s_2 \in \mathbb{R}$, let

$$\zeta_{\mathbf{x},\mathbf{y}}^{(\omega)}(s_1, s_2) := \int_{s_1}^{s_2} \varrho^{(\beta,\omega,\lambda)} \left(i[\tau_{s_1}^{(\omega,\lambda)}(I_{\mathbf{y}}), \tau_s^{(\omega,\lambda)}(I_{\mathbf{x}})] \right) \mathrm{d}s \, . \tag{146}$$

Note that the function $\zeta_{\mathbf{x},\mathbf{y}}^{(\omega)}$ is a map from $\mathbb{R}^2$ to $\mathbb{R}$. By combining (146) with (8)–(9) and (29), we observe that

$$\zeta_{\mathbf{x},\mathbf{y}}^{(\omega)}(s_1, s_2) = \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, s_2 - s_1) = \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{y}, \mathbf{x}, s_1 - s_2) \tag{147}$$

for any $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $\mathbf{x}, \mathbf{y} \in \mathfrak{L}^2$ and $s_1, s_2 \in \mathbb{R}$, while

$$\partial_{s_2} \zeta_{\mathbf{y},\mathbf{x}}^{(\omega)}(s_1, s_2) = \varrho^{(\beta,\omega,\lambda)} \left(i[\tau_{s_1}^{(\omega,\lambda)}(I_{\mathbf{x}}), \tau_{s_2}^{(\omega,\lambda)}(I_{\mathbf{y}})] \right) \, . \tag{148}$$

As a consequence, the assertion follows from (145) and an integration by parts.
(d) is a direct consequence of (30), (63) and Lemma 5.11. ∎

It remains to study the entropic energy increment $\mathbf{S}^{(\omega,\eta\mathbf{A}_l)}$ and the electromagnetic energy $\mathbf{P}^{(\omega,\eta\mathbf{A}_l)}$ defined by (58) and (59), respectively. To this end, it suffices to study the *potential energy difference*

$$\mathbf{P}^{(\omega,\eta\mathbf{A}_l)}(t) - \mathfrak{I}_{\mathrm{d}}^{(\omega,\eta\mathbf{A}_l)}(t) = \rho_t^{(\beta,\omega,\lambda,\eta\mathbf{A}_l)}(W_t^{\eta\mathbf{A}_l}) - \varrho^{(\beta,\omega,\lambda)}(W_t^{\eta\mathbf{A}_l})$$

for all times $t \geq t_0$. This is done in the following lemma:

Lemma 5.13 (Potential energy difference)
For any $\mathbf{A} \in \mathbf{C}_0^\infty$, there is $\eta_0 \in \mathbb{R}^+$ such that, for all $|\eta| \in (0, \eta_0]$ and $l \in \mathbb{R}^+$,

$$\begin{aligned} & \mathbf{P}^{(\omega,\eta\mathbf{A}_l)}(t) - \mathfrak{I}_{\mathrm{d}}^{(\omega,\eta\mathbf{A}_l)}(t) \\ = \; & \frac{\eta^2}{4} \sum_{\mathbf{x},\mathbf{y} \in \mathfrak{K}} \left(\int_{t_0}^t \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x}) \mathrm{d}s \right) \left(\int_{t_0}^t \sigma_{\mathrm{p}}^{(\omega)}(\mathbf{x}, \mathbf{y}, t-s) \, \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{y}) \mathrm{d}s \right) \\ & + \mathcal{O}(\eta^3 l^d) \, , \end{aligned}$$

uniformly for $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$.

Proof: The proof is very similar to the one of Theorem 5.12. In particular, to get the asymptotics, it suffices to observe that, for any $\mathbf{A} \in \mathbf{C}_0^\infty$, there is $\eta_0 \in \mathbb{R}^+$ such that, for all $|\eta| \in (0, \eta_0]$, $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$,

$$\begin{aligned} & \rho_t^{(\beta,\omega,\lambda,\eta\mathbf{A}_l)}(W_t^{\eta\mathbf{A}_l}) - \varrho^{(\beta,\omega,\lambda)}(W_t^{\eta\mathbf{A}_l}) \\ = \; & \int_{t_0}^t \varrho^{(\beta,\omega,\lambda)} \left(i[\tau_s^{(\omega,\lambda)}(W_s^{\eta\mathbf{A}_l}), \tau_t^{(\omega,\lambda)}(W_t^{\eta\mathbf{A}_l})] \right) \mathrm{d}s + \mathcal{O}(\eta^3 l^d) \, , \end{aligned} \tag{149}$$

by (8)–(9), the Dyson–Phillips expansions (137), Lemma 5.11 and tree–decay bounds on multi–commutators [BPK1, Corollary 4.3]. Note that the correction term of order $\mathcal{O}(\eta^3 l^d)$ in (149) is again *uniformly bounded* in $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$.

Then, we use Lemma 5.11 in (149) to obtain

$$\begin{aligned} & \rho_t^{(\beta,\omega,\lambda,\eta\mathbf{A}_l)}(W_t^{\eta\mathbf{A}_l}) - \varrho^{(\beta,\omega,\lambda)}(W_t^{\eta\mathbf{A}_l}) \\ = \; & \frac{\eta^2}{4}\sum_{\mathbf{x},\mathbf{y}\in\mathfrak{K}} \left(\int_{t_0}^t \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x})\mathrm{d}s\right) \int_{t_0}^t \mathrm{d}s_1 \left(\int_{t_0}^{s_1} \mathbf{E}_{s_2}^{\mathbf{A}_l}(\mathbf{y})\mathrm{d}s_2\right) \\ & \times \varrho^{(\beta,\omega,\lambda)}\left(i[\tau_{s_1}^{(\omega,\lambda)}(I_{\mathbf{y}}), \tau_t^{(\omega,\lambda)}(I_{\mathbf{x}})]\right) + \mathcal{O}(\eta^3 l^d) \, , \end{aligned}$$

uniformly for $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$. We then obtain

$$\begin{aligned} & \rho_t^{(\beta,\omega,\lambda,\eta\mathbf{A}_l)}(W_t^{\eta\mathbf{A}_l}) - \varrho^{(\beta,\omega,\lambda)}(W_t^{\eta\mathbf{A}_l}) \\ = \; & \frac{\eta^2}{4}\sum_{\mathbf{x},\mathbf{y}\in\mathfrak{K}} \left(\int_{t_0}^t \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{x})\mathrm{d}s\right) \left(\int_{t_0}^t \zeta_{\mathbf{y},\mathbf{x}}^{(\omega)}(t,s)\, \mathbf{E}_s^{\mathbf{A}_l}(\mathbf{y})\mathrm{d}s\right) + \mathcal{O}(\eta^3 l^d) \, , \end{aligned} \tag{150}$$

by using (146), (148) and an integration by parts. We now invoke Equation (147) in (150) to arrive at the assertion. ■

Therefore, Theorem 4.1 (**Q**) and (**P**) are direct consequences of (58)–(59), (62)–(63), Theorem 5.12 and Lemma 5.13.

5.2.2 Uniformity of Current Linear Response

Following Section 3 we take $\vec{w} := (w_1, \ldots, w_d) \in \mathbb{R}^d$, $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$ and $\mathcal{E}_t := -\partial_t \mathcal{A}_t$ for any $t \in \mathbb{R}$, with $\mathcal{E}_t \vec{w}$ being the space–homogeneous electric field. Then, $\bar{\mathbf{A}} \in \mathbf{C}_0^\infty$ is defined to be the electromagnetic potential such that the value of the electric field equals $\mathcal{E}_t \vec{w}$ at time $t \in \mathbb{R}$ for all $x \in [-1,1]^d$ and $(0,0,\ldots,0)$ for $t \in \mathbb{R}$ and $x \notin [-1,1]^d$. This choice yields rescaled electromagnetic potentials $\eta\bar{\mathbf{A}}_l$ as defined by (17) for $l \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$. Recall that $\mathcal{A}(t,x) := 0$ for all $t \leq t_0$, where $t_0 \in \mathbb{R}$ is any fixed starting time. We also recall that $\{e_k\}_{k=1}^d$ is the canonical orthonormal basis of the Euclidian space $\mathbb{R}^d$.

In this case, the electromagnetic potential energy observable defined by (57) equals

$$W_t^{\eta\bar{\mathbf{A}}_l} = -\sum_{x\in\Lambda_l}\sum_{q\in\{1,\ldots,d\}} 2\,\mathrm{Re}\left[\left(\exp\left(i\eta w_q \int_{t_0}^t \mathcal{E}_s\,\mathrm{d}s\right) - 1\right) a_x^* a_{x+e_q}\right] \in \mathcal{U} \tag{151}$$

for any $l \in \mathbb{R}^+$, $\eta \in \mathbb{R}$, $\vec{w} := (w_1, \ldots, w_d) \in \mathbb{R}^d$, $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$ and $t \in \mathbb{R}$.

The full current density is the sum of the paramagnetic and diamagnetic currents $\mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}$ and $\mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}$ that are respectively defined by (42) and (43). These currents are directly related to the transport coefficients $\Xi_{\mathrm{p},l}^{(\omega)}$ and $\Xi_{\mathrm{d},l}^{(\omega)}$ (cf. (33)–(34)). We show this in two lemmata that yield Theorem 3.3:

Lemma 5.14 (Linear response of paramagnetic currents)
For any $\vec{w} := (w_1, \ldots, w_d) \in \mathbb{R}^d$ and $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$, there is $\eta_0 \in \mathbb{R}^+$ such that, for $|\eta| \in [0, \eta_0]$,

$$\mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) = \eta \int_{t_0}^{t} \left(\Xi_{\mathrm{p},l}^{(\omega)}(t-s)\,\vec{w}\right) \mathcal{E}_s \mathrm{d}s + \mathcal{O}\left(\eta^2\right) ,$$

uniformly for $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$.

Proof: The first assertion is proven by essentially the same arguments as in Section 5.2.1. Indeed, one uses the stationarity (9) of the $(\tau^{(\omega,\lambda)}, \beta)$–KMS state $\varrho^{(\beta,\omega,\lambda)}$, Dyson–Phillips expansions (137) for the non–autonomous dynamics, Lemma 5.11, and tree–decay bounds on multi–commutators [BPK1, Corollary 4.3] as in [BPK1, Lemma 5.10] in order to deduce from (42) the existence of $\eta_0 \in \mathbb{R}^+$ such that, for $|\eta| \in [0, \eta_0]$,

$$\left\{\mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t)\right\}_k = \frac{1}{|\Lambda_l|} \int_{t_0}^{t} \varrho^{(\beta,\omega,\lambda)} \left(i[\tau_s^{(\omega,\lambda)}(W_s^{\eta\bar{\mathbf{A}}_l}), \tau_t^{(\omega,\lambda)}(\mathbb{I}_{k,l})]\right) \mathrm{d}s + \mathcal{O}\left(\eta^2\right) ,$$

uniformly for all $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $k \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$. Then, for $|\eta| \in [0, \eta_0]$, we employ (151) and derive an assertion similar to Lemma 5.11 in order to get

$$\begin{aligned}&\left\{\mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t)\right\}_k \\ &\quad = \frac{\eta}{|\Lambda_l|} \sum_{q\in\{1,\ldots,d\}} \int_{t_0}^{t} \mathrm{d}s_1 \int_{t_0}^{s_1} \mathrm{d}s_2\, \mathcal{E}_{s_2} w_q\, \varrho^{(\beta,\omega,\lambda)} \left(i[\tau_{s_1}^{(\omega,\lambda)}(\mathbb{I}_{q,l}), \tau_t^{(\omega,\lambda)}(\mathbb{I}_{k,l})]\right) \\ &\qquad + \mathcal{O}\left(\eta^2\right) ,\end{aligned}$$

uniformly for all $l, \beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $k \in \{1, \ldots, d\}$ and $t \in \mathbb{R}$. It follows from an integration by parts that

$$\left\{\mathbb{J}_{\mathrm{p}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t)\right\}_k$$
$$= \frac{\eta}{|\Lambda_l|}\int_{t_0}^{t}\sum_{q\in\{1,\ldots,d\}}\left(\int_t^{s_1}\varrho^{(\beta,\omega,\lambda)}\left(i[\mathbb{I}_{k,l},\tau_{s_2-t}^{(\omega,\lambda)}(\mathbb{I}_{q,l})]\right)\mathrm{d}s_2\right)w_q\mathcal{E}_{s_1}\,\mathrm{d}s_1$$
$$+\mathcal{O}\left(\eta^2\right)\ ,$$

which, combined with (33) and (104), yields the assertion. ∎

Lemma 5.15 (Linear response of diamagnetic currents)
For any $\vec{w} := (w_1,\ldots,w_d) \in \mathbb{R}^d$ and $\mathcal{A} \in C_0^\infty(\mathbb{R};\mathbb{R})$, there is $\eta_0 \in \mathbb{R}^+$ such that, for $|\eta| \in [0,\eta_0]$,

$$\mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t) = \eta\left(\Xi_{\mathrm{d},l}^{(\omega)}\vec{w}\right)\int_{t_0}^{t}\mathcal{E}_s\mathrm{d}s + \mathcal{O}\left(\eta^2\right)\ ,$$

uniformly for $l,\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$ and $t \geq t_0$.

Proof: By (9), for any $k \in \{1,\ldots,d\}$, note that

$$\begin{aligned}\left\{\mathbb{J}_{\mathrm{d}}^{(\omega,\eta\bar{\mathbf{A}}_l)}(t)\right\}_k &= \frac{1}{|\Lambda_l|}\varrho^{(\beta,\omega,\lambda)}\left((\tau_{t,t_0}^{(\omega,\lambda,\eta\bar{\mathbf{A}}_l)}-\tau_{t-t_0}^{(\omega,\lambda)})(\mathbf{I}_{k,l}^{\eta\mathbf{A}_l})\right)\\ &\quad+\frac{1}{|\Lambda_l|}\varrho^{(\beta,\omega,\lambda)}(\mathbf{I}_{k,l}^{\eta\mathbf{A}_l})\ ,\end{aligned} \tag{152}$$

while

$$\mathbf{I}_{k,l}^{\eta\mathbf{A}_l} = \eta w_k\left(\int_{t_0}^{t}\mathcal{E}_s\mathrm{d}s\right)\sum_{x\in\Lambda_l}\left(a_{x+e_k}^*a_x + a_x^*a_{x+e_k}\right) + \mathcal{O}(\eta^2 l^d)\ , \tag{153}$$

uniformly for all $\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $k \in \{1,\ldots,d\}$ and $t \in \mathbb{R}$. Therefore, using again Dyson–Phillips expansions (137) for the non–autonomous dynamics, Lemma 5.11, and tree–decay bounds on multi–commutators [BPK1, Corollary 4.3] one deduces the existence of $\eta_0 \in \mathbb{R}^+$ such that, for $|\eta| \in [0,\eta_0]$, the first term in the right hand side of (152) is of order $\mathcal{O}(\eta^2)$, uniformly for $l,\beta \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda \in \mathbb{R}_0^+$, $k \in \{1,\ldots,d\}$ and $t \geq t_0$. Then the assertion follows by combining this property with (34) and (152)–(153). ∎

A Duhamel Two–Point Functions

A.1 Duhamel Two–Point Function on the CAR Algebra

The Duhamel two–point function $(\cdot,\cdot)_{\sim}^{(\omega)}$ is defined by (88), that is,

$$(B_1, B_2)_{\sim}^{(\omega)} \equiv (B_1, B_2)_{\sim}^{(\beta,\omega,\lambda)} := \int_0^\beta \varrho^{(\beta,\omega,\lambda)} \left(B_1^* \tau_{i\alpha}^{(\omega,\lambda)}(B_2) \right) \mathrm{d}\alpha \qquad (154)$$

for any $B_1, B_2 \in \mathcal{U}$. Its name comes from the clear relation to Duhamel's formula, see [Si, Section IV.4] for more details. This sesquilinear form appears in different contexts. For instance, it has been used by Bogoliubov [B1] for finite volume quantum systems in quantum statistical mechanics. It is an useful tool in the first mathematical justification – by Ginibre [G] in 1968 – of the Bogoliubov approximation for the Bose gas. This sesquilinear form is also used in the context of linear response theory, see for instance [BR2, Discussion after Lemma 5.3.16 and Section 5.4]. In fact, it is also named in the literature Bogoliubov or Kubo–Mori *scalar product* as well as the canonical correlation. A detailed analysis of this sesquilinear form for KMS states has been performed by Naudts, Verbeure and Weder in the paper [NVW]. Their aim was to extend to infinite systems some results of linear response theory initiated by Kubo [K] and Mori [M].

Note that our definition of $(\cdot,\cdot)_{\sim}$ is slightly different from the usual one because of the missing normalization factor β^{-1} in front of the integral in (154). Discussions on Duhamel two–point functions and examples of applications can also be found in [MW, H, FB, NV, R, DLS].

A first way to study this sesquilinear form is to use finite volume systems. Indeed, using the Phragmén–Lindelöf theorem [BR2, Proposition 5.3.5] and [BPK1, Theorem A.3] one checks that the formal expression

$$\varrho^{(\beta,\omega,\lambda)} \left(B^* \tau_{i\alpha}^{(\omega,\lambda)}(B) \right) = \varrho^{(\beta,\omega,\lambda)} \left((\tau_{i\alpha/2}^{(\omega,\lambda)}(B))^* \tau_{i\alpha/2}^{(\omega,\lambda)}(B) \right) \geq 0$$

is correct for any $B \in \mathcal{U}$ and all $\alpha \in [0, \beta]$. So $(B_1, B_2) \mapsto (B_1, B_2)_{\sim}$ is a positive semi–definite sesquilinear form on $\mathcal{U}$. It is however important for the study of the conductivity measure to know that this form defines a scalar product. To this end, we invoke the modular theory to have access to functional calculus as it is done in the paper [NVW].

A.2 Duhamel Two–Point Functions on von Neumann Algebras

We consider in all the following subsections an arbitrary strongly continuous one–parameter group $\tau := \{\tau_t\}_{t\in\mathbb{R}}$ of automorphisms of a C^*–algebra $\mathcal{X}$ as well as an arbitrary (τ, β)–KMS state $\varrho \in \mathcal{X}^*$ for some $\beta > 0$. Similar to (154), the Duhamel two–point function $(\cdot, \cdot)_\sim$ on the C^*–algebra $\mathcal{X}$ is defined by

$$(B_1, B_2)_\sim := \int_0^\beta \varrho\left(B_1^* \tau_{i\alpha}(B_2)\right) \mathrm{d}\alpha \ , \qquad B_1, B_2 \in \mathcal{X} \ . \tag{155}$$

We have in mind the example $\mathcal{X} = \mathcal{U}$, $\tau = \tau^{(\omega,\lambda)}$ and $\varrho = \varrho^{(\beta,\omega,\lambda)}$ for $\beta \in \mathbb{R}^+$, $\omega \in \Omega$ and $\lambda \in \mathbb{R}_0^+$, of course.

Any GNS representation of ϱ is denoted by $(\mathcal{H}, \pi, \Psi)$. There is a unique normal state of the von Neumann algebra $\mathfrak{M} := \pi(\mathcal{X})''$, also denoted by $\varrho \in \mathfrak{M}^*$ to simplify notation, with $\rho = \rho \circ \pi$ on $\mathcal{X}$. By [BR2, Corollary 5.3.4], there is a unique σ–weakly continuous $*$–automorphism group on $\mathfrak{M}$, which is again denoted by $\tau = \{\tau_t\}_{t\in\mathbb{R}}$, such that $\tau_t \circ \pi = \pi \circ \tau_t$, $t \in \mathbb{R}$, on $\mathcal{X}$. Moreover, the normal state $\varrho \in \mathfrak{M}^*$ is a (τ, β)–KMS state on $\mathfrak{M}$ and it thus satisfies the KMS (or modular) condition, that is, for any $b_1, b_2 \in \mathfrak{M}$, the map

$$t \mapsto \mathfrak{m}_{b_1,b_2}(t) := \varrho(b_1\tau_t(b_2)) = \langle \Psi, b_1\tau_t(b_2)\Psi\rangle_{\mathcal{H}}$$

from $\mathbb{R}$ to $\mathbb{C}$ extends uniquely to a continuous map $\mathfrak{m}_{b_1,b_2}$ on $\mathbb{R} \times [0, \beta] \subset \mathbb{C}$ which is holomorphic on $\mathbb{R} \times (0, \beta)$ whereas

$$\mathfrak{m}_{b_1,b_2}(i\beta) = \varrho(b_2b_1) \ , \qquad b_1, b_2 \in \mathfrak{M} \ .$$

Here, $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the scalar product of the Hilbert space $\mathcal{H}$. See, e.g., [BR2, Proposition 5.3.7].

Because ϱ is invariant with respect to τ, the $*$–automorphism group τ has a unique representation by conjugation with unitaries $\{U_t\}_{t\in\mathbb{R}} \subset \mathfrak{M}$, i.e.,

$$\tau_t(b) = U_t b U_t^* \ , \qquad t \in \mathbb{R} \ , \ b \in \mathfrak{M} \ ,$$

such that $U_t\Psi = \Psi$. As $t \mapsto \tau_t$ is σ–weakly continuous, the map $t \mapsto U_t$ is strongly continuous. Therefore, the unitary group $\{U_t\}_{t\in\mathbb{R}}$ has an anti–self–adjoint operator $i\mathcal{L}$ as generator, i.e., $U_t = \mathrm{e}^{it\mathcal{L}}$. In particular, $\Psi \in \mathrm{Dom}(\mathcal{L})$ and $\mathcal{L}$ annihilates Ψ, i.e., $\mathcal{L}\Psi = 0$. The operator $\mathcal{L}$ is known in the literature as the *standard Liouvillean* of τ associated with ϱ. The spectral theorem applied to the self–adjoint operator $\mathcal{L}$ ensures the existence of a projection–valued measure E on the real line $\mathbb{R}$ such that

$$\mathcal{L} = \int_{\mathbb{R}} \nu \, \mathrm{d}E(\nu) \ .$$

We now use the (Tomita–Takesaki) modular objects Δ, J of the pair $(\mathfrak{M}, \Psi)$. In particular,

$$J\mathbf{\Delta}^{1/2} (b\Psi) = b^* \Psi \ , \qquad b \in \mathfrak{M} \ . \tag{156}$$

By [P, Proposition 5.11], the modular operator $\mathbf{\Delta}$ is equal to

$$\mathbf{\Delta} = \exp\left(-\beta \mathcal{L}\right) = \int_{\mathbb{R}} \mathrm{e}^{-\beta\nu} \mathrm{d}E(\nu) \tag{157}$$

and $U_t = \mathbf{\Delta}^{-it\beta^{-1}}$.

Now, let the (unbounded) positive operator $\mathfrak{T}$ acting on $\mathcal{H}$ be defined by

$$\mathfrak{T} := \beta^{1/2} \int_{\mathbb{R}} \left(\frac{1 - \mathrm{e}^{-\beta\nu}}{\beta\nu} \right)^{1/2} \mathrm{d}E(\nu) \ . \tag{158}$$

Here,

$$\frac{1 - \mathrm{e}^{-\beta \cdot 0}}{\beta \cdot 0} := 1 \ .$$

The Duhamel two–point function $(\cdot, \cdot)_\sim$ is directly related to this operator:

Theorem A.1 (Duhamel two–point function in the GNS representation)
For any $B_1, B_2 \in \mathcal{X}$,

$$(B_1, B_2)_\sim = \langle \mathfrak{T}\pi(B_1)\Psi, \mathfrak{T}\pi(B_2)\Psi \rangle_{\mathcal{H}} \ .$$

In particular, $(B_1, B_1)_\sim \geq 0$.

Proof: The proof can be found in [NVW, Theorem II.4]. Since it is short, we give it here for completeness. Note first that, for any $b_1, b_2 \in \mathfrak{M}$,

$$\begin{aligned} \left\langle \Psi, b_1 \mathbf{\Delta}^{1/2} b_2 \Psi \right\rangle_{\mathcal{H}} &= \left\langle \mathbf{\Delta}^{1/2} b_1^* \Psi, b_2 \Psi \right\rangle_{\mathcal{H}} = \langle J b_2 \Psi, b_1 \Psi \rangle_{\mathcal{H}} \\ &= \left\langle \mathbf{\Delta}^{1/2} J \mathbf{\Delta}^{1/2} b_2 \Psi, b_1 \Psi \right\rangle_{\mathcal{H}} = \left\langle \Psi, b_2 \mathbf{\Delta}^{1/2} b_1 \Psi \right\rangle_{\mathcal{H}} \ , \end{aligned}$$

where we have used $\mathbf{\Delta} = \mathbf{\Delta}^*$, the anti–unitarity of J, $J^2 = \mathbf{1}$, and $J\mathbf{\Delta}^{1/2}J = \mathbf{\Delta}^{-1/2}$. Using this fact and properties of the map $\mathfrak{m}_{b_1,b_2}$ from $\mathbb{R} \times [0, \beta] \subset \mathbb{C}$

to $\mathbb{C}$ together with the Phragmén–Lindelöf theorem [BR2, Proposition 5.3.5] one shows that, for any $b_1, b_2 \in \mathfrak{M}$,

$$\mathfrak{m}_{b_1,b_2}(i\beta\alpha) = \begin{cases} \langle \Psi, b_1 \mathbf{\Delta}^{\alpha} b_2 \Psi \rangle_{\mathcal{H}} & , \qquad \alpha \in [0,1/2] \ , \\ \langle \Psi, b_2 \mathbf{\Delta}^{1-\alpha} b_1 \Psi \rangle_{\mathcal{H}} & , \qquad \alpha \in [1/2,1] \ . \end{cases}$$

By (155) and (156), it follows that

$$\begin{aligned} (B_1, B_2)_{\sim} &= \beta \int_0^{1/2} \langle \pi(B_1)\Psi, \mathbf{\Delta}^{\alpha} \pi(B_2)\Psi \rangle_{\mathcal{H}} \, \mathrm{d}\alpha \\ &\quad + \beta \int_0^{1/2} \left\langle J\mathbf{\Delta}^{1/2} \pi(B_2)\Psi, \mathbf{\Delta}^{\alpha} J \mathbf{\Delta}^{1/2} \pi(B_1)\Psi \right\rangle_{\mathcal{H}} \mathrm{d}\alpha \ . \end{aligned} \tag{159}$$

Because $J^2 = \mathbf{1}$, $J\mathbf{\Delta}^{\alpha}J = \mathbf{\Delta}^{-\alpha}$ and J is anti–unitary, note that

$$\begin{aligned} & \left\langle J\mathbf{\Delta}^{1/2}\pi(B_2)\Psi, \mathbf{\Delta}^{\alpha} J\mathbf{\Delta}^{1/2}\pi(B_1)\Psi \right\rangle_{\mathcal{H}} \\ = & \left\langle J\mathbf{\Delta}^{\alpha} J\mathbf{\Delta}^{1/2}\pi(B_1)\Psi, \mathbf{\Delta}^{1/2}\pi(B_2)\Psi \right\rangle_{\mathcal{H}} \\ = & \left\langle \mathbf{\Delta}^{-\alpha}\mathbf{\Delta}^{1/2}\pi(B_1)\Psi, \mathbf{\Delta}^{1/2}\pi(B_2)\Psi \right\rangle_{\mathcal{H}} \end{aligned}$$

for all $\alpha \in [0,1/2]$. Therefore, we deduce from (158) and (159) that

$$(B_1, B_2)_{\sim} = \beta \left\langle \pi(B_1)\Psi, \frac{\mathbf{\Delta} - \mathbf{1}}{\ln \mathbf{\Delta}} \pi(B_2)\Psi \right\rangle_{\mathcal{H}} = \langle \mathfrak{T}\pi(B_1)\Psi, \mathfrak{T}\pi(B_2)\Psi \rangle_{\mathcal{H}} \ ,$$

using that

$$\int_0^{1/2} \mathbf{\Delta}^{\alpha} b\Psi \, \mathrm{d}\alpha = \frac{\mathbf{\Delta}^{1/2} - \mathbf{1}}{\ln \mathbf{\Delta}} b\Psi \qquad \text{and} \qquad \int_0^{1/2} \mathbf{\Delta}^{-\alpha} b\Psi \, \mathrm{d}\alpha = \frac{\mathbf{1} - \mathbf{\Delta}^{-1/2}}{\ln \mathbf{\Delta}} b\Psi$$

for any $b \in \mathfrak{M}$. ■

By (158), one checks that $\mathrm{Dom}(\mathbf{\Delta}^{1/2}) \subset \mathrm{Dom}(\mathfrak{T})$ and thus, $\mathfrak{M}\Psi \subset \mathrm{Dom}(\mathfrak{T})$. It is therefore natural to define the Duhamel two–point function, again denoted by $(\cdot,\cdot)_{\sim}$, on the von Neumann algebra $\mathfrak{M} := \pi(\mathcal{X})''$ by

$$(b_1, b_2)_{\sim} := \langle \mathfrak{T} b_1 \Psi, \mathfrak{T} b_2 \Psi \rangle_{\mathcal{H}} \ , \qquad b_1, b_2 \in \mathfrak{M} \ . \tag{160}$$

This sesquilinear form is a scalar product:

Theorem A.2 (Duhamel two–point function as a scalar product)
The sesquilinear form $(\cdot,\cdot)_{\sim}$ is a scalar product of the pre–Hilbert space $\mathfrak{M}$.

Proof: The positivity of the sesquilinear form $(\cdot,\cdot)_\sim$ is clear. Therefore, it only remains to verify that it is non–degenerated. This is proven in [NVW, Lemma II.2.] as follows: First note that 0 is not an eigenvalue of $\mathfrak{T}$. This follows from (158). Indeed, for all $\nu \in \mathbb{R}$,

$$\left(\frac{1-\mathrm{e}^{-\beta\nu}}{\beta\nu}\right)^{1/2} > 0\ .$$

Since ϱ is a (τ,β)–KMS state, the cyclic vector Ψ is also separating for $\mathfrak{M}$, by [BR2, Corollary 5.3.9.]. Therefore, $(b,b)_\sim = 0$ yields $\mathfrak{T}b\Psi = 0$ which in turn implies that $b\Psi = 0$ and $b = 0$. ■

Note that the kernel of π is a closed two–sided ideal. If the C^*–algebra $\mathcal{X}$ is simple (like $\mathcal{U}$), i.e., when $\{0\}$ and $\mathcal{X}$ are the only closed two–sided ideals, it then follows that

$$\ker(\pi) = \{0\}.$$

Using this and Theorem A.2 we deduce that the Duhamel two–point function (155) for $B_1 = B_2 \in \mathcal{X}\backslash\{0\}$ is never zero:

Theorem A.3 (Duhamel two–point function – Strict positivity)
If the C^*–algebra $\mathcal{X}$ is simple then $(B,B)_\sim > 0$ for all non–zero $B \in \mathcal{X}\backslash\{0\}$.

Finally, we observe that it is a priori not clear that the scalar products $(\cdot,\cdot)_\sim$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ are related to each other via some upper or lower bounds. In fact, a combination of Roepstorff's results [R, Eq. (10)] for finite dimensional systems with those of Naudts and Verbeure on von Neumann Algebras yields the so–called *auto–correlation upper bounds* [NV, Theorem III.1], also called Roepstorff's inequality. For self–adjoint observables, these upper bounds read:

Theorem A.4 (Auto–correlation upper bounds for observables)
For any self–adjoint element $b = b^* \in \mathfrak{M}$, $(b,b)_\sim \leq \langle b\Psi, b\Psi\rangle_{\mathcal{H}}$. In particular, for all $B = B^* \in \mathcal{X}$,

$$(B,B)_\sim \leq \varrho(B^2) \leq \|B\|_{\mathcal{X}}^2\ .$$

Proof: This theorem is a particular case of [NV, Theorem III.1], by observing in its proof that $(u-v)\log(u/v)$ should be replaced by u when $u = v$. See also [BR2, Theorem 5.3.17]. ■

Note that the authors derive in [R, NV] further upper and lower bounds related the scalar products $(\cdot,\cdot)_\sim$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. These are however not used in the sequel.

For more details, we refer to [NV] or [BR2, Section 5.3.1]. We only conclude this subsection by an important equality for the Duhamel two–point function $(\cdot,\cdot)_\sim$ which was widely used for finite volume systems. See, e.g., [G, Eq. (2.4)].

This equality does not seem to be proven before for general KMS states. It is a straighforward consequence of Theorem A.1. To this end, denote by δ the generator of the strongly continuous one–parameter group $\tau := \{\tau_t\}_{t\in\mathbb{R}}$ of automorphisms of the C^*–algebra $\mathcal{X}$.

Theorem A.5 (Commutators and Duhamel two–point function)
For all $B_1 \in \mathcal{X}$ and $B_2 \in \mathrm{Dom}(\delta)$,

$$-i(B_1, \delta(B_2))_\sim = \varrho([B_1^*, B_2]) \ .$$

Proof: It is a direct consequence of (156)–(158) and (160): For any $B_1 \in \mathcal{X}$ and $B_2 \in \mathrm{Dom}(\delta)$,

$$\begin{aligned}
-i(B_1, \delta(B_2))_\sim &= \langle \mathfrak{T}\pi(B_1)\Psi, \mathfrak{T}\pi(\delta(B_2))\Psi\rangle_{\mathcal{H}} \\
&= \langle \pi(B_1)\Psi, \pi(B_2)\Psi\rangle_{\mathcal{H}} - \left\langle \boldsymbol{\Delta}^{1/2}\pi(B_1)\Psi, \boldsymbol{\Delta}^{1/2}\pi(B_2)\Psi\right\rangle_{\mathcal{H}} \\
&= \langle \pi(B_1)\Psi, \pi(B_2)\Psi\rangle_{\mathcal{H}} - \langle \pi(B_2^*)\Psi, \pi(B_1^*)\Psi\rangle_{\mathcal{H}} \\
&= \varrho([B_1^*, B_2]) \ .
\end{aligned}$$

See also Theorem A.1. ∎

Corollary A.6 (Duhamel two–point function and generator of dynamics)
For any self–adjoint element $B = B^* \in \mathrm{Dom}(\delta) \subset \mathcal{X}$,

$$(B, \delta(B))_\sim = 0 \qquad \textit{and} \qquad -i\varrho([\delta(B), B]) = (\delta(B), \delta(B))_\sim \geq 0 \ .$$

A.3 Duhamel GNS Representation

In view of Theorem A.2, we denote by $\tilde{\mathcal{H}}$ the completion of $\mathfrak{M}$ w.r.t. the scalar product $(\cdot,\cdot)_\sim$. This Hilbert space is related to any GNS Hilbert space of ϱ by a unitary transformation:

Theorem A.7 (Unitary equivalence of $\mathcal{H}$ and $\tilde{\mathcal{H}}$)
$U_\sim \tilde{\mathcal{H}} = \mathcal{H}$ with $U_\sim$ being the unitary operator defined by $U_\sim b = \mathfrak{T} b \Psi$ for $b \in \mathfrak{M}$.

Proof: Since $\|U_\sim b\|_{\mathcal{H}} = \|b\|_\sim$, the operator $U_\sim$ defined by $U_\sim b = \mathfrak{T}b\Psi$ for $b \in \mathfrak{M}$ has a continuous isometric extension on $\tilde{\mathcal{H}}$. Then, one checks that the range of $\mathfrak{T}$ is dense in $\mathcal{H}$ and is included in the range of $U_\sim$. For more details, see [NVW, Theorem II.3.]. ■

A simple consequence of Theorem A.7 is a GNS representation based on the Duhamel two–point function:

Definition A.8 (Duhamel GNS representation)
The Duhamel GNS representation of the (τ, β)*–KMS state* $\varrho \in \mathcal{X}^*$ *is defined by the triplet* $(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{\Psi})$ *where*

$$\tilde{\Psi} := U_\sim^* \Psi = U_\sim^* \mathfrak{T} \Psi \in \tilde{\mathcal{H}} \qquad \textit{and} \qquad \tilde{\pi}(B) = U_\sim^* \pi(B) U_\sim , \quad B \in \mathcal{X} .$$

If $\mathcal{X}$ *has an identity* $\mathbf{1}$*, then* $\tilde{\Psi} = \pi(\mathbf{1}) \in \mathfrak{M} \subset \tilde{\mathcal{H}}$*.*

This GNS representation of KMS states does not seem – at least to our knowledge – to have been previously used, even if it is a direct consequence of [NVW, Theorem II.3.]. In particular, the name *Duhamel GNS representation* is not standard and it could also be called *Bogoliubov* or *Kubo–Mori* GNS representation in reference to the scalar product $(\cdot, \cdot)_\sim$.

As explained in Section A.2, there is a unique σ–weakly continuous $*$–automorphism group $\tilde{\tau} = \{\tilde{\tau}_t\}_{t\in\mathbb{R}}$ on the von Neumann algebra $\tilde{\mathfrak{M}} := \tilde{\pi}(\mathcal{X})''$, such that $\tau_t = \tilde{\tau}_t \circ \pi$, $t \in \mathbb{R}$. It has a representation by conjugation with unitaries

$$\{\mathrm{e}^{it\tilde{\mathcal{L}}}\}_{t\in\mathbb{R}} \subset \mathfrak{M},$$

the self–adjoint operator $\tilde{\mathcal{L}}$ being equal to

$$\tilde{\mathcal{L}} = U_\sim^* \mathcal{L} U_\sim . \tag{161}$$

Clearly, $\tilde{\Psi} \in \mathrm{Dom}(\tilde{\mathcal{L}})$ and $\tilde{\mathcal{L}}\tilde{\Psi} = 0$. The normal state $\tilde{\varrho} \in \tilde{\mathfrak{M}}^*$ is a $(\tilde{\tau}, \beta)$–KMS state.

At the end of the previous subsection we explain that if the C^*–algebra $\mathcal{X}$ is simple, like the CAR algebra $\mathcal{U}$, then $\pi : \mathcal{X} \to \mathfrak{M}$ is injective and one can see the C^*–algebra $\mathcal{X}$ as a *subspace* of $\tilde{\mathcal{H}}$. In particular, if $\mathcal{X}$ has an identity $\mathbf{1}$, then

$$\tilde{\Psi} = \mathbf{1} \in \mathcal{X} \subset \mathfrak{M} \subset \tilde{\mathcal{H}} .$$

Note additionally that, in this case, for any element $B \in \mathcal{X}$ and time $t \in \mathbb{R}$, one has $\tau_t(B) \in \mathcal{X} \subset \tilde{\mathcal{H}}$ and it is straightforward to check (cf. [NVW, Section III]) that $i\mathcal{L}$ is the generator of a unitary group extending τ to the whole Hilbert space $\tilde{\mathcal{H}}$:

Theorem A.9 (Duhamel GNS representation and dynamics)
Assume $\mathcal{X}$ is simple. Then, for $B \in \mathcal{X} \subset \tilde{\mathcal{H}}$ and $t \in \mathbb{R}$, $\tau_t(B) = \mathrm{e}^{it\tilde{\mathcal{L}}}B$ with $(B, \tilde{\mathcal{L}}B)_\sim = 0$ if $B \in \mathrm{Dom}(\tilde{\mathcal{L}})$.

Proof: See [NVW, Section III]: By Theorem A.7, for any $B \in \mathcal{X} \subset \mathfrak{M} \subset \tilde{\mathcal{H}}$ and $t \in \mathbb{R}$,

$$\begin{aligned}\tau_t(B) &= U_\sim^* \mathfrak{T}\pi\left(\tau_t(B)\right)\Psi = U_\sim^* \mathfrak{T}\mathrm{e}^{it\mathcal{L}}\pi\left(B\right)\Psi \\ &= U_\sim^* \mathrm{e}^{it\mathcal{L}}\mathfrak{T}\pi\left(B\right)\Psi = U_\sim^* \mathrm{e}^{it\mathcal{L}}U_\sim B = \mathrm{e}^{itU_\sim^*\mathcal{L}U_\sim}B\,.\end{aligned}$$

Recall that $(\mathcal{H}, \pi, \Psi)$ is any GNS representation of the (τ, β)–KMS state ϱ and $\mathcal{L}$ is the associated standard Liouvillean. See also (161). The equality $(B, \tilde{\mathcal{L}}B)_\sim = 0$ results from Corollary A.6. ■

Note that Theorem A.9 directly yields the invariance of the norm of $B \in \mathcal{X} \subset \tilde{\mathcal{H}}$ w.r.t. to the group τ acting on the subspace $\mathcal{X} \subset \tilde{\mathcal{H}}$.

Corollary A.10 (Stationarity of the Duhamel norm)
Assume $\mathcal{X}$ is simple. Then, for $B \in \mathcal{X} \subset \tilde{\mathcal{H}}$ and $t \in \mathbb{R}$, $\|\tau_t(B)\|_\sim = \|B\|_\sim$ with $\|\cdot\|_\sim$ denoting the (Duhamel) norm of $\tilde{\mathcal{H}}$ associated with the scalar product $(\cdot,\cdot)_\sim$.

Therefore, by Theorem A.9, we can invoke the spectral theorem in order to analyze the dynamics in relation with the scalar product $(\cdot,\cdot)_\sim$. This is exploited for instance in Theorem 5.4 to extract the conductivity measure from a spectral measure.

Remark A.11 ($\mathcal{U}$ as a pre–Hilbert space)
We identify in all the paper the Duhamel two–point function $(\cdot,\cdot)_\sim$ defined by (154) on the CAR C^*–algebra $\mathcal{U}$ with the scalar product $(\cdot,\cdot)_\sim$ defined by (160) for $\varrho = \varrho^{(\beta,\omega,\lambda)}$ and $\tau = \tau^{(\omega,\lambda)}$ on $\mathfrak{M} := \pi\left(\mathcal{U}\right)'' \subset \tilde{\mathcal{H}}$. Note that $\mathcal{U} \equiv \pi(\mathcal{U}) \subset \mathfrak{M}$ is a pre–Hilbert space w.r.t. $(\cdot,\cdot)_\sim$.

A.4 Duhamel Two–Point Function and Time–Reversal Symmetry

Let $\mathcal{X}$ be a C^*–algebra with unity $\mathbf{1}$ and assume the existence of a map $\Theta : \mathcal{X} \to \mathcal{X}$ with the following properties:

- Θ is antilinear and continuous.
- $\Theta(\mathbf{1}) = \mathbf{1}$ and $\Theta \circ \Theta = \mathrm{Id}_{\mathcal{X}}$.
- $\Theta(B_1 B_2) = \Theta(B_1)\Theta(B_2)$ for all $B_1, B_2 \in \mathcal{X}$.
- $\Theta(B^*) = \Theta(B)^*$ for all $B \in \mathcal{X}$.

Such a map is called a *time–reversal* operation of the C^*–algebra $\mathcal{X}$.

Observe that, for any strongly continuous one–parameter group $\tau := \{\tau_t\}_{t\in\mathbb{R}}$ of automorphisms of $\mathcal{X}$, the family $\tau^\Theta := \{\tau_t^\Theta\}_{t\in\mathbb{R}}$ defined by

$$\tau_t^\Theta := \Theta \circ \tau_t \circ \Theta , \qquad t \in \mathbb{R} ,$$

is again a strongly continuous one–parameter group of automorphisms. Similarly, for any state $\rho \in \mathcal{X}^*$, the linear functional ρ^Θ defined by

$$\rho^\Theta(B) = \overline{\rho \circ \Theta(B)} , \qquad B \in \mathcal{X} ,$$

is again a state. We say that τ and ρ are *time–reversal invariant* if they satisfy $\tau_t^\Theta = \tau_{-t}$ for all $t \in \mathbb{R}$ and $\rho^\Theta = \rho$.

If τ is time–reversal invariant then, for all $\beta > 0$, there is at least one time–reversal invariant (τ, β)–KMS state $\varrho \in \mathcal{X}^*$, provided the set of (τ, β)–KMS states is not empty. This follows from the convexity of the set of KMS states:

Lemma A.12 (Existence of time–reversal invariant (τ, β)–KMS states)
Assume that τ is time–reversal invariant and ϱ is a (τ, β)–KMS state. Then, ρ^Θ is a (τ, β)–KMS state. In particular, $\frac{1}{2}\rho + \frac{1}{2}\rho^\Theta$ is a time–reversal invariant (τ, β)–KMS state.

Proof: For any $t \in \mathbb{R}$ and $B_1, B_2 \in \mathcal{X}$,

$$\rho^\Theta(B_1 \tau_t(B_2)) = \overline{\rho(\Theta(B_1)\tau_{-t}(\Theta(B_2)))} = \rho(\Theta(B_2^*)\tau_t(\Theta(B_1^*))) ,$$

using the stationarity of KMS–states and hermiticity of states. Since ρ is by assumption a (τ, β)–KMS state, the continuous function

$$t \mapsto \mathfrak{m}_{B_1,B_2}(t) := \rho(\Theta(B_2^*)\tau_t(\Theta(B_1^*)))$$

from $\mathbb{R}$ to $\mathbb{C}$ extends uniquely to a continuous map $\mathfrak{m}_{B_1,B_2}$ on $\mathbb{R}\times[0,\beta]\subset\mathbb{C}$ which is holomorphic on $\mathbb{R}\times(0,\beta)$ while, again by stationarity and hermiticity of ρ,

$$\begin{aligned}\mathfrak{m}_{B_1,B_2}(t+i\beta) &= \rho\left(\tau_t\left(\Theta\left(B_1^*\right)\right)\Theta\left(B_2^*\right)\right)\\ &= \rho\left(\Theta\left(B_1^*\right)\Theta\left(\tau_t\left(B_2^*\right)\right)\right)=\rho^{\Theta}\left(\tau_t\left(B_2\right)B_1\right)\end{aligned}$$

for any $t\in\mathbb{R}$ and $B_1,B_2\in\mathcal{X}$. As a consequence, ρ^{Θ} is a (τ,β)–KMS state, see [BR2, Proposition 5.3.7]. ■

This lemma implies that, if ϱ is the unique (τ,β)–KMS state with τ being time–reversal invariant, then ϱ is time–reversal invariant.

Let

$$\mathcal{X}_+:=\{B=B^*\in\mathcal{X}:\Theta(B)=B\}\ ,\quad \mathcal{X}_-:=\{B=B^*\in\mathcal{X}:\Theta(B)=-B\}\ .$$

These spaces are closed real subspaces of $\mathcal{X}$. Furthermore, they are *real* pre–Hilbert spaces w.r.t. the Duhamel two–point function $(\cdot,\cdot)_\sim$ defined by (155).

Lemma A.13 ($\mathcal{X}_\pm$ as real pre–Hilbert spaces)
Assume that τ is time–reversal invariant and ϱ is a time–reversal invariant (τ,β)–KMS state defining the Duhamel two–point function $(\cdot,\cdot)_\sim$. Then, for all $B_1,B_2\in\mathcal{X}_-$ and all $B_3,B_4\in\mathcal{X}_+$,

$$(B_1,B_2)_\sim=(B_2,B_1)_\sim\in\mathbb{R}\quad\textit{and}\quad(B_3,B_4)_\sim=(B_4,B_3)_\sim\in\mathbb{R}\ .$$

Proof: For any $B_1,B_2\in\mathcal{X}_-$, one clearly has

$$(B_1,B_2)_\sim=\left(\Theta\left(B_1\right),\Theta\left(B_2\right)\right)_\sim\ .$$

Thus, we have to prove that

$$\left(\Theta\left(B_1\right),\Theta\left(B_2\right)\right)_\sim=(B_2,B_1)_\sim\ ,\qquad B_1,B_2\in\mathcal{X}_-\ .$$

By the Phragmén–Lindelöf theorem [BR2, Proposition 5.3.5], the stationarity of KMS states and Definition (155), it suffices to show that

$$\varrho\left(\Theta\left(B_1\right)\tau_t(\Theta\left(B_2\right))\right)=\varrho\left(B_2\tau_t(B_1)\right)$$

for all $t\in\mathbb{R}$ and every $B_1,B_2\in\mathcal{X}_-$. In fact, by the time–reversal invariance of ϱ, the stationarity of KMS states and the hermiticity of states,

$$\varrho\left(\Theta\left(B_1\right)\tau_t(\Theta\left(B_2\right))\right)=\overline{\varrho\left(B_1\tau_{-t}(B_2)\right)}=\overline{\varrho\left(\tau_t\left(B_1\right)B_2\right)}=\varrho\left(B_2\tau_t\left(B_1\right)\right)\ .$$

As $(\cdot,\cdot)_\sim$ is a sesquilinear form, we thus have

$$(B_1, B_2)_\sim = \overline{(B_2, B_1)_\sim} = (B_2, B_1)_\sim \in \mathbb{R} \ , \qquad B_1, B_2 \in \mathcal{X}_- \ .$$

The assertion for $\mathcal{X}_+$ is proven in the same way. ■

This lemma can be generalized for time–dependent Duhamel correlation functions. To this end, we show the following assertions:

Lemma A.14 (Commutators and Duhamel correlation functions)
Let ϱ be a (τ,β)–KMS state defining the Duhamel two–point function $(\cdot,\cdot)_\sim$. Then, for any $B_1, B_2 \in \mathcal{X}$ and all $t \in \mathbb{R}$,

$$\int_0^t \varrho\left(i[B_1, \tau_s(B_2)]\right) \mathrm{d}s = (B_1, \tau_t(B_2))_\sim - (B_1, B_2)_\sim \ .$$

Proof: It is an obvious consequence of Theorem A.5. The assertion can also be deduced from [NVW, Theorem II.5]. We give here another proof because some of its arguments are used elsewhere in the paper.

By assumption, for any $B_1, B_2 \in \mathcal{X}$, the map from $\mathbb{R}$ to $\mathbb{C}$ defined by

$$t \mapsto \varrho\left(B_1 \tau_t(B_2)\right)$$

uniquely extends to a continuous map

$$z \mapsto \varrho\left(B_1 \tau_z(B_2)\right)$$

on the strip $\mathbb{R}+i[0,\beta]$, which is holomorphic on $\mathbb{R}+i(0,\beta)$. The KMS property of ϱ, that is,

$$\varrho(B_1 \tau_{t+i\beta}(B_2)) = \varrho(\tau_t(B_2) B_1) \ , \qquad B_1, B_2 \in \mathcal{X} \ , \ t \in \mathbb{R} \ , \tag{162}$$

implies that, for any $B_1, B_2 \in \mathcal{X}$ and $t \in \mathbb{R}$,

$$\varrho\left([B_1, \tau_t(B_2)]\right) = \varrho\left(B_1 \tau_t(B_2)\right) - \varrho\left(B_1 \tau_{t+i\beta}(B_2)\right) \ .$$

As a consequence, by the Cauchy theorem for analytic functions, we obtain that

$$\int_0^t \varrho\left(i[B_1, \tau_s(B_2)]\right) \mathrm{d}s = \int_0^\beta \varrho\left(B_1 \tau_{t+i\alpha}(B_2)\right) \mathrm{d}\alpha - (B_1, B_2)_\sim$$

for any $B_1, B_2 \in \mathcal{X}$ and $t \in \mathbb{R}$. The group property of τ obviously yields

$$\varrho\left(B_1\tau_{t+z}(B_2)\right) = \varrho\left(B_1\tau_z(\tau_t(B_2))\right) \tag{163}$$

for all $z, t \in \mathbb{R}$. On the other hand, the KMS property (162) of ϱ leads to Equation (163) for all $z \in \mathbb{R}+i\beta$. Therefore, we infer from the Phragmén–Lindelöf theorem [BR2, Proposition 5.3.5] that, for any $B_1, B_2 \in \mathcal{X}$, (163) holds true for all $z \in \mathbb{R} + i[0, \beta]$. In particular,

$$\int_0^\beta \varrho\left(B_1\tau_{t+i\alpha}(B_2)\right) \mathrm{d}\alpha = (B_1, \tau_t(B_2))_\sim \, . \tag{164}$$

■

Lemma A.15 (Time–reversal symmetry of commutators)
Assume that τ is time–reversal invariant and ϱ is a time–reversal invariant state. Then, for any $B_1, B_2 \in \mathcal{X}_-$ (or $\mathcal{X}_+$) and all $t \in \mathbb{R}$,

$$\int_0^t \varrho\left(i[B_1, \tau_s(B_2)]\right) \mathrm{d}s = \int_0^{-t} \varrho\left(i[B_1, \tau_s(B_2)]\right) \mathrm{d}s = \int_0^t \varrho\left(i[B_2, \tau_s(B_1)]\right) \mathrm{d}s \, .$$

Proof: The first equality follows from the following assertions: For any $B_1, B_2 \in \mathcal{X}_-$ (or $\mathcal{X}_+$) and $t \in \mathbb{R}$,

$$\begin{aligned}
\int_0^{-t} \varrho\left(i[B_1, \tau_s(B_2)]\right) \mathrm{d}s &= \int_0^{-t} \overline{\varrho \circ \Theta\left(i[B_1, \tau_s(B_2)]\right)} \mathrm{d}s \\
&= -\int_0^{-t} \varrho\left(i[B_1, \tau_{-s}(B_2)]\right) \mathrm{d}s \\
&= \int_0^{t} \varrho\left(i[B_1, \tau_s(B_2)]\right) \mathrm{d}s \, .
\end{aligned}$$

Furthermore, by stationarity of KMS states,

$$\int_0^t \varrho\left(i[B_2, \tau_s(B_1)]\right) \mathrm{d}s = -\int_0^t \varrho\left(i[B_1, \tau_{-s}(B_2)]\right) \mathrm{d}s = \int_0^{-t} \varrho\left(i[B_1, \tau_s(B_2)]\right) \mathrm{d}s$$

for any $B_1, B_2 \in \mathcal{X}_-$ (or $\mathcal{X}_+$) and $t \in \mathbb{R}$. ■

We are now in position to prove a generalization of Lemma A.13:

Theorem A.16 (Symmetries of Duhamel correlation functions)
Assume that τ *is time–reversal invariant and* ϱ *is a time–reversal invariant* (τ, β)*–KMS state defining the Duhamel two–point function* $(\cdot,\cdot)_\sim$*. Then, for all* $B_1, B_2 \in \mathcal{X}_-$ *(or* $\mathcal{X}_+$*) and* $t \in \mathbb{R}$*,*

$$(B_1, \tau_t(B_2))_\sim = (B_1, \tau_{-t}(B_2))_\sim = (B_2, \tau_t(B_1))_\sim \in \mathbb{R} .$$

Proof: By Lemma A.14,

$$(B_1, \tau_t(B_2))_\sim = \int_0^t \varrho\,(i[B_1, \tau_s(B_2)])\,\mathrm{d}s + (B_1, B_2)_\sim$$

for all $B_1, B_2 \in \mathcal{X}_-$ (or $\mathcal{X}_+$) and $t \in \mathbb{R}$. Observe that

$$\varrho\,(i[B_1, \tau_s(B_2)]) \in \mathbb{R} ,$$

for all $B_1, B_2 \in \mathcal{X}_-$ (or $\mathcal{X}_+$) and $s \in \mathbb{R}$, because B_1, B_2 are self–adjoint elements of $\mathcal{X}$. From Lemma A.13, it follows that, for any $B_1, B_2 \in \mathcal{X}_-$ (or $\mathcal{X}_+$) and $t \in \mathbb{R}$,

$$(B_1, \tau_t(B_2))_\sim \in \mathbb{R} \ .$$

Moreover, by Lemmata A.13 and A.15,

$$(B_1, \tau_t(B_2))_\sim = (B_1, \tau_{-t}(B_2))_\sim = (B_2, \tau_t(B_1))_\sim$$

for any $B_1, B_2 \in \mathcal{X}_-$ (or $\mathcal{X}_+$) and $t \in \mathbb{R}$. ■

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