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FAST ITERATION OF COCYCLES OVER ROTATIONS AND COMPUTATION OF HYPERBOLIC BUNDLES

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Abstract. We present numerical algorithms that use small requirements of storage and operations to compute the iteration of cocycles over a rotation. We also show that these algorithms can be used to compute efficiently the stable and unstable bundles and the Lyapunov exponents of the cocycle.

1. **Introduction.** The goal of this paper is to describe efficient algorithms to compute iterations of matrix cocycles over rotations (quasi-periodic cocycles). These quasi-periodic matrix cocycles appear naturally in the study of the variational equations around a quasi-periodic solution [8] and in the study of Schrödinger equations over a quasi-periodic potential [18, 2, 17].

The algorithms we present can compute 2^k iterations of the cocycle at N points by repeating a renormalization step k times. If we denote by $C(N)$ the number of operations of a renormalization step, then the algorithms can compute 2^k iterations of the cocycle at N points requiring only $C(N)k$ operations. Moreover, the storage requirement is proportional to N and independent of k.

In addition, the method we present allows us to compute in a stable way the Lyapunov spectrum and the invariant bundles of the cocycle, by combining the renormalization procedure with the QR method to compute Lyapunov exponents.

Finally, we discuss how the iteration of cocycles can be used to obtain an approximation of the invariant bundles of the stable and unstable splitting by means of a method similar to the power iteration method.

The paper is organized as follows. In Section 2 we review some of the basic concepts on the theory of cocycles. In Section 3 we present the fast algorithms for the iteration of cocycles, which constitute the main result of this paper. In Section 4 we discuss one of the main pitfalls of the iteration of cocycles and how it can be solved. Finally in Section 5 we show how the iteration of cocycles can be applied to the computation of the rank-1 (un)stable bundles.

2. **Some basic facts about cocycles.** In this section, we review some standard notations and some elementary results on the theory of cocycles.

2.1. **Definition of cocycle.** Given a matrix-valued function $M : \mathbb{T}^\ell \to GL(d, \mathbb{R})$ and a vector $\omega \in \mathbb{R}^\ell$, we define $\mathcal{M} : \mathbb{Z} \times \mathbb{T}^\ell \to GL(d, \mathbb{R})$ the *cocycle* over the rotation T_ω, defined as $T_\omega(\theta) = \theta + \omega$, associated to the matrix M by:

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$$\mathcal{M}(n,\theta) = \begin{cases} M(\theta+(n-1)\omega)\cdots M(\theta) & n \geq 1, \\ \mathrm{Id} & n = 0, \\ M^{-1}(\theta+(n+1)\omega)\cdots M^{-1}(\theta) & n \leq -1. \end{cases} \tag{2.1}$$

Equivalently, a cocycle is defined by the recurrence relation:

$$\begin{aligned} \mathcal{M}(0,\theta) &= \mathrm{Id}, \\ \mathcal{M}(1,\theta) &= M(\theta), \\ \mathcal{M}(n+m,\theta) &= \mathcal{M}(n, T_\omega^m(\theta))\mathcal{M}(m,\theta). \end{aligned} \tag{2.2}$$

We will say that M is the generator of $\mathcal{M}$. We omit from the notation of $\mathcal{M}$ the dependence on ω and M when it is clear from the context.

Note that if $M(\mathbb{T}^\ell) \subset G$ where $G \subset GL(d,\mathbb{R})$ is a group, then $\mathcal{M}(\mathbb{Z},\mathbb{T}^\ell) \subset G$. In applications to Mechanics, the group G is the group of symplectic maps.

2.1.1. *Cocycles for continuous time.* Similarly as in the discrete case, given a matrix valued function M and a vector $\omega \in \mathbb{R}^\ell$, a *continuous in time cocycle* $\mathcal{M}(t,\theta)$ is defined to be the unique solution of

$$\begin{aligned} \frac{d}{dt}\mathcal{M}(t,\theta) &= M(\theta+\omega t)\mathcal{M}(t,\theta), \\ \mathcal{M}(0,\theta) &= \mathrm{Id}\,. \end{aligned} \tag{2.3}$$

From the uniqueness part of Cauchy-Lipschitz theorem, we have the following property

$$\begin{aligned} \mathcal{M}(t+s,\theta) &= \mathcal{M}(s,\theta+\omega t)\mathcal{M}(t,\theta), \\ \mathcal{M}(0,\theta) &= \mathrm{Id}\,. \end{aligned} \tag{2.4}$$

Note that (2.3) and (2.4) are the exact analogues of (2.1) and (2.2) in a continuous context. Moreover, if $M(\mathbb{T}^\ell) \subset \mathfrak{G}$, where $\mathfrak{G}$ is the Lie algebra of the Lie group G, then $\mathcal{M}(\mathbb{R},\mathbb{T}^\ell) \subset G$.

2.2. **Some motivations.** In this section we present two situations where cocycles and their asymptotic properties play important roles, which serve as motivation for our study.

2.2.1. *Linearization around quasi-periodic solutions.* Cocycles appear naturally in the study of variational equations, which govern the growth of infinitesimal perturbations around an orbit. If we consider the growth of perturbations around a quasi-periodic orbit, we are lead to quasi-periodic cocycles. Variational equations are crucial in the study of stability properties of a solution or in Newton methods to compute quasi-periodic solutions [8].

Consider a map $F : \mathcal{U} \subset \mathbb{R}^d \mapsto \mathbb{R}^d$. Assume that F has a quasi-periodic solution of frequency $\omega \in \mathbb{R}^d$, given by $x_n = K(n\omega)$, where $K : \mathbb{T}^l \to \mathbb{R}^d$ is the parameterization of the quasi-periodic orbit (sometimes also called the *hull function*). Then,

$$F \circ K = K \circ T_\omega, \tag{2.5}$$

where T_ω denotes the rigid rotation $T_\omega(\theta) = \theta + \omega$.

If we define $M(\theta) = (DF \circ K)(\theta)$, and denote by $\mathcal{M}$ the cocycle associated to the matrix M, the chain rule shows that

$$\mathcal{M}(n,\theta) = DF^n(K(\theta)).$$

Indeed, the chain rule is the recurrence relation (2.2). Hence, iterating a cocycle is the same as integrating forward in time the variational equations.

Remark 1. In the same vein, we note that continuous time cocycles (2.3) appear in the linearization around quasi-periodic solutions of differential equations.

2.2.2. *Schrödinger equations with a quasi-periodic potential.* Quasi-periodic cocycles appear also in the 1-dimensional discrete Schrödinger equations. See [18] for a survey and [17].

The discrete time independent 1-dimensional Schrödinger equation with a quasi-periodic potential V is given by the equation

$$-\psi_{n+1} - \psi_{n-1} + (2 + V(n\omega))\psi_n = E\psi_n \tag{2.6}$$

where $V : \mathbb{T}^l \to \mathbb{R}$, $l \geq 1$, $\omega \in \mathbb{R}^l$ and E is a real parameter usually called the energy.

By setting $u_n = \psi_{n-1}$, it is easy to see that the second order equation (2.6) is equivalent to the system:

$$\begin{pmatrix} \psi \\ u \end{pmatrix}_{n+1} = M(n\omega) \begin{pmatrix} \psi \\ u \end{pmatrix}_n \tag{2.7}$$

with

$$M(\theta) = \begin{pmatrix} 2 - E + V(\theta) & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

Hence the asymptotic behavior of the solutions ψ to (2.6) – and hence whether E is in the spectrum or not – are closely related to the asymptotic properties of the cocycle generated by (2.8).

2.3. **Dichotomy and hyperbolicity of cocycles.** One of the most crucial properties of cocycles is hyperbolicity (or spectral dichotomies) as described in [15, 21, 22, 23, 20].

Definition 1. Given $0 < \lambda < \mu$ we say that a cocycle $\mathcal{M}(n, \theta)$ (resp. $\mathcal{M}(t, \theta)$) has a $\lambda, \mu-$ dichotomy if for every $\theta \in \mathbb{T}^\ell$ there exist a constant $c > 0$ and a splitting depending on θ, such that

$$T\mathbb{R}^d = \mathcal{E}^s \oplus \mathcal{E}^u$$

which is characterized by:

$$\begin{aligned} (x_\theta, v) \in \mathcal{E}^s &\Leftrightarrow |\mathcal{M}(n, \theta)v| \leq c\lambda^n |v| \ , \qquad \forall n \geq 0 \\ (x_\theta, v) \in \mathcal{E}^u &\Leftrightarrow |\mathcal{M}(n, \theta)v| \leq c\mu^n |v| \ , \qquad \forall n \leq 0 \end{aligned} \tag{2.9}$$

(resp.

$$\begin{aligned} (x_\theta, v) \in \mathcal{E}^s &\Leftrightarrow |\mathcal{M}(t, \theta)v| \leq c\lambda^t |v| \ , \qquad \forall t \geq 0 \\ (x_\theta, v) \in \mathcal{E}^u &\Leftrightarrow |\mathcal{M}(t, \theta)v| \leq c\mu^t |v| \ , \qquad \forall t \leq 0). \end{aligned} \tag{2.10}$$

Remark 2. The notation $\mathcal{E}^s$ and $\mathcal{E}^u$ is meant to suggest that an important case is the splitting between stable and unstable bundles. This is the case when $\lambda < 1 < \mu$ and the cocycle is said to be hyperbolic. Nevertheless, the theory developed in this section assumes only the existence of a spectral gap. Note that if M has a λ, μ gap, then, $\tilde{M} = (\lambda\mu)^{-1/2} M$, has a hyperbolic gap and the iterations of $\tilde{M}$ are straightforwardly related to those of M.

In the context of variational equations around quasi-periodic solutions described in Section 2.2.1, the existence of a spectral gap means that at every point x_θ of the quasi-periodic solution $K(\theta)$, there is a splitting so that the vectors grow with appropriate rates λ, μ under iterations of the cocycle. Recall that in this context the generator of the cocycle is just the fundamental matrix of the variational equations, so that the cocycle describes the growth of infinitesimal perturbations.

Remark 3. A system can have several dichotomies. However, Definition 1 will be enough, since we can perform the analysis presented here for each spectral gap.

One fundamental problem for subsequent applications is the computation of the invariant splittings (and, of course, to ensure their existence). This computation of the invariant bundles is closely related to the computation of the iterations of the cocycle.

It is known that the mappings $\theta \to \mathcal{E}^{s,u}_{x_\theta}$ are C^r if $M(\cdot) \in C^r$ for $r \in \mathbb{N} \cup \{\infty, \omega\}$ [11]. This result uses heavily that the cocycles are over a rotation.

Indeed, given a typical vector $(x_\theta, v) \in \mathcal{E}^u$, we expect that, for $n \gg 1$, $\mathcal{M}(n, \theta)v$ will be a vector in $\mathcal{E}^u_{x_{T^n_\omega(\theta)}}$. This property suggests an an analogue of the power method to compute leading eigenvalues of a matrix. Hence, computing large iterates of cocycles is useful and serves as a first motivation for our algorithms to compute iterations fast. Of course, when the unstable directions are two-dimensional, the power method has difficulty computing the second eigenvalue. To overcome this issue, in Section 3.3 we present algorithms that can compute all the Lyapunov exponents using the QR decomposition.

2.4. **Equivalence of cocycles, reducibility.** In this section, we introduce reducibility; a very important property in the theory of cocycles that we will use to develop numerically stable algorithms.

Definition 2. We say the matrix cocycle $\mathcal{M}$ associated to the matrix-valued function $\widetilde{M}$ is equivalent to another cocycle $\mathcal{M}$ associated to the matrix-valued function M if there exists a matrix valued function $U : \mathbb{T}^\ell \to GL(d, \mathbb{R})$ such that

$$\widetilde{M}(\theta) = U(\theta + \omega)^{-1} M(\theta) U(\theta). \tag{2.11}$$

It is easy to check that $\widetilde{\mathcal{M}}$ being equivalent to $\mathcal{M}$ is indeed an equivalence relation.

If $\widetilde{\mathcal{M}}$ is equivalent to a constant cocycle associated to a constant matrix-valued function (i.e. independent of θ), we say that $\widetilde{\mathcal{M}}$ is *reducible*.

When (2.11) holds, we have

$$\widetilde{\mathcal{M}}(n, \theta) = U(\theta + n\omega)^{-1} \mathcal{M}(n, \theta) U(\theta). \tag{2.12}$$

In particular, if M is a constant matrix, we have

$$\widetilde{\mathcal{M}}(n, \theta) = U^{-1}(\theta + n\omega) M^n U(\theta),$$

so that the iterations of reducible cocycles are very easy to compute.

Computing the reduction of a cocycle to a constant cocycle may be difficult in practice. However, we will see in Section 4 that some approximate computations may improve the numerical stability properties of our algorithm in a similar way to the preconditioning methods, standard in numerical analysis.

We will also see that one can alter the numerical stability properties of the iterations of cocycles by choosing appropriately the matrix U in (2.11). In that respect,

it is also important to mention the concept of "quasi-reducibility" introduced by Eliasson [5].

3. Algorithms for fast iteration of cocycles over rotations.

In this section we present the algorithm for fast iteration of cocycles in its simplest form:

Algorithm 1 (Iteration of cocycles)**.** *Given a matrix $M(\theta)$, compute*

$$\widehat{M}(\theta) = M(\theta+\omega)M(\theta). \tag{3.1}$$

Set $\widehat{M} \to M$, $2\omega \to \omega$ and iterate the procedure.

We write $\mathcal{R}M = \widehat{M}$ and $\mathcal{R}\mathcal{M} = \widehat{\mathcal{M}}$ for the cocycle generated by $\mathcal{R}M$. We refer to $\mathcal{R}\mathcal{M} = \widehat{\mathcal{M}}$ as the renormalized cocycle and the procedure $\mathcal{R}$ as the renormalization procedure. The important property is that $\mathcal{R}\mathcal{M}(n,\theta) = \widehat{\mathcal{M}}(n,\theta) = \mathcal{M}(2n,\theta)$.

Therefore, applying k times the renormalization procedure in Algorithm 1, we have $(\mathcal{R}^k\mathcal{M})(n,\theta) = \mathcal{M}(2^k n,\theta)$, so that it amounts to computing 2^k iterates.

3.1. Operation count for the algorithm.

If we discretize the matrix-valued function M by taking N points on $\mathbb{T}^\ell$ (or N Fourier modes) and denote by $C(N)$ the number of operations required to perform a step of Algorithm 1, we can compute 2^k iterates at a cost of $kC(N)$ operations (applying the algorithm k times).

The value of $C(N)$ depends on the details of the computation of (3.1), which involves two main operations: a shift and a matrix multiplication. The first one is diagonal (i.e., the number of operations is $\mathcal{O}(N)$) in Fourier space, while the second one is diagonal in real space. The main difficulty arises from the fact that, if we have points on a equally spaced grid, then $\theta+\omega$ will not be in the same grid. We have at least three options:

1. Store the discretization in real space and compute $M(\theta+\omega)$ by interpolating with nearby points.
2. Store the discretization in real space but compute the shift in Fourier space. To do so, switch to Fourier space (using the Fast Fourier transform) to perform the shift operation and switch back to real space to perform the matrix multiplication.
3. Store the discretization in Fourier space and use the Cauchy formula to perform the product of matrices.

The operation count of each of these options is, respectively,

$$\begin{aligned} C_1(N) &= O(N), \\ C_2(N) &= O(N\log N), \\ C_3(N) &= O(N^2). \end{aligned} \tag{3.2}$$

Note that in the three cases above, the storage requirements for one renormalization step are proportial to N and independent of k.

The above implementations of the renormalization step may have different stability and roundoff properties. We are not aware of any thorough study of these stability or round-off properties (see for instance, [3] for an empirical comparative study of the round-off effects for different methods of multiplying Fourier series).

3.2. **Numerical implementation.** We supply the code (see the version of the paper in `http://www.ma.utexas.edu` for a program that computes iterations of the Shrödinger cocycle (2.8) using Algorithm 1 and the strategy described in step 2 in Section 3.1. Notice that the code can be easily adapted to any other cocycle. The file 'fast2.c' generates a program to compute iterations of the cocyle designed to test for speed. The file 'fast3.c' generates a program to test the correctness of the implementation of the elementary renormalization. When we run the program for representative values of the parameters, namely $\lambda = 0.1$ and $e = 0.2$ on a regular desktop we computed 2^{10} iterations of the cocycle in about 12 seconds, which agree up to the roundoff error with the ones obtained by direct iteration.

3.3. **The QR method.** In this section we present another version of Algorithm 1 that uses the QR decomposition to compute the iterates. It is well known (see for instance [6, 4]), that the QR algorithm is rather stable to compute iterates. One advantage is that, in the case that the spectrum has several gaps, the QR algorithm can compute all the Lyapunov exponents of the cocycle in a stable way. It is interesting to note that the QR method was the basis of the original proof of the multiplicative ergodic theorem [16, 15].

The straightforward version of the QR algorithm consists of the following iteration: given $\mathcal{M}(n, \theta) = Q_n(\theta)R_n(\theta)$ where Q_n is an orthogonal matrix and R_n is an upper triangular one, we compute $M(\theta + \omega n)Q_n(\theta)$ and its QR decomposition $M(\theta + \omega n)Q_n(\theta) = \bar{Q}_n(\theta)\bar{R}_n(\theta)$. Then,

$$\mathcal{M}(n + 1, \theta) = \bar{Q}_n(\theta)Q_n(\theta)\bar{R}_n(\theta)R_n(\theta).$$

Denoting $Q_{n+1}(\theta) = \bar{Q}_n(\theta)Q_n(\theta)$ and $R_{n+1}(\theta) = \bar{R}_n(\theta)R_n(\theta)$, we have that Q_{n+1} is an orthogonal matrix and R_{n+1} is an upper triangular one and we have obtained the QR decomposition of the next iterate.

Notice that using the QR decomposition we can calculate the next iterate of the cocycle by performing only multiplication of orthogonal and triangular matrices and a QR decomposition. Clearly, all these operations are numerically stable (for numerically stable versions of QR, we refer to [7, 1]). Moreover, the above procedure allows us to compute all the Lyapunov exponents. The straightforward iteration is affected by numerical errors and the round-off errors lead to iterations always aligning with the fastest growing eigenvalue.

The above procedure requires a number of operations which is proportional to the number of iterations. By combining Algorithm 1 with the QR decomposition we can compute 2^k iterates at a cost proportional to k and using only numerically stable operations. This result is summarized in the following algorithm:

Algorithm 2 (Fast iteration of cocycles with QR decomposition)**.** *Given $M(\theta)$ and a QR decomposition of $M(\theta)$,*

$$M(\theta) = Q(\theta)R(\theta),$$

perform the following operations:

(1) *Compute* $S(\theta) = R(\theta + \omega)Q(\theta)$
(2) *Compute pointwise a QR decomposition of S,* $S(\theta) = \bar{Q}(\theta)\bar{R}(\theta)$.
(3) *Compute* $\widetilde{Q}(\theta) = Q(\theta + \omega)\bar{Q}(\theta)$
$\widetilde{R}(\theta) = \bar{R}(\theta)R(\theta)$
$\widetilde{M}(\theta) = \widetilde{Q}(\theta)\widetilde{R}(\theta)$
(4) *Set* $M \leftarrow \widetilde{M}$

$$R \leftarrow \widetilde{R}$$
$$Q \leftarrow \widetilde{Q}$$
$$2\omega \leftarrow \omega$$

and iterate the procedure.

Note that all the operations that we perform pointwise (multiplication of matrices and the QR decomposition of matrices) have a cost proportional to N (number of points on the grid) and are numerically stable.

3.4. **The case of 1-dimensional rotations.** In the case of one-dimensional maps, one can be more precise in the description of the method. Indeed, if the frequency ω has a continued fraction expansion

$$\omega = [a_1, a_2, \ldots, a_n, \ldots],$$

it is well known that the denominators q_n of the convergents of ω (i.e. $p_n/q_n = [a_1, \ldots, a_n]$) satisfy

$$q_n = a_n q_{n-1} + q_{n-2},$$
$$q_1 = a_1,$$
$$q_0 = 1.$$

As a consequence, we can consider the following algorithm for this particular case:

Algorithm 3 (Iteration of cocycles $1D$)**.** *Given $\omega = [a_1, \ldots, a_n, \ldots]$ and the cocycle $\mathcal{M}$ over T_ω generated by $M(\theta)$, define $\omega^0 = \omega$, $M^0(\theta) = M(\theta)$ and for $n \geq 1$*

$$M^{(n)}(\theta) = M^{(n-1)}(\theta + (a_n - 1)\omega^{n-1}) \cdots M^{(n-1)}(\theta + \omega^{n-1}) M^{(n-1)}(\theta)$$

is the generator of a cocycle $\mathcal{M}^{(n)}$ over $\omega^n = a_n \omega^{n-1}$

By induction, we have

$$\omega^n = a_n \cdots a_1 \omega \pmod 1$$
$$\mathcal{M}(a_1 \cdots a_n, \theta) = \mathcal{M}^{(n)}(1, \theta)$$

The advantage of this algorithm is that the effective rotation is decreasing to zero so that the iteration of the cocycle is becoming close to the iteration of a constant matrix. This method is somehow reminiscent of some algorithms that have appeared in the mathematical literature [19, 14, 13].

4. **The "straddle the saddle" phenomenon and preconditioning.** The iteration of cocycles has more pitfalls than the the iteration of matrices because the (un)stable bundle depends on the base point x_θ.

In this section we describe a geometric phenomenon that causes some instability in the iteration of cocycles. This instability –which is genuine and affects also the direct iteration method – becomes more dramatic when we apply the fast iteration methods described in Section 3. The phenomenon we will discuss was already observed in [9], but its effect is significantly more drastic in the present algorithm. Fortunately, once the phenomenon is detected, it can be eliminated by a simple manipulation that we describe in Section 4.1.

Since we have the inductive relation,

$$\mathcal{M}(n, \theta) = \mathcal{M}(n-1, \theta + \omega) M(\theta),$$

we can think of computing $\mathcal{M}(n,\theta)$ by applying $\mathcal{M}(n-1,\theta+\omega)$ to the column vectors of $M(\theta)$.

The j^{th}-column of M, which we will denote by $m_j(\theta)$, can be interpreted geometrically as an embedding from $\mathbb{T}^\ell$ to $\mathbb{R}^{2d}$ and is given by $M(\theta)e_j$ where e_j is the j^{th} vector of the canonical basis of $\mathbb{R}^{2d}$. If the stable space of $\mathcal{M}(n-1,\theta+\omega)$ has codimension ℓ or less, there can be points $\theta^* \in \mathbb{T}^\ell$ such that $m_j(\theta^*) \in \mathcal{E}^s_{x_{\theta^*}}$ and such that for every $\theta \neq \theta^*$ we have $m_j(\theta) \notin \mathcal{E}^s_{x_\theta}$.

Clearly, for a fixed θ,

$$m_j^{(n)}(\theta^*) = \mathcal{M}(n-1,\theta^*+\omega)m_j(\theta^*),$$

decreases exponentially as n grows. Nevertheless, for all θ in a neighborhood of θ^* such that $\theta \neq \theta^*$

$$m_j^{(n)}(\theta) = \mathcal{M}(n-1,\theta+\omega)m_j(\theta),$$

grows exponentially as n grows. The direction along which the growth takes place depends on the projection of $m_j(\theta)$ onto $\mathcal{E}^u_{x_{\theta+\omega}}$.

Take for instance the case when $d = 2$, $\ell = 1$ and the stable and unstable directions are one dimensional. Then, the unstable components will have different signs and the vectors $\mathcal{M}(n-1,\theta+\omega)m_j(\theta)$ will align in opposite directions. An illustration of this phenomenon is shown in Figure 1.

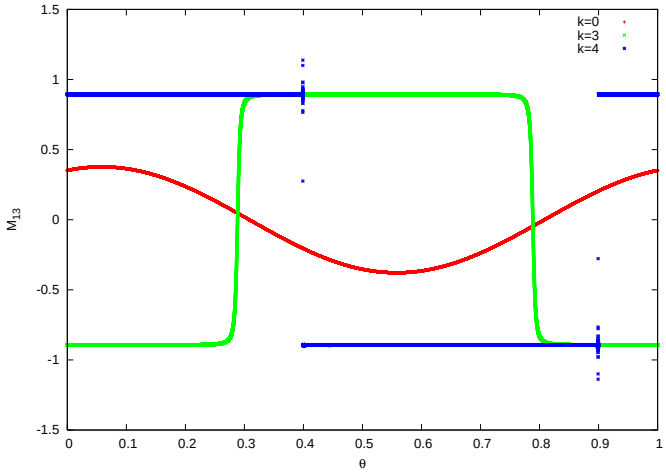

FIGURE 1. The straddle the saddle phenomenon. We plot one of the components of the cocycle $\mathcal{M}(2^k,\theta)$ for the values $k = 0, 3, 4$. The case $k = 0$ was scaled by a factor 200.

The transversal intersection of the range of $m_j(\theta)$ with $\mathcal{E}^s$ is indeed a true phenomenon, and it is a true instability of the method. It cannot be cured by reducing the truncation or round-off errors.

Unfortunately, if $m_j^{(n)}$ is discontinuous as a function of θ, the discretization in Fourier series or the interpolation by splines will be extremely inaccurate and the Algorithm 1 will fail.

This phenomenon is easy to detect when it happens because the derivatives grow exponentially fast in some localized spots.

One important case where the straddle the saddle is unavoidable is when the invariant bundles are non-orientable. This happens near resonances. In [12], it is shown that, by doubling the angle the case of resonances can be studied comfortably because then, non-orientability is the only obstruction to the triviality of the bundle.

4.1. **Eliminating the "straddle the saddle" in the one-dimensional case.** Fortunately, once the phenomenon is detected, it can be eliminated. The main idea is that one can find an equivalent cocycle which does not have the problem (or presents it in a smaller extent).

In more geometric terms we observe that, even if the stable and unstable bundles are geometrically natural objects, the decomposition of a matrix into columns is coordinate dependent. Hence, one can choose a coordinate system which is reasonably close to the stable and unstable bundles, so that if we denote by U the change of coordinates, then the cocycle $\widetilde{\mathcal{M}}$ associated to the matrix

$$\widetilde{M}(\theta) = U(\theta+\omega)^{-1} M(\theta) U(\theta),$$

is close to constant. Notice that his is true only in the one-dimensional case. The picture is by far more involved when the bundles have higher rank.

This may seem somewhat circular, but the circularity can be broken using continuation methods. Given a cocycle which is close to constant, the fast iteration methods work and they allow us to compute the splitting. Once we have computed U for some M, we can use it to precondition the computation of the neighboring M.

5. **Computation of rank-1 stable and unstable bundles using iteration of cocycles.** The algorithms described in the previous section provide a fast way to iterate the cocycle. We will see that this iteration method, which is similar to the power method, gives the dominant eigenvalue $\lambda_{max}(\theta)$ and the corresponding eigenvector $m(\theta)$.

The methods based on iteration strongly rely on the fact that the cocycle has one dominating eigenvalue which is simple.

Consider that we have performed k iterations of the cocycle (assume that we perform scalings at each step) and we have computed $\mathcal{M}(n,\theta)$, with $n = 2^k$. Then, one can obtain the dominant rank-1 bundle from the QR decomposition of the cocycle $\mathcal{M}(n,\theta)$, just taking the column of Q associated to the largest value in the diagonal of the upper triangular matrix R. This provides a vector $m(\theta + 2^k\omega)$ (and therefore $m(\theta)$ by performing a shift of angle $-2^k\omega$) of modulus 1 spanning the unstable manifold. Since,

$$M(\theta)m(\theta) = \lambda_{max}(\theta)m(\theta+\omega),$$

we have

$$\lambda_{max}(\theta) = ([M(\theta)m(\theta)]^T[M(\theta)m(\theta)])^{1/2}.$$

As it is standard in the power method, we perform scalings at each step dividing all the entries in the matrix $M(\theta)$ by the maximum value among the entries of the matrix.

Hence, for the simplest case that there is one dominant eigenvalue, the method produces a section m (spanning the unstable sub-bundle) and a real function λ_{max}, which represents the dynamics on the rank 1 unstable sub-bundle, such that

$$M(\theta)m(\theta) = \lambda_{max}(\theta)m(\theta+\omega).$$

Following [10], under certain non-resonant conditions (which are satisfied in the case of the stable and unstable subspaces) one can reduce the 1-dimensional cocycle associated to to the matrix M and a rotation ω to a constant. Hence, we can look for a positive function p and a constant $\mu \in \mathbb{R}$, such that

$$\lambda_{max}(\theta)p(\theta) = \mu p(\theta + \omega). \tag{5.1}$$

Assuming that $\lambda_{max}(\theta) > 0$ (the case $\lambda_{max}(\theta) < 0$ is analogous), we can take logarithms on both sides of the equation (5.1). This leads to

$$\log \lambda_{max}(\theta) + \log p(\theta) = \log \mu + \log p(\theta + \omega),$$

and taking $\log \mu$ to be the average of $\log \lambda_{max}(\theta)$ the problem reduces to solve the equation for $\log p(\theta)$. Finally, $p(\theta)$ and μ can be obtained just exponentiating.

We note that the results of the above algorithms could well serve as input for the a-posteriori theorems described in [11, 8]. These a-posteriori theorems establish the existence of true invariant splittings provided that the computed solution is approximately invariant with respect to some condition numbers that can be explicitly computed. Hence, these theorems provide criteria to validate the computational results.

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REFERENCES

[1] E. Anderson, Z. Bai, C. Bischof, S. Blackford, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, and D. Sorensen. *LAPACK user's guide. This work is dedicated to Jim Wilkinson. 3rd ed.* Software - Environments - Tools. 9. Philadelphia, PA: SIAM, Society for Industrial and Applied Mathematics. xxi, 407 p. $ 39.00 , 1999.

[2] J. Bourgain. *Green's function estimates for lattice Schrödinger operators and applications*, volume 158 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2005.

[3] Renato Calleja and Rafael de la Llave. A numerically accessible criterion for the breakdown of quasi-periodic solutions and its rigorous justification. *Nonlinearity*, 23(9):2029–2058, 2010.

[4] Luca Dieci and Erik S. Van Vleck. Lyapunov spectral intervals: theory and computation. *SIAM J. Numer. Anal.*, 40(2):516–542 (electronic), 2002.

[5] L. H. Eliasson. Almost reducibility of linear quasi-periodic systems. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 679–705. Amer. Math. Soc., Providence, RI, 2001.

[6] J.-P. Eckmann and D. Ruelle. Ergodic theory of chaos and strange attractors. *Rev. Modern Phys.*, 57(3, part 1):617–656, 1985.

[7] Gene H. Golub and Charles F. Van Loan. *Matrix computations*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, third edition, 1996.

[8] Gemma Huguet, Rafael de la Llave, and Yannick Sire. Computation of whiskered invariant tori and their associated manifolds: new fast algorithms. *Discrete Contin. Dyn. Syst.*, 32(4):1309–1353, 2012.

[9] À. Haro and R. de la Llave. Manifolds on the verge of a hyperbolicity breakdown. *Chaos*, 16(1):013120, 8, 2006.

[10] À. Haro and R. de la Llave. A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: numerical algorithms. *Discrete Contin. Dyn. Syst. Ser. B*, 6(6):1261–1300 (electronic), 2006.
[11] A. Haro and R. de la Llave. A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: rigorous results. *J. Differential Equations*, 228(2):530–579, 2006.
[12] A. Haro and R. de la Llave. A parameterization method for the computation of whiskers in quasi periodic maps: numerical implementation and examples. *SIAM Jour. Appl. Dyn. Syst.*, 6(1):142–207, 2007.
[13] Raphaël Krikorian. C^0-densité globale des systèmes produits-croisés sur le cercle réductibles. *Ergodic Theory Dynam. Systems*, 19(1):61–100, 1999.
[14] Raphaël Krikorian. Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts. *Astérisque*, (259):vi+216, 1999.
[15] Kenneth R. Meyer and George R. Sell. Mel′nikov transforms, Bernoulli bundles, and almost periodic perturbations. *Trans. Amer. Math. Soc.*, 314(1):63–105, 1989.
[16] V. I. Oseledec. A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems. *Trudy Moskov. Mat. Obšč.*, 19:179–210, 1968.
[17] Leonid Pastur and Alexander Figotin. *Spectra of random and almost-periodic operators*, volume 297 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.
[18] J. Puig. Reducibility of linear differential equations with quasi-periodic coefficients: a survey. `http://www.ma.utexas.edu/mp_arc`, 02–246, 2002.
[19] Marek Rychlik. Renormalization of cocycles and linear ODE with almost-periodic coefficients. *Invent. Math.*, 110(1):173–206, 1992.
[20] R.J. Sacker. Existence of dichotomies and invariant splittings for linear differential systems. IV. *J. Differential Equations*, 27(1):106–137, 1978.
[21] R.J. Sacker and G.R. Sell. Existence of dichotomies and invariant splittings for linear differential systems. I. *J. Differential Equations*, 15:429–458, 1974.
[22] R.J. Sacker and G.R. Sell. Existence of dichotomies and invariant splittings for linear differential systems. II. *J. Differential Equations*, 22(2):478–496, 1976.
[23] R.J. Sacker and G.R. Sell. Existence of dichotomies and invariant splittings for linear differential systems. III. *J. Differential Equations*, 22(2):497–522, 1976.

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