SOLVABILITY RELATIONS FOR SOME DIFFUSION EQUATIONS WITH CONVECTION TERMS

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Abstract: Linear second order elliptic equations containing the sum of the two Laplace operators with convection terms or a free Laplacian and a Laplacian with drift are considered in $\mathbb{R}^d$. The corresponding operator L may be non Fredholm, such that solvability conditions for the equation $Lu = f$ are unknown. We obtain solvability conditions in $H^2(\mathbb{R}^d)$ for the non self-adjoint problem via relating it to a self-adjoint Schrödinger type operator, for which solvability relations are derived in our preceding work [16].

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1. Introduction

Reaction-diffusion equations with convective terms have been studied extensively in recent years, in particular with applications to nonlinear propagation phenomena (see e.g. [4], [5],[6], [12], [14]). Rigorous analysis of such equations often relies on the solvability conditions of the corresponding linear problems. Classical results for elliptic equations, namely the so-called Fredholm alternative affirm that they are solvable if and only if the right side is orthogonal to the solutions of the adjoint homogeneous problem. Apparently, it may not be applicable for reaction-diffusion equations in unbounded domains. In such case, solvability relations are not established. In the present work, we study reaction-diffusion equations in the case when the corresponding operator fails to satisfy the Fredholm property and derive for them solvability conditions.

In this work we will study the linear diffusion-advection equation

$$\Delta u + v(\xi).\nabla u + c(\xi)u = f(\xi), \tag{1.1}$$

where ξ is the independent variable, $\xi \in \mathbb{R}^n$, $v(\xi)$ is a given velocity field, $c(\xi)$ is a potential. The dot denotes the scalar product between two vectors. For the usual physical applications u can be temperature or concentration of some substances, and $n = 2$ or 3. In biological applications the space dimension can be more than 3. Let us give two examples.

Consider a population of biological cells. They can be characterized by some intracellular proteins. We can consider the concentration of cells u as a function of intracellular protein concentrations, $u = u(\xi_1, ..., \xi_n)$. In this case, diffusion term describes small random perturbations of intracellular concentrations, advection term shows the rate of change of intracellular concentrations due to reactions or other factors.

The second example concerns populations of animals where the individuals are characterized be their genotype or phenotype. Then the concentration u of individuals can be considered as a function of the variables $\xi_1, ..., \xi_n$ which corresponds to the state of the genome (genes) or some phenotypical characteristics (size, form, and so on). In this example, diffusion term describes small random perturbations of genotype or phenotype in offsprings compared with their parents, advection term describes genetic pressure.

Hence investigation of diffusion-advection equations is justified in any space dimension. We will discuss solvability conditions for such equations taking into account that they may not satisfy the Fredholm property when considered in unbounded domains. We will consider some particular cases of the general equation (1.1) and will impose some conditions on the velocity field and on the potential. In what follows it will be convenient for us to introduce space variables x and y such that $\xi = (x, y)$. The first problem studied in this work is

$$\Delta_x u + v_1(x).\nabla_x u + \Delta_y u + v_2(y).\nabla_y u + c_1(x)u + c_2(y)u = f(x, y), \quad x, y \in \mathbb{R}^3, \tag{1.2}$$

where the functions $c_1(x), c_2(y) : \mathbb{R}^3 \to \mathbb{R}$, the right side is square integrable and the vector fields $v_1(x) = -\nabla_x p_1(x),\ v_2(y) = -\nabla_y p_2(y)$ with $p_{1,2} \in W^{2,\infty}(\mathbb{R}^3)$. The boundedness of the pressure along with its first and second derivatives was established under reasonable assumptions in Lemmas A1 and A2 of [17]. Here and further down $\Delta_x, \Delta_y, \nabla_x, \nabla_y$ stand for the Laplacians and gradients taken with respect to x and y variables.

In the first part of this work we study solvability conditions for problem (1.2). Let us recall that the classical Fredholm solvability condition for the operator equation $Lu = f$ affirms that this problem is solvable if and only if $(\phi_i, f) = 0$ for a finite number of functionals ϕ_i from the space E^* dual to the space E which contains the image of the operator. This solvability condition is applicable if the operator L satisfies the Fredholm property, that is its image is closed, the dimension of the kernel is finite, and the codimension of the image (or the number of solvability conditions) is also finite.

Elliptic equations in unbounded domains satisfy the Fredholm property if and only if the corresponding limiting operators are invertible (see [13]). Let us assume that $c_1(x) + c_2(y) = c_0 + \tilde{c}_1(x) + \tilde{c}_2(y)$, where c_0 is a constant and the functions $\tilde{c}_1(x)$ and $\tilde{c}_2(y)$ converge to zero as $x, y \to \infty$. Then the Fredholm property is satisfied if $c_0 < 0$ and it is not valid if $c_0 \geq 0$. In the latter case the image of the operator L corresponding to the left-hand side of equation (1.2) is not closed, and the solvability conditions are unknown. In the present work, we will establish solvability conditions for the non-Fredholm operator L in the case when c_0 is non-negative. To the best of our knowledge, this is the second result on the solvability conditions of such equations in $\mathbb{R}^n$ with $n > 1$, since the similar problem involving the single Laplace operator with drift was treated in [17]. In the case of $n = 1$, the situation is different and operators without Fredholm property can be studied by introduction of weighted spaces (see [13]) or reducing them to some integro-differential equations (see [4]). Such methods are not applicable when $n > 1$. We will use here our previous results on the solvability conditions for non-Fredholm equations of the Schrödinger type (see [16]), which relied on the spectral theory of self-adjoint operators.

For equation (1.2) the homogeneous formally adjoint problem is given by

$$\Delta_x Q - div_x(v_1(x)Q) + \Delta_y Q - div_y(v_2(y)Q) + c_1(x)Q + c_2(y)Q = 0, \quad (1.3)$$

where div_x and div_y are the divergences computed with respect to x and y variables respectively. We will use the function space

$$\tilde{W}^{2,\infty}(\mathbb{R}^6) := \{Q(x,y) : \mathbb{R}^6 \to \mathbb{C} \mid Q, \nabla Q, \Delta_x Q, \Delta_y Q \in L^\infty(\mathbb{R}^6)\}, \quad (1.4)$$

where $\nabla := \nabla_x + \nabla_y$. Similarly $\Delta := \Delta_x + \Delta_y$. Note that in definition (1.4) we do not require all the second partial derivatives to be bounded, only $\Delta_x Q$ and $\Delta_y Q$. Let us introduce the scalar potential functions

$$V_\alpha(x) := \frac{(\nabla_x p_1(x))^2}{4} - \frac{\Delta_x p_1(x)}{2} - c_1(x),$$

$$U_\beta(y) := \frac{(\nabla_y p_2(y))^2}{4} - \frac{\Delta_y p_2(y)}{2} - c_2(y),$$

assuming that $V_\alpha(x) \to -\alpha$ as $x \to \infty$ and $U_\beta(y) \to -\beta$ as $y \to \infty$ with the nonnegative constants α and β, such that $a := \alpha + \beta > 0$. We write down the corresponding nonhomogeneous Schrödinger equation

$$-\Delta_x z + V(x)z - \Delta_y z + U(y)z - az = g(x,y), \quad (1.5)$$

where

$$g(x,y) := -f(x,y)e^{-\frac{p_1(x)}{2}} e^{-\frac{p_2(y)}{2}}. \quad (1.6)$$

Note that the solutions of equations (1.2) and (1.5) are related via the change of variables

$$u(x,y)=z(x,y)e^{\frac{p_1(x)}{2}}e^{\frac{p_2(y)}{2}}. \tag{1.7}$$

Here the potentials $V(x):=V_\alpha(x)+\alpha$ and $U(y):=U_\beta(y)+\beta$ are assumed to be shallow, short-range and satisfying the conditions analogous to those used in works [15], [16], [17].

Assumption 1. *The potential functions $V(x), U(y):\mathbb{R}^3\to\mathbb{R}$ satisfy the estimates*

$$|V(x)|\le \frac{C}{1+|x|^{3.5+\varepsilon}},\quad |U(y)|\le \frac{C}{1+|y|^{3.5+\varepsilon}}$$

with some $\varepsilon>0$ and $x,y\in\mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}}<1 \quad and \quad \sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}<4\pi.$$

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|U\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|U\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}}<1 \quad and \quad \sqrt{c_{HLS}}\|U\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}<4\pi.$$

Here C denotes a finite positive constant and c_{HLS} given on p.98 of [9] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\Big|\le c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)},\quad f_1\in L^{\frac{3}{2}}(\mathbb{R}^3).$$

In the work the norm of a function $f_1\in L^p(\mathbb{R}^d),\ 1\le p\le\infty,\ d\in\mathbb{N}$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^d)}$. We will be using

$$(f_1(x),f_2(x))_{L^2(\mathbb{R}^d)}:=\int_{\mathbb{R}^d}f_1(x)\bar{f}_2(x)dx,$$

with a slight abuse of notations when the functions involved in the inner product are not square integrable, like for instance $Q(x,y)$ involved in relation (1.8). Indeed, if $f_1(x)\in L^1(\mathbb{R}^d)$ and $f_2(x)$ is bounded, then the integral in the right side of the definition above makes sense. The sphere of radius r in the space of d dimensions centered at the origin will be denoted by S_r^d. Due to the decay at infinity of our potential functions the essential spectrum of the Schrödinger operator $-\Delta_x+V(x)-\Delta_y+U(y)-a$ on $L^2(\mathbb{R}^6)$ involved in the left side of equation (1.5) fills the semi-axis $[-a,\infty)$ (see e.g. [7]) such that there is no finite dimensional isolated kernel and the Fredholm alternative theorem fails to work for problem (1.5). Under our Assumption 1 this Schrödinger operator is self-adjoint and unitarily equivalent to $-\Delta-a$ on $L^2(\mathbb{R}^6)$ via the wave operators (see [1], [8], [11], [15]). The functions of the continuous spectrum satisfy

$$(-\Delta_x+V(x))\varphi_k(x)=k^2\varphi_k(x),\ k\in\mathbb{R}^3,$$

$$(-\Delta_y + U(y))\eta_q(y) = q^2\eta_q(y),\ q \in \mathbb{R}^3,$$

the Lippmann-Schwinger equations for the perturbed plane waves (see e.g. [10] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy,$$

$$\eta_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta_q)(z)dz$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_l(x))_{L^2(\mathbb{R}^3)} = \delta(k-l), \quad k, l \in \mathbb{R}^3,$$

$$(\eta_q(y), \eta_s(y))_{L^2(\mathbb{R}^3)} = \delta(q-s), \quad q, s \in \mathbb{R}^3.$$

Their products $\varphi_k(x)\eta_q(y)$ form a complete system in $L^2(\mathbb{R}^6)$. For the right side of (1.2) we have the following.

Assumption 2. *The function* $f(x,y) \in L^2(\mathbb{R}^6)$ *and* $|x|f(x,y), |y|f(x,y) \in L^1(\mathbb{R}^6)$.

Apparently, the right side of (1.5) defined in (1.6) satisfies the conditions of Assumption 2 as well. Our first main proposition will be as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then problem (1.2) admits a unique solution* $u(x,y) \in H^2(\mathbb{R}^6)$ *if and only if*

$$(f(x,y), Q(x,y))_{L^2(\mathbb{R}^6)} = 0 \tag{1.8}$$

for any $Q(x,y) \in \tilde{W}^{2,\infty}(\mathbb{R}^6)$ *satisfying the homogeneous equation (1.3), where the space* $\tilde{W}^{2,\infty}(\mathbb{R}^6)$ *is defined in (1.4).*

The second problem studied in the article is given by

$$\Delta_x u + \Delta_y u + v(y).\nabla_y u + c(y)u = F(x,y), \tag{1.9}$$

where $x \in \mathbb{R}^n,\ n \in \mathbb{N}$ and $y \in \mathbb{R}^3$. The scalar function $c(y) : \mathbb{R}^3 \to \mathbb{R}$ and the velocity field $v(y) = -\nabla_y p(y)$, assuming that $p(y) \in W^{2,\infty}(\mathbb{R}^3)$. The corresponding adjoint homogeneous equation is

$$\Delta_x Q + \Delta_y Q - div_y(v(y)Q) + c(y)Q = 0. \tag{1.10}$$

The function space used here is given by

$$\tilde{W}^{2,\infty}(\mathbb{R}^{n+3}) = \{Q(x,y) : \mathbb{R}^{n+3} \to \mathbb{C} \mid Q, \nabla Q, \Delta_x Q, \Delta_y Q \in L^\infty(\mathbb{R}^{n+3})\}. \tag{1.11}$$

We introduce the scalar potential function

$$\nu_a(y) := \frac{(\nabla_y p(y))^2}{4} - \frac{\Delta_y p(y)}{2} - c(y).$$

We consider the following two potential situations. The case I) occurs when the dimension $n \in \mathbb{N}$ is arbitrary, $\nu_a(y) \to -a$ as $y \to \infty$ with a constant $a > 0$. Then we define $\nu(y) := \nu_a(y) + a$. In case II) we restrict our attention to the dimension $n = 1$, assuming that $a = 0$, such that $\nu_a(y)$ coincides with $\nu(y) \to 0, \; y \to \infty$. As before, the potential function $\nu(y)$ will be shallow and short-range, such that the corresponding Schrödinger operator $-\Delta_y + \nu(y)$ on $L^2(\mathbb{R}^3)$ is self-adjoint and unitarily equivalent to $-\Delta_y$ via the wave operators. Its functions of the continuous spectrum satisfy

$$(-\Delta_y + \nu(y))\xi_q(y) = q^2 \xi_q(y), \quad q \in \mathbb{R}^3,$$

in the integral formulation the Lippmann-Schwinger equation

$$\xi_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|} (\nu \xi_q)(z) dz$$

and the orthogonality relations

$$(\xi_q(y), \xi_l(y))_{L^2(\mathbb{R}^3)} = \delta(q - l), \quad q, l \in \mathbb{R}^3.$$

We have $\xi_0(y)$ when $q = 0$. The corresponding nonhomogeneous Schrödinger equation is given by

$$-\Delta_x z - \Delta_y z + \nu(y) z - a z = G(x, y) \tag{1.12}$$

with

$$G(x, y) := -F(x, y) e^{-\frac{p(y)}{2}}. \tag{1.13}$$

It can be verified that the solutions of equations (1.9) and (1.12) are related via the change of variables

$$u(x, y) = z(x, y) e^{\frac{p(y)}{2}}. \tag{1.14}$$

For the right side of (1.9) we assume the following.

Assumption 4. *The function* $F(x, y) \in L^2(\mathbb{R}^{n+3})$ *and* $|x|F(x, y), \; |y|F(x, y) \in L^1(\mathbb{R}^{n+3})$.

Obviously, $G(x, y)$ given by (1.13) satisfies the conditions of Assumption 4 as well. Our second main statement is as follows.

Theorem 5. *Let the potential functions* $\nu(y)$ *satisfy Assumption 1 and Assumption 4 holds. Then problem (1.9) has a unique solution* $u(x, y) \in H^2(\mathbb{R}^{n+3})$ *if and only if*

$$(F(x, y), Q(x, y))_{L^2(\mathbb{R}^{n+3})} = 0 \tag{1.15}$$

for any $Q(x,y) \in \tilde{W}^{2,\infty}(\mathbb{R}^{n+3})$ solving the adjoint homogeneous problem (1.10) with the space $\tilde{W}^{2,\infty}(\mathbb{R}^{n+3})$ defined in (1.11).

The similarity with the usual Fredholm solvability conditions here is only formal since the operators involved here do not satisfy the Fredholm property and their ranges are not closed.

The studies of operators without Fredholm property are crucial, for instance for proving the existence in the appropriate functional spaces of stationary and travelling wave solutions of reaction-diffusion equations (see e.g. [2], [3], [17]).

2. Solvability conditions in dimension $n = 6$

We introduce the sequence of infinitely smooth cut-off functions in the space of six dimensions $\{\xi_n\}_{n=1}^{\infty}$, dependent only upon the radial variable such that $\xi_n \equiv 1$ inside the ball $|(x,y)| \leq r_n$, vanishes identically for $|(x,y)| \geq R_n$ and is monotonically decreasing inside the spherical layer $r_n \leq |(x,y)| \leq R_n$. These sequences of radii r_n, R_n tend to infinity as $n \to \infty$ and are properly chosen such that R_n increases at a higher rate. This enables us to achieve $\|\nabla \xi_n\|_{L^2(\mathbb{R}^6)}, \ \|\Delta \xi_n\|_{L^2(\mathbb{R}^6)} \to 0$ as $n \to \infty$. The cut-off functions will be used to perform the limiting argument below since the solutions of the homogeneous problems studied are bounded but may not be decaying at infinity, like for instance the perturbed plane waves $\varphi_k(x)$. The quadratic forms below will be finite since $Q(x,y) \in \tilde{W}^{2,\infty}(\mathbb{R}^6)$ and we integrate over the compact support of ξ_n. Let us procede with proving the solvability conditions for our six dimensional problem.

Proof of Theorem 3. Let us first assume that problem (1.2) possesses a unique solution $u(x,y) \in H^2(\mathbb{R}^6)$ and $Q(x,y) \in \tilde{W}^{2,\infty}(\mathbb{R}^6)$ is a solution of the homogeneous problem (1.3) with the space $\tilde{W}^{2,\infty}(\mathbb{R}^6)$ defined in (1.4). Then we easily arrive at

$$(\Delta_x u + v_1(x).\nabla_x u + \Delta_y u + v_2(y).\nabla_y u + c_1(x)u + c_2(y)u, Q\xi_n)_{L^2(\mathbb{R}^6)} =$$

$$= (f(x,y), Q\xi_n)_{L^2(\mathbb{R}^6)}.$$

Integrating by parts, we easily obtain

$$(\Delta_x u, Q\xi_n)_{L^2(\mathbb{R}^6)} = (u, \xi_n \Delta_x Q)_{L^2(\mathbb{R}^6)} + (u, Q\Delta_x \xi_n)_{L^2(\mathbb{R}^6)} + 2(u, \nabla_x Q.\nabla_x \xi_n)_{L^2(\mathbb{R}^6)},$$

$$(\Delta_y u, Q\xi_n)_{L^2(\mathbb{R}^6)} = (u, \xi_n \Delta_y Q)_{L^2(\mathbb{R}^6)} + (u, Q\Delta_y \xi_n)_{L^2(\mathbb{R}^6)} + 2(u, \nabla_y Q.\nabla_y \xi_n)_{L^2(\mathbb{R}^6)},$$

$$(v_1(x).\nabla_x u, Q\xi_n)_{L^2(\mathbb{R}^6)} = -(u, div_x(v_1(x)Q)\xi_n)_{L^2(\mathbb{R}^6)} - (u, Qv_1(x).\nabla_x \xi_n)_{L^2(\mathbb{R}^6)},$$

$$(v_2(y).\nabla_y u, Q\xi_n)_{L^2(\mathbb{R}^6)} = -(u, div_y(v_2(y)Q)\xi_n)_{L^2(\mathbb{R}^6)} - (u, Qv_2(y).\nabla_y \xi_n)_{L^2(\mathbb{R}^6)}.$$

By adding the terms up, we arrive at

$$(u, [\Delta_x Q + \Delta_y Q - div_x(v_1(x)Q) - div_y(v_2(y)Q) + c_1(x)Q + c_2(y)Q]\xi_n)_{L^2(\mathbb{R}^6)} = 0$$

since $Q(x, y)$ solves the adjoint homogeneous problem (1.3). We estimate the remaining terms using the Schwarz inequality as follows

$$|(u, Q\Delta\xi_n)_{L^2(\mathbb{R}^6)}| \leq \|Q\|_{L^\infty(\mathbb{R}^6)}\|u\|_{L^2(\mathbb{R}^6)}\|\Delta\xi_n\|_{L^2(\mathbb{R}^6)},$$

$$|(u, \nabla Q.\nabla\xi_n)_{L^2(\mathbb{R}^6)}| \leq \|\nabla Q\|_{L^\infty(\mathbb{R}^6)}\|u\|_{L^2(\mathbb{R}^6)}\|\nabla\xi_n\|_{L^2(\mathbb{R}^6)},$$

$$|(u, Qv_1(x).\nabla_x\xi_n)_{L^2(\mathbb{R}^6)}| \leq \|Q\|_{L^\infty(\mathbb{R}^6)}\|v_1\|_{L^\infty(\mathbb{R}^3)}\|u\|_{L^2(\mathbb{R}^6)}\|\nabla_x\xi_n\|_{L^2(\mathbb{R}^6)},$$

$$|(u, Qv_2(y).\nabla_y\xi_n)_{L^2(\mathbb{R}^6)}| \leq \|Q\|_{L^\infty(\mathbb{R}^6)}\|v_2\|_{L^\infty(\mathbb{R}^3)}\|u\|_{L^2(\mathbb{R}^6)}\|\nabla_y\xi_n\|_{L^2(\mathbb{R}^6)},$$

such that the right sides of the all four inequalities above tend to zero as $n \to \infty$. Note that $f(x, y) \in L^1(\mathbb{R}^6)$ by means of Assumption 2 and the Schwarz inequality. Finally, we estimate

$$|(f(x, y), Q\xi_n)_{L^2(\mathbb{R}^6)} - (f(x, y), Q)_{L^2(\mathbb{R}^6)}| \leq \|Q\|_{L^\infty(\mathbb{R}^6)} \int_{|(x,y)|>r_n} |f(x, y)|dxdy,$$

which tends to zero as $n \to \infty$. Thus, we arrive at the desired orthogonality condition (1.8).

To conclude the proof of the theorem, let us assume the opposite, namely that orthogonality relation (1.8) holds. Let us introduce

$$Q_{k,q}(x, y) := e^{-\frac{p_1(x)}{2}}\varphi_k(x)e^{-\frac{p_2(y)}{2}}\eta_q(y), \quad (k, q) \in S^6_{\sqrt{a}} \quad a.e. \tag{2.16}$$

By means of Lemma A3 of [17], the functions of the continuos spectra $\varphi_k(x)$ and $\eta_q(y)$ are bounded along with their gradients and Laplacians. A straightforward computation yields that the functions given by (2.16) satisfy the adjoint homogeneous equation (1.3) and belong to the $\tilde{W}^{2,\infty}(\mathbb{R}^6)$ space defined in (1.4). Therefore,

$$(f(x, y), Q_{k,q}(x, y))_{L^2(\mathbb{R}^6)} = 0, \quad (k, q) \in S^6_{\sqrt{a}} \quad a.e.$$

Hence

$$(g(x, y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0, \quad (k, q) \in S^6_{\sqrt{a}} \quad a.e.$$

By means of the part a) of Theorem 3 of [16] equation (1.5) admits a unique solution $z(x, y) \in H^2(\mathbb{R}^6)$. A straightforward computation yields that $u(x, y) \in H^2(\mathbb{R}^6)$ related to $z(x, y)$ via transform (1.7) solves (1.2). Suppose $u_{1,2}(x, y) \in H^2(\mathbb{R}^6)$ are two solutions of (1.2). Then the difference $z(x, y) := z_1(x, y) - z_2(x, y) \in L^2(\mathbb{R}^6)$, where $z_{1,2}$ are connected to $u_{1,2}$ by means of the variable change (1.7), satisfies the homogeneous equation

$$-\Delta_x z + V(x)z - \Delta_y z + U(y)z - az = 0.$$

Since the operator involved in the left side of the problem above considered on $L^2(\mathbb{R}^6)$ is self adjoint and unitarily equivalent to $-\Delta_x - \Delta_y - a$, it has no nontrivial square integrable zero modes. Therefore, $u_1(x, y) = u_2(x, y)$ a.e. in $\mathbb{R}^6$. ■

3. Solvability conditions in $n+3$ dimensions

We introduce here in the space of $n+3$ dimensions the sequence of smooth cut off functions $\{\xi_m\}_{m=1}^{\infty}$ with the properties analogous to ones used in Section 2.

Proof of Theorem 5. Let us first assume that $u(x,y) \in H^2(\mathbb{R}^{n+3})$ is the unique solution to problem (1.9) and $Q(x,y) \in \tilde{W}^{2,\infty}(\mathbb{R}^{n+3})$ is a solution of the adjoint homogeneous equation (1.10) with the space $\tilde{W}^{2,\infty}(\mathbb{R}^{n+3})$ defined in (1.11). Evidently,

$$(\Delta_x u + \Delta_y u + v(y).\nabla_y u + c(y)u, Q\xi_m)_{L^2(\mathbb{R}^{n+3})} = (F(x,y), Q\xi_m)_{L^2(\mathbb{R}^{n+3})}.$$

$F(x,y) \in L^1(\mathbb{R}^{n+3})$ via Assumption 4 along with the Schwarz inequality. For the right side of the identity above we easily derive

$$|(F(x,y), Q\xi_m)_{L^2(\mathbb{R}^{n+3})} - (F(x,y), Q)_{L^2(\mathbb{R}^{n+3})}| \leq$$

$$\leq \|Q\|_{L^\infty(\mathbb{R}^{n+3})} \int_{|(x,y)|>r_m} |F(x,y)| dxdy \to 0, \quad m \to \infty.$$

Integrating by parts, we easily obtain

$$(\Delta_x u, Q\xi_m)_{L^2(\mathbb{R}^{n+3})} = (u, \xi_m \Delta_x Q)_{L^2(\mathbb{R}^{n+3})} + 2(u, \nabla_x Q.\nabla_x \xi_m)_{L^2(\mathbb{R}^{n+3})} +$$

$$+ (u, Q\Delta_x \xi_m)_{L^2(\mathbb{R}^{n+3})},$$

$$(\Delta_y u, Q\xi_m)_{L^2(\mathbb{R}^{n+3})} = (u, \xi_m \Delta_y Q)_{L^2(\mathbb{R}^{n+3})} + 2(u, \nabla_y Q.\nabla_y \xi_m)_{L^2(\mathbb{R}^{n+3})} +$$

$$+ (u, Q\Delta_y \xi_m)_{L^2(\mathbb{R}^{n+3})},$$

$$(v(y).\nabla_y u, Q\xi_m)_{L^2(\mathbb{R}^{n+3})} = -(u, div_y(v(y)Q)\xi_m)_{L^2(\mathbb{R}^{n+3})} -$$

$$-(u, Qv(y).\nabla_y \xi_m)_{L^2(\mathbb{R}^{n+3})}.$$

Adding the terms up yields

$$(u, [\Delta_x Q + \Delta_y Q - div_y(v(y)Q) + c(y)Q]\xi_m)_{L^2(\mathbb{R}^{n+3})} = 0$$

due to the fact that $Q(x,y)$ solves the adjoint homogeneous problem (1.10). By means of the Schwarz inequality we obtain the bounds

$$|(u, \nabla Q.\nabla \xi_m)_{L^2(\mathbb{R}^{n+3})}| \leq \|u\|_{L^2(\mathbb{R}^{n+3})} \|\nabla Q\|_{L^\infty(\mathbb{R}^{n+3})} \|\nabla \xi_m\|_{L^2(\mathbb{R}^{n+3})},$$

$$|(u, Q\Delta \xi_m)_{L^2(\mathbb{R}^{n+3})}| \leq \|Q\|_{L^\infty(\mathbb{R}^{n+3})} \|u\|_{L^2(\mathbb{R}^{n+3})} \|\Delta \xi_m\|_{L^2(\mathbb{R}^{n+3})},$$

$$|(u, Qv(y).\nabla_y \xi_m)_{L^2(\mathbb{R}^{n+3})}| \leq$$

$$\leq \|Q\|_{L^\infty(\mathbb{R}^{n+3})} \|v(y)\|_{L^\infty(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^{n+3})} \|\nabla_y \xi_m\|_{L^2(\mathbb{R}^{n+3})}.$$

Apparently, the right sides of all these inequalities above tend to zero as $m \to \infty$, such that we obtain the desired orthogonality relation (1.15).

To conclude the proof, we now assume the opposite, such that orthogonality condition (1.15) holds. Let us define in case I) when the dimension $n \in \mathbb{N}$ is arbitrary and $a > 0$

$$Q_{k,n,q}(x,y) := \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{p(y)}{2}} \xi_q(y), \quad (k,q) \in S^{n+3}_{\sqrt{a}} \quad a.e. \tag{3.17}$$

and in case II) when the dimension $n = 1$ and $a = 0$

$$Q_0(x,y) := e^{-\frac{p(y)}{2}} \xi_0(y).$$

Let us consider case I) since in case II) we can exploit the similar ideas. It can be verified that functions (3.17) satisfy the adjoint homogeneous equation (1.10) and belong to the $\tilde{W}^{2,\infty}(\mathbb{R}^{n+3})$ space. Therefore,

$$(F(x,y), Q_{k,n,q}(x,y))_{L^2(\mathbb{R}^{n+3})} = 0.$$

Hence

$$(G(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}} \xi_q(y))_{L^2(\mathbb{R}^{n+3})} = 0, \quad (k,q) \in S^{n+3}_{\sqrt{a}} \quad a.e.$$

By means of the part a) of Theorem 6 of [16], equation (1.12) admits a unique solution $z(x,y) \in H^2(\mathbb{R}^{n+3})$. A straightforward calculation gives us that $u(x,y) \in H^2(\mathbb{R}^{n+3})$ related to $z(x,y)$ via the change of variables (1.14) satisfies (1.9). Let us assume that $u_{1,2}(x,y) \in H^2(\mathbb{R}^{n+3})$ both solve (1.9). Then $z(x,y) := z_1(x,y) - z_2(x,y) \in L^2(\mathbb{R}^{n+3})$, with $z_{1,2}$ connected to $u_{1,2}$ via formula (1.14), is a solution of the homogeneous problem

$$-\Delta_x z - \Delta_y z + \nu(y) z - az = 0.$$

Due to the fact that the operator in the left side of the equation above considered on $L^2(\mathbb{R}^{n+3})$ is self adjoint and unitarily equivalent to $-\Delta_x - \Delta_y - a$, it does not have nontrivial square integrable zero modes. Thus, $u_1(x,y) = u_2(x,y)$ a.e. in $\mathbb{R}^{n+3}$. ■

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