SOLVABILITY CONDITIONS FOR SOME NON FREDHOLM OPERATORS IN A LAYER IN FOUR DIMENSIONS

Vitaly Volpert[1], **Vitali Vougalter**[2]

[1] Institute Camille Jordan, UMR 5208 CNRS, University Lyon 1
Villeurbanne, 69622, France
e-mail: volpert@math.univ-lyon1.fr

[2] Department of Mathematics and Applied Mathematics, University of Cape Town
Private Bag, Rondebosch 7701, South Africa
e-mail: Vitali.Vougalter@uct.ac.za

Abstract: We study solvability in H^2 of certain linear nonhomogeneous elliptic problems involving the sum of the periodic Laplacian and a Schrödinger operator without Fredholm property and prove that under reasonable technical conditions the convergence in L^2 of their right sides implies the existence and the convergence in H^2 of the solutions. We generalize the methods of spectral and scattering theory for Schrödinger type operators from our preceding work [17].

AMS Subject Classification: 35J10, 35P10, 47F05
Key words: solvability conditions, non Fredholm operators, Sobolev spaces

1. Introduction

Consider the problem

$$-\Delta u + V(x)u - au = f \tag{1.1}$$

with $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and $V(x)$ is a function converging to 0 at infinity. When $a \geq 0$, the essential spectrum of the operator $A : E \to F$, which corresponds to the left-hand side of equation (1.1) contains the origin. Consequently, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimensions of its kernel and the codimension of its image are not finite. Elliptic equations containing non-Fredholm operators were studied extensively in recent years (see [16], [17], [18], [19], [20], [22], [23], [24], also [5]) along with their potential applications to the theory of reaction-diffusion equations (see [7], [8]). In the case when $a = 0$ the operator A satisfies the Fredholm property in some properly chosen weighted spaces (see [1],

[2], [3], [4], [5]). However, the case of $a > 0$ is very different and the approach developed in the works above cannot be generalized.

One of the significant questions about problems with non-Fredholm operators concerns their solvability, which are studied in the following setting. Let f_n be a sequence of functions belonging to the image of the operator A, such that $f_n \to f$ in L^2 as $n \to \infty$. Let us denote by u_n a sequence of functions from H^2, such that

$$Au_n = f_n,\ n \in \mathbb{N}.$$

Since the operator A does not satisfy the Fredholm property, the sequence u_n may not be convergent. Let us call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of equation $Au = f$ (see [15]). If this sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this equation. Solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of non-Fredholm operators this convergence may not hold or it can occur in some weaker sense. In this case, solution in the sense of sequences may not imply the existence of the usual solution. Sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions are the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent. In our present work we generalize the results of [25] from the equation involving a single non Fredholm Schrödinger operator to its sum with the one dimensional Laplacian with the periodic boundary conditions. Note that a problem in a layer involving an operator without Fredholm property and with the periodicity on the sides was studied recently in [21] in the context of proving the existence of stationary solutions of a certain nonlocal reaction- diffusion type equation.

In the present work our domain is a product space in four dimensions

$$\Omega := \mathbb{R}^3 \times I = \mathbb{R}^3 \times [0, 2\pi],$$

such that the variables $x \in \mathbb{R}^3$ and $x_\perp \in I = [0, 2\pi]$. Let us consider the equation

$$-\frac{\partial^2 u}{\partial x_\perp^2} - \Delta_x u + V(x)u - au = f(x, x_\perp), \tag{1.2}$$

where $a \geq 0$ is a constant and the right side is square integrable. The cumulative Laplacian operator $\Delta := \dfrac{\partial^2}{\partial x_\perp^2} + \Delta_x$, where Δ_x acts only on the x variable. We will be using the functional space

$$H^2(\Omega) := \Big\{u(x, x_\perp) : \Omega \to \mathbb{C} \mid u(x, x_\perp) \in L^2(\Omega),\ \Delta u(x, x_\perp) \in L^2(\Omega),$$

$$u(x, 0) = u(x, 2\pi),\ \frac{\partial u}{\partial x_\perp}(x, 0) = \frac{\partial u}{\partial x_\perp}(x, 2\pi),\ x \in \mathbb{R}^3\ a.e.\}$$

equipped with the norm

$$\|u\|^2_{H^2(\Omega)} := \|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}.$$

The essential spectrum of our Schrödinger operator with a shallow, short-range potential (see Assumption 1 below) involved in (1.2) fills the semi-axis $[-a, \infty)$ (see e.g. [10]) such that its inverse from $L^2(\mathbb{R}^3)$ to $H^2(\mathbb{R}^3)$ is not bounded. The inner product of two functions

$$(f(x, x_\perp), g(x, x_\perp))_{L^2(\Omega)} := \int_\Omega f(x, x_\perp)\bar{g}(x, x_\perp)dxdx_\perp,$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f \in L^1(\Omega)$ and g is bounded, then evidently the integral considered above makes sense, like for instance in the case of functions involved in the orthogonality conditions of Theorem 2 below. Similarly, we will use

$$(f(x), g(x))_{L^2(\mathbb{R}^3)} := \int_{\mathbb{R}^3} f(x)\bar{g}(x)dx,$$

$$(f(x_\perp), g(x_\perp))_{L^2(I)} := \int_0^{2\pi} f(x_\perp)\bar{g}(x_\perp)dx_\perp.$$

The sphere of radius $r > 0$ in $\mathbb{R}^3$ centered at the origin will be denoted by S_r^3.

Let us make the following technical assumptions on the scalar potential $V(x)$ analogously to those stated in Assumption 1.1 of [17] (see also [18], [19]).

Assumption 1. *The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the estimate*

$$|V(x)| \leq \frac{C}{1 + |x|^{3.5+\delta}}$$

with some $\delta > 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \quad \textit{and} \quad \sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here and further down C stands for a finite positive constant and c_{HLS} given on p.98 of [12] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\left| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f_1(x)f_1(y)}{|x-y|^2}dxdy \right| \leq c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

According to Lemma 2.3 of [17], under Assumption 1 above on the potential function, the operator $-\Delta_x + V(x) - a$ on $L^2(\mathbb{R}^3)$ is self-adjoint and unitarily equivalent to $-\Delta_x - a$ via the wave operators (see [11], [14])

$$\Omega^\pm := s - \lim_{t\to\mp\infty} e^{it(-\Delta_x+V)}e^{it\Delta_x},$$

where the limit is understood in the strong L^2 sense (see e.g. [13] p.34, [6] p.90). Hence $-\Delta_x + V(x) - a$ on $L^2(\mathbb{R}^3)$ has only the essential spectrum $\sigma_{ess}(-\Delta_x +$

$V(x) - a) = [-a, \ \infty)$. By means of the spectral theorem, its functions of the continuous spectrum satisfying

$$[-\Delta_x + V(x)]\varphi_k(x) = k^2 \varphi_k(x), \quad k \in \mathbb{R}^3, \tag{1.3}$$

in the integral formulation the Lippmann-Schwinger equation for the perturbed plane waves (see e.g. [13] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} (V\varphi_k)(y) dy \tag{1.4}$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3 \tag{1.5}$$

form the complete system in $L^2(\mathbb{R}^3)$. In particular, when the vector $k = 0$, we have $\varphi_0(x)$. We denote the generalized Fourier transform with respect to these functions using the tilde symbol as

$$\tilde{f}(k) := (f(x), \varphi_k(x))_{L^2(\mathbb{R}^3)}, \ k \in \mathbb{R}^3.$$

The integral operator involved in (1.4) is being designated as

$$(Q\varphi)(x) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} (V\varphi)(y) dy, \quad \varphi \in L^\infty(\mathbb{R}^3).$$

Let us consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. Under Assumption 1, according to Lemma 2.1 of [17] the operator norm $\|Q\|_\infty < 1$, in fact it is bounded above by a quantity independent of k which is expressed in terms of the appropriate $L^p(\mathbb{R}^3)$ norms of the potential function $V(x)$. Our first main statement is as follows.

Theorem 2. *Let Assumption 1 hold,* $f(x, x_\perp) \in L^2(\Omega), \ |x| f(x, x_\perp) \in L^1(\Omega)$ *and* $f(x, 0) = f(x, 2\pi)$ *for* $x \in \mathbb{R}^3$ *a.e.*

a) When $a = 0$*, equation (1.2) admits a unique solution* $u(x, x_\perp) \in H^2(\Omega)$ *if and only if the orthogonality condition*

$$(f(x, x_\perp), \frac{1}{\sqrt{2\pi}} \varphi_0(x))_{L^2(\Omega)} = 0 \tag{1.6}$$

holds.

b) When $a = n_0^2, \ n_0 \in \mathbb{N}$ *equation (1.2) possesses a unique solution* $u(x, x_\perp) \in H^2(\Omega)$ *if and only if the orthogonality relations*

$$(f(x, x_\perp), \varphi_0(x) \frac{e^{inx_\perp}}{\sqrt{2\pi}})_{L^2(\Omega)} = 0, \quad n = \pm n_0, \tag{1.7}$$

$$(f(x,x_\perp),\varphi_k(x)\frac{e^{inx_\perp}}{\sqrt{2\pi}})_{L^2(\Omega)}=0, \quad k\in S^3_{\sqrt{n_0^2-n^2}} \quad a.e., \quad |n|\leq n_0-1 \tag{1.8}$$

hold.

c) When $n_0^2<a<(n_0+1)^2$, $n_0\in\mathbb{Z}^+=\{\mathbb{N}\}\cup\{0\}$ equation (1.2) has a unique solution $u(x,x_\perp)\in H^2(\Omega)$ if and only if the orthogonality condition

$$(f(x,x_\perp),\varphi_k(x)\frac{e^{inx_\perp}}{\sqrt{2\pi}})_{L^2(\Omega)}=0, \quad k\in S^3_{\sqrt{a-n^2}} \quad a.e., \quad |n|\leq n_0 \tag{1.9}$$

holds.

Note that orthogonality conditions from (1.6) to (1.9) involve the functions of the continuous spectrum of our Schrödinger operator, as distinct from the Limiting Absorption Principle in which one needs to orthogonalize to the standard Fourier harmonics (see e.g. Lemma 2.3 and Proposition 2.4 of [9]).

In the second part of the article we consider the sequence of equations corresponding to problem (1.2), namely

$$-\frac{\partial^2 u_m}{\partial x_\perp^2}-\Delta_x u_m+V(x)u_m-au_m=f_m(x,x_\perp), \quad m\in\mathbb{N}, \tag{1.10}$$

where $a\geq 0$ is a constant. Our second main result is as follows.

Theorem 3. *Let Assumption 1 hold, $m\in\mathbb{N}$, such that $f_m(x,0)=f_m(x,2\pi)$ for $x\in\mathbb{R}^3$ a.e., $f_m\in L^2(\Omega)$ and $|x|f_m(x,x_\perp)\in L^1(\Omega)$, such that $f_m\to f$ in $L^2(\Omega)$ and $|x|f_m(x,x_\perp)\to|x|f(x,x_\perp)$ in $L^1(\Omega)$ as $m\to\infty$. Let in the cases a), b) and c) of Theorem 2 the corresponding orthogonality conditions (1.6), (1.7), (1.8) and (1.9) hold for all f_m. Then problems (1.2) and (1.10) admit unique solutions $u(x,x_\perp)\in H^2(\Omega)$ and $u_m(x,x_\perp)\in H^2(\Omega)$ respectively, such that $u_m(x,x_\perp)\to u(x,x_\perp)$ in $H^2(\Omega)$ as $m\to\infty$.*

Our final technical statement will be helpful in establishing the result of the theorem above.

Lemma 4. *Let the assumptions of Theorem 3 hold. Then for every $n\in\mathbb{Z}$ we have*

$$\|\nabla_k(\tilde{f}_{m,n}(k)-\tilde{f}_n(k))\|_{L^\infty(\mathbb{R}^3)}\to 0, \quad m\to\infty$$

with $\tilde{f}_n(k)$ and $\tilde{f}_{m,n}(k)$ defined in formulas (2.11) and (3.21) respectively.

First of all, let us turn our attention to establishing the solvability conditions for problem (1.2).

2. Proof of the solvability conditions in a layer in four dimensions

Proof of Theorem 2. Note that according to our assumptions, the potential function involved in equation (1.2) is bounded and the right side of (1.2) is square integrable. Therefore, if we find a solution $u(x, x_\perp) \in L^2(\Omega)$ of problem (1.2), it will belong to $H^2(\Omega)$ as well.

Suppose equation (1.2) admits two solutions $u_1(x, x_\perp), u_2(x, x_\perp) \in H^2(\Omega)$. Then their difference function $w(x, x_\perp) := u_1(x, x_\perp) - u_2(x, x_\perp) \in H^2(\Omega)$ as well and solves the homogeneous problem

$$-\frac{\partial^2 w}{\partial x_\perp^2} - \Delta_x w + V(x)w - aw = 0.$$

Let us use the standard Fourier series expansion

$$w(x, x_\perp) = \sum_{n=-\infty}^{\infty} w_n(x)\frac{e^{inx_\perp}}{\sqrt{2\pi}}.$$

Clearly, $\|w\|^2_{L^2(\Omega)} = \sum_{n=-\infty}^{\infty} \|w_n\|^2_{L^2(\mathbb{R}^3)}$, such that $w_n(x) \in L^2(\mathbb{R}^3)$ for $n \in \mathbb{Z}$. We easily arrive at

$$-\Delta_x w_n(x) + V(x)w_n(x) = (a - n^2)w_n(x), \quad n \in \mathbb{Z}.$$

As discussed before, the Schrödinger operator on $L^2(\mathbb{R}^3)$ involved in the left side of the equation above has only the essential spectrum and no square integrable bound states. Hence, $w_n(x) = 0$ a.e. in $\mathbb{R}^3$ for $n \in \mathbb{Z}$. Therefore, $u_1(x, x_\perp) = u_2(x, x_\perp)$ a.e. in Ω.

We will be using the cumulative transform with $k \in \mathbb{R}^3$ and $n \in \mathbb{Z}$ as

$$\tilde{f}_n(k) := (f(x, x_\perp), \varphi_k(x)\frac{e^{inx_\perp}}{\sqrt{2\pi}})_{L^2(\Omega)} = \int_{\mathbb{R}^3} dx \int_0^{2\pi} dx_\perp f(x, x_\perp)\bar{\varphi}_k(x)\frac{e^{-inx_\perp}}{\sqrt{2\pi}}. \tag{2.11}$$

For the right side of (1.2) we estimate from above its norm using the Schwarz inequality as

$$\|f\|_{L^1(\Omega)} \leq \sqrt{\int_{|x|\leq 1} dx \int_0^{2\pi} dx_\perp |f(x, x_\perp)|^2}\sqrt{\int_{|\hat{x}|\leq 1} d\hat{x} \int_0^{2\pi} d\hat{x}_\perp} +$$

$$+\int_{|x|>1} dx \int_0^{2\pi} dx_\perp |x||f(x, x_\perp)| \leq C\|f\|_{L^2(\Omega)} + \||x|f\|_{L^1(\Omega)} < \infty$$

according to our assumptions. Thus, $f \in L^1(\Omega)$. By applying (2.11) to both sides of problem (1.2), we arrive at

$$\tilde{u}_n(k) = \frac{\tilde{f}_n(k)}{n^2 + k^2 - a}, \quad k \in \mathbb{R}^3, \quad n \in \mathbb{Z}, \quad a \geq 0. \tag{2.12}$$

First of all, let us turn our attention to the case a) of the theorem when $a = 0$. Then (2.12) yields

$$\tilde{u}_n(k) = \frac{\tilde{f}_n(k)}{n^2 + k^2}, \quad k \in \mathbb{R}^3, \quad n \in \mathbb{Z}.$$

Clearly

$$\|u\|^2_{L^2(\Omega)} = \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_0(k)|^2}{|k|^4} + \sum_{n\in\mathbb{Z},\ n\neq 0} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_n(k)|^2}{(n^2 + k^2)^2}. \tag{2.13}$$

Obviously, for the second term in the right side of (2.13) we have the upper bound of $\|f\|^2_{L^2(\Omega)} < \infty$ as assumed. To study the first term in the right side of (2.13), we introduce the auxiliary problem

$$-\Delta_x v_0(x) + V(x) v_0(x) = f_0(x) \tag{2.14}$$

with $f_0(x) = (f(x, x_\perp), \frac{1}{\sqrt{2\pi}})_{L^2(I)}$ and $\tilde{f}_0(k) = (f_0(x), \varphi_k(x))_{L^2(\mathbb{R}^3)},\ k \in \mathbb{R}^3$.

Evidently, $\tilde{v}_0(k) = \frac{\tilde{f}_0(k)}{k^2}$, such that the norm

$$\|v_0\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \frac{|\tilde{f}_0(k)|^2}{|k|^4} dk,$$

which equals to the first term in the right side of (2.13). Let us use the standard Fourier series expansion

$$f(x, x_\perp) := \sum_{n=-\infty}^{\infty} f_n(x) \frac{e^{inx_\perp}}{\sqrt{2\pi}},$$

such that

$$\|f\|^2_{L^2(\Omega)} = \sum_{n=-\infty}^{\infty} \|f_n\|^2_{L^2(\mathbb{R}^3)} < \infty \tag{2.15}$$

and therefore, $f_0(x) \in L^2(\mathbb{R}^3)$. We estimate

$$\||x|f_0(x)\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |x||(f(x, x_\perp), \frac{1}{\sqrt{2\pi}})_{L^2(I)}| dx \le \frac{1}{\sqrt{2\pi}} \||x|f\|_{L^1(\Omega)} < \infty$$

due to our assumption. Hence $|x|f_0(x) \in L^1(\mathbb{R}^3)$. Theorem 1.2 of [17] gives us the necessary and sufficient solvability condition of equation (2.14) in $L^2(\mathbb{R}^3)$, namely $(f_0(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0$, which is equivalent to (1.6).

Then we consider the case b) of the theorem, such that $a = n_0^2,\ n_0 \in \mathbb{N}$ and (2.12) yields

$$\tilde{u}_n(k) = \frac{\tilde{f}_n(k)}{n^2 + k^2 - n_0^2}, \quad k \in \mathbb{R}^3, \quad n \in \mathbb{Z}.$$

Let us express the norm

$$\|u\|^2_{L^2(\Omega)} = \sum_{n=\pm n_0} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_n(k)|^2}{|k|^4} +$$

$$+ \sum_{|n|>n_0} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_n(k)|^2}{(n^2+k^2-n_0^2)^2} + \sum_{|n|\le n_0-1} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_n(k)|^2}{(n^2+k^2-n_0^2)^2}. \tag{2.16}$$

We estimate the second term in the right side of (2.16) from above by $\|f\|^2_{L^2(\Omega)} < \infty$ according to one of our assumptions. To study the first term in the right side of (2.16) we introduce following auxiliary problem

$$-\Delta_x v_n(x) + V(x) v_n(x) = f_n(x), \quad n = \pm n_0. \tag{2.17}$$

By applying the generalized Fourier transform to both sides of (2.17), we arrive at $\tilde{v}_n(k) = \dfrac{\tilde{f}_n(k)}{k^2}$, such that the norm

$$\|v_n(x)\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_n(k)|^2}{|k|^4}.$$

Note that by means of (2.15) we have $f_n(x) \in L^2(\mathbb{R}^3)$ for $n \in \mathbb{Z}$. Let us estimate the norm

$$\||x| f_n(x)\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |x| |(f(x, x_\perp), \frac{e^{inx_\perp}}{\sqrt{2\pi}})_{L^2(I)}| dx \le \frac{1}{\sqrt{2\pi}} \||x| f\|_{L^1(\Omega)} < \infty$$

as assumed. Hence $|x| f_n(x) \in L^1(\mathbb{R}^3), \ n \in \mathbb{Z}$. By means of Theorem 1.2 of [17], the necessary and sufficient solvability condition of problem (2.17) in $L^2(\mathbb{R}^3)$ is given by $(f_n(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0, \ n = \pm n_0$, which is equivalent to (1.7).

Let us use the following auxiliary problem for the studies of the third term in the right side of (2.16)

$$-\Delta_x v_n(x) + V(x) v_n(x) - (n_0^2 - n^2) v_n(x) = f_n(x), \quad |n| \le n_0 - 1. \tag{2.18}$$

Note that $f_n(x) \in L^2(\mathbb{R}^3)$ and $|x| f_n(x) \in L^1(\mathbb{R}^3)$ as discussed above. Application of the generalized Fourier transform to both sides of (2.18) yields

$$\tilde{v}_n(k) = \frac{\tilde{f}_n(k)}{k^2 - (n_0^2 - n^2)},$$

such that

$$\|v_n(x)\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_n(k)|^2}{(k^2 + n^2 - n_0^2)^2}.$$

Theorem 1.2 of [17] gives us the necessary and sufficient solvability condition of equation (2.18) in $L^2(\mathbb{R}^3)$, namely

$$(f_n(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \; k \in S^3_{\sqrt{n_0^2-n^2}} \; a.e., \; |n| \leq n_0 - 1,$$

which is equivalent to orthogonality relation (1.8).

Finally, we turn our attention to the case c) of the theorem, when $n_0^2 < a < (n_0+1)^2, \; n_0 \in \mathbb{Z}^+ = \{\mathbb{N}\} \cup \{0\}$. Let us write the norm of our solution

$$\|u\|^2_{L^2(\Omega)} = \sum_{|n|\leq n_0} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_n(k)|^2}{(k^2+n^2-a)^2} + \sum_{|n|\geq n_0+1} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_n(k)|^2}{(k^2+n^2-a)^2}. \quad (2.19)$$

The second term in the right side of (2.19) can be trivially estimated from above by $\dfrac{1}{((n_0+1)^2-a)^2}\|f\|^2_{L^2(\Omega)} < \infty$ as assumed.

Let us introduce the following auxiliary equation for the purpose of studying the first term in the right side of (2.19), namely

$$-\Delta_x v_n(x) + V(x)v_n(x) - (a-n^2)v_n(x) = f_n(x), \quad |n| \leq n_0, \quad (2.20)$$

such that for its right side $f_n(x) \in L^2(\mathbb{R}^3)$ and $|x|f_n(x) \in L^1(\mathbb{R}^3)$ (see above). When applying the generalized Fourier transform to both sides of problem (2.20), we obtain

$$\tilde{v}_n(k) = \frac{\tilde{f}_n(k)}{k^2+n^2-a},$$

and therefore

$$\|v_n(x)\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_n(k)|^2}{(k^2+n^2-a)^2}.$$

According to Theorem 1.2 of [17], the necessary and sufficient solvability condition of equation (2.20) in $L^2(\mathbb{R}^3)$ is given by

$$(f_n(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0, \; k \in S^3_{\sqrt{a-n^2}} \; a.e., \; |n| \leq n_0,$$

which is equivalent to orthogonality condition (1.9). ∎

3. Solvability in the sense of sequences

Proof of Theorem 3. Under the assumptions of Theorem 3 by means of Theorem 2 problem (1.10) admits a unique solution $u_m(x, x_\perp) \in H^2(\Omega), \; m \in \mathbb{N}$. We have $f_m(x, x_\perp) \in L^1(\Omega), \; m \in \mathbb{N}$ (see the proof of Theorem 2). Let us estimate the norm via the Schwarz inequality

$$\|f_m - f\|_{L^1(\Omega)} \leq \sqrt{\int_{|x|\leq 1} dx \int_0^{2\pi} dx_\perp |f_m - f|^2} \sqrt{\int_{|x|\leq 1} dx \int_0^{2\pi} dx_\perp +}$$

$$+\int_{|x|>1}dx\int_0^{2\pi}dx_\perp|x||f_m-f|\le C\|f_m-f\|_{L^2(\Omega)}+$$
$$+\||x|f_m-|x|f\|_{L^1(\Omega)}\to 0,\quad m\to\infty$$

due to our assumptions. Hence $f_m\to f$ in $L^1(\Omega)$ as $m\to\infty$ and the limiting function $f(x,x_\perp)\in L^1(\Omega)$ as well. There is a subsequence $f_{m_k}\to f$ pointwise a.e. in Ω and therefore

$$f(x,0)=lim_{k\to\infty}f_{m_k}(x,0)=lim_{k\to\infty}f_{m_k}(x,2\pi)=f(x,2\pi)$$

a.e. in $\mathbb{R}^3$. Let us assume that the orthogonality condition

$$(f_m,w)_{L^2(\Omega)}=0,\quad m\in\mathbb{N}$$

holds for some $w(x,x_\perp)\in L^\infty(\Omega)$. Then we easily obtain

$$|(f,w)_{L^2(\Omega)}|=|(f-f_m,w)_{L^2(\Omega)}|\le\|f-f_m\|_{L^1(\Omega)}\|w\|_{L^\infty(\Omega)}\to 0,\quad m\to\infty,$$

such that $(f,w)_{L^2(\Omega)}=0$ as well. Note that via Corollary 2.2 of [17] the functions of the continuous spectrum of our Schrödinger operator are bounded and the argument above gives us that orthogonality conditions (1.6), (1.7), (1.8) and (1.9) valid for $f_m,\ m\in\mathbb{N}$ by assumption, will hold for the limiting function f as well. Then the limiting problem (1.2) has a unique solution $u(x,x_\perp)\in H^2(\Omega)$.

From equations (1.2) and (1.10) we easily deduce the inequality for $m\in\mathbb{N}$

$$\|\Delta(u_m-u)\|_{L^2(\Omega)}\le\|f_m-f\|_{L^2(\Omega)}+\|V\|_{L^\infty(\mathbb{R}^3)}\|u_m-u\|_{L^2(\Omega)}+a\|u_m-u\|_{L^2(\Omega)}.$$

Hence under our assumptions it will be sufficient to prove that $u_m\to u$ in $L^2(\Omega)$ as $m\to\infty$, which will imply that $u_m\to u$ in $H^2(\Omega)$ as $m\to\infty$ as well.

Let us first consider the case of $a=0$ and apply the cumulative Fourier transform to both sides of (1.10). We arrive at

$$\tilde{u}_{m,n}(k)=\frac{\tilde{f}_{m,n}(k)}{n^2+k^2},\quad k\in\mathbb{R}^3,\quad n\in\mathbb{Z},\quad m\in\mathbb{N}$$

with

$$\tilde{f}_{m,n}(k):=(f_m(x,x_\perp),\varphi_k(x)\frac{e^{inx_\perp}}{\sqrt{2\pi}})_{L^2(\Omega)}. \tag{3.21}$$

Let us express the norm

$$\|u_m-u\|^2_{L^2(\Omega)}=\int_{\mathbb{R}^3}dk\frac{|\tilde{f}_{m,0}(k)-\tilde{f}_0(k)|^2}{|k|^4}+\sum_{n\neq 0}\int_{\mathbb{R}^3}dk\frac{|\tilde{f}_{m,n}(k)-\tilde{f}_n(k)|^2}{(n^2+k^2)^2}. \tag{3.22}$$

The second term in the right side of (3.22) can be easily estimated from above by $\|f_m - f\|^2_{L^2(\Omega)} \to 0,\ m \to \infty$ as assumed. The first term in the right side of (3.22) can be expressed as

$$\int_{|k|\leq 1} dk \frac{|\tilde{f}_{m,0}(k) - \tilde{f}_0(k)|^2}{|k|^4} + \int_{|k|>1} dk \frac{|\tilde{f}_{m,0}(k) - \tilde{f}_0(k)|^2}{|k|^4}. \tag{3.23}$$

We easily derive the upper bound for the second expression in (3.23) as

$$\int_{|k|>1} dk |\tilde{f}_{m,0}(k) - \tilde{f}_0(k)|^2 \leq \|f_m - f\|^2_{L^2(\Omega)} \to 0,\ m \to \infty$$

due to our assumption. To estimate the remaining term we express

$$\tilde{f}_0(k) = \tilde{f}_0(0) + \int_0^{|k|} \frac{\partial \tilde{f}_0}{\partial |s|}(|s|, \omega) d|s|.$$

Here and further down ω will denote the angle variables on the sphere. Similarly

$$\tilde{f}_{m,0}(k) = \tilde{f}_{m,0}(0) + \int_0^{|k|} \frac{\partial \tilde{f}_{m,0}}{\partial |s|}(|s|, \omega) d|s|.$$

Note that $\tilde{f}_{m,0}(0)$ vanishes as assumed and $\tilde{f}_0(0) = 0$ as well, which can be obtained by taking $m \to \infty$ as discussed before. Using the formulas above, we easily estimate

$$\frac{|\tilde{f}_{m,0}(k) - \tilde{f}_0(k)|}{|k|^2} \leq \|\nabla_k(\tilde{f}_{m,0}(k) - \tilde{f}_0(k))\|_{L^\infty(\mathbb{R}^3)} \frac{1}{|k|}.$$

Then we trivially obtain the upper bound for the first expression in (3.23) as

$$4\pi \|\nabla_k(\tilde{f}_{m,0}(k) - \tilde{f}_0(k))\|^2_{L^\infty(\mathbb{R}^3)} \to 0, \quad m \to \infty$$

by means of Lemma 4, such that $u_m \to u$ in $L^2(\Omega)$ as $m \to \infty$ in the case of $a = 0$.

Then we turn our attention to the situation when $a = n_0^2,\ n_0 \in \mathbb{N}$. Thus, we have

$$\tilde{u}_{m,n}(k) = \frac{\tilde{f}_{m,n}(k)}{n^2 + k^2 - n_0^2}, \quad k \in \mathbb{R}^3, \quad n \in \mathbb{Z}, \quad m \in \mathbb{N}.$$

Let us write the norm

$$\|u_m - u\|^2_{L^2(\Omega)} = \sum_{n=\pm n_0} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_{m,n}(k) - \tilde{f}_n(k)|^2}{|k|^4} +$$

$$+ \sum_{|n|>n_0} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_{m,n}(k) - \tilde{f}_n(k)|^2}{(n^2 + k^2 - n_0^2)^2} + \sum_{|n|\leq n_0 - 1} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_{m,n}(k) - \tilde{f}_n(k)|^2}{(n^2 + k^2 - n_0^2)^2}. \tag{3.24}$$

The second term in the right side of (3.24) can be trivially estimated from above by $\|f_m - f\|^2_{L^2(\Omega)} \to 0,\ m \to \infty$. Let us write the first term in the right side of (3.24) as

$$\sum_{n=\pm n_0} \int_{|k|\leq 1} dk \frac{|\tilde{f}_{m,n}(k) - \tilde{f}_n(k)|^2}{|k|^4} + \sum_{n=\pm n_0} \int_{|k|>1} dk \frac{|\tilde{f}_{m,n}(k) - \tilde{f}_n(k)|^2}{|k|^4}. \quad (3.25)$$

The second expression in (3.25) can be trivially estimated from above by $\|f_m - f\|^2_{L^2(\Omega)} \to 0,\ m \to \infty$ as assumed. To study the first term in (3.25), we will use the expansions

$$\tilde{f}_n(k) = \tilde{f}_n(0) + \int_0^{|k|} \frac{\partial \tilde{f}_n}{\partial |s|}(|s|, \omega) d|s|,$$

$$\tilde{f}_{m,n}(k) = \tilde{f}_{m,n}(0) + \int_0^{|k|} \frac{\partial \tilde{f}_{m,n}}{\partial |s|}(|s|, \omega) d|s|.$$

Note that in the formula above $\tilde{f}_{m,n}(0) = 0,\ n = \pm n_0$ as assumed and $\tilde{f}_n(0) = 0$, which can be obtained via the limiting argument as $m \to \infty$ as discussed before. This yields

$$\frac{|\tilde{f}_{m,n}(k) - \tilde{f}_n(k)|}{|k|^2} \leq \frac{\|\nabla_k(\tilde{f}_{m,n}(k) - \tilde{f}_n(k))\|_{L^\infty(\mathbb{R}^3)}}{|k|},$$

which enables us to estimate from above the first term in (3.25) by

$$4\pi \sum_{n=\pm n_0} \|\nabla_k(\tilde{f}_{m,n}(k) - \tilde{f}_n(k))\|^2_{L^\infty(\mathbb{R}^3)} \to 0, \quad m \to \infty$$

via Lemma 4. Clearly, we have the trivial inequality for $|n| \leq n_0 - 1$

$$(k^2 - (n_0^2 - n^2))^2 \geq (|k| - \sqrt{n_0^2 - n^2})^2 (n_0^2 - n^2),$$

such that we have the upper bound for the last term in the right side of (3.24) as

$$\sum_{|n|\leq n_0-1} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_{m,n}(k) - \tilde{f}_n(k)|^2}{(n_0^2 - n^2)(|k| - \sqrt{n_0^2 - n^2})^2}.$$

For technical purposes, let us introduce the following set of spherical layers in the space of three dimensions

$$A_{n,\sigma} := \{k \in \mathbb{R}^3 \mid \sqrt{n_0^2 - n^2} - \sigma \leq |k| \leq \sqrt{n_0^2 - n^2} + \sigma\}, \quad |n| \leq n_0 - 1$$

with $0 < \sigma < \sqrt{n_0^2 - n^2}$. Thus, it remains to estimate the sum

$$\sum_{|n|\leq n_0-1} \frac{1}{n_0^2 - n^2} \int_{\mathbb{R}^3} dk \frac{|\tilde{f}_{m,n}(k) - \tilde{f}_n(k)|^2}{(|k| - \sqrt{n_0^2 - n^2})^2} \chi_{A_{n,\sigma}} +$$

$$+\sum_{|n|\le n_0-1}\frac{1}{n_0^2-n^2}\int_{\mathbb{R}^3}dk\frac{|\tilde{f}_{m,n}(k)-\tilde{f}_n(k)|^2}{(|k|-\sqrt{n_0^2-n^2})^2}\chi_{A_{n,\sigma}{}^c}. \tag{3.26}$$

Here and further down χ_A denotes the characteristic function of a set A and A^c stands for the complement of A in the space of three dimensions. For the second term in (3.26) we have the upper bound of

$$\sum_{|n|\le n_0-1}\frac{1}{(n_0^2-n^2)\sigma^2}\int_{\mathbb{R}^3}dk|\tilde{f}_{m,n}(k)-\tilde{f}_n(k)|^2\le\frac{1}{\sigma^2(2n_0-1)}\|f_m-f\|^2_{L^2(\Omega)}\to 0$$

as $m\to\infty$ according to our assumption. Hence it remains to estimate

$$\frac{1}{2n_0-1}\sum_{|n|\le n_0-1}\int_{\mathbb{R}^3}dk\frac{|\tilde{f}_{m,n}(k)-\tilde{f}_n(k)|^2}{(|k|-\sqrt{n_0^2-n^2})^2}\chi_{A_{n,\sigma}}. \tag{3.27}$$

For this purpose we express

$$\tilde{f}_n(k)=\tilde{f}_n(\sqrt{n_0^2-n^2},\omega)+\int_{\sqrt{n_0^2-n^2}}^{|k|}\frac{\partial\tilde{f}_n}{\partial|s|}(|s|,\omega)d|s|,$$

$$\tilde{f}_{m,n}(k)=\tilde{f}_{m,n}(\sqrt{n_0^2-n^2},\omega)+\int_{\sqrt{n_0^2-n^2}}^{|k|}\frac{\partial\tilde{f}_{m,n}}{\partial|s|}(|s|,\omega)d|s|.$$

Evidently, for $|n|\le n_0-1$ we have $\tilde{f}_{m,n}(\sqrt{n_0^2-n^2},\omega)=0$ due to our assumption and $\tilde{f}_n(\sqrt{n_0^2-n^2},\omega)$ vanishes as well, which can be obtained by letting $m\to\infty$ as discussed before. Then the expansions above will give us

$$\frac{|\tilde{f}_{m,n}(k)-\tilde{f}_n(k)|}{||k|-\sqrt{n_0^2-n^2}|}\le\|\nabla_k(\tilde{f}_{m,n}(k)-\tilde{f}_n(k))\|_{L^\infty(\mathbb{R}^3)}.$$

Finally, for (3.27) we derive the upper bound of

$$\frac{1}{2n_0-1}\sum_{|n|\le n_0-1}\|\nabla_k(\tilde{f}_{m,n}(k)-\tilde{f}_n(k))\|^2_{L^\infty(\mathbb{R}^3)}C_{n,\sigma}\to 0,\quad m\to\infty$$

via Lemma 4. Here $C_{n,\sigma}:=\frac{4\pi}{3}\{(\sqrt{n_0^2-n^2}+\sigma)^3-(\sqrt{n_0^2-n^2}-\sigma)^3\}$. Hence we arrive at $u_m\to u$ in $L^2(\Omega)$ as $m\to\infty$ when $a=n_0^2,\ n_0\in\mathbb{N}$.

We conclude the proof with the studies of the situation when $n_0^2<a<(n_0+1)^2$ with $n_0\in\mathbb{Z}^+=\{\mathbb{N}\}\cup\{0\}$. Evidently

$$\tilde{u}_{m,n}(k)=\frac{\tilde{f}_{m,n}(k)}{n^2+k^2-a},\quad k\in\mathbb{R}^3,\quad n\in\mathbb{Z},\quad m\in\mathbb{N}.$$

Let us express the norm $\|u_m-u\|^2_{L^2(\Omega)}$ as the sum

$$\sum_{|n|\le n_0}\int_{\mathbb{R}^3}dk\frac{|\tilde{f}_{m,n}(k)-\tilde{f}_n(k)|^2}{(n^2+k^2-a)^2}+\sum_{|n|\ge n_0+1}\int_{\mathbb{R}^3}dk\frac{|\tilde{f}_{m,n}(k)-\tilde{f}_n(k)|^2}{(n^2+k^2-a)^2}. \qquad (3.28)$$

Apparently, the second term in (3.28) can be estimated from above by

$$\frac{1}{((n_0+1)^2-a)^2}\|f_m-f\|^2_{L^2(\Omega)}\to 0,\ m\to\infty$$

by our assumption. We will be using the trivial inequality for $|n|\le n_0$

$$(k^2-(a-n^2))^2\ge(a-n^2)(|k|-\sqrt{a-n^2})^2.$$

Let us introduce the set of spherical layers in the space of three dimensions

$$B_{n,\sigma}:=\{k\in\mathbb{R}^3\mid \sqrt{a-n^2}-\sigma\le|k|\le\sqrt{a-n^2}+\sigma\},\quad |n|\le n_0$$

with $0<\sigma<\sqrt{a-n^2}$. Then the first term in (3.28) can be bounded from above by the sum

$$\frac{1}{a-n_0^2}\sum_{|n|\le n_0}\int_{\mathbb{R}^3}dk\frac{|\tilde{f}_{m,n}(k)-\tilde{f}_n(k)|^2}{(|k|-\sqrt{a-n^2})^2}\chi_{B_{n,\sigma}}+$$
$$+\frac{1}{a-n_0^2}\sum_{|n|\le n_0}\int_{\mathbb{R}^3}dk\frac{|\tilde{f}_{m,n}(k)-\tilde{f}_n(k)|^2}{(|k|-\sqrt{a-n^2})^2}\chi_{B^c_{n,\sigma}}. \qquad (3.29)$$

For the second term in (3.29) we have the upper bound of

$$\frac{1}{\sigma^2(a-n_0^2)}\|f_m-f\|^2_{L^2(\Omega)}\to 0,\ m\to\infty$$

as assumed. To estimate the remaining term in (3.29), we will use the representation formulas

$$\tilde{f}_n(k)=\tilde{f}_n(\sqrt{a-n^2},\omega)+\int_{\sqrt{a-n^2}}^{|k|}\frac{\partial\tilde{f}_n}{\partial|s|}(|s|,\omega)d|s|,$$
$$\tilde{f}_{m,n}(k)=\tilde{f}_{m,n}(\sqrt{a-n^2},\omega)+\int_{\sqrt{a-n^2}}^{|k|}\frac{\partial\tilde{f}_{m,n}}{\partial|s|}(|s|,\omega)d|s|.$$

Note that $\tilde{f}_{m,n}(\sqrt{a-n^2},\omega)=0$ for $|n|\le n_0$ via (1.9) and $\tilde{f}_n(\sqrt{a-n^2},\omega)$ vanishes as well which can be obtained by letting $m\to\infty$ as discussed before. Hence

$$\frac{|\tilde{f}_{m,n}(k)-\tilde{f}_n(k)|}{||k|-\sqrt{a-n^2}|}\le\|\nabla_k(\tilde{f}_{m,n}(k)-\tilde{f}_n(k))\|_{L^\infty(\mathbb{R}^3)},$$

such that the first term in (3.29) can be bounded from above by

$$\frac{1}{a-n_0^2}\sum_{|n|\le n_0}\|\nabla_k(\tilde{f}_{m,n}(k)-\tilde{f}_n(k))\|^2_{L^\infty(\mathbb{R}^3)}D_{n,\sigma}\to 0, \quad m\to\infty$$

via Lemma 4. Here $D_{n,\sigma}:=\frac{4}{3}\pi\{(\sqrt{a-n^2}+\sigma)^3-(\sqrt{a-n^2}-\sigma)^3\}$. Therefore, $u_m\to u$ in $L^2(\Omega)$ as $m\to\infty$ in the case of $n_0^2<a<(n_0+1)^2$ with $n_0\in\mathbb{Z}^+=\{\mathbb{N}\}\cup\{0\}$ as well. ∎

We conclude the paper with establishing the result of the technical Lemma 4 used in the proof of Theorem 3 above.

Proof of Lemma 4. We have $f_n(x)\in L^2(\mathbb{R}^3)$ and $|x|f_n(x)\in L^1(\mathbb{R}^3)$ for $n\in\mathbb{Z}$ as discussed before. We use the standard Fourier series expansion

$$f_m(x,x_\perp)=\sum_{n=-\infty}^{\infty}f_{m,n}(x)\frac{e^{inx_\perp}}{\sqrt{2\pi}}, \quad m\in\mathbb{N}$$

such that

$$\|f_m(x,x_\perp)\|^2_{L^2(\Omega)}=\sum_{n=-\infty}^{\infty}\|f_{m,n}(x)\|^2_{L^2(\mathbb{R}^3)}<\infty$$

according to our assumption. Hence $f_{m,n}(x)\in L^2(\mathbb{R}^3)$ for $n\in\mathbb{Z}$ and $m\in\mathbb{N}$. Let us estimate the norm

$$\||x|f_{m,n}(x)\|_{L^1(\mathbb{R}^3)}=\int_{\mathbb{R}^3}dx|x||(f_m(x,x_\perp),\frac{e^{inx_\perp}}{\sqrt{2\pi}})_{L^2(I)}|\le$$

$$\le\frac{1}{\sqrt{2\pi}}\||x|f_m(x,x_\perp)\|_{L^1(\Omega)}<\infty$$

as assumed. Therefore, $|x|f_{m,n}(x)\in L^1(\mathbb{R}^3)$ for $n\in\mathbb{Z}$ and $m\in\mathbb{N}$. Moreover,

$$\|f_{m,n}(x)-f_n(x)\|^2_{L^2(\mathbb{R}^3)}=\int_{\mathbb{R}^3}dx|\int_0^{2\pi}[f_m(x,x_\perp)-f(x,x_\perp)]\frac{e^{-inx_\perp}}{\sqrt{2\pi}}dx_\perp|^2\le$$

$$\le\|f_m-f\|^2_{L^2(\Omega)}\to 0, \quad m\to\infty$$

due to our assumption and the Schwarz inequality. Thus $f_{m,n}(x)\to f_n(x)$ in $L^2(\mathbb{R}^3)$ as $m\to\infty$. Furthermore,

$$\||x|f_{m,n}(x)-|x|f_n(x)\|_{L^1(\mathbb{R}^3)}=\int_{\mathbb{R}^3}dx|x||(f_m(x,x_\perp)-f(x,x_\perp),\frac{e^{inx_\perp}}{\sqrt{2\pi}})_{L^2(I)}|\le$$

$$\le\frac{1}{\sqrt{2\pi}}\||x|f_m-|x|f\|_{L^1(\Omega)}\to 0,\ m\to\infty$$

as assumed, such that $|x|f_{m,n}(x)\to|x|f_n(x)$ in $L^1(\mathbb{R}^3)$ as $m\to\infty$ for each $n\in\mathbb{Z}$. Then the statement of the lemma follows from Lemma 3.4 of [25]. ∎

References

[1] C. Amrouche, V. Girault, J. Giroire, *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator. An approach in weighted Sobolev spaces*, J. Math, Pures Appl., **76** (1997), 55–81.

[2] C. Amrouche, F. Bonzom, Mixed exterior Laplace's problem, J. Math. Anal. Appl., **338** (2008), 124–140.

[3] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Holderienne pour des opérateurs différentiels elliptiques dans R^n*, J. Math. Pures Appl., **72** (1993), 105–119.

[4] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Hölderienne pour le problème extérieur de Dirichlet*, Comm. Partial Differential Equations, **26** (2001), No. 1-2, 315-334.

[5] N. Benkirane, *Propriété d'indice en théorie Holderienne pour des opérateurs elliptiques dans R^n*, CRAS, **307**, Série I (1988), 577–580.

[6] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Springer-Verlag, Berlin (1987).

[7] A. Ducrot, M. Marion, V. Volpert, *Systemes de réaction-diffusion sans propriété de Fredholm*, CRAS, **340** (2005), 659–664.

[8] A. Ducrot, M. Marion, V. Volpert, *Reaction-diffusion problems with non Fredholm operators*, Advances Diff. Equations, **13** (2008), No. 11-12, 1151–1192.

[9] M. Goldberg, W. Schlag, *A limiting absorption principle for the three-dimensional Schrödinger equation with L^p potentials*, Int. Math. Res. Not., (2004), No. 75, 4049–4071.

[10] B.L.G. Jonsson, M. Merkli, I.M. Sigal, F. Ting, *Applied Analysis*, In preparation.

[11] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann., **162** (1965/1966), 258–279.

[12] E. Lieb, M. Loss, *Analysis. Graduate Studies in Mathematics*, **14**, American Mathematical Society, Providence (1997).

[13] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, III: Scattering Theory*, Academic Press (1979).

[14] I. Rodnianski, W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math., **155** (2004), No. 3, 451–513.

[15] V. Volpert. Elliptic partial differential equations. Volume 1. Fredholm theory of elliptic problems in unbounded domains. Birkhauser, 2011.

[16] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski, *Solvability conditions for elliptic problems with non-Fredholm operators*, Appl.Math., **29** (2002), No. 2, 219–238.

[17] V. Vougalter, V. Volpert, *Solvability conditions for some non Fredholm operators*, Proc. Edinb. Math. Soc. (2), **54** (2011), No.1, 249–271

[18] V. Vougalter, V. Volpert. *On the solvability conditions for some non Fredholm operators,* Int. J. Pure Appl. Math., **60** (2010), No. 2, 169–191.

[19] V. Vougalter, V. Volpert. *On the solvability conditions for the diffusion equation with convection terms*, Commun. Pure Appl. Anal., **11** (2012), No. 1, 365–373.

[20] V. Vougalter, V. Volpert. *Solvability relations for some non Fredholm operators,* Int. Electron. J. Pure Appl. Math., **2** (2010), No. 1, 75–83.

[21] V. Vougalter, V. Volpert. *On the existence of stationary solutions for some non-Fredholm integro-differential equations,* Doc. Math., **16** (2011), 561–580.

[22] V. Volpert, V. Vougalter. *On the solvability conditions for a linearized Cahn-Hilliard equation,* Rend. Istit. Mat. Univ. Trieste, **43** (2011), 1–9.

[23] V. Vougalter, V. Volpert. *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order,* Math. Model. Nat. Phenom., **7** (2012), No. 2, 146–154.

[24] V. Vougalter, V. Volpert. *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2** (2012), No.4, 473–496.

[25] V. Volpert, V. Vougalter. *Solvability in the sense of sequences to some non-Fredholm operators,* Electron. J. Differential Equations, **2013** (2013), No.160, 1–16.