ON SOME SHARP SPECTRAL INEQUALITIES FOR SCHRÖDINGER OPERATORS ON SEMI-AXIS

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ABSTRACT. In this paper we obtain sharp Lieb-Thirring inequalities for a Schrödinger operator on semi-axis with a matrix potential and show how they can be used to other related problems. Among them are spectral inequalities on star graphs and spectral inequalities for Schrödinger operators on half-spaces with Robin boundary conditions.

1. INTRODUCTION

Let us consider a self-adjoint Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta - V, \tag{1.1}$$

where V is a real-valued function. If the potential function V decays rapidly enough, then the spectrum of the operator H typically is absolutely continuous on $[0, \infty)$. If V has a non-trivial positive part, then H might have finite or infinite number of negative eigenvalues $\{-\lambda_n(H)\}$. If the number of negative eigenvalues is infinite, the point zero is the only possible accumulating point. The inequalities

$$\sum_n \lambda_n^\gamma \leq \frac{R_{\gamma,d}}{(2\pi)^d} \iint_{\mathbb{R}^{2d}} (|\xi|^2 - V(x))_-^\gamma \, d\xi dx \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}} \, dx \tag{1.2}$$

are known as Lieb-Thirring bounds. Here and in the following, $V_\pm = (|V| \pm V)/2$ denote the positive and negative parts of the function V.

It is known that the inequality (1.2) holds true with some finite constants if and only if $\gamma \geq 1/2$, $d = 1$; $\gamma > 0$, $d = 2$ and $\gamma \geq 0$, $d \geq 3$. There are examples showing that (1.2) fails for $0 \leq \gamma < 1/2$, $d = 1$ and $\gamma = 0$, $d = 2$.

Almost all the cases except for $\gamma = 1/2$, $d = 1$ and $\gamma = 0$, $d \geq 3$ were justified in the original paper of E.H.Lieb and W.Thirring [LT]. The critical case $\gamma = 0$, $d \geq 3$ is known as the Cwikel-Lieb-Rozenblum inequality, see [Cw, L, Roz]. It was also proved in [Fe, LY, Con] and very recently by R. Frank [Fr] using Rumin's approach. The remaining case $\gamma = 1/2$, $d = 1$ was verified by T.Weidl in [W1].

The sharp value of the constants $R_{\gamma,d} = 1$ in (1.2) are known for the case $\gamma \geq 3/2$ in all dimensions and it was first proved in [LT] and [AizL] for $d = 1$ and later in

1991 *Mathematics Subject Classification.* Primary: 35P15; Secondary: 81Q10.

Key words and phrases. Schrödinger operators, Lieb-Thirring inequalities.

[LW1, LW2] for any dimension. In this case

$$L_{\gamma,d} = L^{cl}_{\gamma,d} := (2\pi)^{-d} \int_{\mathbb{R}^d} (1 - |\xi|^2)^{\gamma}_{+} \, d\xi.$$

The only other case where the sharp value of the constant $R_{\gamma,d}$ is known is the case $R_{1/2,1} = 2$.

In this paper we consider a one-dimensional systems of Schrödinger operators acting in $L^2(\mathbb{R}_+, \mathbb{C}^N)$, $\mathbb{R}_+ = (0,\infty)$, defined by

$$\mathcal{H}\varphi(x) = \Big(-\frac{d^2}{dx^2} \otimes \mathbb{I} - V(x) \Big) \varphi(x), \qquad \varphi'(0) - \mathfrak{S}\, \varphi(0) = 0, \tag{1.3}$$

where $\mathbb{I}$ is the $N \times N$ identity matrix, V is a Hermitian $N \times N$ matrix-function and $\mathfrak{S}$ is a $N \times N$ Hermitian matrix.
Assuming that the potential V generates only a discrete negative spectrum, we denote by $\{-\lambda_n\}$ the negative eigenvalues of $\mathcal{H}$.

One of the main results of this paper is the following

Theorem 1.1. *Let* $\operatorname{Tr} V^2 \in L^1(\mathbb{R}_+)$, $V \geq 0$. *Then the negative spectrum of the operator* $\mathcal{H}$ *defined in* (1.3) *is discrete and the following Lieb-Thirring inequality for its eigenvalues* $\{-\lambda_n\}$ *holds*

$$\frac{3}{4}\lambda_1 \operatorname{Tr} \mathfrak{S} + \frac{1}{2}\,(2\varkappa_1 - N)\,\lambda_1^{3/2} + \sum_{n=2}^{\infty} \varkappa_n \lambda_n^{3/2} \leq \frac{3}{16} \int_0^\infty \operatorname{Tr} V^2(x)\, dx + \frac{1}{4} \operatorname{Tr} \mathfrak{S}^3, \tag{1.4}$$

where $\varkappa_n$ *is the multiplicity of the eigenvalue* $-\lambda_n$.

Examples.
1. *Let* $V \equiv 0$ *and* $N = 1$. *Then the boundary value problem*

$$-\varphi''(x) = -\lambda\varphi(x), \qquad \varphi'(0) - \sigma\varphi(0) = 0, \quad \sigma < 0,$$

has only one L^2*-solution*

$$\varphi(x) = C\, e^{-\sqrt{\lambda}\, x}, \qquad -\sqrt{\lambda} = \sigma.$$

In this case the inequality (1.4) *becomes saturated,* $\frac{3}{4}\,\sigma^3 - \frac{1}{2}\,\sigma^3 \leq \frac{1}{4}\,\sigma^3$.

2. *Let* $N = 2$, $V \equiv 0$ *and*

$$\mathfrak{S} = \begin{pmatrix} \sigma & 0 \\ 0 & -\alpha\sigma \end{pmatrix}, \qquad \sigma < 0$$

2a) *If $\alpha \geq 0$ then the boundary value problem* (1.3) *has one negative eigenvalue $-\lambda$ of multiplicity one satisfying the identity $-\sqrt{\lambda} = \sigma$. In this case $2\varkappa_1 - N = 0$ and the inequality* (1.4) *becomes*

$$3\lambda \operatorname{Tr} \mathfrak{S} = 3\lambda\, \sigma\, (1-\alpha) \leq (1-\alpha^3)\, \sigma^3 = \operatorname{Tr} \mathfrak{S}^3,$$

or

$$3(\alpha - 1) \leq \alpha^3 - 1.$$

which holds true for any $\alpha \geq 0$.
2b) *If $-1 < \alpha < 0$, then the problem* (1.3) *has two eigenvalues satisfying $-\sqrt{\lambda_1} = \sigma$ and $-\sqrt{\lambda_2} = -\alpha\sigma$ and* (1.4) *is reduced to*

$$3(\alpha - 1) - 4\alpha^3 \leq \alpha^3 - 1.$$

2c) *Finally, if $\alpha = -1$, then $-\sqrt{\lambda_1} = \sigma$ is of multiplicity $\varkappa_1 = 2$ and* (1.4) *becomes identity.*

Note that if $\operatorname{Tr} \mathfrak{S}^3 \leq 0$, then the inequality (1.4) implies

$$\frac{3}{4}\lambda_1 \operatorname{Tr} \mathfrak{S} + \frac{1}{2}\,(2\varkappa_1 - N)\,\lambda_1^{3/2} + \sum_{n=2}^{\infty} \varkappa_n \lambda_n^{3/2} \leq \frac{3}{16}\int_0^\infty \operatorname{Tr} V^2(x)\, dx. \tag{1.5}$$

The latter allows us to use the standard Aizenman-Lieb arguments [AizL] and derive

Corollary 1.2. *Let $\operatorname{Tr} \mathfrak{S}^3 \leq 0$, $V \geq 0$ and $\operatorname{Tr} V^{\gamma+1/2}(x) \in L^1(0,\infty)$. Then for any $\gamma \geq 3/2$ we have*

$$\begin{aligned}\frac{\mathcal{B}(\gamma - 3/2, 2)}{\mathcal{B}(\gamma - 3/2, 5/2)}\, \frac{3}{4}\, \lambda_1^{\gamma - 1/2} \operatorname{Tr} \mathfrak{S} + \frac{1}{2}\,(2\varkappa_1 - N)\,\lambda_1^\gamma + \sum_{n=2}^{\infty} \varkappa_n \lambda_n^\gamma& \\ \leq L^{cl}_{\gamma,1} \int_0^\infty \operatorname{Tr}\,(V(x))^{\gamma+1/2}\, dx&,\end{aligned}$$

where by $\mathcal{B}(p,q)$ we denote the classical Beta function

$$\mathcal{B}(p,q) = \int_0^1 (1-t)^{q-1} t^{p-1}\, dt.$$

Corollary 1.3. *If $\mathfrak{S} = 0$, then* (1.3) *can be identified with the Neumann boundary value problem and we obtain*

$$\frac{1}{2}\,(2\varkappa_1 - N)\,\varkappa_1 \lambda_1^\gamma + \sum_{n=2}^{\infty} \varkappa_n \lambda_n^\gamma \leq L^{cl}_{\gamma,1} \int_0^\infty \operatorname{Tr}\,(V(x))^{\gamma+1/2}\, dx, \quad \gamma \geq 3/2.$$

Remark.
Note that in the scalar case $N = 1$ we obtain

$$\frac{1}{2}\,\lambda_1^\gamma + \sum_{n=2}^\infty \lambda_n^\gamma \le L_{\gamma,1}^{cl} \int_0^\infty V^{\gamma+1/2}(x)\,dx, \qquad \gamma \ge 3/2, \tag{1.6}$$

which means that the semi-classical inequality holds true for all eigenvalues starting from $n = 2$ and that in the latter inequality the Neumann boundary condition affects only the first eigenvalue.

If $V \ge 0$ is a diagonal $N \times N$ matrix-function, then the operator $\mathcal{H}$ could be interpreted as a Schrödinger operator on a star graph with N edges; the matrix $\mathfrak{S}$ describes a vertex coupling without the Dirichlet component [Ku]. In such a case we obtain:

Theorem 1.4. *Let $V \ge 0$ be a diagonal $N \times N$ matrix-function and let $\mathfrak{S}$ be a Hermitian matrix. Then the operator* (1.3) *can be identified with a Schrödinger operator on a star graph with N semi-infinite edges and its negative spectrum satisfies the inequality* (1.4).

If both $V \ge 0$ and $\mathfrak{S}$ are diagonal $N \times N$ matrices, then the negative spectrum of the operator $\mathcal{H}$ is the union of the eigenvalues from each channel and we obtain

Theorem 1.5. *Let $V \ge 0$, $\operatorname{Tr} V^2 \in L^1(0,\infty)$ and let V and $\mathfrak{S}$ be diagonal $N \times N$ matrices with entries v_j and σ_j, $j = 1, \dots N$, respectively. Then the negative eigenvalues of the operator $\mathcal{H}$ defined in* (1.3)*, satisfy the inequality*

$$\frac{3}{4}\sum_{j=1}^N \lambda_{j1}\,\sigma_j + \frac{1}{2}\,\sum_{j=1}^N \lambda_{j1}^{3/2} + \sum_{j=1}^N \sum_{n=2}^\infty \lambda_{jn}^{3/2} \le \frac{3}{16}\int_0^\infty \operatorname{Tr} V^2(x)\,dx + \frac{1}{4}\operatorname{Tr}\mathfrak{S}^3, \tag{1.7}$$

where $-\lambda_{jn}$ are negative eigenvalues of operators h_j defined by

$$h_j\psi(x) = \frac{d^2}{dx^2}\psi(x) - v_j(x)\psi(x), \qquad \psi'(0) - \sigma_j\psi(0) = 0.$$

Remark. *Note that the inequality* (1.7) *is much more precise than* (1.4) *due to the diagonal structure of the operator $\mathcal{H}$. In* (1.7) *all N first eigenvalues generated by each channel are affected by the Robin boundary conditions, whereas in* (1.4) *only the first one, see Example 2b).*

Finally we give an example how our results could be applied for spectral estimates of multi-dimensional Schrödinger operators.
Let $\mathbb{R}^d_+ = \{x = (x_1, x') : x_1 > 0, x' \in \mathbb{R}^{d-1}\}$ and let H be a Schrödinger operator in $L^2(\mathbb{R}^d_+)$ with the Neumann boundary conditions

$$H\psi = -\Delta\psi - V\psi = -\lambda\psi, \qquad \frac{\partial}{\partial x_1}\psi(0,x') = 0. \tag{1.8}$$

The following result could be obtained by a “lifting” argument with respect to dimension, see [L], [LT]:

Theorem 1.6. *Let $V \geq 0$ and $V \in L^{\gamma+d/2}$, $\gamma \geq 3/2$. Then for the negative eigenvalues $\{-\lambda_n\}$ of the operator* (1.8) *we have*

$$\sum_n \lambda_n^\gamma \leq L^{cl}_{\gamma,d} \int_{\mathbb{R}^d_+} V^{\gamma+d/2}(x)\, dx + \frac{1}{2}\, L^{cl}_{\gamma,d-1} \int_{\mathbb{R}^{d-1}} \mu_1^{\gamma+(d-1)/2}(x')\, dx' \leq 2\, L^{cl}_{\gamma,d} \int_{\mathbb{R}^d_+} V^{\gamma+d/2}\, dx. \quad (1.9)$$

Here $\mu_1(x')$ is the ground state energy for the operator $-d^2/dx_1^2 - V(x_1, x')$ in $L^2(\mathbb{R}_+)$ with the Neumann boundary condition at zero.

Remark. *A similar inequality could be obtained by extending the operator* (1.8) *to the whole space $L^2(\mathbb{R}^d)$ with the symmetrically reflected potential. However, applying then the known Lieb-Thirring inequalities, we would have the constant $2^{\gamma+d/2}$ instead of 2 in* (1.9).

2. Some auxiliary results

In this Section we assume that the matrix-function V is compactly supported, $\operatorname{supp} V \subset [a, b]$ for some $a, b : 0 < a < b < \infty$ and adapt the arguments from [BL] to the case of semiaxis.

We begin with stating a well-known fact concerning the ground state of the operator (1.3).

Lemma 2.1. *Let $-\lambda < 0$ be the ground state energy of the operator $\mathcal{H}$ and let $\varphi(x) = \{\varphi_k\}_{k=0}^N$ be a $L^2(\mathbb{R}_+, \mathbb{C}^N)$-vector-function satisfying the equation*

$$\mathcal{H}\varphi(x) = -\frac{d^2}{dx^2}\varphi(x) - V(x)\varphi(x) = -\lambda\, \varphi(x), \qquad \varphi'(0) - \mathfrak{S}\varphi(0) = 0, \quad (2.1)$$

and such that the $2N$ vector $(\varphi(0), \varphi'(0))$ is not trivial. Then $\varphi(x) \neq 0$, $x \in \mathbb{R}_+$, and the ground state energy multiplicity is at most N.

Proof. Suppose that $\varphi(x_0) = 0$ for some $x_0 > 0$. Consider the continuous function

$$\tilde\varphi(x) = \begin{cases} \varphi(x), & x < x_0 \\ 0, & x \geq x_0. \end{cases}$$

This function is non-trivial, belongs to the Sobolev space $H^1(\mathbb{R}_+, \mathbb{C}^N)$ and satisfies

$$\int_{\mathbb{R}_+} \left(|\tilde\varphi'|^2 - (V\tilde\varphi, \tilde\varphi)_{\mathbb{C}^N}\right) dx = \int_0^{x_0} \left(|\varphi'|^2 - (V\varphi, \varphi)_{\mathbb{C}^N}\right) dx$$
$$= \int_0^{x_0} (-\varphi'' - V\varphi, \varphi)_{\mathbb{C}^N}\, dx = -\lambda \int_0^{x_0} |\varphi|^2\, dx = -\lambda \int_{\mathbb{R}_+} |\tilde\varphi|^2\, dx.$$

Therefore $\tilde\varphi$ minimizes the closed quadratic form associated with $\mathcal{H}$. Thus by the variational principle $\tilde\varphi$ belongs to the domain of $\mathcal{H}$ and solves the Cauchy problem pointwise. However, since $\tilde\varphi(x) = 0$ for $x \geq x_0$ it also solves the backward Cauchy problem with zero initial data at x_0 and by uniqueness must vanish everywhere. This contradicts the non-triviality of $\tilde{\varphi}$ for $x < x_0$. $\square$

Similarly to [BL] let us introduce a (not necessary L^2) fundamental $N \times N$-matrix-solution $M(x)$ of the equation (2.1), where $-\lambda$ is the ground state energy for the operator $\mathcal{H}$, so M satisfies the equation

$$-\frac{d^2}{dx^2}M(x) - V(x)M(x) = -\lambda\, M(x), \qquad M'(0) - \mathfrak{S}M(0) = 0. \tag{2.2}$$

Denoting $M(0) = A$ and $M'(0) = B$, $B - \mathfrak{S}\, A = 0$, we shall always assume that the matrix A is invertible.
By using Lemma 2.1 we obtain that the matrix-function $M(x)$ is invertible for any $x \in \mathbb{R}_+$ and thus we can consider

$$F(x) = M'(x)\, M^{-1}(x). \tag{2.3}$$

Lemma 2.2. *The matrix function $F(x)$ satisfies the following properties:*

- *$F(x)$ is Hermitian for any $x \in \mathbb{R}_+$.*
- *$F(x)$ is independent of the choice of the matrices A, B, satisfying the equation $B - \mathfrak{S}A = 0$ and*
$$F(0) = B\, A^{-1} = \mathfrak{S}.$$
- *F satisfies the matrix Riccati equation*
$$F'(x) + F^2(x) + V(x) = \lambda \mathbb{I}. \tag{2.4}$$

Proof. From the Wronskian identity

$$\frac{d}{dx}W(x) := \frac{d}{dx}\Big(M^*(x)\, M'(x)\, - (M^*(x))'\, M(x)\Big) = 0$$

we obtain

$$W(x) = M^*(x)\, M'(x) - (M^*(x))'\, M(x) = \text{const}.$$

Since $M(0) = A$ and $M'(0) = B$, using the fact that $\mathfrak{S}$ is Hermitian we find

$$\begin{aligned} W(0) &= M^*(0)\, M'(0) - (M^*(0))'\, M(0) \\ &= A^*\Big(B\, A^{-1} - (A^*)^{-1}B^*\Big)\, A = A^*(\mathfrak{S} - \mathfrak{S}^*)A = 0. \end{aligned}$$

Thus

$$W(x) = M^*(x)\, M'(x) - (M^*(x))'\, M(x) = 0.$$

Multiplying the latter identity by M^{-1} from the right and by $(M^{-1})^*$ from the left we obtain $F(x) = F^*(x)$. Moreover

$$\begin{aligned} &F' + F^2 = (M'\, M^{-1})' + (M'\, M^{-1})^2 \\ &= M''M^{-1} - M'\, M^{-1}M'M^{-1} + M'\, M^{-1}M'M^{-1} = (\lambda - V)M\, M^{-1} = \lambda\, \mathbb{I} - V. \end{aligned}$$

□

Next, we analyze the behavior of the matrices $F(x)$ and their eigenvalues and eigenvectors as $x \to \infty$. For $x > b$ any solution of the differential equation (2.2) can be written as

$$\begin{aligned} M(x) &= \cosh(\sqrt{\lambda}(x-b))M(b) + \frac{1}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}(x-b))M'(b) \\ &= \left(\cosh(\sqrt{\lambda}(x-b))\mathbb{I} + \frac{1}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}(x-b))F(b)\right)M(b). \end{aligned} \tag{2.5}$$

With the help of this representation we show

Lemma 2.3. *For all $x \geq b$ it holds $F(x) = f(x, F(b))$, where*

$$f(x,\mu) = \sqrt{\lambda}\,\frac{\sqrt{\lambda}\tanh(\sqrt{\lambda}(x-b)) + \mu}{\sqrt{\lambda} + \mu\tanh(\sqrt{\lambda}(x-b))}. \tag{2.6}$$

Proof. In view of (2.5) we have

$$M'(x) = \left(\sqrt{\lambda}\sinh(\sqrt{\lambda}(x-b))\mathbb{I} + \cosh(\sqrt{\lambda}(x-b))F(b)\right)M(b),$$

$$(M(x))^{-1} = (M(b))^{-1}\left(\cosh(\sqrt{\lambda}(x-b))\mathbb{I} + \frac{1}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}(x-b))F(b)\right)^{-1}.$$

It remains to insert these expressions in the definition $F(x) = M'(x)(M(x))^{-1}$ and to apply the spectral theorem for the Hermitian matrix $F(b)$. □

Note that $f(x,\mu)$ is strictly monotone in μ. As a direct consequence of Lemma 2.3 we conclude, that the eigenvectors of the matrix $F(x)$ are independent of x for $x \geq b$ as vectors in $\mathbb{C}^N$. Moreover, the eigenvalues of F may or may not depend on x outside the support of V depending on if they correspond to growing or decaying solutions.

Corollary 2.4. *Each eigenvalue μ_k of $F(b)$ gives rise to a continuous eigenvalue branch $\mu_k(x) = f(x, \mu_k(b))$. In particular, we have*

$$\mu_k(x) = -\sqrt{\lambda} \quad \textit{iff} \quad \mu_k(b) = -\sqrt{\lambda},$$

and

$$\lim_{x\to\infty}\mu_k(x) = \sqrt{\lambda} \quad \textit{iff} \quad \mu_k(b) \neq -\sqrt{\lambda}.$$

The limit in the last expression is achieved exponentially fast.

Remark. *There is a one-to-one correspondence between the $\varkappa_1$-dimensional space of ground states for $\mathcal{H}$ and a $\varkappa_1$-dimensional eigenspace of $F(b)$ corresponding to the eigenvalue $-\sqrt{\lambda}$. Indeed, since $M(x)$ is a fundamental system of the solutions of the Cauchy problem* (2.1) *and $F(b)$ is invertible, any particular solution φ of*

(2.1) *can be represented as* $\varphi(x) = F(x)(F(b))^{-1}\,\nu$ *with some* $\nu \in \mathbb{C}^N$*. Hence, by* (2.5)

$$\begin{aligned}\varphi_\nu(x) &= \cosh(\sqrt{\lambda}(x-b))\,\nu + \frac{1}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}(x-b))F(b)\,\nu \\ &= \frac{1}{2\sqrt{\lambda}}\,e^{\sqrt{\lambda}(x-b)}\Big(\sqrt{\lambda}\nu + F(b)\,\nu\Big) - \frac{1}{2\sqrt{\lambda}}\,e^{-\sqrt{\lambda}(x-b)}\Big(\sqrt{\lambda}\nu - F(b)\,\nu\Big). \qquad (2.7)\end{aligned}$$

This function becomes an L^2*-eigenfunction of* $\mathcal{H}$*, if and only if* $F(b)\,\nu = -\sqrt{\lambda}\,\nu$*.*

3. Proofs of the main results

Proof of Theorem 1.1 .
Let now $-\lambda_1$ be the ground state energy of the operator $\mathcal{H}$ with multiplicity $\varkappa_1 \le N$, let $M_1(x)$ be a fundamental system of solutions corresponding the eigenvalue $-\lambda_1$ and $F_1 = {M_1}'\,M_1^{-1}$. We consider the operator

$$Q_1 = \frac{d}{dx}\otimes\mathbb{I} - F_1(x)$$

and its adjoint

$$Q_1^* = -\frac{d}{dx}\otimes\mathbb{I} - F_1(x)$$

in $L^2(\mathbb{R}^+,\mathbb{C}^N)$. Using Riccati's equation (2.4) we obtain the following factorization of the original operator $\mathcal{H}$

$$Q_1^*Q_1 = -\frac{d^2}{dx^2}\otimes\mathbb{I} + {F_1}'(x) + (F_1(x))^2 = \mathcal{H} + \lambda_1\mathbb{I}.$$

Consider

$$Q_1Q_1^* = -\frac{d^2}{dx^2}\otimes\mathbb{I} - V(x) - 2{F_1}'(x) + \lambda_1\mathbb{I} = \mathcal{H} - 2{F_1}'(x) + \lambda_1\mathbb{I}.$$

Note that non-zero eigenvalues of $Q_1^*Q_1$ and $Q_1Q_1^*$ are same. However, while the vector-eigenfunctions φ defined in (2.7) satisfy the boundary conditions

$$\varphi'(0) - \mathfrak{S}\varphi(0) = 0,$$

the vector-eigenfunctions of $Q_1Q_1^*$ satisfy the Dirichlet boundary condition at 0.

Indeed, if φ is a vector-eigenfunction of $Q_1^*Q_1$ satisfying $\varphi'(0) - \mathfrak{S}\varphi(0) = 0$ then $\psi = Q_1\varphi$ is an eigenfunction of $Q_1Q_1^*$ and

$$\psi(0) = (Q_1\varphi)(0) = \varphi'(0) - F_1(0)\varphi(0) = 0.$$

Next, let us verify that the kernel $\ker Q_1^*$ is trivial, and consequently, $0 \notin \operatorname{spec}(Q_1Q_1^*)$. Indeed, assume for a moment that there is a non-trivial vector-function ψ satisfying the Dirichlet boundary conditions at $x = 0$ and such that

$$Q_1Q_1^*\psi = 0. \qquad (3.1)$$

Then

$$(Q_1Q_1^*\psi,\psi) = \|Q_1^*\psi\| = 0.$$

However, $Q_1^*\phi = 0$ if and only if $\psi'(x) = F(x)\psi(x)$ for all $x \in \mathbb{R}_+$ and, in particular, $\psi'(0) = F(0)\psi(0) = 0$. Since ψ satisfies the equation (3.1) together with $\psi(0) = \psi'(0) = 0$ we obtain that $\psi \equiv 0$.

Hence, the negative spectra of $\mathcal{H}$ and $\mathcal{H} - 2{F_1}'$ coincide except for the spectral value of the ground state energy, which does not belong to the spectrum of $\mathcal{H} - 2{F_1}'$ anymore. We emphasize that even in the case of a $\varkappa_1$-fold degenerate ground state $-\lambda_1 = -\lambda_2 = \dots = -\lambda_{\varkappa_1}$ of $\mathcal{H}$, this commutation method removes all these eigenvalues $-\lambda_1, -\lambda_2, \dots, -\lambda_{\varkappa_1}$.

Therefore the spectral problem for the operator (1.3) is reduced to the operator in $L^2(\mathbb{R}_+)$

$$\mathcal{H}_1\psi = \Big(-\frac{d^2}{dx^2}\otimes I - V(x) - 2{F_1}'\Big)\,\psi = -\lambda\psi \qquad \psi(0) = 0.$$

Let us extend V by zero to the negative semi-axis. Using then the variational principle we can apply the well-known Lieb-Thirring inequalities for 1D Schrödinger operators with matrix-valued potentials (see [LW1], [BL]) and obtain

$$\begin{aligned}\sum_{n=2}^\infty \varkappa_n\,\lambda_n^{3/2} &\le \frac{3}{16}\int_0^\infty \operatorname{Tr}\,(V(x) + 2{F_1}'(x))^2\,dx \\ &= \frac{3}{16}\int_0^\infty \operatorname{Tr}\,\Big(V^2(x) + 4F_1'(x)\big(V(x) + F_1'(x)\big)\Big)\,dx.\end{aligned}$$

Using the Riccati equation (2.4), the fact that the matrix $\lim_{x\to\infty} F(x)$ has the eigenvalue $-\sqrt{\lambda_1}$ of multiplicity $\varkappa_1$ and the eigenvalue $\sqrt{\lambda_1}$ of multiplicity $N - \varkappa_1$ and that $F(0) = \mathfrak{S}$, we finally arrive at

$$\begin{aligned}\sum_{n=2}^\infty \varkappa_n\,\lambda_n^{3/2} &\le \frac{3}{16}\int_0^\infty \operatorname{Tr}\,\Big(V^2(x) + 4F_1'(x)(\lambda_1 - F_1^2(x)\Big)\,dx \\ &= \frac{3}{16}\int_0^\infty \operatorname{Tr}\,V^2(x)\,dx + \frac{3}{4}\,\lambda_1\operatorname{Tr}\,F_1(x)\Big|_0^\infty - \frac{1}{4}\operatorname{Tr}\,F_1^3(x)\Big|_0^\infty \\ &= \frac{3}{16}\int_0^\infty \operatorname{Tr}\,V^2(x)\,dx + \frac{3}{4}\,\lambda_1\Big(-\varkappa_1\sqrt{\lambda_1} + (N-\varkappa_1)\sqrt{\lambda_1} - \operatorname{Tr}\mathfrak{S}\Big) \\ &\qquad - \frac{1}{4}\Big(-\varkappa_1\lambda_1^{3/2} + (N-\varkappa_1)\lambda_1^{3/2} - \operatorname{Tr}\mathfrak{S}^3\Big) \\ &= \frac{3}{16}\int_0^\infty \operatorname{Tr}\,V^2(x)\,dx - \frac{1}{2}\,(2\varkappa_1 - N)\lambda_1^{3/2} - \frac{3}{4}\,\lambda_1\operatorname{Tr}\mathfrak{S} + \frac{1}{4}\operatorname{Tr}\mathfrak{S}^3.\end{aligned}$$

Finally using standard arguments we can consider the closure of the latter inequality from the class of compactly supported potentials to the class $L^2(\mathbb{R}_+, \mathbb{C}^N\times\mathbb{C}^N)$. The proof of Theorem 1.1 is complete.

Proof of Corollary 1.2.

Let us denote by $\lambda_n = \lambda_n(V)$ the eigenvalues of the Schrödinger operator with the potential V. Then by using the variational principle and the inequality (1.5) we find that for any $\gamma > 3/2$

$$\begin{aligned}
&\mathcal{B}(\gamma-3/2,2)\,\frac{3}{4}\,\mathrm{Tr}\,\mathfrak{S}\,\lambda_1^{\gamma-1/2}(V)\\
&\qquad + \mathcal{B}(\gamma-3/2,5/2)\,\Big(\frac{1}{2}\,(2\varkappa_1 - N)\,\lambda_1^\gamma(V) + \sum_{n=2}^\infty \varkappa_n \lambda_n^\gamma(V)\Big)\\
&\qquad\qquad = \int_0^\infty \Big(\frac{3}{4}\,\mathrm{Tr}\,\mathfrak{S}\,(\lambda_1(V)-t)_+\\
&\qquad + \frac{1}{2}\,(2\varkappa_1 - N)(\lambda_1(V)-t)_+^{3/2} + \sum_{n=2}^\infty \varkappa_n(\lambda_n(V)-t)_+^{3/2}\Big)t^{\gamma-5/2}\,dt\\
&\qquad\qquad \le \int_0^\infty \Big(\frac{3}{4}\,\mathrm{Tr}\,\mathfrak{S}\,(\lambda_1((V-t)_+)\\
&\qquad + \frac{1}{2}\,(2\varkappa_1 - N)(\lambda_1((V-t)_+)^{3/2} + \sum_{n=2}^\infty \varkappa_n(\lambda_n((V-t)_+)^{3/2}\Big)t^{\gamma-5/2}\,dt\\
&\qquad\qquad \le \frac{3}{16}\int_0^\infty\int_0^\infty \mathrm{Tr}\,(V(x)-t)_+^2\,t^{\gamma-5/2}\,dtdx\\
&\qquad\qquad\qquad = \mathcal{B}(\gamma-3/2,3)\,\frac{3}{16}\int_0^\infty \mathrm{Tr}\,V^{\gamma+1/2}(x)\,dx.
\end{aligned}$$

Dividing by $\mathcal{B}(\gamma-3/2,5/2)$ and noting that

$$\frac{3}{16}\,\frac{\mathcal{B}(\gamma-3/2,3)}{\mathcal{B}(\gamma-3/2,5/2)} = L^{cl}_{\gamma+1/2,1}$$

we complete the proof.

Proof of Theorem 1.6.
Let $\{\mu_j(x')\}$ be eigenvalues of the Neumann problem for the Schrödinger operator

$$-\frac{d^2}{dx_1^2}\psi(x_1,x') - V(x_1,x')\,\psi(x_1,x') = -\mu(x')\psi(x_1,x')$$

considering x' as a parameter.
For any $\gamma \ge 3/2$ and $d \ge 1$ let us apply the operator version of the Lieb-Thirring inequality (see [LW1]) with respect to $\mathbb{R}^{d-1}$ and obtain

$$\sum_n \lambda_n^\gamma \le L^{cl}_{\gamma,d-1}\int_{\mathbb{R}^{d-1}}\sum_j \mu_j^{\gamma+(d-1)/2}(x')\,dx'.$$

By using (1.6) we find

$$\sum_j \mu_j^{\gamma+(d-1)/2}(x') \le \frac{1}{2}\,\mu_1^{\gamma+(d-1)/2}(x') + L^{cl}_{\gamma+(d-1)/2,1}\int_0^\infty V^{\gamma+d/2}(x_1,x')\,dx_1$$
$$\le 2\,L^{cl}_{\gamma+(d-1)/2,1}\int_0^\infty V^{\gamma+d/2}(x_1,x')\,dx_1.$$

Noticing that

$$L^{cl}_{\gamma,d-1}L^{cl}_{\gamma+(d-1)/2,1} = L^{cl}_{\gamma,d}$$

we obtain the proof.

Acknowledgment. P.E. is grateful for the hospitality extended to him in *Institut Mittag-Leffler* where a part of the work was done. The research was supported in part by the *Czech Science Foundation* under the project P203/11/0701.

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