On Gevrey solutions of threefold singular nonlinear partial differential equations

Alberto Lastra, Stéphane Malek and Javier Sanz

July 26, 2013

Abstract

We study Gevrey asymptotics of the solutions to a family of threefold singular nonlinear partial differential equations in the complex domain. We deal with both Fuchsian and irregular singularities, and allow the presence of a singular perturbation parameter. By means of the Borel-Laplace summation method, we construct sectorial actual holomorphic solutions which turn out to share a same formal power series as their Gevrey asymptotic expansion in the perturbation parameter. This result rests on the Malgrange-Sibuya theorem, and it requires to prove that the difference between two neighboring solutions is exponentially small, what in this case involves an asymptotic estimate for a particular Dirichlet-like series.

Key words: Nonlinear partial differential equations, singular perturbations, formal power series, Borel-Laplace transform, Borel summability, Gevrey asymptotic expansions.
2010 MSC: 35C10, 35C20.

1 Introduction

We study a family of threefold singular nonlinear partial differential equations of the following form

$$
(1) \quad \left((z\partial_z + 1)^{r_1} \epsilon^{r_3}(t^2\partial_t + t)^{r_2} + 1\right)\partial_z^S X(t,z,\epsilon) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{s,\kappa_0,\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X)(t,z,\epsilon)
$$
$$
+ P(t,z,\epsilon,X(t,z,\epsilon)),
$$

for given initial conditions

$$
(2) \quad (\partial_z^j X)(t,0,\epsilon) = \varphi_j(t,\epsilon), \quad 0 \le j \le S-1.
$$

The elements r_1, r_2, r_3, S are nonnegative integers (i.e. belong to $\mathbb{N} = \{0,1,\ldots\}$) with $r_2, S \ge 1$. $\mathcal{S}$ consists of a finite number of tuples (s, κ_0, κ_1) verifying $\kappa_1 < S$. The coefficients $b_{s,\kappa_0,\kappa_1}(z,\epsilon)$ of the linear part in (1) belong to $\mathcal{O}\{z,\epsilon\}$ for every $(s, \kappa_0, \kappa_1) \in \mathcal{S}$. Here, $\mathcal{O}\{z,\epsilon\}$ stands for the set of holomorphic functions near the origin of $\mathbb{C}^2$ in the variables (z,ϵ). In addition to this, $P(t,z,\epsilon,X)$ is a polynomial in the variables t and X with coefficients belonging to $\mathcal{O}\{z,\epsilon\}$. In this problem, ϵ plays the role of a complex perturbation parameter near $0 \in \mathbb{C}$. The initial data $\varphi_j(t,\epsilon)$ in (2) are assumed to be holomorphic functions on a product of two sectors with vertex at the origin of $\mathbb{C}$ and finite radius.

The family of problems considered in this work constitutes a generalization of the one studied by the second author in [26] in mainly three respects. The first improvement concerns the fuchsian operator $(z\partial_z + 1)^{r_1}$ which now appears on the lefthand side of the main equation. The research of fuchsian singularities in the framework of partial differential equations is widely developed, we provide [1], [7], [19], [21], [30], [34] and [39] as examples of references in this direction. The second point under consideration has to do with the irregular operator $(t^2\partial_t + t)^{r_2}$. In the present work, a wider choice of r_2 is allowed with respect to [26], where only $r_2 = 1$ was considered. The family of equations provided by (1) rests on the class of partial differential equations with an irregular singularity at $t = 0$ (in the sense of [29]). The solutions of such equations and their asymptotic properties are studied in [10], [11], [29], [32], [33] among others. The last generalization deals with the freedom on the choice of the powers of the complex perturbation parameter ϵ, which in [26] was taken to be $r_3 = 1$.

The present work lies within the framework of the asymptotic analysis of singular perturbations of initial value problems

$$(3) \qquad \epsilon L_2(t, z, \partial_t, \partial_z)[u(t, z, \epsilon)] + L_1(u(t, z, \epsilon)) = 0,$$

where L_2 is a linear differential operator and L_1 is a nonlinear differential operator, for given initial data $(\partial_z^j u)(t, 0, \epsilon) = h_j(t, \epsilon)$, $0 \le j \le \nu - 1$ belonging to some function spaces. Most of the results one can find in the literature are related to the study of (3) when ϵ is real and L_2 is an elliptic or a hyperbolic operator of second order which may act on real spaces of functions such as infinitely smooth functions $\mathcal{C}^\infty(\mathbb{R}^d)$ or Sovolev spaces $H^s(\mathbb{R}^d)$. These results provide sufficient conditions for a solution $u(t, x, \epsilon)$ of (3) to admit an asymptotic expansion of the form

$$u(t, x, \epsilon) = \sum_{i=0}^{n-1} w_i(t, x)\epsilon^i + R_n(t, x, \epsilon),$$

giving bounds for every remainder R_n, and they are based on semi-group operator methods (see [17]), the maximum principle and energy integrals estimates (see [22], [31]), or fixed point theorems for the nonlinear equations (see [18], [22]). In [23] a general survey is exhibited on singular perturbations under both, asymptotic and numerical points of view. Although the papers by M. Canalis-Durand, J. Mozo-Fernandez and R. Schäfke [9], and by the second author [25, 26] consider these type of problems when working with a complex perturbation parameter ϵ, with solutions in spaces of analytic functions, and for partial differential equations which are singular, as far as we know no treatment can be found in the literature dealing with a singularly perturbed partial differential equation which additionally involves a Fuchsian and an irregular singularity.

The main aim of this paper is to construct actual holomorphic solutions $X(t, z, \epsilon)$ of (1)-(2) and to get conditions for existence and uniqueness of the asymptotic expansion

$$X(t, z, \epsilon) = \sum_{\kappa=0}^{n-1} H_\kappa(t, z)\frac{\epsilon^\kappa}{\kappa!} + R_n(t, z, \epsilon),$$

where the remainder $R_n(t, z, \epsilon)$ is bounded in terms of a Gevrey sequence of certain order $\alpha > 0$, it is to say, there exist $C, M > 0$ such that

$$|R_n(t, z, \epsilon)| \le C M^n n!^\alpha |\epsilon|^n, \quad n = 1, 2, \ldots$$

for every ϵ on a sector, uniformly in (t, z) on a product of a sector and an appropriate disc centered at 0. In this case, compared to [26], the Gevrey order 1 will turn out to become

$(r_1 + r_2)/r_3$, owing to the positive power of ϵ and the appearance of the fuchsian and irregular operators $(z\partial_z + 1)^{r_1}$ and $(t^2\partial_t + t)^{r_2}$, respectively.

The strategy followed is similar to the one shown in [26], which in turn rests on the one in [9], by M. Canalis-Durand, J. Mozo-Fernández and R. Schäfke. It consists of transforming the problem (1), by means of the linear map $t \mapsto t/\epsilon^{r_3/r_2}$, into an auxiliary regularly perturbed nonlinear partial differential equation which has an irregular singularity at $t = 0$, preserving the fuchsian singularity at $z = 0$, see (34). This transformation induces the existence of poles in the coefficients of the new problem with respect to ϵ at 0.

The solution of (1)-(2) is constructed by handling a resummation procedure of formal power series, known as κ-summability. This process is widely used when working with Gevrey asymptotic expansions of analytic solutions of linear and nonlinear differential equations with irregular singularities. See, for example, [4], [8], [14], [16], [28], [35], [36]. The formal solution of the auxiliary problem (34),

$$\hat{Y}(t, z, \epsilon) = \sum_{m \geq 0} Y_m(z, \epsilon) \frac{t^m}{m!}$$

is such that, for every ϵ, its Borel transform of order 1 with respect to t,

$$V_\epsilon(\tau, z) = \sum_{\beta \geq 0} V_{\beta,\epsilon}(z) \frac{\tau^\beta}{(\beta!)^2}, \quad V_{\beta,\epsilon}(z) := Y_\beta(z, \epsilon),$$

satisfies a new singular Cauchy problem, see (14)-(15). This turns out to be a non-linear convolution integro-differential Cauchy problem with rational coefficients in τ, holomorphic in (τ, z) near the origin and meromorphic in ϵ with poles at 0.

For suitably chosen initial data (see Proposition 3), there exists $\rho > 0$ such that the function $V(\tau, z, \epsilon) := V_\epsilon(\tau, z)$ defines a holomorphic function on $S_d \times D(0, \rho) \times \mathcal{E}$, where S_d is a suitable open sector with vertex at the origin, infinite radius, bisecting direction d and small opening, $D(0, \rho) := \{z \in \mathbb{C} : |z| < \rho\}$ and $\mathcal{E}$ is a sector of finite radius with vertex at the origin. In addition to this, one can prove that $V_\beta(z, \epsilon) := V_{\beta,\epsilon}(z)$ verifies adequate estimates on the variable τ for every $\beta \geq 0$ so that Laplace transform $\mathcal{L}^d$ can be applied, leading to a solution of (34)-(35). While in [26] the only forbidden direction is π, here we come up with some other more (see Assumption (A.1)). However, Assumption (B) in [26] remains unchanged in the present paper, so that every equation studied in [26] fits in the family taken into account now. For every $\epsilon \in \mathcal{E}$, the function

$$(t, z) \mapsto Y_{d,\epsilon}(t, z) = \sum_{\beta \geq 0} \mathcal{L}_\tau^d(V_\beta(\tau, \epsilon))(t) \frac{z^\beta}{\beta!}$$

defines a holomorphic function in a sector of finite radius depending on ϵ, times a neighborhood of the origin (see Theorem 1). It is worth saying that the bounds verified by the coefficients $V_\beta(z, \epsilon)$ are obtained by means of fixed point arguments in certain well chosen weighted Banach spaces of holomorphic functions, see Section 2.

Moreover, if one chooses appropriate initial conditions and a finite family of sectors $\{\mathcal{E}_i\}_{i \in I}$ whose union forms a good covering (see Definition 3), $X_i(t, z, \epsilon) := Y_{d,\epsilon}(\epsilon^{\frac{r_3}{r_2}} t, z)$ turns out to be a holomorphic and bounded solution of the main problem (1)-(2) on $\mathcal{T} \times D(0, \frac{\rho}{2}) \times \mathcal{E}_i$, for every $i \in I$, where $\mathcal{T}$ is an open sector with vertex at 0 and finite radius. Actually, the difference $|X_i(t, z, \epsilon) - X_{i+1}(t, z, \epsilon)|$ tends to 0 in the variable ϵ faster than $\exp(-M_i/|\epsilon|^{A_1})$, for $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$, uniformly in the other variables and where $M_i > 0$ and $A_1 = r_3/(r_1 + r_2)$ (see Theorem 2). The

procedure followed at this stage rests on a careful estimation of Dirichlet-like series of the form

$$(4) \qquad \sum_{\kappa \geq 0} e^{-\frac{1}{(\kappa+1)^\alpha}\frac{1}{\epsilon}} a^\kappa,$$

where $0 < a < 1$, $\alpha > 0$ and ϵ is small. It is worth mentioning that general Dirichlet series

$$\sum_{n=0}^{\infty} a_n e^{-\lambda_n z}$$

have been thoroughly studied in the case when $\{\lambda_n\}_{n=0}^{\infty}$ is an increasing sequence of real numbers tending to $+\infty$ (see [20, 38, 2]), or a sequence of complex numbers with $|\lambda_n| \to \infty$ (see [24]), but (4) does not fit in these situations. Also, there is a well-known theory about almost periodic functions, introduced by H. Bohr (see [6, 5, 13]), which are the uniform limits in $\mathbb{R}$ of exponential polynomials $\sum_{k=1}^{n} a_k e^{i s_k x}$, where the values s_k belong to the so-called spectrum $\Lambda \subset \mathbb{R}$. However, in our case we would be rather interested in the asymptotic behaviour of the sum when x tends to ∞ in the positive imaginary axis. Our technique will finally rest on Euler-Maclaurin formula, Watson's Lemma and the equivalence between null Gevrey asymptotics and exponential smallness.

The main result of the present work (Theorem 4) establishes the existence of a formal power series

$$\hat{X}(\epsilon) = \sum_{\kappa \geq 0} H_\kappa \frac{\epsilon^\kappa}{\kappa!},$$

with coefficients in the Banach space of holomorphic and bounded funcions on $\mathcal{T} \times D(0, \frac{\rho}{2})$ with the supremum norm, which is the formal solution of (1). $\hat{X}$ is the Gevrey asymptotic expansion of order $(r_1 + r_2)/r_3$, of the functions X_i on $\mathcal{E}_i$, for every $i \in I$. This last result is based on a cohomological criterion of B. Malgrange and Y. Sibuya (Theorem (**MS**)).

The paper is organized as follows.

Section 2 is devoted to the study of the behavior of several operators acting on the elements in a family of weighted Banach spaces. These results are applied in Section 3 when searching for the solution of a parameter-depending nonlinear convolution differential Cauchy problem with singular coefficients.

After a brief introduction from the classical theory of Borel-Laplace transforms, and after establishing some commutation formulas with multiplication and integro-differential operators in Section 4.1, we focus our attention on finding a solution of a nonlinear Cauchy problem with irregular and regular singularities and whose coefficients have polar singularities. This is performed in Section 4.2. The link between the Cauchy problem solved in Section 3 and this one is established by means of Borel-Laplace transform on the corresponding solutions.

In Section 5.1 we construct actual solutions X_i, $i \in I$, of our initial problem and prove exponential flatness of the difference of two of these solutions with respect to ϵ in the intersection of their domain of definition, uniformly in the other variables.

Finally, in Section 5.2 we conclude with the main result of the present work, leading to the existence of a formal power series of the variable ϵ with coefficients in an appropriate Banach space of functions which formally solves equation (1) and is the Gevrey asymptotic expansion of the functions X_i of a certain order, for every $i \in I$.

2 Weighted Banach spaces of holomorphic functions on sectors

In what follows, for an open set $U \subset \mathbb{C}$, $\mathcal{O}(U)$ denotes the set of complex holomorphic functions in U. We consider an open sector $\mathcal{E}$ with vertex at the origin and finite radius $r_\mathcal{E} > 0$, and also an open sector S_d centered at 0 with infinite radius and bisecting direction $d \in \mathbb{R}$. Let $(\rho_\beta)_{\beta \geq 0}$ be a sequence of positive real numbers. In the following, Ω_β stands for the set $S_d \cup D(0, \rho_\beta)$, where $D(0, \rho_\beta)$ is the open disc centered at 0 with radius ρ_β, for every $\beta \geq 0$.

Throughout this section, b and r are fixed positive real numbers with $b > 1$. In addition to this, r_1 and r_2 stand for fixed nonnegative integers with $r_2 \geq 1$ and σ is a positive real number.

The following definition of the norms in these weighted Banach spaces heavily rests on the one appearing in [26]. These norms were appropriate modifications of those defined by O. Costin in [15] and C. Stenger and the second author in [27].

Definition 1. *For every $\beta \geq 0$ and $\epsilon \in \mathcal{E}$ $F_{\beta,\epsilon,\sigma,\Omega_\beta}$ denotes the vector space of holomorphic functions v defined in Ω_β such that*

$$\|v(\tau)\|_{\beta,\epsilon,\sigma,\Omega_\beta} := \sup_{\tau \in \Omega_\beta} \left\{ |v(\tau)| \left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}} \right) e^{-\frac{\sigma}{|\epsilon|^r} r_b(\beta)|\tau|} \right\} < \infty,$$

where $r_b(\beta) := \sum_{n=0}^{\beta} \frac{1}{(n+1)^b}$.

Assumption (A):

A.1 Sector S_d is such that

$$\left(\frac{2k+1}{r_2} \right) \pi, \quad k = 0, \ldots, r_2 - 1$$

differs from the arguments of the elements in $\overline{S_d} \setminus \{0\}$.

A.2 $\rho_\beta := \frac{1}{2(\beta+1)^{\frac{r_1}{r_2}}}$ for every $\beta \geq 0$.

Remark: When S_d verifies Assumption A.1, the roots of the polynomial $\tau \mapsto (\beta+1)^{r_1} \tau^{r_2} + 1$, which are the complex numbers $\frac{1}{(\beta+1)^{\frac{r_1}{r_2}}} e^{i\pi \left(\frac{2k+1}{r_2} \right)}$ for $k = 0, \ldots, r_2 - 1$, do not belong to $\overline{S_d}$. This is crucial in the following

Lemma 1. *Under Assumption (A), a constant $C_1 > 0$ (only depending on S_d and r_2) exists such that*

$$\left| \frac{1}{(\beta+1)^{r_1} \tau^{r_2} + 1} \right| \leq C_1,$$

for every $\beta \geq 0$ and every $\tau \in \Omega_\beta$.

Proof. Let $\beta \geq 0$. It is straightforward to derive that

$$(5) \qquad \frac{1}{(\beta+1)^{r_1} \tau^{r_2} + 1} = \sum_{k=0}^{r_2-1} \frac{A_{k,\beta}}{\tau - \frac{e^{i\pi \left(\frac{2k+1}{r_2} \right)}}{(\beta+1)^{\frac{r_1}{r_2}}}},$$

for every $\tau \in \Omega_\beta$, where

$$A_{k,\beta} = \frac{1}{r_2} e^{-i\pi \left(\frac{r_2-1}{r_2} \right)(2k+1)} (\beta+1)^{-\frac{r_1}{r_2}}, \quad k = 0, \ldots, r_2 - 1.$$

A similar argument as the one followed in the proof of Lemma 7 in [25] allows us to derive the existence of a positive constant $C_1 > 0$, not depending on β, such that

$$
(6) \qquad \left| \tau - \frac{e^{i\pi\left(\frac{2k+1}{r_2}\right)}}{(\beta+1)^{\frac{r_1}{r_2}}} \right| \geq \frac{C_1}{(\beta+1)^{\frac{r_1}{r_2}}},
$$

for every $\tau \in \Omega_\beta$ and $k = 0, \ldots, r_2 - 1$. From (5) and (6) we conclude. $\qquad \square$

The next lemmas are devoted to the behavior of the elements in the Banach space introduced in Definition 1 under certain operators. Let $\rho > 0$. The first lemma involves the integral operator ∂_τ^{-1} defined for every $v \in \mathcal{O}(S_d \cup D(0,\rho))$ by $\partial_\tau^{-1} v(\tau) := \int_0^\tau v(\tau_1) d\tau_1$, for $\tau \in S_d \cup D(0,\rho)$. More generally, for any $\kappa_0 \geq 1$, we define the operator $\partial_\tau^{-\kappa_0}$ by

$$
(7) \qquad \partial_\tau^{-\kappa_0} v(\tau) := \int_0^\tau \int_0^{\tau_1} \cdots \int_0^{\tau_{\kappa_0-1}} v(\tau_{\kappa_0}) d\tau_{\kappa_0} d\tau_{\kappa_0-1} \cdots d\tau_1, \quad \tau \in S_d \cup D(0,\rho),
$$

for every $v \in \mathcal{O}(S_d \cup D(0,\rho))$.

Remark: The change of parametrization in (7) given by $\tau_j = h_j \tau_{j-1}$ for $1 \leq j \leq \kappa_0$, with $\tau_0 = \tau$ and $0 \leq h_j \leq 1$ for $j = 1, \ldots, \kappa_0$ allows us to write

$$
(8) \qquad \partial_\tau^{-\kappa_0} v(\tau) = \tau^{\kappa_0} \int_0^1 \cdots \int_0^1 v(h_{\kappa_0} \cdots h_1 \tau) M_{\kappa_0}(h_1, \ldots, h_{\kappa_0-1}) dh_{\kappa_0} \cdots dh_1,
$$

for every $v \in \mathcal{O}(S_d \cup D(0,\rho))$. Here, $M_{\kappa_0}(h_1, \ldots, h_{\kappa_0-1})$ is a monic monomial in $h_1, \ldots, h_{\kappa_0-1}$ for $\kappa_0 \geq 2$, while $M_1 \equiv 1$.

Lemma 2. *Let $s_1, s_2, \kappa_0, \kappa_1, \beta_2, \beta, S \geq 0$ be nonnegative integers. We assume*

$$
\beta_2 \leq \beta \quad \text{and} \quad \kappa_1 < S.
$$

Then, for every $\epsilon \in \mathcal{E}$, the operator $\dfrac{\tau^{s_1}}{\epsilon^{s_2}} \partial_\tau^{-\kappa_0} : F_{\beta_2+\kappa_1,\epsilon,\sigma,\Omega_{\beta_2+\kappa_1}} \to F_{\beta+S,\epsilon,\sigma,\Omega_{\beta+S}}$ is bounded. Moreover, one has

$$
\left\| \frac{\tau^{s_1}}{\epsilon^{s_2}} \partial_\tau^{-\kappa_0} V_{\beta_2+\kappa_1,\epsilon}(\tau) \right\|_{\beta+S,\epsilon,\sigma,\Omega_{\beta+S}} \leq \left\| V_{\beta_2+\kappa_1,\epsilon}(\tau) \right\|_{\beta_2+\kappa_1,\epsilon,\sigma,\Omega_{\beta_2+\kappa_1}} |\epsilon|^{r(s_1+\kappa_0)-s_2}
$$

$$
(9)
$$
$$
\times \left[\left(\frac{(s_1+\kappa_0)e^{-1}}{\sigma(S-\kappa_1)} \right)^{s_1+\kappa_0} (\beta+S+1)^{b(s_1+\kappa_0)} + \left(\frac{(s_1+\kappa_0+2)e^{-1}}{\sigma(S-\kappa_1)} \right)^{s_1+\kappa_0+2} (\beta+S+1)^{b(s_1+\kappa_0+2)} \right],
$$

for every $V_{\beta_2+\kappa_1,\epsilon} \in F_{\beta_2+\kappa_1,\epsilon,\sigma,\Omega_{\beta_2+\kappa_1}}$.

Remark: If $s_1 + \kappa_0 = 0$, the expression above must be understood to be the limit when $s_1 + \kappa_0$ tends to 0. The case $\kappa_0 = 0$ can be studied separately. Both can be directly checked from the definition of the norms.

Proof. Let $\epsilon \in \mathcal{E}$ and $V_{\beta_2+\kappa_1,\epsilon} \in F_{\beta_2+\kappa_1,\epsilon,\sigma,\Omega_{\beta_2+\kappa_1}}$. We assume $\kappa_0 \neq 0$. From (8) we have

$$
\left| \frac{\tau^{s_1}}{\epsilon^{s_2}} \partial_\tau^{-\kappa_0} V_{\beta_2+\kappa_1,\epsilon}(\tau) \right| = \left| \frac{\tau^{s_1+\kappa_0}}{\epsilon^{s_2}} \int_0^1 \cdots \int_0^1 V_{\beta_2+\kappa_1,\epsilon}(h_{\kappa_0} \cdots h_1 \tau) \left(1 + \frac{|h_{\kappa_0} \cdots h_1 \tau|^2}{|\epsilon|^{2r}} \right) \right.
$$

$$
\left. \times e^{-\frac{\sigma}{|\epsilon|^r} r_b(\beta_2+\kappa_1)|h_{\kappa_0} \cdots h_1 \tau|} \frac{e^{\frac{\sigma}{|\epsilon|^r} r_b(\beta_2+\kappa_1)|h_{\kappa_0} \cdots h_1 \tau|}}{\left(1 + \frac{|h_{\kappa_0} \cdots h_1 \tau|^2}{|\epsilon|^{2r}} \right)} M_{\kappa_0}(h_1, \ldots, h_{\kappa_0-1}) dh_{\kappa_0} \cdots dh_1 \right|,
$$

for every $\tau \in \Omega_{\beta_2 + \kappa_1}$. From this expression we get

$$\left| \frac{\tau^{s_1}}{\epsilon^{s_2}} \partial_\tau^{-\kappa_0} V_{\beta_2 + \kappa_1, \epsilon}(\tau) \right| \left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}} \right) e^{-\frac{\sigma}{|\epsilon|^r} r_b(\beta + S)|\tau|}$$

$$\leq \frac{|\tau|^{s_1 + \kappa_0}}{|\epsilon|^{s_2}} \|V_{\beta_2 + \kappa_1, \epsilon}(\tau)\|_{\beta_2 + \kappa_1, \epsilon, \sigma, \Omega_{\beta_2 + \kappa_1}} \left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}} \right) e^{-\frac{\sigma}{|\epsilon|^r} (r_b(\beta + S) - r_b(\beta_2 + \kappa_1))|\tau|}.$$

By definition of r_b one has

$$r_b(\beta + S) - r_b(\beta_2 + \kappa_1) \geq \frac{\beta + S - (\beta_2 + \kappa_1)}{(\beta + S + 1)^b} \geq \frac{S - \kappa_1}{(\beta + S + 1)^b}.$$

This fact and the application of the classical estimates

$$(10) \qquad \sup_{x \geq 0} x^{m_1} e^{-m_2 x} = \left(\frac{m_1}{m_2} \right)^{m_1} e^{-m_1}, \quad m_1, m_2 > 0,$$

to the bounds achieved lead us to (9). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3. *Let $\beta \geq 0$ be an integer and $\epsilon \in \mathcal{E}$. For all integers $\beta_1, \beta_2 \geq 0$ such that $\beta_1 + \beta_2 = \beta$ and $V_\epsilon(\tau) \in F_{\beta_1, \epsilon, \sigma, \Omega_{\beta_1}}$, $W_\epsilon(\tau) \in F_{\beta_2, \epsilon, \sigma, \Omega_{\beta_2}}$ one has $(V_\epsilon \star W_\epsilon)(\tau) := \int_0^\tau V_\epsilon(\tau - s) W_\epsilon(s) ds$ belongs to $F_{\beta, \epsilon, \sigma, \Omega_\beta}$. Moreover, there exists a universal constant $C_2 > 0$ such that*

$$\left\| \int_0^\tau V_\epsilon(\tau - s) W_\epsilon(s) ds \right\|_{\beta, \epsilon, \sigma, \Omega_\beta} \leq C_2 |\epsilon|^r \|V_\epsilon(\tau)\|_{\beta_1, \epsilon, \sigma, \Omega_{\beta_1}} \|W_\epsilon(\tau)\|_{\beta_2, \epsilon, \sigma, \Omega_{\beta_2}}.$$

Proof. Let $\beta_1, \beta_2, \beta, \epsilon$ as in the statements of Lemma 3. Let $V_\epsilon(\tau) \in F_{\beta_1, \epsilon, \sigma, \Omega_{\beta_1}}$ and $W_\epsilon(\tau) \in F_{\beta_2, \epsilon, \sigma, \Omega_{\beta_2}}$. It is worth pointing out that $\Omega_\beta \subseteq \Omega_{\beta_1} \cap \Omega_{\beta_2}$.

From the fact that

$$\left| \int_0^\tau V_\epsilon(\tau - s) W_\epsilon(s) ds \right| = \left| \int_0^\tau V_\epsilon(\tau - s) \left(1 + \frac{|\tau - s|^2}{|\epsilon|^{2r}} \right) e^{-\frac{\sigma}{|\epsilon|^r} r_b(\beta_1)|\tau - s|} \right.$$

$$\left. \times W_\epsilon(s) \left(1 + \frac{|s|^2}{|\epsilon|^{2r}} \right) e^{-\frac{\sigma}{|\epsilon|^r} r_b(\beta_2)|s|} \frac{e^{\frac{\sigma}{|\epsilon|^r} (r_b(\beta_1)|\tau - s| + r_b(\beta_2)|s|)}}{\left(1 + \frac{|\tau - s|^2}{|\epsilon|^{2r}} \right) \left(1 + \frac{|s|^2}{|\epsilon|^{2r}} \right)} ds \right|$$

for every $\tau \in \Omega_\beta$, we deduce that

$$\left| \int_0^\tau V_\epsilon(\tau - s) W_\epsilon(s) ds \right| \leq \|V_\epsilon(\tau)\|_{\beta_1, \epsilon, \sigma, \Omega_{\beta_1}} \|W_\epsilon(\tau)\|_{\beta_2, \epsilon, \sigma, \Omega_{\beta_2}}$$

$$\times \int_0^1 \frac{|\tau| e^{\frac{\sigma |\tau|}{|\epsilon|^r} (r_b(\beta_1)(1 - h) + r_b(\beta_2)h)}}{\left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}} (1 - h)^2 \right) \left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}} h^2 \right)} dh,$$

for $\tau \in \Omega_\beta$.

In order to conclude, we only have to prove the existence of a universal constant $C_2 > 0$ verifying
(11)

$$I(|\tau|, |\epsilon|, \beta, \beta_1, \beta_2) = \left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}} \right) e^{-\frac{\sigma}{|\epsilon|^r} r_b(\beta)|\tau|} \int_0^1 \frac{|\tau| e^{\frac{\sigma |\tau|}{|\epsilon|^r} (r_b(\beta_1)(1 - h) + r_b(\beta_2)h)}}{\left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}} (1 - h)^2 \right) \left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}} h^2 \right)} dh \leq |\epsilon|^r C_2,$$

for every $\tau \in \Omega_\beta$. Since r_b is increasing, one has $r_b(\beta_1)(1-h) + r_b(\beta_2)h \le r_b(\beta)$ from where we deduce

$$I(|\tau|, |\epsilon|, \beta, \beta_1, \beta_2) \le J(|\tau|, |\epsilon|) = \int_0^1 \frac{\left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}}\right)|\tau|}{\left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}}(1-h)^2\right)\left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}}h^2\right)} dh.$$

We have

$$(12) \qquad \frac{J(|\epsilon|^r|\tau|, |\epsilon|)}{|\epsilon|^r} = \int_0^1 \frac{(1 + |\tau|^2)|\tau|}{(1 + |\tau|^2(1-h)^2)(1 + |\tau|^2 h^2)} dh.$$

From Corollary 4.9 in [15] we derive the right-hand side in (12) is a bounded function of τ. Lemma 3 follows from

$$\sup_{|\tau| \ge 0} \frac{J(|\tau|, |\epsilon|)}{|\epsilon|^r} = \sup_{|\tau| \ge 0} \frac{J(|\epsilon|^r|\tau|, |\epsilon|)}{|\epsilon|^r} \le C_2,$$

where C_2 does not depend on $\epsilon \in \mathcal{E}$. $\qquad \square$

Let $V \in \mathcal{O}(S_d \cup D(0, \rho))$. We put $(V(\tau))^{\star 1} := V(\tau)$. For every $\ell \ge 2$ we define $(V(\tau))^{\star \ell} := V(\tau) \star (V(\tau))^{\star(\ell-1)}$. By recursion, one can prove the following

Corollary 1. *Let $\beta \in \mathbb{N}$, $\ell_1 \in \mathbb{N}$ with $\ell_1 \ge 2$. We also fix $\epsilon \in \mathcal{E}$ and we take $\beta_1, \ldots, \beta_{\ell_1} \in \mathbb{N}$ such that $\beta_1 + \ldots + \beta_{\ell_1} = \beta$ and $V_{\beta_j, \epsilon} \in F_{\beta_j, \epsilon, \sigma, \Omega_{\beta_j}}$ for every $j = 1, \ldots, \ell_1$. Then, $V_{\beta_1, \epsilon} \star \cdots \star V_{\beta_{\ell_1}, \epsilon} \in F_{\beta, \epsilon, \sigma, \Omega_\beta}$. Moreover,*

$$\left\| V_{\beta_1, \epsilon}(\tau) \star \cdots \star V_{\beta_{\ell_1}, \epsilon}(\tau) \right\|_{\beta, \epsilon, \sigma, \Omega_\beta} \le C_2^{\ell_1 - 1} |\epsilon|^{r(\ell_1 - 1)} \left\| V_{\beta_1, \epsilon}(\tau) \right\|_{\beta_1, \epsilon, \sigma, \Omega_{\beta_1}} \cdots \left\| V_{\beta_{\ell_1}, \epsilon}(\tau) \right\|_{\beta_{\ell_1}, \epsilon, \sigma, \Omega_{\beta_{\ell_1}}},$$

for a universal constant $C_2 > 0$.

Corollary 2. *Let $\beta, m_1, \ell_0 \ge 0$ be integers. For every $\epsilon \in \mathcal{E}$, the operator $\frac{1}{\epsilon^{m_1}} \partial_\tau^{-\ell_0}$ from $F_{\beta, \epsilon, \sigma, \Omega_\beta}$ into itself is bounded. Moreover, there exists a positive constant C_3 (only depending on σ) such that*

$$\left\| \frac{1}{\epsilon^{m_1}} \partial_\tau^{-\ell_0} V_\epsilon(\tau) \right\|_{\beta, \epsilon, \sigma, \Omega_\beta} \le C_3 |\epsilon|^{r\ell_0 - m_1} \left\| V_\epsilon(\tau) \right\|_{\beta, \epsilon, \sigma, \Omega_\beta},$$

for every $V_\epsilon \in F_{\beta, \epsilon, \sigma, \Omega_\beta}$.

Proof. If $\ell_0 = 0$, the result is straightforward. We consider the case when $\ell_0 \ge 1$. Let us fix $\epsilon \in \mathcal{E}$.

Let $\chi_{\mathbb{C}}(\tau) \equiv 1$ for every $\tau \in \mathbb{C}$. One can check

$$\partial_\tau^{-\ell_0} V_\epsilon(\tau) = \left(\chi_{\mathbb{C}}(\tau)\right)^{\star \ell_0} \star V_\epsilon(\tau).$$

From Corollary 1 we deduce that

$$\left\| \frac{1}{\epsilon^{m_1}} \partial_\tau^{-\ell_0} V_\epsilon(\tau) \right\|_{\beta, \epsilon, \sigma, \Omega_\beta} \le C_2^{\ell_0} |\epsilon|^{r\ell_0 - m_1} \left\| \chi_{\mathbb{C}}(\tau) \right\|_{0, \epsilon, \sigma, \Omega_0}^{\ell_0} \left\| V_\epsilon(\tau) \right\|_{\beta, \epsilon, \sigma, \Omega_\beta},$$

for a universal positive constant C_2. Finally, the estimates in (10) allow us to write

$$\left\| \chi_{\mathbb{C}}(\tau) \right\|_{0, \epsilon, \sigma, \Omega_0} = \sup_{\tau \in S_d \cup D(0, \frac{1}{2})} \left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}}\right) e^{-\frac{\sigma}{|\epsilon|^r}|\tau|} \le 1 + \left(\frac{2e^{-1}}{\sigma}\right)^2,$$

from where we conclude. $\qquad \square$

Lemma 4. *Let β, β' be nonnegative integers such that $\beta' \leq \beta$. For every $\epsilon \in \mathcal{E}$ one has $F_{\beta',\epsilon,\sigma,\Omega_{\beta'}} \subseteq F_{\beta,\epsilon,\sigma,\Omega_\beta}$. In addition to this, for every $V_\epsilon \in F_{\beta',\epsilon,\sigma,\Omega_{\beta'}}$ one derives*

$$\|V_\epsilon(\tau)\|_{\beta,\epsilon,\sigma,\Omega_\beta} \leq \|V_\epsilon(\tau)\|_{\beta',\epsilon,\sigma,\Omega_{\beta'}}.$$

Here, we are giving the same name to a function defined in $\Omega_{\beta'}$ and its restriction to Ω_β.

Proof. The result directly follows from the definition of the norms, bearing in mind that $\Omega_\beta \subseteq \Omega_{\beta'}$ and that r_b is increasing. $\qquad\square$

Lemma 5. *Let $h(\tau, \epsilon) \in \mathcal{O}(\Omega_0 \times \mathcal{E})$ such that there exists $M > 0$ satisfying*

$$\sup_{(\tau,\epsilon)\in\Omega_0\times\mathcal{E}} |h(\tau,\epsilon)| \leq M.$$

Then,

$$\|h(\tau,\epsilon)V_\epsilon(\tau)\|_{\beta,\epsilon,\sigma,\Omega_\beta} \leq M \|V_\epsilon(\tau)\|_{\beta,\epsilon,\sigma,\Omega_\beta},$$

for every $\beta \geq 0$, $\epsilon \in \mathcal{E}$ and $V_\epsilon \in F_{\beta,\epsilon,\sigma,\Omega_\beta}$.

Proof. It is an immediate consequence of the definition of the norm. $\qquad\square$

3   A global Cauchy problem

In this section, we study the behavior of the solution of the forthcoming auxiliary Cauchy problem (14)-(15). Our main aim is to determine its solution in the form

$$(13) \qquad V(\tau, z, \epsilon) = \sum_{\beta \geq 0} V_\beta(\tau, \epsilon) \frac{z^\beta}{\beta},$$

defined on appropriate domains, and to obtain suitable estimates for the coefficients in (13) in terms of the norm in Definition 1.

We keep the same notation as in the previous section. In particular, the values of $r > 0$, $\sigma > 0$, $b > 1$ and $r_1, r_2, S \in \mathbb{N}$, with $r_1 \geq 0$ and $r_2, S \geq 1$, are the same.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be finite subsets of $\mathbb{N}^2$.

For every $(\kappa_0, \kappa_1) \in \mathcal{A}_1$, $I_{(\kappa_0,\kappa_1)}$ is a finite subset of $\mathbb{N}^2$. We assume $\kappa_1 < S$ for every $(\kappa_0, \kappa_1) \in \mathcal{A}_1$. For each $(s_1, s_2) \in I_{(\kappa_0,\kappa_1)}$ and every $\beta \in \mathbb{N}$, $a_{s_1,s_2,\kappa_0,\kappa_1,\beta}(\tau, \epsilon)$ stands for a bounded holomorphic function defined on $\Omega_0 \times \mathcal{E}$. For every $(\kappa_0, \kappa_1) \in \mathcal{A}_1$, we define the formal power series

$$a_{(\kappa_0,\kappa_1)}(\tau, z, \epsilon) := \sum_{(s_1,s_2)\in I_{(\kappa_0,\kappa_1)}} \Big(\sum_{\beta \geq 0} a_{s_1,s_2,\kappa_0,\kappa_1,\beta}(\tau, \epsilon) \tau^{s_1} \epsilon^{-s_2} \frac{z^\beta}{\beta!} \Big).$$

For every $(\ell_0, \ell_1) \in \mathcal{A}_2$, $J_{(\ell_0,\ell_1)}$ is a finite subset of $\mathbb{N}$. For a given $m_1 \in J_{(\ell_0,\ell_1)}$ and $\beta \in \mathbb{N}$, $\alpha_{m_1,\ell_0,\ell_1,\beta}(\tau, \epsilon)$ stands for a bounded holomorphic function defined on $\Omega_0 \times \mathcal{E}$. For every $(\ell_0, \ell_1) \in \mathcal{A}_2$, we put

$$\alpha_{(\ell_0,\ell_1)}(\tau, z, \epsilon) := \sum_{m_1\in J_{(\ell_0,\ell_1)}} \Big(\sum_{\beta \geq 0} \alpha_{m_1,\ell_0,\ell_1,\beta}(\tau, \epsilon) \epsilon^{-m_1} \frac{z^\beta}{\beta!} \Big).$$

For any fixed $\epsilon \in \mathcal{E}$, we consider the following Cauchy problem

$$((z\partial_z + 1)^{r_1}\tau^{r_2} + 1)\partial_z^S V_\epsilon(\tau, z) = \sum_{(\kappa_0,\kappa_1)\in\mathcal{A}_1} a_{(\kappa_0,\kappa_1)}(\tau, z, \epsilon)\partial_\tau^{-\kappa_0}\partial_z^{\kappa_1} V_\epsilon(\tau, z)$$

$$(14) \qquad\qquad + \sum_{(\ell_0,\ell_1)\in\mathcal{A}_2,\ell_1\geq 2} \alpha_{(\ell_0,\ell_1)}(\tau, z, \epsilon)\partial_\tau^{-\ell_0}(V_\epsilon(\tau, z))^{\star\ell_1},$$

for given initial data

$$(15) \qquad\qquad (\partial_z^j V_\epsilon)(\tau, 0) = V_{j,\epsilon}(\tau) \in F_{j,\epsilon,\sigma,\Omega_j}, \quad 0 \leq j \leq S - 1.$$

Proposition 1. *We work under assumption **(A)** on the sets Ω_β for $\beta \geq 0$. For every $\epsilon \in \mathcal{E}$, there exists a formal power series solution of (14)-(15),*

$$(16) \qquad\qquad V_\epsilon(\tau, z) = \sum_{\beta\geq 0} V_{\beta,\epsilon}(\tau)\frac{z^\beta}{\beta!} \in \mathcal{O}(\Omega_\beta)[[z]],$$

whose coefficients $V_{\beta,\epsilon}$ belong to $F_{\beta,\epsilon,\sigma,\Omega_\beta}$ for every $\beta \geq 0$, $\epsilon \in \mathcal{E}$. Moreover, these coefficients verify the recursion formula

$$\frac{V_{\beta+S,\epsilon}(\tau)}{\beta!} = \sum_{(\kappa_0,\kappa_1)\in\mathcal{A}_1} \sum_{(s_1,s_2)\in I_{(\kappa_0,\kappa_1)}} \sum_{\beta_1+\beta_2=\beta} \frac{a_{s_1,s_2,\kappa_0,\kappa_1,\beta_1}(\tau,\epsilon)}{((\beta+1)^{r_1}\tau^{r_2}+1)\beta_1!}\tau^{s_1}\epsilon^{-s_2}\frac{\partial_\tau^{-\kappa_0}(V_{\beta_2+\kappa_1,\epsilon}(\tau))}{\beta_2!}$$

$$(17)$$

$$+ \sum_{(\ell_0,\ell_1)\in\mathcal{A}_2,\ell_1\geq 2} \sum_{m_1\in J_{(\ell_0,\ell_1)}} \sum_{\beta_0+\beta_1+...+\beta_{\ell_1}=\beta} \frac{\alpha_{m_1,\ell_0,\ell_1,\beta_0}(\tau,\epsilon)\epsilon^{-m_1}\partial_\tau^{-\ell_0}(V_{\beta_1,\epsilon}(\tau)\star\cdots\star V_{\beta_{\ell_1},\epsilon}(\tau))}{((\beta+1)^{r_1}\tau^{r_2}+1)\beta_0!\beta_1!\cdots\beta_{\ell_1}!},$$

for every $\beta \geq 0$, $\tau \in \Omega_{\beta+S}$ and $\epsilon \in \mathcal{E}$.

Proof. Let $\beta \geq 0$, $\tau \in \Omega_{\beta+S}$ and $\epsilon \in \mathcal{E}$. Substituting (16) into equation (14) we deduce the left-hand side is

$$(18) \qquad ((z\partial_z + 1)^{r_1}\tau^{r_2} + 1)\partial_z^S V_\epsilon(\tau, z) = \sum_{\beta\geq 0}((\beta+1)^{r_1}\tau^{r_2}+1)V_{\beta+S,\epsilon}(\tau)\frac{z^\beta}{\beta!}.$$

Each term from the first sum in the right-hand side of (14) becomes

$$(19) \quad \left(\sum_{(s_1,s_2)\in I_{(\kappa_0,\kappa_1)}} \sum_{\beta\geq 0} a_{s_1,s_2,\kappa_0,\kappa_1,\beta}(\tau,\epsilon)\tau^{s_1}\epsilon^{-s_2}\frac{z^\beta}{\beta!}\right) \cdot \left(\sum_{\beta\geq 0}\partial_\tau^{-\kappa_0}V_{\beta+\kappa_1,\epsilon}(\tau)\frac{z^\beta}{\beta!}\right)$$

$$= \sum_{(s_1,s_2)\in I_{(\kappa_0,\kappa_1)}} \sum_{\beta\geq 0}\left(\sum_{\beta_1+\beta_2=\beta} a_{s_1,s_2,\kappa_0,\kappa_1,\beta_1}(\tau,\epsilon)\tau^{s_1}\epsilon^{-s_2}\partial_\tau^{-\kappa_0}V_{\beta_2+\kappa_1,\epsilon}(\tau)\frac{\beta!}{\beta_1!\beta_2!}\right)\frac{z^\beta}{\beta!}.$$

Every term from the last sum in the right-hand side of (14) turns into

$$\left(\sum_{m_1\in J_{(\ell_0,\ell_1)}} \sum_{\beta\geq 0}\alpha_{m_1,\ell_0,\ell_1,\beta}(\tau,\epsilon)\epsilon^{-m_1}\frac{z^\beta}{\beta!}\right)$$

$$\times \left(\sum_{\beta\geq 0}(\sum_{\beta_1+...+\beta_{\ell_1}=\beta}\frac{\beta!}{\beta_1!\cdots\beta_{\ell_1}!}\partial_\tau^{-\ell_0}(V_{\beta_1,\epsilon}(\tau)\star\cdots\star V_{\beta_{\ell_1},\epsilon}(\tau))\frac{z^\beta}{\beta!}\right)$$

(20)
$$= \sum_{m_1 \in J_{(\ell_0,\ell_1)}} \sum_{\beta \geq 0} \left(\sum_{\beta_0+\beta_1+\ldots+\beta_{\ell_1}=\beta} \frac{\beta! \alpha_{m_1,\ell_0,\ell_1,\beta_0}(\tau,\epsilon)}{\beta_0! \beta_1! \cdots \beta_{\ell_1}!} \epsilon^{-m_1} \partial_\tau^{-\ell_0} (V_{\beta_1,\epsilon}(\tau) \star \cdots \star V_{\beta_{\ell_1},\epsilon}(\tau)) \right) \frac{z^\beta}{\beta!}.$$

The recursion (17) immediately follows from (18), (19) and (20). The domain of definition of the elements in $(V_{\beta,\epsilon})_{\beta \geq 0}$ comes from the denominator $((\beta+1)^{r_1}\tau^{r_2}+1)$ appearing in the recursion. In fact, it could be taken larger than Ω_β, but this does not make any difference to our final purpose. Furthermore, Lemma 1, Lemma 2, Corollary 1, Corollary 2, Lemma 4 and Lemma 5 provide that the coefficients $V_{\beta,\epsilon}$ belong to $F_{\beta,\epsilon,\sigma,\Omega_\beta}$ for every $\beta \geq 0$ and $\epsilon \in \mathcal{E}$.
□

Remark: One can check that if the initial conditions $V_j(\tau,\epsilon) := V_{j,\epsilon}(\tau)$ $j = 0,\ldots,S-1$, are holomorphic functions on $\Omega_j \times \mathcal{E}$ for every $j = 0,\ldots,S-1$, then $V_\beta(\tau,\epsilon) := V_{\beta,\epsilon}(\tau)$ are also holomorphic functions defined in $\Omega_\beta \times \mathcal{E}$ for every $\beta \geq 0$, due to the way they are constructed from recursion formula (17).

For every $\beta \in \mathbb{N}$ and $\epsilon \in \mathcal{E}$ we put

(21)
$$w_\beta(\epsilon) = \|V_{\beta,\epsilon}(\tau)\|_{\beta,\epsilon,\sigma,\Omega_\beta}.$$

A recursion estimate for the elements w_β is obtained in the next

Lemma 6. *Under Assumption (A) and*

(22)
$$r(s_1 + \kappa_0) \geq s_2, \qquad r(\ell_0 + \ell_1 - 1) \geq m_1, \qquad \ell_1 \geq 2$$

for every $(\kappa_0,\kappa_1) \in \mathcal{A}_1$, $(s_1,s_2) \in I_{(\kappa_0,\kappa_1)}$, $(\ell_0,\ell_1) \in \mathcal{A}_2$ and $m_1 \in J_{(\ell_0,\ell_1)}$, there exist $D_1, D_2 > 0$ (depending on S_d, r_2, $r_\mathcal{E}$ and the elements in the finite sets $\mathcal{A}_1$, $\mathcal{A}_2$ and the corresponding $I_{(\kappa_0,\kappa_1)}$, $J_{(\ell_0,\ell_1)}$) such that

(23)
$$\frac{w_{\beta+S}(\epsilon)}{\beta!} \leq \sum_{(\kappa_0,\kappa_1) \in \mathcal{A}_1} \sum_{(s_1,s_2) \in I_{(\kappa_0,\kappa_1)}} \sum_{\beta_1+\beta_2=\beta} D_1 \frac{A_{s_1,s_2,\kappa_0,\kappa_1,\beta_1}}{\beta_1!} \frac{w_{\beta_2+\kappa_1}(\epsilon)}{\beta_2!} (\beta+S+1)^{b(s_1+\kappa_0+2)}$$
$$+ \sum_{(\ell_0,\ell_1) \in \mathcal{A}_2, \ell_1 \geq 2} \sum_{m_1 \in J_{(\ell_0,\ell_1)}} \sum_{\beta_0+\ldots+\beta_{\ell_1}=\beta} D_2 \frac{B_{m_1,\ell_0,\ell_1,\beta_0}}{\beta_0!} \frac{w_{\beta_1}(\epsilon)}{\beta_1!} \cdots \frac{w_{\beta_{\ell_1}}(\epsilon)}{\beta_{\ell_1}!},$$

for every $\beta \geq 0$ and $\epsilon \in \mathcal{E}$, where

(24)
$$A_{s_1,s_2,\kappa_0,\kappa_1,\beta_1} = \sup_{(\tau,\epsilon) \in \Omega_0 \times \mathcal{E}} |a_{s_1,s_2,\kappa_0,\kappa_1,\beta_1}(\tau,\epsilon)|,$$
$$B_{m_1,\ell_0,\ell_1,\beta_0} = \sup_{(\tau,\epsilon) \in \Omega_0 \times \mathcal{E}} |\alpha_{m_1,\ell_0,\ell_1,\beta_0}(\tau,\epsilon)|.$$

Proof. Let $\beta \geq 0$ and $\epsilon \in \mathcal{E}$. Take $\|\cdot\|_{\beta+S,\epsilon,\sigma,\Omega_{\beta+S}}$ on the two sides of the equality (17). A direct application of Lemma 1, Lemma 2, Corollary 1, Corollary 2, Lemma 4, Lemma 5 and the assumptions (22) allow us to write, for some $\tilde{C}_1 > 0$,

$$\frac{w_{\beta+S}(\epsilon)}{\beta!} \leq \sum_{(\kappa_0,\kappa_1) \in \mathcal{A}_1} \sum_{(s_1,s_2) \in I_{(\kappa_0,\kappa_1)}} \sum_{\beta_1+\beta_2=\beta} C_1 \frac{A_{s_1,s_2,\kappa_0,\kappa_1,\beta_1}}{\beta_1!} \frac{w_{\beta_2+\kappa_1}(\epsilon)}{\beta_2!}$$
$$\times |\epsilon|^{r(s_1+\kappa_0)-s_2} \tilde{C}_1 (\beta+S+1)^{b(s_1+\kappa_0+2)}$$
$$+ \sum_{(\ell_0,\ell_1) \in \mathcal{A}_2, \ell_1 \geq 2} \sum_{m_1 \in J_{(\ell_0,\ell_1)}} \sum_{\beta_0+\ldots+\beta_{\ell_1}=\beta} \frac{B_{m_1,\ell_0,\ell_1,\beta_0}}{\beta_0!} C_3 C_2^{\ell_1-1} |\epsilon|^{r(\ell_0+\ell_1-1)-m_1} \frac{w_{\beta_1}(\epsilon)}{\beta_1!} \cdots \frac{w_{\beta_{\ell_1}}(\epsilon)}{\beta_{\ell_1}!},$$

from which (23) easily follows. □

Proposition 2. *Under the same hypotheses as in Lemma 6, let us define*

$$
A_{s_1,s_2,\kappa_0,\kappa_1}(x) := \sum_{\beta \geq 0} A_{s_1,s_2,\kappa_0,\kappa_1,\beta} \frac{x^\beta}{\beta!} \in \mathbb{C}[[x]],
$$

(25)

$$
B_{m_1,\ell_0,\ell_1}(x) := \sum_{\beta \geq 0} B_{m_1,\ell_0,\ell_1,\beta} \frac{x^\beta}{\beta!} \in \mathbb{C}[[x]],
$$

where the coefficients $A_{s_1,s_2,\kappa_0,\kappa_1,\beta}, B_{m_1,\ell_0,\ell_1,\beta}$ are given by (24). For every $\epsilon \in \mathcal{E}$ we consider the Cauchy problem

$$
\partial_x^S u(x,\epsilon) = \sum_{(\kappa_0,\kappa_1) \in \mathcal{A}_1} \sum_{(s_1,s_2) \in I_{(\kappa_0,\kappa_1)}} D_1(x\partial_x + S + 1)^{b(s_1+\kappa_0+2)} \left(A_{s_1,s_2,\kappa_0,\kappa_1}(x)\partial_x^{\kappa_1} u(x,\epsilon) \right)
$$

(26)

$$
+ \sum_{(\ell_0,\ell_1) \in \mathcal{A}_2, \ell_1 \geq 2} \sum_{m_1 \in J_{(\ell_0,\ell_1)}} D_2 B_{m_1,\ell_0,\ell_1}(x)(u(x,\epsilon))^{\ell_1},
$$

with initial conditions

(27)
$$
(\partial_x^j u)(0,\epsilon) = w_j(\epsilon), \quad 0 \leq j \leq S - 1,
$$

where the w_j are the ones defined in (21).

  Then, the problem (26)-(27) has a unique formal solution

(28)
$$
u(x,\epsilon) = \sum_{\beta \geq 0} u_\beta(\epsilon) \frac{x^\beta}{\beta!} \in \mathbb{R}[[x]],
$$

and the sequence $(u_\beta(\epsilon))_{\beta \geq 0}$ verifies

(29)
$$
w_\beta(\epsilon) \leq u_\beta(\epsilon)
$$

for every $\beta \geq 0$.

Proof. From the initial conditions fixed in (27) we have $u_\beta(\epsilon) = w_\beta(\epsilon)$ for every $0 \leq \beta \leq S - 1$. By inserting (28) into (26), and using (25), we obtain a recursion formula for the coefficients:

$$
\frac{u_{\beta+S}(\epsilon)}{\beta!} = \sum_{(\kappa_0,\kappa_1) \in \mathcal{A}_1} \sum_{(s_1,s_2) \in I_{(\kappa_0,\kappa_1)}} \sum_{\beta_1+\beta_2=\beta} D_1 \frac{A_{s_1,s_2,\kappa_0,\kappa_1,\beta_1}}{\beta_1!} \frac{u_{\beta_2+\kappa_1}(\epsilon)}{\beta_2!}(\beta + S + 1)^{b(s_1+\kappa_0+2)}
$$

(30)
$$
+ \sum_{(\ell_0,\ell_1) \in \mathcal{A}_2, \ell_1 \geq 2} \sum_{m_1 \in J_{(\ell_0,\ell_1)}} \sum_{\beta_0+\ldots+\beta_{\ell_1}=\beta} D_2 \frac{B_{m_1,\ell_0,\ell_1,\beta_0}}{\beta_0!} \frac{u_{\beta_1}(\epsilon)}{\beta_1!} \cdots \frac{u_{\beta_{\ell_1}}(\epsilon)}{\beta_{\ell_1}!},
$$

for every $\beta \geq 0$. So, it is clear that the $u_\beta(\epsilon)$, $\beta \geq S$, are uniquely determined real numbers, and that the series (28) so defined will certainly be a formal solution of (26)-(27). By comparing (23) to (30), one recursively obtains the inequalities (29). □

  This section concludes with the effective detection of the solution of (14)-(15) and some bounds satisfied by its coefficients with respect to the variable z.

Proposition 3. *Under Assumption **(A)**, let $V_j(\tau, \epsilon) := V_{j,\epsilon}(\tau)$, as in (15), be holomorphic functions defined in $\Omega_j \times \mathcal{E}$, $j = 0, \ldots, S - 1$. Moreover, assume that:*

(i) *The functions $A_{s_1,s_2,\kappa_0,\kappa_1}(x)$, $B_{m_1,\ell_0,\ell_1}(x)$ in (25) belong to $\mathbb{C}\{x\}$ for every $(\kappa_0, \kappa_1) \in \mathcal{A}_1$, $(s_1, s_2) \in I_{(\kappa_0,\kappa_1)}$, $(\ell_0, \ell_1) \in \mathcal{A}_2$ and $m_1 \in J_{(\ell_0,\ell_1)}$.*

(ii) *$S > b(s_1 + \kappa_0 + 2) + \kappa_1$ for every $(\kappa_0, \kappa_1) \in \mathcal{A}_1$, and $(s_1, s_2) \in I_{(\kappa_0,\kappa_1)}$.*

(iii) *$r(s_1 + \kappa_0) \geq s_2$, $r(\ell_0 + \ell_1 - 1) \geq m_1$ and $\ell_1 \geq 2$ for every $(\ell_0, \ell_1) \in \mathcal{A}_2$ and $m_1 \in J_{(\ell_0,\ell_1)}$, all $(\kappa_0, \kappa_1) \in \mathcal{A}_1$ and $(s_1, s_2) \in I_{(\kappa_0,\kappa_1)}$.*

Then, there exists $\delta > 0$ such that, whenever $w_j(\epsilon) = \|V_{j,\epsilon}(\tau)\|_{j,\epsilon,\sigma,\Omega_j} < \delta$ for every $0 \leq j \leq S - 1$ and $\epsilon \in \mathcal{E}$, the problem (14)-(15) has a solution

$$V(\tau, z, \epsilon) = \sum_{\beta \geq 0} V_\beta(\tau, \epsilon) \frac{z^\beta}{\beta!},$$

holomorphic in $S_d \times D(0, \rho) \times \mathcal{E}$ for some $\rho > 0$, and verifying that $V_\beta(\tau, \epsilon) \in \mathcal{O}(\Omega_\beta \times \mathcal{E})$ for every $\beta \geq 0$. In addition to this, there exists $M > 0$ such that

$$\tag{31} \sum_{\beta \geq 0} \|V_\beta(\tau, \epsilon)\|_{\beta,\epsilon,\sigma,\Omega_\beta} \frac{\rho^\beta}{\beta!} \leq M,$$

for every $\epsilon \in \mathcal{E}$.

Proof. From the classical theory of existence of solutions of nonlinear ODEs with complex parameters (see [12]), there exists $\delta > 0$ such that whenever $w_j(\epsilon) < \delta$ for every $0 \leq j \leq S - 1$ and $\epsilon \in \mathcal{E}$, one has that the unique formal series solution of (26)-(27), $u(x, \epsilon) = \sum_{\beta \geq 0} u_\beta(\epsilon) x^\beta / \beta!$, belongs to $\mathbb{C}\{x\}$, with a radius of convergence $\rho > 0$ independent of $\epsilon \in \mathcal{E}$. Moreover, there exists $M > 0$ such that

$$\tag{32} \sum_{\beta \geq 0} u_\beta(\epsilon) \frac{\rho^\beta}{\beta!} \leq M$$

for every $\epsilon \in \mathcal{E}$. Now observe that, by Proposition 1 and the remark following its proof, Cauchy problem (14)-(15) has a formal solution $\sum_{\beta \geq 0} V_\beta(\tau, \epsilon) z^\beta / \beta!$ such that $V_\beta \in F_{\beta,\epsilon,\sigma,\Omega_\beta}$ for every $\beta \geq 0$. In this situation, Proposition 2 ensures that for every $\beta \geq 0$ we have $\|V_\beta(\tau, \epsilon)\|_{\beta,\epsilon,\sigma,\Omega_\beta} =: w_\beta(\epsilon) \leq u_\beta(\epsilon)$, so that (31) is a clear consequence of (32). From here one can easily deduce that $\sum_{\beta \geq 0} V_\beta(\tau, \epsilon) z^\beta / \beta!$ indeed defines a holomorphic solution of (14)-(15) in $S_d \times D(0, \rho) \times \mathcal{E}$. $\square$

4 Analytic solutions of a threefold singular Cauchy problem

4.1 Laplace transform and asymptotic expansions

We recall the definition and main properties of Borel-Laplace summation process when considering formal power series with coefficients in a Banach space. For more details, see [4].

Definition 2. *Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a complex Banach space. A formal power series*

$$\hat{X}(t) = \sum_{j=0}^{\infty} \frac{a_j}{j!} t^j \in \mathbb{E}[[t]]$$

is 1-summable with respect to t in the direction $d \in [0, 2\pi)$ if

1.- the formal Borel transform of $\hat{X}$,

$$\mathcal{B}(\hat{X})(\tau) := \sum_{j=0}^{\infty} \frac{a_j}{(j!)^2} \tau^j \in \mathbb{E}[[\tau]]$$

is absolutely convergent for $|\tau| < \rho$, for some $\rho > 0$, and

2.- $\mathcal{B}(\hat{X})(\tau)$ can be analytically continued with respect to τ in a sector $S_{d,\delta} := \{\tau \in \mathbb{C}^{\star} : |d - arg(\tau)| < \delta\}$ for some $\delta > 0$. Moreover, there exist $C, K > 0$ such that

$$\left\|\mathcal{B}(\hat{X})(\tau)\right\|_{\mathbb{E}} \leq Ce^{K|\tau|}, \quad \tau \in S_{d,\delta}.$$

If this happens, then the Laplace transform of order 1 of $\mathcal{B}(\hat{X})(\tau)$ in the direction d is given by

$$\mathcal{L}^d(\mathcal{B}(\hat{X}))(t) := t^{-1} \int_{L_\gamma} \mathcal{B}(\hat{X})(\tau) e^{-\frac{\tau}{t}} d\tau, \quad L_\gamma = \mathbb{R}_+ e^{i\gamma} \subseteq S_{d,\delta} \cup \{0\},$$

where γ depends on t and it is chosen so that $\cos(\gamma - arg(t)) \geq \delta_1 > 0$ for some positive constant δ_1. Under these settings, $\mathcal{L}^d(\mathcal{B}(\hat{X}))$ is well-defined for

(33) $$t \in S_{d,\theta,R} := \{t \in \mathbb{C}^{\star} : |t| < R, |d - arg(t)| < \theta/2\},$$

for any $\pi < \theta < \pi + 2\delta$ and $0 < R < \delta_1/K$, and it is known as the 1-sum of $\hat{X}(t)$ in the direction d. It turns out to be a bounded holomorphic function in $S_{d,\theta,R}$, and to admit $\hat{X}(t)$ as its Gevrey asymptotic expansion of order 1 with respect to t in $S_{d,\theta,R}$, meaning that for every $\theta_1 < \theta$, one can find $C, M > 0$ with

$$\left\|\mathcal{L}^d(\mathcal{B}(\hat{X}))(t) - \sum_{p=0}^{n-1} \frac{a_p}{p!} t^p\right\|_{\mathbb{E}} \leq CM^n n! |t|^n,$$

for every $n \geq 1$ and $t \in S_{d,\theta_1,R}$.

One can state several algebraic properties on the formal Borel transformation. Direct computations when inserting a formal power series into the expressions below lead to prove the following

Proposition 4. For every $\hat{X}(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}$, $\hat{G}(t) = \sum_{n \geq 0} b_n \frac{t^n}{n!} \in \mathbb{E}[[t]]$, one has the following (formal) equalities:

$$(\tau \partial_\tau^2 + \partial_\tau)(\mathcal{B}(\hat{X})(\tau)) = \mathcal{B}(\partial_t \hat{X}(t))(\tau), \qquad\qquad \partial_\tau^{-1}(\mathcal{B}(\hat{X}))(\tau) = \mathcal{B}(t\hat{X}(t))(\tau),$$

$$\tau \mathcal{B}(\hat{X})(\tau) = \mathcal{B}((t^2 \partial_t + t)\hat{X}(t))(\tau), \qquad \int_0^\tau (\mathcal{B}\hat{X})(\tau - s)(\mathcal{B}\hat{G})(s) ds = \mathcal{B}(t\hat{X}(t)\hat{G}(t))(\tau).$$

4.2 Analytic solutions of a singular Cauchy problem

Let $S \geq 1$ be an integer. We set $r_1, r_2 \in \mathbb{N}$ with $r_1 \geq 0$ and $r_2 \geq 1$. Let $b > 1$, $\sigma > 0$ and $r > 0$.

We fix $d \in \mathbb{R}$. Let S_d be a sector with infinite radius and bisecting direction d and $\Omega_j := S_d \cup D(0, \rho_j)$, for $0 \leq j \leq S - 1$, which verify Assumption **(A)**. $\mathcal{E}$ stands for a fixed open sector with finite radius $r_{\mathcal{E}} > 0$. We also fix $\gamma \in [0, 2\pi)$ such that $\mathbb{R}_+ e^{i\gamma} \subseteq S_d \cup \{0\}$.

Let $\mathcal{S}$ (resp. $\mathcal{N}$) be a finite subset of $\mathbb{N}^3$ (resp. of $\mathbb{N}^2$). For every $(s, \kappa_0, \kappa_1) \in \mathcal{S}$, and $(\ell_0, \ell_1) \in \mathcal{N}$, $b_{s,\kappa_0,\kappa_1}(z, \epsilon)$, $c_{\ell_0,\ell_1}(z, \epsilon)$ are holomorphic bounded functions defined on $D(0, \rho) \times D(0, \epsilon_0)$, for some $\rho, \epsilon_0 > 0$.

For every $\epsilon \in \mathcal{E}$, we consider the following Cauchy problem

$$
\left((z\partial_z + 1)^{r_1}(t^2\partial_t + t)^{r_2} + 1\right) \partial_z^S Y_{d,\epsilon}(t, z) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{s,\kappa_0,\kappa_1}(z, \epsilon)\epsilon^{r(\kappa_0-s)}t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}Y_{d,\epsilon})(t, z)
$$

$$
(34) \qquad\qquad\qquad + \sum_{(\ell_0,\ell_1)\in\mathcal{N}} c_{\ell_0,\ell_1}(z, \epsilon)\epsilon^{-r(\ell_0+\ell_1-1)}t^{\ell_0+\ell_1-1}(Y_{d,\epsilon}(t, z))^{\ell_1}
$$

for given initial conditions

$$
(35) \qquad\qquad (\partial_z^j Y_{d,\epsilon})(t, 0) = Y_{d,\epsilon,j}(t), \quad 0 \le j \le S - 1.
$$

The initial conditions $Y_{d,\epsilon,j}(t)$, $0 \le j \le S - 1$ are constructed in the following way: for every $0 \le j \le S - 1$, let $V_j(\tau, \epsilon)$ be a holomorphic function defined in $\Omega_j \times \mathcal{E}$. Moreover, assume there exists $\delta > 0$ with

$$
(36) \qquad\qquad \sup_{\epsilon\in\mathcal{E}} \|V_j(\tau, \epsilon)\|_{j,\epsilon,\sigma,\Omega_j} < \delta, \quad 0 \le j \le S - 1.
$$

Then,

$$
Y_{d,\epsilon,j}(t) := \mathcal{L}_\tau^d(V_j(\tau, \epsilon))(t),
$$

where Laplace transform is taken with respect to variable τ in $V_j(\tau, \epsilon)$. From (33) in Definition 2, $t \mapsto Y_{d,\epsilon,j}(t)$ defines a holomorphic function for all $t = |t|e^{i\theta}$ such that $\cos(\theta - \gamma) \ge \delta_1 > 0$ and $|t| < |\epsilon|^r \frac{\delta_1}{\sigma\xi(b)}$, for some $\delta_1 > 0$, where $\xi(b) = \sum_{n\ge0} \frac{1}{(n+1)^b}$.

We now introduce a new condition, namely

Assumption (B):

$$
s \ge 2\kappa_0, \quad S > b(s - \kappa_0 + 2) + \kappa_1 \text{ for every } (s, \kappa_0, \kappa_1) \in \mathcal{S}, \qquad \ell_1 \ge 2 \text{ for every } (\ell_0, \ell_1) \in \mathcal{N}.
$$

Remark: We find it adequate to clarify the role played by the assumptions above. While the first inequality has to do with the condition appearing in the incoming Lemma 8, the second and third ones are related to the conditions asked on the parameters appearing in Proposition 3, when reducing the Cauchy problem (34)-(35) to the auxiliary one studied in Section 3.

The next result establishes the solution of (34)-(35) by means of the properties of Borel transform and Proposition 3.

Theorem 1. *Under Assumptions* **(A)** *and* **(B)**, *let the initial data (35) be constructed as above. Then, for every $\epsilon \in \mathcal{E}$, the problem (34)-(35) admits a holomorphic solution $(t, z) \mapsto Y_{d,\epsilon}(t, z)$ defined in*

$$
S_{d,\theta,|\epsilon|^r\frac{\delta_1}{\sigma\xi(b)}} \times D(0, \frac{\rho}{2}),
$$

for any fixed $\pi < \theta < \pi + 2\tilde{\delta}$.

Proof. Let $\epsilon \in \mathcal{E}$. By means of the formal Borel transform with respect to t applied to equation (34), and taking into account Proposition 4, this equation formally turns into

$$
\left((z\partial_z + 1)^{r_1}\tau^{r_2} + 1\right) \partial_z^S V_\epsilon(\tau, z) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{s,\kappa_0,\kappa_1}(z, \epsilon)\epsilon^{r(\kappa_0-s)}\partial_\tau^{-s}(\tau\partial_\tau^2 + \partial_\tau)^{\kappa_0}\partial_z^{\kappa_1}V_\epsilon(\tau, z)
$$

$$
(37) \qquad\qquad\qquad + \sum_{(\ell_0,\ell_1)\in\mathcal{N}} c_{\ell_0,\ell_1}(z, \epsilon)\epsilon^{-r(\ell_0+\ell_1-1)}\partial_\tau^{-\ell_0}(V_\epsilon(\tau, z))^{\star\ell_1}.
$$

Let $V_{j,\epsilon}(\tau) := V_j(\tau, \epsilon)$ for every $0 \le j \le S-1$. We consider

$$(38) \qquad (\partial_z^j V_\epsilon)(\tau, 0) = V_{j,\epsilon}(\tau) \in \mathcal{O}(\Omega_j), \quad 0 \le j \le S-1,$$

the initial conditions associated to the equation (37). The equation (37) can be suitably rewritten thanks to the two following technical lemmas. Their proof can be found in [26], Lemma 5 and Lemma 6, so we omit them.

Lemma 7. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $u : \Omega \to \mathbb{C}$ a holomorphic function. For every $\kappa_0 \ge 0$, one has*

$$(\tau \partial_\tau^2 + \partial_\tau)^{\kappa_0} u(\tau) = \sum_{\kappa=\kappa_0}^{2\kappa_0} a_{\kappa,\kappa_0} \tau^{\kappa-\kappa_0} \partial_\tau^\kappa u(\tau),$$

for some constants $a_{\kappa,\kappa_0} \in \mathbb{N}$, $\kappa_0 \le \kappa \le 2\kappa_0$.

Lemma 8. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $u : \Omega \to \mathbb{C}$ a holomorphic function. We also fix $a, b, c \in \mathbb{N}$ such that $a \ge b$ and $a \ge c$. Let $\Delta = a + b - c$. Then,*

$$\partial_\tau^{-a}(\tau^b \partial_\tau^c u(\tau)) = \sum_{(b',c') \in \mathcal{O}_\Delta} \alpha_{b',c'} \tau^{b'} \partial_\tau^{c'} u(\tau),$$

where $\mathcal{O}_\Delta$ is a finite subset of $\mathbb{Z}^2$ such that $b' - c' = \Delta$, $b' \ge 0$, $c' \le 0$ for every $(b', c') \in \mathcal{O}_\Delta$, and where $\alpha_{b',c'} \in \mathbb{Z}$.

From Assumption **(B)**, equation (37) is rewritten as

$$((z\partial_z + 1)^{r_1} \tau^{r_2} + 1) \partial_z^S V_\epsilon(\tau, z) = \sum_{(s,\kappa_0,\kappa_1) \in \mathcal{S}} b_{s,\kappa_0,\kappa_1}(z,\epsilon) \epsilon^{r(\kappa_0-s)} \sum_{(r',p') \in \mathcal{O}_{s-\kappa_0}} \alpha_{r',p'} \tau^{r'} \partial_\tau^{-p'} \partial_z^{\kappa_1} V_\epsilon(\tau, z)$$

$$(39) \qquad\qquad + \sum_{(\ell_0,\ell_1) \in \mathcal{N}} c_{\ell_0,\ell_1}(z,\epsilon) \epsilon^{-r(\ell_0+\ell_1-1)} \partial_\tau^{-\ell_0}(V_\epsilon(\tau, z))^{\star \ell_1},$$

where $\mathcal{O}_{s-\kappa_0}$ is a finite subset of $\mathbb{N}^2$ such that for every $(r', p') \in \mathcal{O}_{s-\kappa_0}$ one has $r' + p' = s - \kappa_0$, and $\alpha_{r',p'} \in \mathbb{Z}$.

We consider the Cauchy problem (39)-(38). From (36), we have $V_{j,\epsilon} \in F_{j,\epsilon,\sigma,\Omega_j}$ for $0 \le j \le S-1$. Moreover, (36) assures the verification of the first hypothesis in Proposition 3. In addition to this, the remaining hypotheses in Proposition 3 are being verified from Assumption **(B)** and the relationship among the indices involved. From Proposition 3 we learn that, as long as δ in (36) is small enough, there is a holomorphic function on $S_d \times D(0,\rho) \times \mathcal{E}$ for some $\rho > 0$,

$$(40) \qquad V_d(\tau, z, \epsilon) := \sum_{\beta \ge 0} V_\beta(\tau, \epsilon) \frac{z^\beta}{\beta!},$$

which solves (39)-(38). In addition to this, one has $V_\beta \in \mathcal{O}(\Omega_\beta \times \mathcal{E})$ and

$$(41) \qquad \sum_{\beta \ge 0} \|V_\beta(\tau, \epsilon)\|_{\beta,\epsilon,\sigma,\Omega_\beta} \frac{\rho^\beta}{\beta!} < +\infty.$$

Let $\beta \ge 0$. From (41) and the definition of the norms involved (see Definition 1), there exists $C > 0$ (not depending on ϵ nor β) such that

$$(42) \qquad |V_\beta(\tau, \epsilon)| \le C\beta! \left(\frac{1}{\rho}\right)^\beta \left(1 + \frac{|\tau|^2}{|\epsilon|^{2r}}\right)^{-1} e^{\frac{\sigma}{|\epsilon|^r} r_b(\beta)|\tau|},$$

for every $\tau \in \Omega_\beta$ and $\epsilon \in \mathcal{E}$. If $t = |t|e^{i\theta}$, we deduce that

$$\left| \int_{L_\gamma} V_\beta(\tau, \epsilon) e^{-\frac{\tau}{t}} d\tau \right| \leq \int_0^\infty |V_\beta(se^{i\gamma}, \epsilon)| e^{-\frac{s}{|t|} \cos(\gamma - \theta)} ds$$

$$\leq \int_0^\infty C\beta! \left(\frac{1}{\rho} \right)^\beta e^{\left(\frac{\sigma}{|\epsilon|^r} \xi(b) - \frac{\delta_1}{|t|} \right) s} ds,$$

when departing from t such that $\cos(\gamma - \theta) \geq \delta_1 > 0$. If $|t| < |\epsilon|^r \frac{\delta_1}{\sigma\xi(b)}$, then the previous integral converges. This implies that for every $\epsilon \in \mathcal{E}$, $\mathcal{L}_\tau^d(V_\beta(\tau, \epsilon))(t)$ is well defined for

$$t \in S_{d,\theta,|\epsilon|^r \frac{\delta_1}{\sigma\xi(b)}},$$

for any $\pi < \theta < \pi + 2\tilde{\delta}$. Moreover, for every $\epsilon \in \mathcal{E}$,

$$(t, z) \mapsto Y_{d,\epsilon}(t, z) := \sum_{\beta \geq 0} \mathcal{L}_\tau^d(V_\beta(\tau, \epsilon))(t) \frac{z^\beta}{\beta!}$$

defines a holomorphic function on $S_{d,\theta,|\epsilon|^r \frac{\delta_1}{\sigma\xi(b)}} \times D(0, \frac{\rho}{2})$, for every $\pi < \theta < \pi + 2\tilde{\delta}$. From the fact that $V_d(\tau, z, \epsilon)$ is a solution of (37)-(38) and from Proposition 4 we conclude that $(t, z) \mapsto Y_{d,\epsilon}(t, z)$ is a solution of (34)-(35) for every $\epsilon \in \mathcal{E}$. $\qquad\square$

5 Formal series solutions and Gevrey asymptotic expansions in a complex parameter for a threefold Cauchy problem

5.1 Analytic solutions in a complex parameter for a Cauchy problem

Let $S \geq 1$ be an integer. We fix $r_1, r_2, r_3 \in \mathbb{N}$ with

$$r_1 \geq 0, \quad r_2 \geq 1, \quad \text{and} \quad r_3 \geq 1,$$

and we put

$$r := \frac{r_3}{r_2}.$$

We recall the definition of a good covering.

Definition 3. *Let $\mathcal{E}_i$ be an open sector with vertex at 0 and finite radius ϵ_0 for every $0 \leq i \leq \nu - 1$. We assume that $\mathcal{E}_i \cap \mathcal{E}_{i+1} \neq \emptyset$ for $0 \leq i \leq \nu - 1$ (we define $\mathcal{E}_\nu := \mathcal{E}_0$). When a family of sectors $\{\mathcal{E}_i\}_{0 \leq i \leq \nu - 1}$ constructed as before verifies $\cup_{i=0}^{\nu-1} \mathcal{E}_i = \mathcal{U} \setminus \{0\}$, for some neighborhood $\mathcal{U}$ of $0 \in \mathbb{C}$, it is known as a good covering in $\mathbb{C}^\star$.*

Definition 4. *Let $\{\mathcal{E}_i\}_{0 \leq i \leq \nu - 1}$ be a good covering in $\mathbb{C}^\star$. Let $\mathcal{T}$ be an open sector with vertex at 0 and finite positive radius $r_\mathcal{T}$. We also fix a family of open sectors*

$$U_{d_i, \theta, \epsilon_0^r r_\mathcal{T}} = \{t \in \mathbb{C}^\star : |t| < \epsilon_0^r r_\mathcal{T}, |d_i - arg(t)| < \frac{\theta}{2}\},$$

with $d_i \in [0, 2\pi)$ for $0 \leq i \leq \nu - 1$, and $\theta > \pi$, with the following properties:

1.- for every i, $0 \leq i \leq \nu - 1$, one has $arg(d_i) \neq \pi \frac{(2k+1)}{r_2}$, $0 \leq k \leq r_2 - 1$, and

2.- *for every* $0 \le i \le \nu - 1$, $t \in \mathcal{T}$ *and* $\epsilon \in \mathcal{E}_i$, *one has* $\epsilon^r t \in U_{d_i,\theta,\epsilon_0^r r \mathcal{T}}$.

Under the previous settings, we say the family $\{(U_{d_i,\theta,\epsilon_0 r \mathcal{T}})_{0 \le i \le \nu-1}, \mathcal{T}\}$ *is associated to the good covering* $\{\mathcal{E}_i\}_{0 \le i \le \nu-1}$.

Let us consider a good covering in $\mathbb{C}^\star$, $\{\mathcal{E}_i\}_{0 \le i \le \nu-1}$.

Let $\mathcal{S}$ (resp. $\mathcal{N}$) be a finite subset of $\mathbb{N}^3$ (resp. $\mathbb{N}^2$). For any fixed $(s, \kappa_0, \kappa_1) \in \mathcal{S}$ and $(\ell_0, \ell_1) \in \mathcal{N}$, let $b_{s,\kappa_0,\kappa_1}, c_{\ell_0,\ell_1}$ be holomorphic bounded functions on a polydisc $D(0, \rho) \times D(0, \epsilon_0)$, for some $\rho > 0$.

For each $0 \le i \le \nu - 1$, we study the Cauchy problem

$$\left((z\partial_z + 1)^{r_1} \epsilon^{r_3} (t^2 \partial_t + t)^{r_2} + 1\right) \partial_z^S X_i(t, z, \epsilon) = \sum_{(s,\kappa_0,\kappa_1) \in \mathcal{S}} b_{s,\kappa_0,\kappa_1}(z, \epsilon) t^s \left(\partial_t^{\kappa_0} \partial_z^{\kappa_1} X_i\right)(t, z, \epsilon)$$

$$(43) \qquad\qquad + \sum_{(\ell_0,\ell_1) \in \mathcal{N}} c_{\ell_0,\ell_1}(z, \epsilon) t^{\ell_0 + \ell_1 - 1} (X_i(t, z, \epsilon))^{\ell_1},$$

for given initial conditions

$$(44) \qquad\qquad (\partial_z^j X_i)(t, 0, \epsilon) = \varphi_{i,j}(t, \epsilon), \quad 0 \le j \le S - 1,$$

where the functions $\varphi_{i,j}(t, \epsilon)$ are constructed as follows:
We take a family of sectors $\{\{U_{d_i,\theta,\epsilon_0^r r \mathcal{T}}\}_{0 \le i \le \nu-1}, \mathcal{T}\}$ associated to the good covering $\{\mathcal{E}_i\}_{0 \le i \le \nu-1}$.
For short, we write U_{d_i} instead of $U_{d_i,\theta,\epsilon_0^r r \mathcal{T}}$. Let $0 \le i \le \nu - 1$.
For every $0 \le j \le S - 1$, we consider a holomorphic function $V_{U_{d_i},\mathcal{E}_i,j}$ defined in $\Omega_{j,i} \times \mathcal{E}_i$, where

$$\Omega_{j,i} := S_{d_i} \cup D(0, \rho_j)$$

and S_{d_i}, ρ_j are as in Assumption $(\mathbf{A})$.
We assume this function verifies:

a) there exist $\delta, \sigma > 0$ with

$$(45) \qquad\qquad \sup_{\epsilon \in \mathcal{E}_i} \left\| V_{U_{d_i},\mathcal{E}_i,j}(\tau, \epsilon) \right\|_{j,\epsilon,\sigma,\Omega_{j,i}} < \delta.$$

b) There exists a function $\tau \in D(0, \rho_j) \mapsto V_j(\tau, \epsilon)$ such that $V_{U_{d_i},\mathcal{E}_i,j}(\tau, \epsilon) \equiv V_j(\tau, \epsilon)$ for every $0 \le i \le \nu - 1$, $\epsilon \in \mathcal{E}_i$, $\tau \in D(0, \rho_j)$, so that one has $V_{U_{d_i},\mathcal{E}_i,j}(\tau, \epsilon) = V_{U_{d_{i+1}},\mathcal{E}_{i+1},j}(\tau, \epsilon)$ for every $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$, $\tau \in D(0, \rho_j)$.

Let γ_i such that $L_{\gamma_i} = \mathbb{R}_+ e^{\gamma_i \sqrt{-1}} \subseteq S_{d_i} \cup \{0\}$. The function

$$(46) \qquad\qquad \varphi_{i,j}(t, \epsilon) = Y_{d_i,\epsilon,j}(\epsilon^r t) := \frac{1}{\epsilon^r t} \int_{L_{\gamma_i}} V_{U_{d_i},\mathcal{E}_i,j}(\tau, \epsilon) e^{-\frac{\tau}{\epsilon^r t}} d\tau,$$

is well defined for every $0 \le j \le S - 1$. $\varphi_{i,j}$ turns out to be a holomorphic function defined on $\mathcal{T} \times \mathcal{E}_i$.

Theorem 2. *Let the initial data (44) be constructed as above. Under Assumption $(\mathbf{A})$ on the sets $\Omega_{j,i}$, and Assumption $(\mathbf{B})$ on the constants appearing in (43), if δ in (45) is small enough the problem (43)-(44) has a holomorphic and bounded solution $X_i(t, z, \epsilon)$ on $(\mathcal{T} \cap D(0, h')) \times D(0, \frac{\rho}{2}) \times \mathcal{E}_i$, for every $0 \le i \le \nu - 1$, for some $h' > 0$.*

Moreover, for every $0 \le i \le \nu - 1$ there exist constants $0 < h'' < h'$, $K_i, M_i > 0$ (not depending on ϵ), such that

$$(47) \qquad \sup_{\substack{t \in \mathcal{T} \cap D(0, h'') \\ z \in D(0, \frac{\rho}{2})}} |X_{i+1}(t, z, \epsilon) - X_i(t, z, \epsilon)| \le K_i \exp\left(-\frac{M_i}{|\epsilon|^{r_3/(r_1 + r_2)}}\right),$$

for every $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$ (where, by convention, $X_\nu := X_0$).

Proof. Let $0 \le i \le \nu$ and $\epsilon \in \mathcal{E}_i$. We consider the Cauchy problem (34), with initial conditions

$$(48) \qquad (\partial_z^j Y_{d_i, \epsilon})(t, 0) = Y_{d_i, \epsilon, j}(t), \quad 0 \le j \le S - 1.$$

From our hypotheses, Theorem 1 shows that the problem (34)-(48) has a solution $(t, z) \mapsto Y_{d_i, \epsilon}(t, z)$, which is holomorphic and bounded in the set $U_{d_i, \theta, h'|\epsilon|^r} \times D(0, \frac{\rho}{2})$ for some $h' > 0$ (not depending on ϵ).

We put $X_i(t, z, \epsilon) = Y_{d_i, \epsilon}(\epsilon^r t, z)$ which defines a holomorphic and bounded function on $(\mathcal{T} \cap D(0, h')) \times D(0, \frac{\rho}{2}) \times \mathcal{E}_i$. We note that, by following the construction of X_i and applying Hartog's Theorem, one can see that X_i is holomorphic with respect to ϵ. Also, one easily checks that $X_i(t, z, \epsilon)$ solves the problem (43)-(44) on $(\mathcal{T} \cap D(0, h')) \times D(0, \frac{\rho}{2}) \times \mathcal{E}_i$.

In the second part of the proof, we obtain the estimates in (47).

Let $0 \le i \le \nu$. Following the procedure of construction for $X_i(t, z, \epsilon)$, one can write

$$X_i(t, z, \epsilon) = \sum_{\beta \ge 0} X_{i, \beta}(t, \epsilon) \frac{z^\beta}{\beta!},$$

where

$$X_{i, \beta}(t, \epsilon) = \frac{1}{\epsilon^r t} \int_{L_{\gamma_i}} V_{U_{d_i}, \mathcal{E}_i, \beta}(\tau, \epsilon) e^{-\frac{\tau}{\epsilon^r t}} d\tau,$$

for every $(t, z, \epsilon) \in (\mathcal{T} \cap D(0, h')) \times D(0, \rho/2) \times \mathcal{E}_i$. Here, the sequence $(V_{U_{d_i}, \mathcal{E}_i, \beta}(\tau, \epsilon))_{\beta \ge 0}$ consists of the coefficients of the series in (40), and

$$V_{U_{d_i}, \mathcal{E}_i}(\tau, z, \epsilon) := \sum_{\beta \ge 0} V_{U_{d_i}, \mathcal{E}_i, \beta}(\tau, \epsilon) \frac{z^\beta}{\beta!}$$

is the solution of the auxiliary problem (14)-(15), which is constructed in Proposition 3.

From assumption b) on the initial conditions and the recurrence formula (17), for every $\beta \ge 0$ we derive that the functions $V_{U_{d_i}, \mathcal{E}_i, \beta}$, $0 \le i \le \nu - 1$, define a holomorphic function $V_\beta(\tau, \epsilon)$ such that $V_\beta(\tau, \epsilon) = V_{U_{d_i}, \mathcal{E}_i, \beta}(\tau, \epsilon)$ for every $\tau \in D(0, \rho_\beta)$, $\epsilon \in \mathcal{E}_i$. For $\beta \ge 0$, $(t, \epsilon) \in (\mathcal{T} \cap D(0, h')) \times (\mathcal{E}_i \cap \mathcal{E}_{i+1})$, this yields

$$X_{i+1, \beta}(t, \epsilon) - X_{i, \beta}(t, \epsilon) = \frac{1}{\epsilon^r t} \left(\int_{L_{\frac{\rho_\beta}{2}, \gamma_{i+1}}} V_{U_{d_{i+1}}, \mathcal{E}_{i+1}, \beta}(\tau, \epsilon) e^{-\frac{\tau}{\epsilon^r t}} d\tau \right.$$

$$(49) \qquad \left. - \int_{L_{\frac{\rho_\beta}{2}, \gamma_i}} V_{U_{d_i}, \mathcal{E}_i, \beta}(\tau, \epsilon) e^{-\frac{\tau}{\epsilon^r t}} d\tau + \int_{C(\frac{\rho_\beta}{2}, \gamma_i, \gamma_{i+1})} V_\beta(\tau, \epsilon) e^{-\frac{\tau}{\epsilon^r t}} d\tau \right),$$

where $L_{\frac{\rho_\beta}{2}, \gamma_{i+1}} := [\frac{\rho_\beta}{2}, +\infty) e^{\sqrt{-1} \gamma_{i+1}}$, $L_{\frac{\rho_\beta}{2}, \gamma_i} := [\frac{\rho_\beta}{2}, +\infty) e^{\sqrt{-1} \gamma_i}$, and $C(\frac{\rho_\beta}{2}, \gamma_i, \gamma_{i+1})$ is an arc of circle with radius $\frac{\rho_\beta}{2}$ connecting $\frac{\rho_\beta}{2} e^{\sqrt{-1} \gamma_{i+1}}$ and $\frac{\rho_\beta}{2} e^{\sqrt{-1} \gamma_i}$ with a well chosen orientation.

First, we give estimates for $I_1 = |(\epsilon^r t)^{-1} \int_{L_{\frac{\rho_\beta}{2},\gamma_{i+1}}} V_{U_{d_{i+1}},\mathcal{E}_{i+1,\beta}}(\tau,\epsilon)e^{-\frac{\tau}{\epsilon^r t}}d\tau|$. Direction γ_{i+1} was chosen depending on $\epsilon^r t$. In fact, one can affirm there exists $\delta_1 > 0$ with $\cos(\gamma_{i+1} - \arg(\epsilon^r t)) \geq \delta_1$, for every $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$ and $t \in \mathcal{T} \cap D(0,h')$. From (42), we obtain

$$I_1 \leq |\epsilon^r t|^{-1} \int_{\rho_\beta/2}^\infty C_{i+1}\beta! \left(\frac{1}{\rho}\right)^\beta \left(1 + \frac{h^2}{|\epsilon|^{2r}}\right)^{-1} \exp(\frac{\sigma}{|\epsilon|^r}r_b(\beta)h - \frac{h}{|\epsilon|^r|t|}\cos(\gamma_{i+1} - \arg(\epsilon^r t)))dh$$

$$\leq |\epsilon^r t|^{-1} \int_{\rho_\beta/2}^\infty C_{i+1}\beta! \left(\frac{1}{\rho}\right)^\beta \exp\left((\sigma\xi(b) - \frac{\delta_1}{|t|})\frac{h}{|\epsilon|^r}\right)dh$$

$$= |t|^{-1}C_{i+1}\beta! \left(\frac{1}{\rho}\right)^\beta \frac{\exp\left((\sigma\xi(b) - \frac{\delta_1}{|t|})\frac{\rho_\beta}{2|\epsilon|^r}\right)}{\frac{\delta_1}{|t|} - \sigma\xi(b)},$$

for some $C_{i+1} > 0$. Moreover, let δ_2 be a positive constant such that $\delta_2 < \delta_1$, and take $h'' := \frac{\delta_1 - \delta_2}{\sigma\xi(b)} > 0$. Then, for $t \in \mathcal{T} \cap D(0,h')$ with $|t| < h''$ and every $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$ one derives

$$(50) \qquad I_1 \leq \delta_2^{-1}C_{i+1}\beta! \left(\frac{1}{\rho}\right)^\beta \exp(-\delta_2\frac{\rho_\beta}{2h'}\frac{1}{|\epsilon|^r}).$$

Estimates for $I_2 = |(\epsilon^r t)^{-1} \int_{L_{\frac{\rho_\beta}{2},\gamma_i}} V_{U_{d_i},\mathcal{E}_{i,\beta}}(\tau,\epsilon)e^{-\frac{\tau}{\epsilon^r t}}d\tau|$ follow from similar calculations. In this step we arrive at the existence of $C_i > 0$ such that

$$(51) \qquad |I_2| \leq \delta_2^{-1}C_i\beta! \left(\frac{1}{\rho}\right)^\beta \exp(-\frac{\delta_2}{2h'}\frac{\rho_\beta}{|\epsilon|^r}),$$

for every $t \in \mathcal{T} \cap D(0,h'')$ and $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$.

Finally, we study $I_3 = |\epsilon^r t|^{-1} \left|\int_{C(\frac{\rho_\beta}{2},\gamma_i,\gamma_{i+1})} V_\beta(\tau,\epsilon)e^{-\frac{\tau}{\epsilon^r t}}d\tau\right|$. From (42), we have

$$I_3 \leq |\epsilon^r t|^{-1} \left|\int_{\gamma_{i+1}}^{\gamma_i} C\beta! \left(\frac{1}{\rho}\right)^\beta \left(1 + \frac{\rho_\beta^2}{4|\epsilon|^{2r}}\right)^{-1} e^{\sigma r_b(\beta)\frac{\rho_\beta}{2|\epsilon|^r}} e^{-\frac{\rho_\beta}{2}\frac{1}{|\epsilon^r t|}\cos(\theta - \arg(\epsilon^r t))}\frac{\rho_\beta}{2}d\theta\right|.$$

By construction, the arc of circle $C(\frac{\rho_\beta}{2},\gamma_i,\gamma_{i+1})$ is chosen so that $\cos(\theta - \arg(\epsilon^r t)) \geq \delta_1$, for every $\theta \in [\gamma_i,\gamma_{i+1}]$ (if $\gamma_i < \gamma_{i+1}$), $\theta \in [\gamma_{i+1},\gamma_i]$ (if $\gamma_{i+1} < \gamma_i$), for every $t \in \mathcal{T}$ and all $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$. From the previous inequality we derive

$$I_3 \leq |\gamma_{i+1} - \gamma_i|C\beta! \left(\frac{1}{\rho}\right)^\beta \frac{\rho_\beta}{2}\frac{1}{|\epsilon^r t|} \exp\left(-\left(\frac{\delta_1}{|t|} - \sigma\xi(b)\right)\frac{\rho_\beta}{2}\frac{1}{|\epsilon|^r}\right)$$

$$\leq |\gamma_{i+1} - \gamma_i|C\beta! \left(\frac{1}{\rho}\right)^\beta \frac{\rho_\beta}{2}\frac{1}{|\epsilon^r t|} \exp\left(-\frac{\delta_2}{4}\frac{\rho_\beta}{|\epsilon|^r|t|}\right) \exp\left(-\frac{\delta_2}{|\epsilon|^r h'}\frac{\rho_\beta}{4}\right),$$

for all $t \in \mathcal{T} \cap D(0,h')$, $|t| < \frac{\delta_1 - \delta_2}{\sigma\xi(b)}$ and every $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$.

From (10) we derive

$$(52) \qquad I_3 \leq |\gamma_{i+1} - \gamma_i|C\beta! \left(\frac{1}{\rho}\right)^\beta \frac{2e^{-1}}{\delta_2} \exp\left(-\frac{\delta_2}{|\epsilon|^r h'}\frac{\rho_\beta}{4}\right),$$

for $t \in \mathcal{T} \cap D(0,h')$, $|t| < \frac{\delta_1 - \delta_2}{\sigma\xi(b)}$ and every $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$.

Taking into account (49), (50), (51) and (52) we arrive at

$$|X_{i+1}(t,z,\epsilon) - X_i(t,z,\epsilon)| \le \sum_{\beta \ge 0} |X_{i+1,\beta}(t,\epsilon) - X_{i,\beta}(t,\epsilon)| \frac{|z|^\beta}{\beta!}$$

$$(53) \qquad \le \frac{C_i + C_{i+1}}{\delta_2} \sum_{\beta \ge 0} e^{-\frac{\delta_2}{2h'} \frac{\rho_\beta}{|\epsilon|^r}} \left(\frac{1}{2}\right)^\beta + |\gamma_{i+1} - \gamma_i| C \frac{2e^{-1}}{\delta_2} \sum_{\beta \ge 0} e^{-\frac{\delta_2}{4h'} \frac{\rho_\beta}{|\epsilon|^r}} \left(\frac{1}{2}\right)^\beta,$$

for every $t \in \mathcal{T} \cap D(0,h')$, $|t| < \frac{\delta_1 - \delta_2}{\sigma \xi(b)}$, $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$ and $z \in D(0, \frac{\rho}{2})$.

The particular choice of $\rho_\beta := 1/(2(\beta+1)^{r_1/r_2})$, made in Assumption **(A)**, makes it necessary to work with the so called general Dirichlet series and apply the following

Lemma 9. *Let $0 < a < 1$ and $\alpha > 0$. There exist $K, M > 0$ and $\delta > 0$ such that*

$$\sum_{\kappa \ge 0} e^{-\frac{1}{(\kappa+1)^\alpha} \frac{1}{\epsilon}} a^\kappa \le K \exp\left(-M\epsilon^{-\frac{1}{\alpha+1}}\right),$$

for every $\epsilon \in (0, \delta]$.

The proof of this lemma is postponed until the end of the current section not to interfere the line of arguments.

From (53) and Lemma 9 we deduce that there exist $K_i, M_i > 0$ (not depending on ϵ) such that

$$|X_{i+1}(t,z,\epsilon) - X_i(t,z,\epsilon)| \le K_i e^{-\frac{M_i}{|\epsilon|^{r_3/(r_1+r_2)}}},$$

for every $t \in \mathcal{T} \cap D(0,h')$, $|t| \le \frac{\delta_1 - \delta_2}{\sigma \xi(b)}$, $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$, if ϵ_0 is small enough, and $|z| \le \frac{\rho}{2}$, from where we conclude. $\qquad \square$

The proof of Lemma 9 rests on the following results.

Lemma 10 (Watson's Lemma. Exercise 4, page 16 in [3])**.** *Let $b > 0$ and $f : [0,b] \to \mathbb{C}$ be a continuous function having the formal expansion $\sum_{n \ge 0} a_n t^n \in \mathbb{C}[[t]]$ as its asymptotic expansion of Gevrey order $\kappa > 0$ at 0, meaning there exist $C, M > 0$ such that*

$$\left| f(t) - \sum_{n=0}^{N-1} a_n t^n \right| \le CM^N N!^\kappa |t|^N,$$

for every $N \ge 1$ and $t \in [0, \delta]$, for some $0 < \delta < b$.

Then, the function

$$I(x) = \int_0^b f(s) e^{-\frac{s}{x}} ds$$

admits the formal power series $\sum_{n \ge 0} a_n n! x^{n+1} \in \mathbb{C}[[x]]$ as its Gevrey asymptotic expansion of Gevrey order $\kappa + 1$ at 0, it is to say, there exist $\tilde{C}, \tilde{K} > 0$ such that

$$\left| I(x) - \sum_{n=0}^{N-1} a_n n! x^{n+1} \right| \le \tilde{C} \tilde{K}^{N+1} (N+1)!^{1+\kappa} |x|^{N+1},$$

for every $N \ge 0$ and $x \in [0, \delta']$ for some $0 < \delta' < b$.

Lemma 11 (Exercise 3, page 18 in [3])**.** *Let $\delta, q > 0$, and $\psi : [0, \delta] \to \mathbb{C}$ be a continuous function. The following assertions are equivalent:*

1. There exist $C, M > 0$ such that $|\psi(x)| \leq CM^n n!^q |x|^n$, for every $n \in \mathbb{N}$, $n \geq 0$ and $x \in [0, \delta]$.

2. There exist $C', M' > 0$ such that $|\psi(x)| \leq C' e^{-M'/x^{\frac{1}{q}}}$, for every $x \in (0, \delta]$.

Proof of Lemma 9: Let $f : [0, +\infty) \to \mathbb{R}$ be a $\mathcal{C}^1$ function. From the Euler-Maclaurin formula, one has

$$(54) \qquad \sum_{\kappa=0}^{n} f(\kappa) = \frac{1}{2}(f(0) + f(n)) + \int_0^n f(t)dt + \int_0^n B_1(t - \lfloor t \rfloor)f'(t)dt,$$

for every $n \in \mathbb{N}$, where B_1 is the Bernoulli polynomial $B_1(s) = s - \frac{1}{2}$. Here, $\lfloor \cdot \rfloor$ stands for the floor function.

Let $\epsilon > 0$. If we choose f in (54) to be $f(s) = e^{-\frac{1}{(s+1)^\alpha} \frac{1}{\epsilon}} a^s$, for $s \geq 0$, one has

$$\sum_{\kappa=0}^{n} e^{-\frac{1}{(\kappa+1)^\alpha} \frac{1}{\epsilon}} a^\kappa = \frac{1}{2}(e^{-\frac{1}{\epsilon}} + e^{-\frac{1}{(n+1)^\alpha} \frac{1}{\epsilon}} a^n)$$

$$+ \int_0^n e^{-\frac{1}{(t+1)^\alpha} \frac{1}{\epsilon}} a^t \left[1 + B_1(t - \lfloor t \rfloor) \left(\frac{\alpha}{\epsilon(t+1)^{\alpha+1}} + \ln(a) \right) \right] dt,$$

for every $n \in \mathbb{N}$. Taking the limit when n tends to infinity in the previous expression we arrive at an equality for a convergent series:

$$\sum_{\kappa \geq 0} e^{-\frac{1}{(\kappa+1)^\alpha} \frac{1}{\epsilon}} a^\kappa$$

$$= \frac{1}{2} e^{-\frac{1}{\epsilon}} + \int_0^\infty \left[e^{-\frac{1}{(t+1)^\alpha} \frac{1}{\epsilon}} a^t + B_1(t - \lfloor t \rfloor) e^{-\frac{1}{(t+1)^\alpha} \frac{1}{\epsilon}} a^t \left(\frac{\alpha}{\epsilon(t+1)^{\alpha+1}} + \ln(a) \right) \right] dt$$

$$\leq \frac{1}{2} e^{-\frac{1}{\epsilon}} + \int_0^\infty e^{-\frac{1}{(t+1)^\alpha} \frac{1}{\epsilon}} a^t dt + \frac{1}{2} \int_0^\infty e^{-\frac{1}{(t+1)^\alpha} \frac{1}{\epsilon}} a^t \left(\frac{\alpha}{\epsilon(t+1)^{\alpha+1}} + |\ln(a)| \right) dt.$$

Let $I_1(\epsilon)$ (resp. $I_2(\epsilon)$) be the first (resp. second) integral appearing on the right hand side of the preceding inequality. The proof is reduced to demonstrate the existence of $\delta > 0$ and $K, M > 0$ such that

$$(55) \qquad I_1(\epsilon), I_2(\epsilon) \leq K e^{-\frac{M}{\epsilon^{1/(\alpha+1)}}},$$

for every $\epsilon \in (0, \delta)$. The change of variable $s := \frac{1}{(t+1)^\alpha}$ in $I_1(\epsilon)$ and $I_2(\epsilon)$ allows us to write

$$I_1(\epsilon) = \int_0^1 e^{-\frac{s}{\epsilon}} f_1(s)ds, \text{ with } f_1(s) := \frac{1}{a\alpha} \frac{1}{s^{\frac{1}{\alpha}+1}} e^{\ln(a)\frac{1}{s^{1/\alpha}}}, 0 < s \leq 1,$$

and

$$I_2(\epsilon) = \int_0^1 e^{-\frac{s}{\epsilon}} f_{2,1}(s)ds + \frac{1}{\epsilon} \int_0^1 e^{-\frac{s}{\epsilon}} f_{2,2}(s)ds,$$

with $f_{2,1}(s) := |\ln(a)| f_1(s)$ and $f_{2,2}(s) := \frac{1}{a} e^{\ln(a)\frac{1}{s^{1/\alpha}}}$, $0 < s \leq 1$, for every $\epsilon > 0$. It is clear that $|f_1(s)|$, $|f_{2,1}(s)|$ and $|f_{2,2}(s)|$ may be estimated by $C' \exp\left(-M'/s^{1/\alpha}\right)$ for some $C', M' > 0$ and for every $s \in (0, 1]$. From Lemma 11, we get that $f_1, f_{2,1}, f_{2,2}$ have $\hat{0}$ as their formal asymptotic expansion of Gevrey order α. Lemma 10 indicates that both $I_1(\epsilon)$ and $\epsilon I_2(\epsilon)$ admit $\hat{0}$ as their Gevrey asymptotic expansion of order $\alpha+1$, and the same is true for $I_2(\epsilon)$. Again, by Lemma 11, one derives that there exist $C'', M'' > 0$ such that $|I_1(\epsilon)|$ and $|I_2(\epsilon)|$ can be upper bounded by $C'' e^{-\frac{M''}{\epsilon^{1/(\alpha+1)}}}$, from where we can easily conclude (55). $\qquad \square$

5.2 Existence of formal series solutions in the compex parameter for the threefold singular problem

This final subsection is devoted to prove the main result of the present work, the existence of a formal power series $\hat{X}(t, z, \epsilon)$ which asymptotically represents the solution of problem (43)-(44) in a precise sense, for every $0 \leq i \leq \nu - 1$. Indeed, under the same notation as in the previous subsection, the formal power series $\hat{X}(t, z, \epsilon)$ belongs to $\mathcal{O}((\mathcal{T} \cap D(0, h'')) \times D(0, \frac{\rho}{2}))[[\epsilon]]$, and $X_i(t, z, \epsilon)$, solution of $(43) - (44)$ admits $\hat{X}$ as its Gevrey asymptotic expansion of order $\frac{r_1 + r_2}{r_3}$ on $\mathcal{E}_i$, for every $0 \leq i \leq \nu - 1$.

The proof rests on a cohomological criterion for summability of formal series with coefficients in a Banach space, known in the literature as Malgrange-Sibuya theorem. For a reference, see [4], [37].

Theorem 3. (MS)

Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a complex Banach space over $\mathbb{C}$. Let $\{\mathcal{E}_i\}_{0 \leq i \leq \nu - 1}$ be a good covering in $\mathbb{C}^{\star}$. For every $0 \leq i \leq \nu - 1$, let G_i be a holomorphic function from $\mathcal{E}_i$ into $\mathbb{E}$, and let the cocycle $\Delta_i(\epsilon) := G_{i+1}(\epsilon) - G_i(\epsilon)$ be a holomorphic function from $Z_i := \mathcal{E}_i \cap \mathcal{E}_{i+1}$ into $\mathbb{E}$ (with the convention that $\mathcal{E}_\nu = \mathcal{E}_0$ and $G_\nu = G_0$). We assume that:

1. *$G_i(\epsilon)$ is bounded as $\epsilon \in \mathcal{E}_i$ tends to 0, for every $0 \leq i \leq \nu - 1$,*

2. *Δ_i has an exponential decreasing of order $s > 0$ on Z_i, for every $0 \leq i \leq \nu - 1$, meaning there exist $C_i, A_i > 0$ such that*

$$\|\Delta_i(\epsilon)\|_{\mathbb{E}} \leq C_i e^{-\frac{A_i}{|\epsilon|^{1/s}}},$$

for every $\epsilon \in Z_i$ and $0 \leq i \leq \nu - 1$.

Then, there exists a formal power series $\hat{G}(\epsilon) \in \mathbb{E}[[\epsilon]]$ such that $G_i(\epsilon)$ admits $\hat{G}(\epsilon)$ as its asymptotic expansion of Gevrey order s on $\mathcal{E}_i$, for every $0 \leq i \leq \nu - 1$.

The last assertion in Theorem 3 means that if we write $\hat{G}(\epsilon) = \sum_{n \geq 0} G_n \epsilon^n$ and fix $0 \leq i \leq \nu - 1$, then for every proper and bounded subsector T_i of $\mathcal{E}_i$ and all $N \geq 1$, one has

$$\left\| G_i(\epsilon) - \sum_{n=0}^{N-1} G_n \epsilon^n \right\|_{\mathbb{E}} \leq K M^N N!^s |\epsilon|^N, \quad \epsilon \in T_i$$

for some positive constants $K, M > 0$, not depending on N.

We now state the main result of this paper. Let $\mathbb{E}$ be the Banach space of holomorphic and bounded functions defined on $(\mathcal{T} \cap D(0, h'')) \times D(0, \frac{\rho}{2})$, equipped with the supremum norm, for $h'', \rho > 0$ and $\mathcal{T}$ as in Theorem 2.

Theorem 4. *Under the hypotheses made on Theorem 2, if $h'' > 0$ is small enough, there exists a formal power series*

$$\hat{X}(t, z, \epsilon) := \sum_{\kappa \geq 0} H_\kappa(t, z) \frac{\epsilon^\kappa}{\kappa!} \in \mathbb{E}[[\epsilon]],$$

which formally solves the threefold singular problem (43)-(44) and is the Gevrey asymptotic expansion of order $\frac{r_1 + r_2}{r_3}$ of the $\mathbb{E}$-valued function $\epsilon \in \mathcal{E}_i \mapsto X_i(t, z, \epsilon)$ constructed in Theorem 2, for every $0 \leq i \leq \nu - 1$.

Proof. Let us consider the family $(X_i(t,z,\epsilon))_{0\leq i\leq \nu-1}$ constructed in Theorem 2. For every $0 \leq i \leq \nu-1$ and $\epsilon \in \mathcal{E}$, we put $G_i(\epsilon) := (t,z) \mapsto X_i(t,z,\epsilon)$, which belongs to the space $\mathbb{E}$. From (47), we derive the cocycle $\Delta_i = G_{i+1}(\epsilon) - G_i(\epsilon)$ is exponentially decreasing of order $\frac{r_1+r_2}{r_3}$ on the set $Z_i = \mathcal{E}_i \cap \mathcal{E}_{i+1}$, for every $0 \leq i \leq \nu-1$.

From Theorem **(MS)**, one can guarantee the existence of $\hat{G}(\epsilon) \in \mathbb{E}[[\epsilon]]$, series of asymptotic expansion of order $\frac{r_1+r_2}{r_3}$ of $G_i(\epsilon)$, on $\mathcal{E}_i$, for $0 \leq i \leq \nu-1$. Let us define

$$\hat{G}(\epsilon) := \hat{X}(t,z,\epsilon) = \sum_{\kappa \geq 0} H_\kappa(t,z)\frac{\epsilon^\kappa}{\kappa!}.$$

It only rests to prove that $\hat{X}$ is a formal solution of (43)-(44). From the fact that $X_i(\epsilon)$ admits $\hat{X}(\epsilon)$ as its asymptotic expansion at 0 on $\mathcal{E}_i$, one has

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon \in \mathcal{E}_i}} \sup_{(t,z)\in(\mathcal{T}\cap D(0,h''))\times D(0,\rho/2)} \left|\partial_\epsilon^\ell X_i(t,z,\epsilon) - H_\ell(t,z)\right| = 0, \tag{56}$$

for every $0 \leq i \leq \nu-1$ and $\ell \geq 0$.

We fix $0 \leq i \leq \nu-1$. By construction, X_i satisfies (43)-(44). We differentiate in the equality (43) ℓ times with respect to ϵ. By means of Leibniz's rule, we deduce that $\partial_\epsilon^\ell X_i(t,z,\epsilon)$ satisfies

$$\sum_{h=1}^{\min\{\ell,r_3\}} \frac{\ell!}{(\ell-h)!h!}r_3(r_3-1)\cdot\ldots\cdot(r_3-h+1)\epsilon^{r_3-h}(z\partial_z+1)^{r_1}(t^2\partial_t+t)^{r_2}\partial_z^S\partial_\epsilon^{\ell-h}X_i(t,z,\epsilon)$$

$$+ \epsilon^{r_3}(z\partial_z+1)^{r_1}(t^2\partial_t+t)^{r_2}\partial_z^S\partial_\epsilon^\ell X_i(t,z,\epsilon) + \partial_z^S\partial_\epsilon^\ell X_i(t,z,\epsilon)$$

$$= \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}}\left(\sum_{h_1+h_2=\ell}\frac{\ell!}{h_1!h_2!}(\partial_\epsilon^{h_1}b_{s,\kappa_0,\kappa_1}(z,\epsilon))t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}\partial_\epsilon^{h_2}X_i(t,z,\epsilon))\right)$$

$$+ \sum_{(\ell_0,\ell_1)\in\mathcal{N}}\left(\sum_{h_0+\ldots+h_{\ell_1}=\ell}\frac{\ell!}{h_0!\cdots h_{\ell_1}!}(\partial_\epsilon^{h_0}c_{\ell_0,\ell_1}(z,\epsilon))t^{\ell_0+\ell_1-1}\prod_{j=1}^{\ell_1}\partial_\epsilon^{h_j}X_i(t,z,\epsilon)\right), \tag{57}$$

for every $\ell \geq 0$ and $(t,z,\epsilon) \in (\mathcal{T}\cap D(0,\frac{\rho}{2}))\times D(0,\frac{\rho}{2})\times\mathcal{E}_i$. We take $\ell \geq r_3$ and let $\epsilon \to 0$ in (57). From (56) one gets a recursion formula verified by the elements in $(H_\ell)_{\ell\geq 0}$ given by

$$(z\partial_z+1)^{r_1}(t^2\partial_t+t)^{r_2}\partial_z^S\left(\frac{H_{\ell-r_3}(t,z)}{(\ell-r_3)!}\right) + \partial_z^S\left(\frac{H_\ell(t,z)}{\ell!}\right)$$

$$= \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}}\left(\sum_{h_1+h_2=\ell}\frac{(\partial_\epsilon^{h_1}b_{s,\kappa_0,\kappa_1})(z,0)}{h_1!}t^s\partial_t^{\kappa_0}\partial_z^{\kappa_1}\left(\frac{H_{h_2}(t,z)}{h_2!}\right)\right)$$

$$+ \sum_{(\ell_0,\ell_1)\in\mathcal{N}}\left(\sum_{h_0+\ldots+h_{\ell_1}=\ell}\frac{(\partial_\epsilon^{h_0}c_{\ell_0,\ell_1})(z,0)}{h_0!}t^{\ell_0+\ell_1-1}\prod_{j=1}^{\ell_1}\left(\frac{H_{h_j}(t,z)}{h_j!}\right)\right), \tag{58}$$

for every $\ell \geq r_3$, and $(t,z) \in (\mathcal{T}\cap D(0,h''))\times D(0,\rho/2)$. From holomorphy of every $b_{s,\kappa_0,\kappa_1}(z,\epsilon)$ and $c_{\ell_0,\ell_1}(z,\epsilon)$ with respect to ϵ near the origin, we have

$$b_{s,\kappa_0,\kappa_1}(z,\epsilon) = \sum_{h\geq 0}\frac{(\partial_\epsilon^h b_{s,\kappa_0,\kappa_1})(z,0)}{h!}\epsilon^h, \quad c_{\ell_0,\ell_1}(z,\epsilon) = \sum_{h\geq 0}\frac{(\partial_\epsilon^h c_{\ell_0,\ell_1})(z,0)}{h!}\epsilon^h, \tag{59}$$

for every (z, ϵ) in a neighborhood of the origin in $\mathbb{C}^2$. In order to conclude, from recursion (58) and (59), we deduce $\hat{X}(t, z, \epsilon) = \sum_{\kappa \geq 0} H_\kappa(t, z) \frac{\epsilon^\kappa}{\kappa!}$ is a formal solution of (43)-(44). $\qquad \square$

References

[1] S. Alinhac, *Problèmes de Cauchy pour des opérateurs singuliers.* (French) Bull. Soc. Math. France 102 (1974), 289–315.

[2] T. Apostol, Modular functions and Dirichlet series in number theory, Springer, first edition 1976, second edition, 1990.

[3] W. Balser, From divergent power series to analytic functions. Theory and application of multisummable power series. Lecture Notes in Mathematics, **1582**. Springer-Verlag, Berlin, 1994.

[4] W. Balser, Formal power series and linear systems of meromorphic ordinary differential equations, Springer-Verlag, New-York, 2000.

[5] A. S. Besicovitch, Almost periodic functions, Dover, New York, 1954.

[6] H. Bohr, Almost periodic functions, Chelsea, New York, 1947.

[7] A. Bove, J. Lewis, C. Parenti, *Propagation of singularities for fuchsian operators.* Lecture Notes in Mathematics, 984. Springer-Verlag, Berlin, 1983. ii+161 pp.

[8] B. Braaksma, *Multisummability of formal power series solutions of nonlinear meromorphic differential equations.* Ann. Inst. Fourier (Grenoble) 42 (1992), no. 3, 517–540.

[9] M. Canalis-Durand, J. Mozo-Fernandez, R. Schäfke, *Monomial summability and doubly singular differential equations*, J. Differential Equations 233 (2007), no. 2, 485–511.

[10] H. Chen, Z. Luo, H. Tahara, *Formal solutions of nonlinear first order totally characteristic type PDE with irregular singularity.* Ann. Inst. Fourier (Grenoble) 51 (2001), no. 6, 1599-1620.

[11] H. Chen, Z. Luo, C. Zhang, *On the summability of formal solutions for a class of nonlinear singular PDEs with irregular singularity.* Recent progress on some problems in several complex variables and partial differential equations, 53–64, Contemp. Math., 400, Amer. Math. Soc., Providence, RI, 2006.

[12] E. A. Coddington, N. Levinson, Theory of ordinary differential equations, MacGraw-Hill, New York, 1955.

[13] C. Corduneanu, Almost periodic functions, Wiley Interscience, New York, 1961.

[14] O. Costin, Asymptotics and Borel summability, Chapman Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 141. CRC Press, Boca Raton, FL, 2009.

[15] O. Costin, S. Tanveer, *Existence and uniqueness for a class of nonlinear higher-order partial differential equations in the complex plane.* Comm. Pure Appl. Math. 53 (2000), no. 9, 1092-1117.

[16] J. Écalle, Les fonctions résurgentes. Publications Mathématiques d'Orsay, 1981.

[17] H. Fattorini, *The hyperbolic singular perturbation problem: an operator theoretic approach.* J. Differential Equations 70 (1987), no. 1, 1–41.

[18] R. Geel, E. de Jager, *Hyperbolic singular perturbations of nonlinear first order differential equations.* Differential equations and applications (Proc. Third Scheveningen Conf., Scheveningen, 1977), pp. 75–83, North-Holland Math. Stud., 31, North-Holland, Amsterdam-New York, 1978.

[19] R. Gérard, H. Tahara, *Singular nonlinear partial differential equations.* Aspects of Mathematics. Friedr. Vieweg and Sohn, Braunschweig, 1996. viii+269 pp.

[20] G. H. Hardy, M. Riesz, The general theory of Dirichlet series, Cambridge University Press, 1915.

[21] K. Igari, *Removable singularities of holomorphic solutions of partial differential equations.* J. Math. Soc. Japan 56 (2004), no. 1, 87–113.

[22] E. de Jager, F. Jiang, The theory of singular perturbations. North-Holland Series in Applied Mathematics and Mechanics, 42. north-Holland Publishing Co., Amsterdam, 1996. xii+340 pp.

[23] M. K. Kadalbajoo, K. C. Patidar, *Singularly perturbed problems in partial differential equations: a survey.* Appl. Math. Comput. 134 (2003), no. 2-3, 371–429.

[24] A. F. Leont'ev, Entire functions. Series of exponentials (in Russian), Nauka, Moscow, 1983.

[25] S. Malek, *On Gevrey functions solutions of partial differential equations with fuchsian and irregular singularities*, J. Dyn. Control Syst 15 (2009), no. 2.

[26] S. Malek, *On the summability of formal solutions for doubly nonlinear partial differential equations*, J. Dyn. Control Syst. 18 (2012), no. 1, 45–82.

[27] S. Malek, C. Stenger, *Complex singularity analysis of holomorphic solutions of linear partial differential equations*, to appear in Adv. Dyn. Syst. Appl. (2011).

[28] B. Malgrange, *Sommation des séries divergentes.* Expo. Math. **13** (1995), no. 2–3, 163–222.

[29] T. Mandai, *Existence and nonexistence of null-solutions for some non-Fuchsian partial differential operators with T-dependent coefficients.* Nagoya Math. J. 122 (1991), 115–137.

[30] T. Mandai, *The method of Frobenius to Fuchsian partial differential equations.* J. Math. Soc. Japan 52 (2000), no.3, 645–672.

[31] J. Melenk, C. Schwab, *Analytic regularity for a singularly perturbed problem.* SIAM J. Math. Anal. 30 (1999), no. 2, 379–400.

[32] S. Ouchi, *Multisummability of formal solutions of some linear partial differential equations.* J. Differential Equations 185 (2002), no.2, 513–549.

[33] S. Ouchi, *Multisummability of formal power series solutions of nonlinear partial differential equations in complex domains* Asymptot. Anal. 47 (2006), no.3-4, 187–225.

[34] P. Pongérard, C. Wagschal, *Opérateurs de Fuchs non linéaires.* (French) [Nonlinear Fuchs operators] Ann. Fac. Sci. Toulouse Math. (6) 16 (2007), no.2, 303–329.

[35] J-P. Ramis, *Dévissage Gevrey.* Journées Singulières de Dijon (Univ. Dijon, Dijon, 1978), pp.4, 173–204, Astérisque, 59–60, Soc. Math. France, Paris, 1978.

[36] J-P. Ramis, Y. Sibuya, *A new proof of multisummability of formal solutions of nonlinear meromorphic differential equations.* Ann. Inst. Fourier (Grenoble) 44 (1994), no. 3, 811–848.

[37] Y. Sibuya, Linear differential equations in the complex domain: problems of analytic continuation. Translations of Mathematical Monographs, 82. American Mathematical Society, Providence, RI, 1990.

[38] E. C. Titchmarsh, The theory of functions, Oxford University Press, second edition, 1939.

[39] H. Yamane, *Global Fuchsian Cauchy problem.* J. Math. Sci. Univ. Tokyo 7 (2000), no. 1, 147–162.

Authors' Affiliation:

Alberto Lastra
Departamento de Física y Matemáticas
Edificio de Ciencias
Universidad de Alcalá
Alcalá de Henares, Madrid, Spain
E-mail: alberto.lastra@uah.es

Stéphane Malek
UFR de Mathématiques Pures et Appliquées
Cité Scientifique - Bât. M2
59655 Villeneuve d'Ascq Cedex, France
E-mail: malek@math.univ-lille1.fr

Javier Sanz
Departamento de Álgebra. Análisis Matemático, Geometría y Topología
Facultad de Ciencias
Universidad de Valladolid
47011 Valladolid, Spain
E-mail: jsanzg@am.uva.es