ABSOLUTELY CONTINUOUS SPECTRUM OF THE SCHRÖDINGER OPERATOR WITH A POTENTIAL REPRESENTABLE AS A SUM OF THREE FUNCTIONS WITH SPECIAL PROPERTIES

OLEG SAFRONOV

1. Main results

We study the absolutely continuous spectrum of a Schrödinger operator

$$H = -\Delta + V + V_+ + \alpha V_0, \tag{1}$$

acting in the space $L^2(\mathbb{R}^d)$. Here, V, V_+ and V_0 are real valued potentials; α is a real parameter.

Definition. We say that the absolutely continuous spectrum of the operator H is essentially supported by a set containing $[0,\infty)$, if the spectral projection $E(\Omega)$ of H corresponding to any set $\Omega \subset [0,\infty)$ is different from zero $E(\Omega) \neq 0$ as soon as the Lebesgue measure of Ω is positive.

While the potential V_0 is a function of $x \in \mathbb{R}^d$, we shall also study the dependence of V_0 on the spherical coordinates $r = |x|$ and $\theta = x/|x|$. Therefore, sometimes the value of V_0 at $x \in \mathbb{R}^d$ will be denoted by $V_0(r,\theta)$. Even less often the radial variable will be denoted by ρ. Let

$$W_0(r,\theta) = \int_0^r V_0(\rho,\theta)d\rho, \qquad \forall r > 0. \tag{2}$$

Assume that W_0 belongs to the space $\mathcal{H}^1_{\rm loc}(\mathbb{R}^d)$ of functions having (generalized) locally square integrable derivatives. Suppose that

$$\int_{\mathbb{R}^d} \frac{|\nabla W_0|^2}{|x|^{d-1}} dx < \infty. \tag{3}$$

Note that in $d = 1$, condition (3) turns into

$$\int_{\mathbb{R}} |V_0|^2 dx < \infty. \tag{4}$$

Operators with such potentials were studied in the work of Deift and Killip [3], the main result of which states that absolutely continuous spectrum of the operator $-d^2/dx^2 + V_0$ covers the positive half-line $[0,\infty)$, if V_0 satisfies (4).

While V and V_0 are assumed to be bounded

$$V \in L^\infty(\mathbb{R}^d), \qquad V_0 \in L^\infty(\mathbb{R}^d),$$

the potential $V_+ \geq 0$ does not have to be bounded on $\mathbb{R}^d$ globally:

$$V_+ \in L^\infty_{loc}(\mathbb{R}^d).$$

Nevertheless, we would like to have some control on the behavior of the function V_+ at the infinity, so we impose the condition

$$\lim_{|x|\to\infty} \frac{V_+(x)}{\exp(\epsilon|x|)} = 0, \qquad \forall \epsilon > 0. \tag{5}$$

We shall also suppose that

$$\int_{\mathbb{R}^d} \frac{V_+}{|x|^{d-1}}\, dx < \infty. \tag{6}$$

The function V will be oscillating, because we shall assume that there exists a vector potential $Q : \mathbb{R}^d \to \mathbb{R}^d$, such that

$$V = |x|^{(d-1)/2} \mathrm{div}\, Q, \qquad \text{and } Q \in L^2(\mathbb{R}^d). \tag{7}$$

The next condition shows clearly that our assumptions about V and V_+ are not independent:

$$V_+ \geq \tau \Big(|x|^{d-1} |Q|^2 + \frac{(d-1)}{2} |x|^{(d-3)/2} |Q| \Big), \qquad \text{for some} \quad \tau > 1, \tag{8}$$

which implies, in particular, that Q is locally bounded in the region $\mathbb{R}^d \setminus \{0\}$.

Our main result is the following

Theorem 1.1. *Let V and V_+ obey conditions* (5)-(8). *Let V_0 be a real potential satisfying conditions* (2) *and* (3). *Then the absolutely continuous spectrum of the operator* (1) *is essentially supported by a set containing* $[0, \infty)$ *for almost every $\alpha \in \mathbb{R}$.*

Since the potential V_+ might be unbounded, the operator H can not be defined as the sum of two operators $-\Delta$ and $V + V_+ + \alpha V_0$. Instead of that, one defines H as the self-adjoint operator corresponding to the quadratic form

$$\mathfrak{h}[u] = \int_{\mathbb{R}^d} |\nabla u|^2\, dx + \int_{\mathbb{R}^d} \Big(V(x) + V_+(x) + \alpha V_0 \Big) |u|^2\, dx.$$

The domain $d(\mathfrak{h})$ of this quadratic form consists of all $\mathcal{H}^1$-functions for which the integral

$$\int_{\mathbb{R}^d} V_+(x) |u|^2\, dx < \infty$$

is finite. The quadratic form $\mathfrak{h}$ generates the sesquilinear form (which is denoted by the same symbol):

$$\mathfrak{h}[u, v] = \frac{1}{4} \Big(\mathfrak{h}[u+v] - \mathfrak{h}[u-v] + i(\mathfrak{h}[u+iv] - \mathfrak{h}[u-iv]) \Big).$$

Obviously, $\mathfrak{h}[u] = \mathfrak{h}[u, u]$.

According to the general theory of self-adjoint operators, a function $u \in d(\mathfrak{h})$ belongs to the domain of H, if and only if there exists $w \in L^2(\mathbb{R}^d)$ such that

$$\mathfrak{h}[u, v] = (w, v)$$

for all v from the domain of the quadratic form. Moreover, in this case, $Hu = w$.

The potentials for which the presence of the a.c. spectrum is already established can be conditionally divided in the three groups. These three groups correspond to our choice of the functions V, V_+ and V_0. Functions of the form V can be called oscillating functions. Such potentials are studied in the paper [8], which suggested the idea to study the a.c. spectrum of operator families for the first time. Potentials of the form V_0 were studied in [22]. Finally, conditions that are imposed on V_+ were first introduced in [11]. In the present paper, we interpolate between the three separate cases. The main technical difficulty appearing on our way is that the dependence of the spectral measure of the operator H on the potential is not linear.

One of the reasons why such an interpolation might be important is the following. Let Ω be a bounded smooth domain in $\mathbb{R}^{d-1}$. Suppose that

$$V \in L^2 \Big(\Omega \times \mathbb{R} \Big) \cap L^\infty \Big(\Omega \times \mathbb{R} \Big), \qquad V = \overline{V}.$$

Then V is representable as a sum of two real-valued bounded functions

$$V = V_1 + V_2,$$

such that $V_2 \in L^2$ depends only on the last variable and $V_1 = \operatorname{div} Q$ with $Q \in L^2\Big(\Omega \times \mathbb{R}\Big)$. Therefore, potentials of the form $V_1 + \alpha V_2$ are similar to potentials considered in Theorem 1.1. This might be helpful in understanding of the structure of the a.c. spectrum of a Schrödinger operator on the strip $\Omega \times \mathbb{R}$.

Remark. We do not claim in Theorem 1.1, that the essential support of the absolutely continuous spectrum is $[0,\infty)$, because of the following reason. One could easily construct a potential V satisfying

$$\int_{\mathbb{R}^d} \frac{V^2}{|x|^{d-1}} dx < \infty, \qquad d \geq 3,$$

and such that the a.c. spectrum of $-\Delta + V$ is $[-1,\infty)$.

In fact, this construction is very simple. Let $\Gamma \subset \mathbb{R}^d$ be the straight line $\{x : \ x = te, \quad t \in \mathbb{R}\}$, where e is a fixed non-zero vector. Define V setting

$$V(x) = \begin{cases} -A, & \text{if} \quad \operatorname{dist}\{x, \Gamma\} < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now choose $A > 0$ so that the bottom of the spectrum of the operator $-\Delta + V$ is -1. This is a model where the variables can be separated. The corresponding waves propagate in direction of the vector e and are therefore not spherically symmetric.

We conclude this section by giving an interesting example of applicaion of Theorem 1.1. Let ϕ be a smooth compactly supported real function on $\mathbb{R}^d$. Let ω_n be bounded independent identically distributed random variables, $n \in \mathbb{Z}^d$. Assume that all odd moments of the random variables are equal to zero:

$$\mathbb{E}\Big[\omega_n^{2j+1}\Big] = 0, \qquad \text{for all} \quad j \in \mathbb{N} = \{0,1,2,\dots\}.$$

Set now

$$V(x) = (1+|x|)^{-1/2-\varepsilon} \sum_{n\in\mathbb{Z}^d} \omega_n \phi(x-n), \qquad \varepsilon > 0, \tag{9}$$

and define V_+ by

$$V_+(x) = \frac{1}{(1+|x|)(1+\log_+|x|)^p}, \qquad p > 1. \tag{10}$$

Theorem 1.2. *Let V and V_+ be defined by (9) and (10). Let V_0 be a real bounded potential such that W_0 in (2) obeys (3). Then for almost every choice of ω_n, the a.c. spectrum of $H = -\Delta + V + V_+ + \alpha V_0$ is essentially supported by a set containing the positive half-line $[0,\infty)$ for almost every α.*

Proof. Representation (7) with Q satisfying

$$|Q(x)| \leq \frac{C}{(1+|x|)^{1/2+\varepsilon_1}}, \qquad \varepsilon_1 > 0,$$

was established in [6]. Even though, the constant C in this bound depends on the choice of random variables, we still have (8) outside of a large ball $\{x : \ |x| > R\}$. The radius of this ball depends on ω_n as well. However, once it is finite, we can remove a compactly supported part of V so that (8) will hold everywhere. In order to complete the proof, it is enough to notice that the latter operation does not change the absolutely continuous spectrum (see Proposition 5.1). $\square$

Notations. Throughout the text, $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of a complex number z. The notation $\mathbb{S}$ stands for the unit sphere in $\mathbb{R}^d$. Its area is denoted by $|\mathbb{S}|$.

2. Entropy

Let μ be a positive finite Borel measure on the real line $\mathbb{R}$. As any other measure it is decomposed uniquely into a sum of three terms

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sc},$$

where the first term is pure point, the second term is absolutely continuous and the last term is a continuous but singular measure on $\mathbb{R}$. Obviously, $\mu(-\infty, \lambda)$ is a monotone function of λ, therefore, it is differentiable almost everywhere. In particular, the limit

$$\mu'(\lambda) = \lim_{\epsilon \to 0} \frac{\mu(\lambda - \epsilon, \lambda + \epsilon)}{2\epsilon}$$

exists for almost every $\lambda \in \mathbb{R}$. It is also clear that

$$\mu_{ac}(\Omega) = \int_\Omega \mu'(\lambda)\, d\lambda, \qquad \forall \Omega \subset \mathbb{R},$$

which means $\mu' = \mu'_{ac}$.

Let $\Omega_0 = \{\lambda : \ \mu'(\lambda) > 0\}$ A measurable set $\Omega \subset \mathbb{R}$ is called an essetial support of μ_{ac}, if the Lebesgue measure of the symmetric difference

$$\Omega_0 \bigtriangleup \Omega := \Bigl(\Omega_0 \setminus \Omega\Bigr) \cup \Bigl(\Omega \setminus \Omega_0\Bigr)$$

is zero. So, an essential support of μ_{ac} coincides with the set where $\mu' > 0$ up to a set of measure zero. As we see, the study of the essential support of the a.c. part of the measure μ is reduced to the study of the set $\Omega_0 = \{\lambda : \ \mu'(\lambda) > 0\}$. Let Ω be a measurable set. One of the ways to show that $\mu'(\lambda) > 0$ for almost every $\lambda \in \Omega$ relies on the study of the quantity

$$S_\Omega(\mu) := \int_\Omega \log \mu'(\lambda)\, d\lambda.$$

Due to Jenssen's inequality, $S_\Omega < \infty$, if $|\Omega| < \infty$. So, the entropy can diverge only to the negative infinity.

But if $|\Omega| < \infty$ and

$$S_\Omega(\mu) > -\infty,$$

then

$$\mu'(\lambda) > 0, \qquad \text{a.e. on } \Omega.$$

Very often one can obtain an estimate for μ' by an analytic function from below. In this case we will use the following statement

Proposition 2.1. *Let a function $F(\lambda) \neq 0$ be analytic in the neighborhood of an interval $[a, b] \subset \mathbb{R}$. Suppose that*

$$\mu'(\lambda) > c_0 |F(\lambda)|^2, \qquad \text{for all } \lambda \in \Omega \subset [a, b]. \tag{11}$$

Then

$$S_\Omega(\mu) := \int_\Omega \log \mu'(\lambda)\, d\lambda \geq C > -\infty,$$

where the constant $C = C(c_0, F, \Omega)$ depends on c_0, F and Ω.

The proof is left to the reader as an exercise. We only mention that zeros of an analytic function are always isolated zeros of a finite order.

In applications to Schrödinger operators, one often has an estimate of the form (11) for a sequence of measures μ_n that converges to μ weakly

$$\mu_n \to \mu \qquad \text{(weakly)}.$$

In this situation, one can still derive a certain information about the limit measure μ from the information about μ_n.

Definition. Let ρ, ν be finite Borel measures on a compact Hausdorff space, X. We define the entropy of ρ relative to ν by

$$S(\rho|\nu) = \begin{cases} -\infty, & \text{if } \rho \text{ is not } \nu-\text{ac} \\ -\int_X \log(\frac{d\rho}{d\nu})d\rho, & \text{if } \rho \text{ is } \nu-\text{ac}. \end{cases} \tag{12}$$

Theorem 2.1. (cf.[10]) *The entropy $S(\rho|\nu)$ is jointly upper semi-continuous in ρ and ν with respect to the weak topology. That is, if $\rho_n \to \rho$ and $\nu_n \to \nu$ as $n \to \infty$, then*

$$S(\rho|\nu) \geq \limsup_{n\to\infty} S(\rho_n|\nu_n).$$

Now, we will use this theorem in order to prove the following statement.

Proposition 2.2. *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic in the neighborhood of $[a, b]$. Let μ_n be a sequence of positive finite measures on the real line $\mathbb{R}$ converging to μ weakly. Suppose that*

$$\mu_n'(\lambda) > c_0|F(\lambda)|^2, \qquad \text{for all } \lambda \in \Omega_n \subset [a, b],$$

where the measurable sets Ω_n satisfy

$$\Big|[a, b] \setminus \Omega_n\Big| \leq \varepsilon.$$

Then $\mu'(\lambda) > 0$ on a subset of $[a, b]$ whose measure is not smaller than $b - a - \varepsilon$

Proof. Let us denote the characteristic function of the set Ω_n by χ_n. Since L^2-norms of χ_n are uniformly bounded, this sequence of functions has a weakly convergent subsequence. Therefore without loss of generality, one can assume that

$$\chi_n \to \chi, \qquad \text{weakly in} \quad L^2(\mathbb{R}).$$

This, of cause, implies that the corresponding measures $\chi_n d\lambda$ also converge weakly to $\chi d\lambda$. Even though, $\mathbb{R}$ is not compact, we can still use Theorem 2.1 and show (see [21]) that

$$\int_{\mathbb{R}} \log\Big(\frac{\mu'(\lambda)}{\chi(\lambda)}\Big)\, \chi(\lambda)\, d\lambda \geq \limsup_{n\to\infty} \int_{\mathbb{R}} \log\Big(\frac{\mu_n'(\lambda)}{\chi_n}\Big)\, \chi_n(\lambda)\, d\lambda > -\infty$$

Even though this means that $\mu' > 0$ on the support of the function χ, we still need to know how big this set is. For that purpose we estimate the integral

$$\int_a^b \chi(\lambda)\, d\lambda = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_n(\lambda)\, d\lambda \geq b - a - \varepsilon.$$

If we combine this with the fact that $0 \leq \chi \leq 1$, then we will realize that the Lebesgue measure of the support of the function χ is not smaller than $b - a - \varepsilon$. $\square$

Since we deal with a family of operators depending on a parameter α, we also need a modification of the previous statement, suitable in the case when measures depend on the parameter α as well. Let

$\mathfrak{S}$ be the sigma-algebra of Borel subsets of $\mathbb{R}$. By an α-dependent family of positive finite measures on $\mathbb{R}$, we mean a function

$$\mu : \mathfrak{S} \times \mathbb{R} \mapsto [0, \infty),$$

such that $\mu(\cdot, \alpha)$ is a positive measure for each $\alpha \in \mathbb{R}$.

Proposition 2.3. *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic in the neighborhood of $[a, b]$. Let $\mu_n(\cdot, \alpha)$ be a sequence of α-dependent families of positive finite measures on $\mathbb{R}$ converging to $\mu(\cdot, \alpha)$ weakly for every $\alpha \in \mathbb{R}$. Suppose that the derivatives of μ_n with respect to $d\lambda$ satisfy*

$$\mu_n'(\lambda, \alpha) > c_0|F(\lambda)|^2, \qquad \text{for all } (\lambda, \alpha) \in \Omega_n \subset [a, b] \times [\alpha_1, \alpha_2],$$

where the measurable sets Ω_n obey

$$\Big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega_n\Big| \leq \varepsilon.$$

Then $\mu'(\lambda, \alpha) > 0$ on a subset of $[a, b] \times [\alpha_1, \alpha_2]$ whose measure is not smaller than $(b-a)(\alpha_2-\alpha_1)-\varepsilon$.

The proof of this statement is a counterpart of the proof of Proposition 2.2 and it is left to the reader as an exercise. A similar statement is proven in [21].

We conclude this section by a discussion of the situation when derivative of a measure can be estimated by the square of an analytic function.

Proposition 2.4. *Let $a < b$ and let $\alpha_1 < \alpha_2$. Let $F(\lambda) \neq 0$ be a function analytic in the neighborhood of $[a, b]$. Let $\mu(\cdot, \alpha)$ be an α-dependent family of positive finite measures on $\mathbb{R}$. Suppose that the derivatives of μ with respect to $d\lambda$ satisfy the estimate*

$$\mu'(\lambda, \alpha) \geq |F(\lambda)|^2(1 - \Psi(\lambda, \alpha)), \qquad \text{where } \int_{\alpha_1}^{\alpha_2}\int_a^b |\Psi(\lambda, \alpha)|\, d\lambda d\alpha \leq \varepsilon/2.$$

Then

$$\mu'(\lambda, \alpha) \geq \frac{1}{2}|F(\lambda)|^2, \qquad \text{for all } (\lambda, \alpha) \in \Omega \subset [a, b] \times [\alpha_1, \alpha_2],$$

where the measurable set Ω obeys

$$\Big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega\Big| \leq \varepsilon. \tag{13}$$

Prrof. Due to Chebyshev's inequality,

$$\Psi(\lambda, \alpha) \leq 1/2$$

on a set Ω satisfying (13). $\square$

3. Spectral measures

The theory of spectral measures is based on the theory of the usual scalar measure and on the properties of orthogonal projections. We deal only with spectral measures on the real line $\mathbb{R}$. Let $\mathfrak{S}$ be the sigma-algebra of Borel subsets of $\mathbb{R}$, let $\mathfrak{H}$ be a separable Hilbert space and let $\mathcal{P}(\mathfrak{H})$ be the set of all operators of orthogonal projection acting in $\mathfrak{H}$.

Definition. By a spectral measure on $\mathbb{R}$ we mean a map $E : \mathfrak{S} \mapsto \mathcal{P}(\mathfrak{H})$ having the following properties:

1. *Sigma-additivity:* If $\{\delta_n\}$ is a countable (or finite) collection of disjoint Borel sets, then

$$E\Big(\cup_n \delta_n\Big) f = \sum_n E(\delta_n) f, \qquad \forall f \in \mathfrak{H}.$$

2. *Completeness:* $E(\mathbb{R}) = I$.

Every spectral measure generates a family of finite positive measures defined for $f \in \mathfrak{H}$ by

$$\mu_f(\delta) := (E(\delta)f, f). \tag{14}$$

We are going to study the relation between the properties of the measures E and μ_f.

Definition. A Borel set $X \subset \mathbb{R}$ is called an essential support of the absolutely continuous part of a spectral measure E, if

1. There exists a set $Y \subset \mathbb{R} \setminus X$ of zero Lebesgue measure such that
$$E\Big(\mathbb{R} \setminus X\Big) = E(Y).$$
2. For any Borel subset $\delta \subset X$, it holds $E(\delta) \neq 0$ as soon as the Lebesgue measure of δ is positive.

Since this definition does not require understanding of what the absolutely continuous part of E is, we do not provide its description here.

Proposition 3.1. *Let $[a, b]$ be a finite interval of the real line $\mathbb{R}$. Let E be a spectral measure on $\mathbb{R}$ and let μ_f be the family of measures defined by* (14)*. Suppose that for any $\varepsilon > 0$ there exists a Borel subset $\delta \subset [a, b]$ and $f \in \mathfrak{H}$ such that*

$$|\delta| > b - a - \varepsilon, \qquad \text{and} \quad \mu_f'(\lambda) > 0 \quad \text{a.e. on} \quad \delta. \tag{15}$$

Then the interval $[a, b]$ is contained in an essential support of the absolutely continuous part of the measure E.

Proof. Assume the opposite: that $[a, b]$ is not contained in any essential support of the measure E. This means that there exists a Borel subset $\Omega \subset [a, b]$ such that

$$|\Omega| > 0, \quad \text{but} \quad E(\Omega) = 0.$$

Let now $\varepsilon < |\Omega|$. Select δ having the property (15). If δ and Ω were disjoint, then the measure of their union would satisfy the estimate

$$|\delta \cup \Omega| = |\delta| + |\Omega| \geq b - a - \varepsilon + |\Omega| > b - a.$$

This would contradict the condition $\delta \cup \Omega \subset [a, b]$. Consequently, $\delta \cap \Omega \neq \emptyset$. Moreover, this argument proves that

$$|\delta \cap \Omega| > 0, \tag{16}$$

because otherwise they would become disjoint after one removes a set of measure zero. Combining (14) with (16), we obtain that

$$(E(\delta \cap \Omega)f, f) = \int_{\delta \cap \Omega} d\mu_f(\lambda) > 0.$$

On the other hand one can easily show that $E(\delta \cap \Omega) = E(\delta)E(\Omega)$, which implies that $E(\Omega) \neq 0$. This contradict our assumption. $\square$

Since we deal with a family of operators depending on a parameter α, we will also need a modification of the previous statement, suitable in the case when spectral measures depend on the parameter α as well. Let $\mathfrak{S}$ be the sigma-algebra of Borel subsets of $\mathbb{R}$ and let $\mathcal{P}(\mathfrak{H})$ be the set of all orthogonal projections acting in $\mathfrak{H}$. By an α-dependent family of spectral measures on $\mathbb{R}$, we mean a function

$$E : \mathfrak{S} \times \mathbb{R} \mapsto \mathcal{P}(\mathfrak{H}),$$

such that $E(\cdot, \alpha)$ is a spectral measure for each $\alpha \in \mathbb{R}$. Sometimes we use a different notation for a family of spectral measures:

$$E_\alpha(\cdot) = E(\cdot, \alpha).$$

Proposition 3.2. *Let $[a,b]$ and $[\alpha_1,\alpha_2]$ be two finite intervals of the real line $\mathbb{R}$. Let E_α be a family of spectral measures on $\mathbb{R}$ depending on $\alpha\in\mathbb{R}$. Suppose that for any $\varepsilon>0$ there exists a Borel subset $\delta\subset[a,b]\times[\alpha_1,\alpha_2]$ satisfying*

$$|[a,b]\times[\alpha_1,\alpha_2]\setminus\delta|\le\varepsilon, \tag{17}$$

and having the property that there exists a function $f:[\alpha_1,\alpha_2]\mapsto\mathfrak{H}$ such that

$$\frac{d(E_\alpha(\lambda)f(\alpha),f(\alpha))}{d\lambda}>0 \quad \text{for a.e.} \quad (\lambda,\alpha)\in\delta. \tag{18}$$

Then the interval $[a,b]$ is contained in an essential support of the absolutely continuous part of the measure E_α for almost every $\alpha\in[\alpha_1,\alpha_2]$.

Proof. Let $\varepsilon>0$ and let $\chi_\varepsilon(\lambda,\alpha)$ be the characteristic function of the set $[a,b]\times[\alpha_1,\alpha_2]\setminus\delta$, where δ is the same as above. Define $e_\varepsilon(\alpha)$ by

$$e_\varepsilon(\alpha):=\int_a^b\chi_\varepsilon(\lambda,\alpha)\,d\lambda.$$

According to Proposition 3.1, it is sufficient to show that $\inf_\varepsilon[e_\varepsilon(\alpha)]=0$ for almost every $\alpha\in[\alpha_1,\alpha_2]$. Suppose the opposite: that

$$\inf_{\varepsilon>0}e_\varepsilon(\alpha)>0, \qquad \text{on a set } \Omega\subset[\alpha_1,\alpha_2] \qquad \text{with } |\Omega|>0.$$

This would mean that there exists a positive (measurable) function $\psi>0$ such that

$$e_\varepsilon(\alpha)\ge\psi(\alpha), \qquad \text{on } \Omega.$$

The latter would imply that

$$\varepsilon\ge|[a,b]\times[\alpha_1,\alpha_2]\setminus\delta|=\int_{\alpha_1}^{\alpha_2}e_\varepsilon(\alpha)\,d\alpha\ge\int_\Omega\psi(\alpha)\,d\alpha>0.$$

This contradicts the fact that $\varepsilon>0$ can be arbitrarily small. $\square$

Spectral theorem

Let E be a spectral measure in a Hilbert space $\mathfrak{H}$ defined on Borel subsets of $\mathbb{R}$. One can define the inegral

$$H = \int_{\mathbb{R}} t\, dE(t) \tag{19}$$

representing a self-adjoint operator in $\mathfrak{H}$. The domain of definition of this operator is the set $D(H)$ defined by

$$D(H) = \{f \in \mathfrak{H} :\ \int_{\mathbb{R}} t^2 d\mu_f(t) < \infty\},$$

where μ_f is defined in the same way as in (14). In order to understand the relation (19), we define

$$H_n := \sum_{k=-n^2}^{n^2} \frac{k}{n} E\Big([\frac{k}{n}, \frac{k+1}{n})\Big)$$

for any positive integer n. If $f \in D(H)$, then $H_n f$ is a Cauchy sequence and we set

$$Hf = \lim_{n\to\infty} H_n f.$$

It turns out that any self-adjoint operator admits such a representation.

Theorem 3.1. *Let H be a self-adjoint operator in a separable Hilbert space $\mathfrak{H}$. There exists a unique spectral measure E in $\mathfrak{H}$ defined on Borel subsets of* $\mathbb{R}$*, such that* (19) *holds.*

The measure E is called the spectral measure of the operator H.

Now, given a self-adjoint operator H and a real valued Borel-measurable function ϕ on $\mathbb{R}$, we can define $\phi(H)$, by setting

$$\phi(H) = \int_{\mathbb{R}} t\, d\tilde{E}(t),$$

where $\tilde{E}$ is the spectral measure satisfying

$$\tilde{E}(\delta) = E(\phi^{-1}(\delta)), \qquad \forall \delta \in \mathfrak{S}.$$

If $\phi = \phi_1 + i\phi_2$ is a complex valued function, we set $\phi(H) = \phi_1(H) + i\phi_2(H)$. This definition is consistent with other definitions of functions of an operator. In particular, if $\operatorname{Im} z \neq 0$ and $n \in \mathbb{N}$, then

$$(H - z)^{-n} = \phi(H), \qquad \text{with} \qquad \phi(t) = (t - z)^{-n}.$$

Moreover, one can also show that if ϕ is bounded, then

$$(\phi(H)f, f) = \int_{\mathbb{R}} \phi(t)\, d\mu_f(t), \qquad \forall f \in \mathfrak{H}. \tag{20}$$

In particular,

$$((H - z)^{-1} f, f) = \int_{-\infty}^{\infty} \frac{d\mu_f(t)}{t - z}, \qquad \operatorname{Im} z \neq 0. \tag{21}$$

which implies that

$$\operatorname{Im}((H - z)^{-1} f, f) = \pi \int_{-\infty}^{\infty} \mathcal{P}_\varepsilon(\lambda, t) d\mu_f(t), \qquad z = \lambda + i\varepsilon, \quad \varepsilon > 0, \tag{22}$$

where $\mathcal{P}_\varepsilon$ is the Poisson kernel,

$$\mathcal{P}_\varepsilon(\lambda, t) = \frac{\varepsilon}{\pi((t-\lambda)^2 + \varepsilon^2)}.$$

There is a beautiful theory of integrals (22) based on the Hardy-Littlewood maximal inequality. One of the main results in this theory is the following statement.

Theorem 3.2. *Let μ be a positive finite Borel measure on the real line $\mathbb{R}$ and let*

$$g_\varepsilon(\lambda) = \int_{-\infty}^{\infty} \mathcal{P}_\varepsilon(\lambda, t) d\mu(t), \qquad \varepsilon > 0.$$

Then $g_\varepsilon(\lambda) \to \mu'(\lambda)$ as $\varepsilon \to 0$ for almost every $\lambda \in \mathbb{R}$.

This means in particular, that the limit is finite almost everywhere. Using this fact, one can aslo show that if

$$\phi_\varepsilon(\lambda) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{t - \lambda - i\varepsilon}, \qquad \varepsilon > 0,$$

then the limit

$$\lim_{\varepsilon \to 0} \phi_\varepsilon(\lambda)$$

exists for almost every $\lambda \in \mathbb{R}$. For that purpose, one introduces

$$F(z) = \exp(i\phi_\varepsilon(\lambda)), \qquad \text{where} \qquad z = \frac{\lambda + i\varepsilon - i}{\lambda + i\varepsilon + i}.$$

After that one notices that $F(z)$ is a bounded analytic function in the unit disc and such functions have finite boundary values a.e. on the unit circle. One should take additional care of the possibility that the boundary values of $F(z)$ are zeros. But this is excluded by Theorem 3.2.

The said above allows us to use the following notation. For a selfadjoint operator $H = H^*$ in a Hilbert space $\mathfrak{H}$ and a vector $f \in \mathfrak{H}$ the expression $((H - \lambda - i0)^{-1} f, f)$ is always understood as the limit

$$\Big((H - \lambda - i0)^{-1} f, f\Big) = \lim_{\varepsilon \to 0} \Big((H - \lambda - i\varepsilon)^{-1} f, f\Big), \qquad \varepsilon > 0, \ \lambda \in \mathbb{R}.$$

Note that relation (21) implies that this limit exists for almost every $\lambda \in \mathbb{R}$.

The following simple and very well known statement plays very important role in the proof of Theorem 1.1..

Lemma 3.1. *Let B be a self-adjoint operator in a separable Hilbert space $\mathfrak{H}$ and let $g \in \mathfrak{H}$. Then the function*

$$\eta(k) := \operatorname{Im} \Big((B - k - i0)^{-1} g, g\Big) \geq 0,$$

is integrable over $\mathbb{R}$. Moreover,

$$\int_{-\infty}^{\infty} \eta(k) dk \leq \pi ||g||^2.$$

and

$$\int_{-\infty}^{\infty} \frac{\eta(k)}{k^2 + 1} dk \leq \pi ||(B^2 + I)^{-1/2} g||^2.$$

Proof. Let E be the spectral measure of the operator B. Then

$$((B - z)^{-1} g, g) = \int_{-\infty}^{\infty} \frac{d(E(t)g, g)}{t - z}, \qquad \operatorname{Im} z \neq 0.$$

Accoriding to Theorem 3.2,

$$\pi^{-1} \eta(k) = \frac{d(E(k)g, g)}{dk},$$

which implies that for any positive bounded Borel measurable function $\phi : \mathbb{R} \mapsto \mathbb{R}_+$,

$$\pi^{-1} \int_{\mathbb{R}} \phi(k) \eta(k) \, dk \leq \int_{\mathbb{R}} \phi(k) d(E(k)g, g).$$

Note now that formula (20) written for an operator H and a vector f can be also written for the operator B and the vector g. The latter would imply that

$$\pi^{-1}\int_{\mathbb{R}}\phi(k)\eta(k)\,dk \le (\phi(B)g,g).$$

It remains to take either $\phi = 1$ or $\phi(k) = (k^2+1)^{-1}$. $\square$

4. Proof of Theorem 1.1

Our proof is based on the relation between the derivative of the spectral measure and the so called scattering amplitude. While the spectral measure is defined for any self-adjoint operator, the scattering amplitude will be introduced only for a Schrödinger operator. Let f be a vector in the Hilbert space $\mathfrak{H}$ and H be a self-adjoint operator in $\mathfrak{H}$. According to (21), the quadratic form of the resolvent of H can be written as a Cauchy integral

$$((H-z)^{-1}f,f) = \int_{-\infty}^{\infty}\frac{d\mu(t)}{t-z}, \qquad \operatorname{Im} z \ne 0.$$

(Here we write μ instead of μ_f omitting the subindex.) The measure μ in this representation is called the spectral measure of H corresponding to the element f.

Proposition 4.1. *Let μ be the spectral measure of a self-adjoint operator H corresponding to an element f. Then*

$$\mu'(\lambda) = \pi^{-1}\lim_{\varepsilon\to 0}\operatorname{Im}\Big((H-\lambda-i\varepsilon)^{-1}f,f\Big), \qquad \varepsilon > 0.$$

Let us introduce the scattering amplitude. First of all, assume that the support of the potentials V and V_0 are compact. Take any compactly supported function f. For any $z \in \mathbb{C}\setminus\mathbb{R}$, we introduce k setting

$$k^2 = z, \ \operatorname{Im} k > 0.$$

In these notations (see [25], p. 40-42),

$$(H-z)^{-1}f = \frac{e^{ik|x|}}{|x|^{(d-1)/2}}\Big(A_f(k,\theta) + O(|x|^{-1})\Big), \qquad \text{as } |x|\to\infty, \ \ \theta = \frac{x}{|x|}. \tag{23}$$

This asymptotic relation is valid even when k is real. In this case, the left hand side is understood as a limit. If $d = 3$, then this asymtotic formula follows from the fact that

$$(H-z)^{-1}f = \int_{\mathbb{R}^d}\frac{e^{ik|x-y|}}{4\pi|x-y|}f_0(y)dy, \qquad \text{where } f_0 = f - (V + V_+ + \alpha V_0)(H-z)^{-1}f.$$

If $d \ne 3$ then (23) is still valid but is less obvious. Note that according to (23),

$$(H-z)^{-1}f = \phi_1(x) + \phi_2(x),$$

where

$$\phi_1(x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}}A_f(k,\theta), \qquad \text{and} \qquad \limsup_{z\to\lambda+i0}||\phi_2||^2 < \infty.$$

Therefore,

$$\mu'(\lambda) = \pi^{-1}\lim_{z\to\lambda+i0}\operatorname{Im}((H-z)^{-1}f,f) = \pi^{-1}\lim_{z\to\lambda+i0}\operatorname{Im} z||(H-z)^{-1}f||^2$$

implies that (see [25], p. 40-42, again)

$$\pi\mu'(\lambda) = \sqrt{\lambda}\int_{\mathbb{S}}|A_f(k,\theta)|^2\,d\theta, \qquad k^2 = \lambda > 0. \tag{24}$$

Formula (24) is a very important equality that relates the absolutely continuous spectrum to so-called extended states. The rest of the proof will be devoted to a lower estimate of $|A_f(k,\theta)|$.

Consider first the case $d = 3$. For our purposes, it is sufficient to assume that f is the characteristic function of the unit ball. Traditionally, H is viewed as an operator obtained by a perturbation of

$$H_0 = -\Delta.$$

In its turn, $(H-z)^{-1}$ can be viewed as an operator obtained by a perturbation of $(H_0-z)^{-1}$. The theory of such perturbations is often based on the second resolvent identity

$$(H-z)^{-1} = (H_0-z)^{-1} - (H-z)^{-1}\,(V + V_+ + \alpha V_0)(H_0-z)^{-1}, \tag{25}$$

which turns out to be useful for our reasoning. As a consequence of (25), we obtain that

$$A_f(k,\theta) = F(k) - A_g(k,\theta), \qquad z = k^2 + i0,\ k > 0, \tag{26}$$

where

$$g(x) = \ (V(x) + V_+(x) + \alpha V_0(x))(H_0 - z)^{-1} f$$

and $F(k)$ is defined by

$$(H_0 - z)^{-1} f = e^{ik|x|}\frac{F(k)}{|x|^{(d-1)/2}}, \qquad \text{for } |x| > 1 \qquad (\text{recall that } d = 3). \tag{27}$$

Without loss of generality, one can assume that $V(x) = V_+(x) = V_0(x) = 0$ inside the unit ball. In this case,

$$g = F(k)h_k, \quad \text{where} \quad h_k(x) = \ (V + V_+ + \alpha V_0)e^{ik|x|}|x|^{(1-d)/2}. \tag{28}$$

According to (26),

$$2\int_{\mathbb{S}} |A_f(k,\theta)|^2\,d\theta \geq |F(k)|^2|\mathbb{S}| - 2\int_{\mathbb{S}} |A_g(k,\theta)|^2\,d\theta,$$

which can be written in the form

$$2\pi\mu'(\lambda) \geq |F(k)|^2\Big(|\mathbb{S}|\sqrt{\lambda} - 2\,\mathrm{Im}\,\Big((H-z)^{-1}h_k,\, h_k\Big)\Big), \qquad z = \lambda + i0, \tag{29}$$

due to (24) and to Proposition 4.1 combined with (28). Therefore, in order to establish the presence of the absolutely continuous spectrum, we need to show that the quantity $\mathrm{Im}\,\Big((H-z)^{-1}h_k,\, h_k\Big)$ is small.

Proposition 4.2. *Let $\varepsilon > 0$. Suppose that $\lambda_1 > 0$. Let*

$$\Psi(\lambda,\alpha) := \frac{2}{|\mathbb{S}|\sqrt{\lambda}}\mathrm{Im}\,\Big((H - \lambda - i0)^{-1}h_k,\, h_k\Big), \qquad k = \sqrt{\lambda} > 0. \tag{30}$$

If

$$\int_{\alpha_1}^{\alpha_2}\int_{\lambda_1}^{\lambda_2} \Psi(\lambda,\alpha)\,d\lambda d\alpha \leq \varepsilon/2,$$

then

$$4\pi\mu'(\lambda) \geq |F(k)|^2|\mathbb{S}|\sqrt{\lambda}$$

on a set $\Omega \subset [\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2]$ satisfying

$$|[\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2]\setminus\Omega| \leq \varepsilon.$$

This statement is a direct consequence of Proposition 2.4. The next result plays the key role in the proof of Theorem 1.1.

Lemma 4.1. *Let $\lambda_1 > 0$. Let Ψ be the same as in (30). Assume also that W_0 is compactly supported. Then there is a positive constant C_0 that depends only on the edges of the intervals $[\lambda_1, \lambda_2]$ and $[\alpha_1, \alpha_2]$ such that*

$$\int_{\alpha_1}^{\alpha_2}\int_{\lambda_1}^{\lambda_2} \Psi(\lambda,\alpha)d\lambda d\alpha \le C_0\Big(\int_{\mathbb{R}^d} \frac{|\nabla W_0|^2}{|x|^{d-1}}dx + \int_{\mathbb{R}^d} \frac{V_+}{|x|^{d-1}}dx\Big). \tag{31}$$

Proof. Observe that $\mathrm{Im}\,((H-z)^{-1}h_k, h_k)$ is a positive quadratic form (in h_k). Therefore,

$$\mathrm{Im}\,\Big((H-z)^{-1}h_k,\, h_k\Big) \le 2\,\mathrm{Im}\,\Big((H-z)^{-1}h_k^+,\, h_k^+\Big) + 2\,\mathrm{Im}\,\Big((H-z)^{-1}h_k^-,\, h_k^-\Big)$$

where the vectors $h_k^\pm$ are defined by

$$h_k^-(x) = \alpha V_0 e^{ik|x|}|x|^{(1-d)/2}, \qquad h_k^+(x) = (V+V_+)e^{ik|x|}|x|^{(1-d)/2}.$$

Consider first the function

$$\eta_0(k,\alpha) := \frac{k}{\alpha^2}\mathrm{Im}\,\Big((H-z)^{-1}h_k^-,\, h_k^-\Big) \ge 0, \qquad z = (k+i0)^2.$$

Obviously, η_0 is positive for all real $k \ne 0$, because $z = k^2 \pm i0$ if $\pm k > 0$. This is very convenient. Since $\eta_0 > 0$, we can conclude that η_0 is small on a rather large set if the integral of this function is small. That is why we will try to estimate

$$J := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta_0(k,\alpha)}{(\alpha^2+k^2)(k^2+1)(|\alpha|+|k|)}\,|k||\alpha|\,dk d\alpha = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta_0(k,tk)}{(k^2+1)(t^2+1)(|t|+1)}\,|t| dk dt. \tag{32}$$

Proposition 4.3. *Let J be the quantity, defined in (32). Let $H_\varepsilon = -\Delta + V + V_+ + \varepsilon I$ and let B be the bounded selfadjoint operator defined by*

$$B = H_\varepsilon^{-1/2}\Big(-2i\frac{\partial}{\partial r} - \frac{i(d-1)}{|x|} + tV_0\Big)H_\varepsilon^{-1/2}, \qquad t \in \mathbb{R},\ \varepsilon > 0,$$

where r denotes the radial variable in spherical coordinates. Then

$$J \le \pi \liminf_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{(1+t^2)(|t|+1)}||B^{-1}H_\varepsilon^{-1/2}v||^2\,|t|dt \tag{33}$$

where $v = V_0|x|^{(1-d)/2}$.

Proof. The reader can easily establish that B is not only self-adjoint but bounded for $\varepsilon > 0$. Let us introduce

$$\eta_\varepsilon(k,\alpha) = \frac{k}{\alpha^2}\mathrm{Im}\,\Big((H+\varepsilon-z)^{-1}h_k^-,\, h_k^-\Big).$$

Note that

$$\eta_0(k,\alpha) = \lim_{\varepsilon\to 0}\eta_\varepsilon(k,\alpha) \qquad \text{a.e. on } \mathbb{R}\times\mathbb{R},$$

which, due to Fatou's lemma, implies that

$$J \le \liminf_{\varepsilon\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta_\varepsilon(k,\alpha)}{(\alpha^2+k^2)(k^2+1)(|\alpha|+|k|)}\,|k||\alpha|\,dk d\alpha.$$

Now we look at the values of η_ε at the points (k,α) that belong to the line $\alpha = kt$. It turns out that

$$\eta_\varepsilon(k,kt) = \mathrm{Im}\,\Big((B + 1/k - i0)^{-1}H_\varepsilon^{-1/2}v,\, H_\varepsilon^{-1/2}v\Big). \tag{34}$$

In order to justify (34) at least formally, one has to introduce the operator U of multiplication by the function $\exp(ik|x|)$. Using this notation, we can represent η_ε in the following form

$$\eta_\varepsilon(k,tk) = k\mathrm{Im}\Big(U^{-1}(H+\varepsilon-z)^{-1}Uv,\, v\Big).$$

Since we deal with a unitary equivalence of operators, we can employ the formula

$$U^{-1}(H+\varepsilon-z)^{-1}U = (U^{-1}HU+\varepsilon-z)^{-1}.$$

On the other hand, since H is a differential operator and U is an operator of multiplication, the commutator $[H,U] := HU-UH$ can be easily found

$$\Big[H,U\Big] = kU\Big(-2i\frac{\partial}{\partial r} - \frac{i(d-1)}{|x|} + k\Big).$$

The latter equality implies that

$$U^{-1}HU+\varepsilon-z = H_\varepsilon + k\Big(-2i\frac{\partial}{\partial r} - \frac{i(d-1)}{|x|} + tV_0\Big) = H_\varepsilon^{1/2}(I+kB)H_\varepsilon^{1/2}.$$

Consequently,

$$kU^{-1}(H+\varepsilon-z)^{-1}U = H_\varepsilon^{-1/2}(B+1/k)^{-1}H_\varepsilon^{-1/2}. \tag{35}$$

Let us have a look at the formula (34). If k belongs to the upper half plane then so does $-1/k$. Since B is a self-adjoint operator, $\pi^{-1}\eta_\varepsilon(k,kt)$ coincides with the derivative of the spectral measure of the operator B corresponding to the element $H_\varepsilon^{-1/2}v$. According to Lemma 3.1, the latter observation implies that

$$\int_{-\infty}^{\infty} \frac{\eta_\varepsilon(k,kt)}{(1+k^2)}dk \le \pi\Big((B^2+I)^{-1}H_\varepsilon^{-1/2}v, H_\varepsilon^{-1/2}v\Big),$$

which leads to

$$\int_{-\infty}^{\infty} \frac{\eta_\varepsilon(k,kt)}{(1+k^2)}dk \le \pi\Big(B^{-1}H_\varepsilon^{-1/2}v, B^{-1}H_\varepsilon^{-1/2}v\Big) = \pi||B^{-1}H_\varepsilon^{-1/2}v||^2.$$

The statement of the proposition follows now by Fatou's lemma. $\square$

Our further arguments will be related to the estimate of the quantity in the right hand side of (33).

Proposition 4.4. *There is a positive constant C, such that*

$$\lim_{\varepsilon\to 0} ||B^{-1}H_\varepsilon^{-1/2}v||^2 \le C\Big(\int_{\mathbb{R}^d} \frac{|\nabla W_0|^2}{|x|^{d-1}}dx + \frac{1}{|t|}\int_{\mathbb{R}^d}\frac{V_+}{|x|^{d-1}}dx\Big). \tag{36}$$

Moreover, the limit in the left hand side of (36) *is uniform in t.*

Proof. In order to prove this estimate, we use the representation

$$B^{-1}H_\varepsilon^{-1/2} = H_\varepsilon^{1/2}T^{-1}, \tag{37}$$

where $T \subset T^*$ is the first order differential operator, defined by

$$T = -2i\frac{\partial}{\partial r} - \frac{i(d-1)}{|x|} + tV_0, \qquad D(T) = D(H_\varepsilon^{1/2}).$$

The representation (37) is a simple consequence of the fact that $B = H_\varepsilon^{-1/2}TH_\varepsilon^{-1/2}$. The study of the basic properties of the operator T is rather simple, because one can derive an explicit formula for its inverse. For that purpose, one needs to recall the theory of ordinary differential equations, which says that the equation

$$y' + p(t)y = f(t), \qquad y = y(t),\ t\in\mathbb{R},$$

is equivalent to the relation

$$\Big(e^{\int p\,dt}y\Big)' = e^{\int p\,dt}f.$$

Put differently,

$$y' + p(t)y = e^{-\int p\,dt}\Big(e^{\int p\,dt}y\Big)'.$$

This gives us a clear idea of how to handle the operator T. Let U_0 and U_1 be the operators of multiplication by $|x|^{(d-1)/2}$ and by $\exp(2^{-1}itW_0)$, then

$$T = -2iU_1^{-1}U_0^{-1}\Big[\frac{\partial}{\partial r}\Big]U_0U_1, \qquad \text{and} \qquad T^{-1} = \frac{i}{2}U_1^{-1}U_0^{-1}\Big[\frac{\partial}{\partial r}\Big]^{-1}U_0U_1.$$

Since $[\frac{\partial}{\partial r}]^{-1}$ means just the simple integration with respect to r and $\partial W_0/\partial r = V_0$,

$$\begin{gathered} T^{-1}v = \frac{i}{2}e^{-2^{-1}itW_0}|x|^{-(d-1)/2}\int_0^r e^{2^{-1}itW_0}V_0dr = \\ \frac{1}{t}e^{-2^{-1}itW_0}|x|^{-(d-1)/2}(e^{2^{-1}itW_0}-1) = \frac{1}{t}|x|^{-(d-1)/2}(1-e^{-2^{-1}itW_0}). \end{gathered} \tag{38}$$

Note, that $T^{-1}v$ turns out to be compactly supported, which leaves no doubt about the relation $v \in D(T^{-1})$. Combining (37) with (38), we conclude that

$$\begin{gathered} \lim_{\varepsilon\to 0}||B^{-1}H_\varepsilon^{-1/2}v||^2 = \lim_{\varepsilon\to 0}||H_\varepsilon^{1/2}T^{-1}v||^2 = \\ ||\nabla T^{-1}v||^2 + \int_{\mathbb{R}^d}(V+V_+)|T^{-1}v|^2dx + \lim_{\varepsilon\to 0}\varepsilon||T^{-1}v||^2 \le \\ C\Big(\int_{\mathbb{R}^d}\frac{|W_0|^2}{|x|^{d+1}}dx + +\int_{\mathbb{R}^d}\frac{|\nabla W_0|^2}{|x|^{d-1}}dx + \frac{1}{|t|}\int_{\mathbb{R}^d}|Q|^2dx + \frac{1}{|t|}\int_{\mathbb{R}^d}\frac{V_+}{|x|^{d-1}}dx\Big). \end{gathered}$$

In order to complete the proof, it is sufficient to use the Hardy inequality

$$\int_{\mathbb{R}^d}\frac{|W_0|^2}{|x|^{d+1}}dx + \le 4\int_{\mathbb{R}^d}\frac{|\nabla W_0|^2}{|x|^{d-1}}dx,$$

and recall that

$$|Q|^2 \le \frac{V_+}{|x|^{d-1}}. \qquad \square$$

We remind the reader that (33), (36) are needed to estimate the quantity J from (32). We can conclude now that the following statement holds.

Proposition 4.5. *Let J be the quantity* (32). *Then there exists a constant $C > 0$ such that*

$$J \le C\Big(\int_{\mathbb{R}^d}\frac{|\nabla W_0|^2}{|x|^{d-1}}dx + \int_{\mathbb{R}^d}\frac{V_+}{|x|^{d-1}}dx\Big).$$

We shall now obtain an integral estimate for the function η_0^+ defined by

$$\eta_0^+(k,\alpha) := k\,\mathrm{Im}\Big((H-z)^{-1}h_k^+,\, h_k^+\Big) \ge 0, \qquad z = (k+i0)^2.$$

It is clear that $\eta_0^+(k,\alpha)$ is positive for all real $k \ne 0$. Since $\eta_0^+ > 0$, we can conclude that η_0^+ is small on a rather large set if the following integral

$$J^+ := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\eta_0^+(k,\alpha)}{|k|(k^2+\alpha^2)}\,dk\,d\alpha \tag{39}$$

is small.

Proposition 4.6. *Let $H_\varepsilon = -\Delta + V + V_+ + \varepsilon$ and let $v_+ = |x|^{-(d-1)/2}(V + V_+)$. Then the quantity J_+ from* (39) *satisfies the relation*

$$J^+ \leq \pi^2 \liminf_{\varepsilon\to 0} ||H_\varepsilon^{-1/2} v_+||^2, \qquad \varepsilon > 0.$$

Proof. We employ the same tricks as before. Let us introduce

$$\eta_\varepsilon^+(k,\alpha) := k\mathrm{Im}\Big((H+\varepsilon - z)^{-1} h_k^+,\, h_k^+\Big).$$

Since

$$\eta_0^+(k,\alpha) = \lim_{\varepsilon\to 0} \eta_\varepsilon^+(k,\alpha) \qquad \text{a.e. on } \mathbb{R}\times\mathbb{R},$$

we conclude according to Fatou's lemma, that

$$J^+ \leq \liminf_{\varepsilon\to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta_\varepsilon^+(k,\alpha)}{|k|(k^2+\alpha^2)}\, dk\, d\alpha.$$

We also set $\alpha = kt$ and represent η_ε^+ in the form

$$\eta_\varepsilon^+(k,kt) = \mathrm{Im}\Big((B + 1/k - i0)^{-1} H_\varepsilon^{-1/2} v_+,\, H_\varepsilon^{-1/2} v_+\Big) \tag{40}$$

where B is the same as before

$$B = H_\varepsilon^{-1/2}\Big(-2i\frac{\partial}{\partial r} - \frac{i(d-1)}{|x|} + tV_0\Big) H_\varepsilon^{-1/2}.$$

The symbol r in the latter formula denotes the radial variable $r = |x|$. Recall that (40) was already justified.

Let us look at the formula (40). Since B is a self-adjoint operator, $\pi^{-1}\eta_\varepsilon^+(k,kt)$ coincides with the derivative of the spectral measure of the operator B corresponding to the element $H_\varepsilon^{-1/2} v_+$. According to Lemma 3.1,

$$\int_{-\infty}^{\infty} \eta_\varepsilon^+(k,kt)\, k^{-2} dk \leq \pi ||H_\varepsilon^{-1/2} v_+||^2.$$

The latter inequality implies that J_+ from (39) satisfies the relation

$$J^+ \leq \pi^2 \liminf_{\varepsilon\to 0} ||H_\varepsilon^{-1/2} v_+||^2. \qquad \square$$

In order to proceed further, we have to establish two inequalities.

Proposition 4.7. *The operator H_ε satisfies*

$$H_\varepsilon \geq (\tau^{-1} - 1)\Delta + \varepsilon \tag{41}$$

and

$$H_\varepsilon \geq (1 - \tau^{-1}) V_+ + \varepsilon. \tag{42}$$

Proof. Both estimates follow from the simple fact that

$$-t^{-1}\Delta + \operatorname{div} Q_0 + t|Q_0|^2 = (it^{-1/2}\nabla + t^{1/2}Q_0)^*(it^{-1/2}\nabla + t^{1/2}Q_0) \geq 0, \qquad \forall t > 0,$$

for any smooth compactly supported vector potential Q_0. In particular, if we take $t = \tau$ and set $Q_0 = |x|^{(d-1)/2} Q$, then we will obtain (41). Inequality (42) is obtained in the case $t = 1$. $\square$

Proposition 4.8. *The quantity* (39) *satisfies the estimate*

$$J^+ \le C \int_{\mathbb{R}^d} \frac{V_+}{|x|^{d-1}} dx. \tag{43}$$

where the constant C depends on τ but is otherwise independent of the potentials V, V_+ and V_0.

Proof. Define $v_1 = V|x|^{-(d-1)/2}$ and $v_2 = V_+|x|^{-(d-1)/2}$. Since $v_+ = v_1 + v_2$, we obtain that

$$J^+ \le \pi^2 \liminf_{\varepsilon\to 0} ||H_\varepsilon^{-1/2} v_+||^2 \le 2\pi^2 \liminf_{\varepsilon\to 0}\Big(||H_\varepsilon^{-1/2} v_1||^2 + ||H_\varepsilon^{-1/2} v_2||^2\Big) \le$$
$$2\pi^2 \liminf_{\varepsilon\to 0}\Big(||\big((\tau^{-1}-1)\Delta+\varepsilon\big)^{-1/2} v_1||^2 + ||\big((1-\tau^{-1})V_+ + \varepsilon\big)^{-1/2} v_2||^2\Big) \le$$
$$C_\tau\Big(\int_{\mathbb{R}^d} \frac{|W(\xi)|^2}{|\xi|^2} d\xi + \int_{\mathbb{R}^d} \frac{V_+}{|x|^{d-1}} dx\Big).$$

where W is the Fourier transform of the function $V|x|^{-(d-1)/2}$. In order to complete the proof of Proposition 4.8, it is sufficient to note that

$$\int_{\mathbb{R}^d} \frac{|W(\xi)|^2}{|\xi|^2} d\xi \le \int_{\mathbb{R}^d} \frac{V_+}{|x|^{d-1}} dx,$$

because obviously,

$$\int_{\mathbb{R}^d} \frac{|W(\xi)|^2}{|\xi|^2} d\xi \le \int_{\mathbb{R}^d} |Q|^2 dx. \qquad \square$$

Lemma 4.1 follows now from Propositions 4.5, 4.8 and the fact that there exists a constant $C > 0$ depending on the edges of the intervals $[\alpha_1, \alpha_2]$ and $[\lambda_1, \lambda_2]$, such that

$$\int_{\alpha_1}^{\alpha_2} \int_{\lambda_1}^{\lambda_2} \Psi(\lambda, \alpha) d\lambda d\alpha \le C(J + J_+)$$

5. End of the proof of Theorem 1.1

Now, let us complete the proof of Theorem 1.1 and mention what ingredients were missing. We need to transfer the estimates obtained in the previous section to the case of potentials with infinite supports. We will need the following statement from the Scattering theory.

Proposition 5.1. *Let $\tilde V$, $\tilde V_+$ and $\tilde V_0$ be three real valued locally bounded functions on $\mathbb{R}^d$, that are equal to V, V_+ and V_0 in the region $|x| > R$ for some $R > 0$. Let E and $\tilde E$ be the spectral measures of the operators H and $\tilde H := -\Delta + \tilde V + \tilde V_+ + \alpha \tilde V_0$. Then for any $\tilde f \in L^2(\mathbb{R}^d)$ there exists a function $f \in L^2(\mathbb{R}^d)$ such that the absolutely continuous parts of the positive measures*

$$\mu(\cdot) = (E(\cdot)f, f) \qquad \text{and} \qquad \tilde\mu(\cdot) = (\tilde E(\cdot)\tilde f\, \tilde f)$$

coincide.

The proof of this statement can be found in [13]. According to this result, we can assume without loss of generality that the integrals (3) and (6) are small. Recall that similar integrals appear in (31).

Proposition 5.2. *Let V, V_+ and V_0 be the same as in Theorem 1.1. For any $\varepsilon > 0$ there exists a region $\{x : |x| > R\}$ where V, V_+ and V_0 are equal to locally bounded functions $\tilde V$, $\tilde V_+$ and $\tilde V_0$, such that*

1) the function

$$\tilde W_0(r, \theta) := \int_0^r \tilde V_0(\rho, \theta) d\rho \tag{44}$$

satisfies the estimate

$$\int_{\mathbb{R}^d} \frac{|\nabla W_0|^2}{|x|^{d-1}} dx < \varepsilon;$$

2) $\tilde V = |x|^{(d-1)/2} \operatorname{div} \tilde Q$, *with* $\tilde Q \in L^2(\mathbb{R}^d)$*;*
3) $\tilde V_+ \geq \tau\Big(|x|^{d-1}|\tilde Q|^2 + \frac{(d-1)}{2}|x|^{(d-3)/2}|\tilde Q|\Big)$ *and*

$$\int_{\mathbb{R}^d} \frac{\tilde V_+}{|x|^{d-1}}\, dx < \varepsilon.$$

The proof of this statement is left to the reader as an exercise.

Approximations.

Now we will approximate $\tilde V$, $\tilde V_+$ and $\tilde V_0$ from Proposition 5.2 by compactly supported functions. Let us first describe our choice of compactly supported functions V_n^0 approximating the given potential V_0. Let us choose a spherically symmetric function $\zeta \in \mathcal{H}^1(\mathbb{R}^d)$ such that

$$\zeta(x) = \begin{cases} 1, & \text{if} \quad |x| < 1; \\ 0, & \text{if} \quad |x| > 2. \end{cases}$$

Assume for simplicity that $0 \leq \zeta \leq 1$ and $|\nabla \zeta| \leq 1$. Define

$$\zeta_n(x) = \zeta(x/n).$$

Note that $\nabla \zeta_n \neq 0$ only in the spherical layer $\{\, x : \ n \leq |x| \leq 2n\}$. Moreover $|\nabla \zeta_n| \leq 1/n$, which leads to the estimate

$$|\nabla \zeta_n(x)| \leq 2/|x|.$$

Our approximations of V will be the functions V_n defined as

$$V_n^0 = \frac{\partial}{\partial r}\,(\zeta_n \tilde W_0), \tag{45}$$

where $\tilde W_0$ is the function from (44). Thus, approximations of V_0 by V_n^0 correspond to approximations of $\tilde W_0$ by

$$W_n^0 := \zeta_n \tilde W_0. \tag{46}$$

Observe that, in this case,

$$\int_{\mathbb{R}^d} \frac{|\nabla W_n^0(x)|^2}{|x|^{d-1}} dx \leq \int_{|x|<2n} \frac{2|\nabla \tilde W_0(x)|^2 + 8|x|^{-2}|\tilde W_0(x)|^2}{|x|^{d-1}} dx \leq 34 \int_{\mathbb{R}^d} \frac{|\nabla \tilde W_0(x)|^2}{|x|^{d-1}} dx.$$

Therefore,

$$\sup_n \int_{\mathbb{R}^d} \frac{|\nabla W_n^0(x)|^2}{|x|^{d-1}} dx < 34\varepsilon. \tag{47}$$

The potential $\tilde V$ will be approximated by the sequence $V_n := |x|^{(d-1)/2} \operatorname{div}\Big(\zeta_n \tilde Q\Big)$. Finally, approximations of $\tilde V_+$ will be the functions $V_n^+ := \zeta_n \tilde V_+$. Obviously, $V_n^+ \geq \tau\Big(|x|^{d-1}|\zeta_n \tilde Q|^2 + \frac{(d-1)}{2}|x|^{(d-3)/2}|\zeta_n \tilde Q|\Big)$ and

$$\sup_n \int_{\mathbb{R}^d} \frac{V_n^+}{|x|^{d-1}}\, dx < \varepsilon. \tag{48}$$

Define the measures μ_n setting

$$((-\Delta + V_n + V_n^+ + \alpha V_n^0 - z)^{-1} f, f) = \int_{-\infty}^{\infty} \frac{d\mu_n(t)}{t - z}, \qquad \operatorname{Im} z \neq 0,$$

where f is the characteristic function of the unit ball $\{x : |x| < 1\}$. Combining Lemma 4.1 with Proposition 4.2 we obtain that conditions (47) and (48) guarantee that $\mu_n' > 0$ on a rather large set of pairs (λ, α).

Lemma 5.1. *Let μ_n be as described above. Let C_0 be the constant from the inequality* (31)*. Then*

$$\pi \mu_n'(\lambda) \geq |F(\sqrt{\lambda})|^2 \frac{|\mathbb{S}|\sqrt{\lambda}}{4}, \qquad \forall (\lambda, \alpha) \in \Omega_n, \tag{49}$$

where the set $\Omega_n \subset [\lambda_1, \lambda_2] \times [\alpha_1, \alpha_2]$ satisfies

$$|[\lambda_1, \lambda_2] \times [\alpha_1, \alpha_2] \setminus \Omega_n| \leq 35 C_0 \varepsilon.$$

Observe that $\tilde{V}$, $\tilde{V}_+$ and $\tilde{V}_0$ are the pointwise limits of the sequences V_n, V_n^+ and V_n^0. We will also show (see Section 6) that

$$\mu_n \to \tilde{\mu} \qquad \text{as } n \to \infty \tag{50}$$

weakly. Here $\tilde{\mu}$ is the spectral measure of $-\Delta + \tilde{V} + \tilde{V}_+ + \alpha \tilde{V}_0$ constructed for the same element f as before.

The following statement follows directly from Proposition 2.3.

Proposition 5.3. *Let $\varepsilon > 0$ and let $\tilde{V}$, $\tilde{V}_+$ and $\tilde{V}_0$ be the same as in Proposition 5.2. Let also C_0 be the constant from the inequality* (31)*. Then*

$$\tilde{\mu}'(\lambda) > 0 \tag{51}$$

on a set $\Omega \subset [\lambda_1, \lambda_2] \times [\alpha_1, \alpha_2]$ satisfying

$$|[\lambda_1, \lambda_2] \times [\alpha_1, \alpha_2] \setminus \Omega| \leq 35 C_0 \varepsilon.$$

Finally, combining this statement with Propositions 3.2 and 5.1, we obtain that the essential support of the absolutely continuous part of the spectral measure E of the operator H contains the interval $[\lambda_1, \lambda_2]$ for almost every $\alpha \in [\alpha_1, \alpha_2]$. It remains to note that these intervals are arbitrary. This completes the proof of Theorem 1.1 for $d = 3$.

Now, if $d \neq 3$, then equality of the form (27) is incorrect. The ratio in the right hand side is only the asymptotics of the function in the left hand side, so (27) holds only up to terms of smaller order. If we want to avoid the difficulty of dealing with these terms, we need to replace $H_0 = -\Delta$ by the operator

$$H_0 = -\Delta - \frac{\kappa_d \tilde{\chi}}{|x|^2} P_0, \qquad \kappa_d = \Big(\frac{d-2}{2}\Big)^2 - \frac{1}{4},$$

where P_0 is the projection onto the space of spherically symmetric functions and $\tilde{\chi}$ is the characteristic function of the compliment of the unit ball. Provided that H_0 is defined as above, relation (27) holds without terms of smaller order and all proofs of the statements in this paper can be repeated literally. (see [21] for details)

In conclusion of this section, we would like to draw your attention to the papers [1]-[2], [5]-[9], [12], [14]-[23] which contain an important work on the absolutely continuous spectrum of multi-dimensional Schrödinger operators. Surveys of these results are given in [4] and [19].

6. Weak convergence of spectral measures

The proof of convergence of spectral measures is based on the fact that Green's function of the Scrödinger operator operator decays exponentially fast. Since all potentials appearing in the previous sections are bounded from below, without loss of generality we can assume that they are positive.

Proposition 6.1. *Let V be a locally bounded positive potential. Let f be the characteristic function of the unit ball $\{x: \ |x|<1\}$. Then for any z with $\operatorname{Im} z \neq 0$, the function*

$$u = (-\Delta + V - z)^{-1} f$$

satisfies

$$\int_{|x|>r} |u|^2 dx \leq C \exp(-\epsilon r), \qquad r>1, \tag{52}$$

with some C and $\epsilon>0$ independent of R. Moreover, the parameter ϵ and the constant C are separated from zero and infinity correspondingly when z belongs to a compact set in the open upper half-plane.

Proof. Denote $A := -\Delta + V$. It is easy to see, that the resolvent operator $(A-z)^{-1}$ can be considered as a continuous map from L^2 to $\mathcal{H}^1$. Denote the square of the norm of this map by C_0. In particular, we have

$$||\nabla u||^2 + ||u||^2 \leq C_0 ||f||^2, \qquad \forall f \in L^2. \tag{53}$$

Note, that inequality (52) is interesting only for large values of R. So, we shall assume that R is large. Take a family of functions $\zeta_R \in C_0^\infty(\mathbb{R}^d)$ having the following properties

$$\zeta_R(x) = \begin{cases} 1, & \text{if} \qquad |x|>R, \\ 0, & \text{if} \qquad |x|<R-L \end{cases}$$

with some fixed $L>0$. We can select L so large that

$$2|\nabla \zeta_R| + |\Delta \zeta_R| \leq \frac{1}{2C_0}. \tag{54}$$

The supports of the functions f and ζ do not intersect, if R is large enough. Therefore we have

$$\begin{aligned} \big(A^{1/2}\zeta_R u,\, A^{1/2}\phi\big) - z\Big(\zeta_R u, \phi\Big) = \\ \int \nabla\bar\phi \nabla(\zeta_R u) dx + \int (V-z)\zeta_R u \bar\phi dx = \\ -\int 2\bar\phi \nabla\zeta_R \nabla u dx - \int (\Delta\zeta_R) u \bar\phi dx, \qquad \forall \phi \in D((A+I)^{1/2}), \end{aligned}$$

which means that

$$(A-z)\zeta_R u = -2\nabla\zeta_R \nabla u - (\Delta\zeta_R) u.$$

The latter relation together with (53) and (54) leads to

$$\int_{|x|>R} \Big(|\nabla u|^2 + |u|^2\Big) dx \leq \frac{1}{2} \int_{R-L<|x|<R} \Big(|\nabla u|^2 + |u|^2\Big) dx.$$

We are going to use this inequality to estimate the norm of the function in the spherical layer of a larger radius by the norm of u in the layer of a smaller radius. The corresponding constant will be equal to $1/2$. A recursive application of this inequality leads to multiplication of the constants, which means that we will get $1/2^n$ in front of the integral in the right hand side. The only restriction on R

that we have is that $R > L + 1$. For instance, if $L > 1$, then we can take $R = 2L$. Consequently, we obtain

$$\int_{(n+1)L<|x|<(n+2)L}\Bigl(|\nabla u|^2 + |u|^2\Bigr)dx \le \frac{1}{2^n}\int_{L<|x|<2L}\Bigl(|\nabla u|^2 + |u|^2\Bigr)dx, \qquad \forall n \in \mathbb{N}. \tag{55}$$

It is an exercise to the reader to prove that (55) implies

$$\int_{|x|>(n+2)L}\Bigl(|\nabla u|^2 + |u|^2\Bigr)dx \le \frac{1}{2^n}\int_{L<|x|<2L}\Bigl(|\nabla u|^2 + |u|^2\Bigr)dx, \qquad \forall n \in \mathbb{N}.$$

Thus,

$$\int_{|x|>r}\Bigl(|\nabla u|^2 + |u|^2\Bigr)dx \le C_0 \exp\Bigl(-\ln 2\frac{(r-3L)}{L}\Bigr)||f||^2$$

for $r > 3L$. It remains to note that since $C_0 \le ||(A+I)^{1/2}(A-z)^{-1}||^2$ it can be estimated by the maximum of the function

$$\psi(t) = (t+1)^{1/2}|t-z|^{-1}, \qquad t > 0.$$

This maximum is separated from zero on compact sets in the upper half-plane. The parameter L can be expressed in terms of C_0 explicitly and behaves as $a \cdot C_0$ with some universal constant a. This completes the proof of the statement. $\square$

Corollary 6.1. *Let V be a positive locally bounded potential and let $A = -\Delta + V$. Let also f be the characteristic function of the unit ball. Then for any z with $\operatorname{Im} z \neq 0$, the function*

$$u = (A-z)^{-1}f$$

satisfies

$$\int_{|x|>1}\exp(\epsilon|x|)|u|^2 dx \le C, \qquad \epsilon > 0, \tag{56}$$

with some C and ϵ separated from infinity and zero (correspondingly) when z runs over a compact set in the open upper half-plane.

The exponential decay of Green's function is crucial for convergence of spectral measures.

Let V be a positive locally bounded potential satisfying

$$\lim_{n\to\infty}\frac{V(x)}{\exp(\epsilon|x|)} = 0, \qquad \forall \epsilon > 0,$$

and let V_n be the sequence of compactly supported bounded functions, such that

$$\lim_{n\to\infty}\sup_x\Bigl\{(V(x) - V_n(x))^2\exp(-\epsilon|x|)\Bigr\} = 0, \qquad \forall \epsilon > 0. \tag{57}$$

Define measures μ_n setting

$$((-\Delta + V_n - z)^{-1}f, f) = \int_{-\infty}^{\infty}\frac{d\mu_n(t)}{t-z}, \qquad \operatorname{Im} z \neq 0, \tag{58}$$

where f is the characteristic function of the unit ball $\{x : \ |x| < 1\}$. Define also the measure μ by

$$((-\Delta + V - z)^{-1}f, f) = \int_{-\infty}^{\infty}\frac{d\mu(t)}{t-z}, \qquad \operatorname{Im} z \neq 0, \tag{59}$$

Proposition 6.2. *The sequence of measures μ_n converges to μ weakly:*

$$\mu_n \to \mu \qquad \text{as } n \to \infty.$$

Proof. Since any compacly supported continuous function can be approximated by a finite linear combinations of functions of the form $\operatorname{Im} 1/(t-z)$ in $C(\mathbb{R})$-topology, due to (59) and (58), it is sufficient to prove that

$$((-\Delta+V_n-z)^{-1}f,f)\to((-\Delta+V-z)^{-1}f,f),\qquad \text{as } n\to\infty,$$

uniformly on compact sets in the upper half-plane. The latter simply follows from the fact that

$$(-\Delta+V_n-z)^{-1}f-(-\Delta+V-z)^{-1}f=(-\Delta+V_n-z)^{-1}(\tilde V-V_n)(-\Delta+V-z)^{-1}f$$

converges to zero in L^2. Indeed, since the norm of $(-\Delta+V_n-z)^{-1}$ is bounded, it is enough to show that $(V-V_n)u$ converges to zero for $u=(-\Delta+V-z)^{-1}f$. But

$$||(V-V_n)u||^2\le\sup_x\Big\{(V-V_n)^2\exp(-\epsilon|x|)\Big\}\int_{|x|>1}\exp(\epsilon|x|)|u|^2dx$$

which tends to zero due to (56) and (57). This completes the proof of the proposition. □

References

[1] J. Bourgain: *On random Schrödinger operators on* $\mathbb{Z}^2$, Discrete Contin. Dyn. Syst, **8**, No. 1, 2002, 1–15.

[2] J. Bourgain: *Random lattice Schrödinger operators with decaying potential: some multidimensional phenomena*, Geometric Aspects of Functional Analysis: Israel Seminar 2001–2002 (V. D. Milman and G. Schechtman, Eds.), Lecture Notes in Mathematics, **1807**, Springer, Berlin, 2003, pp. 70–98.

[3] P.Deift and R.Killip: *On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials*. Comm. Math. Phys. **203** (1999), no. 2, 341–347.

[4] S. Denisov and A. Kiselev: *Spectral properties of the Schrödinger operators with decaying potentials*, Proceedings of Symposia in Pure Mathematics **76**, AMS 2007

[5] S. Denisov: *On the absolutely continuous spectrum of Dirac operator*. Comm. Partial Differential Equations **29** (2004), no. 9-10, 1403–1428.

[6] S.Denisov: *Absolutely continuous spectrum of multidimensional Schrödinger operator*. Int. Math. Res. Not. 2004, no. 74, 3963–3982.

[7] S.Denisov: *On the preservation of absolutely continuous spectrum for Schrödinger operators*. J. Funct. Anal. **231** (2006), no. 1, 143–156.

[8] S. Denisov: *Schrödinger operators and associated hyperbolic pencils*, J.Func. Anal. **254** (2008), 2186–2226

[9] S.Denisov: *Wave Propagation through sparse potential barriers*, Communications on Pure and Applied Mathematics **61** (2008), Issue 2, 156–185

[10] R. Killip and B. Simon: *Sum rules for Jacobi matrices and their application to spectral theory*, Annals of Math. **158** (2003), 253-321.

[11] A. Laptev, S. Naboko and O. Safronov : *A Szegö condition for a multidimensional Schrödinger operator*, J. Funct. Analysis **219** (2005), no. 2, 285-305

[12] A.Laptev, S.Naboko and O.Safronov: *Absolutely continuous spectrum of Schrödinger operators with slowly decaying and oscillating potentials*. Comm. Math. Phys. **253** (2005), no. 3, 611–631

[13] A. Pushnitski, *Spectral shift function of the Schrodinger operator in the large coupling constant limit*, Comm. in PDE **25**, no 3,4 (2000), 703-736.

[14] G. Perelman: *Stability of the absolutely continuous spectrum for multidimensional Schrödinger operators*, Int Math Res Notices (2005) No 37, 2289-2313.

[15] O.Safronov: *On the absolutely continuous spectrum of multi-dimensional Schrödinger operators with slowly decaying potentials*. Comm. Math. Phys. **254** (2005), no. 2, 361–366.

[16] O.Safronov: *Multi-dimensional Schrödinger operators with some negative spectrum*. J. Funct. Anal. **238** (2006), no. 1, 327–339

[17] O.Safronov: *Multi-dimensional Schrödinger operators with no negative spectrum*. Ann. Henri Poincare **7** (2006), no. 4, 781–789

[18] O. Safronov: *Absolutely continuous spectrum of one random elliptic operator*, J. Funct. Anal. **225** (2008), no.3, 755-767

[19] O. Safronov: *Absolutely continuous spectrum of multi-dimensional Schrödinger operators with slowly decaying potentials*, special issue of AMS Translations, in honor of M.S. Birman's 80th birthday

[20] O. Safronov and G. Stolz: *Absolutely continuous spectrum of Schrödinger operators with potentials slowly decaying inside a cone*, Journal of Math. Analysis and Appl.,**326** (2007) 192-208

[21] O. Safronov: : *Absolutely continuous spectrum of a one-parametric family of Schrödinger operators*, St. Petersburg Math. J. **24**, No 6.

[22] O. Safronov: *Absolutely continuous spectrum of a typical Schrödinger operator with a slowly decaying potential*, to appear

[23] O. Safronov: *Lower bounds on the eigenvalues of the Schrodinger operator and the spectral conservation law*. Journal of Mathematical Science, **166**, No. 3, 2010.

[24] B.Simon: *Schrödinger operators in the twenty-first century*. Mathematical physics 2000, 283–288, Imp. Coll. Press, London, 2000.

[25] D.R. Yafaev: *Scattering theory: Old and New results*, Lecture notes in mahematics 1735, Springer-Verlag, Berlin 2000