# LOWER SEMICONTINUITY OF FUNCTIONALS OF FRACTIONAL TYPE AND APPLICATIONS TO NONLOCAL EQUATIONS WITH CRITICAL SOBOLEV EXPONENT

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Abstract. In the present paper we study the weak lower semicontinuity of the functional

$$\Phi_{\lambda,\gamma}(u) := \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{\gamma}{2}\Big(\int_\Omega |u(x)|^{2^*}\,dx\Big)^{2/2^*},$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^n$, $n>2s$, $s\in(0,1)$, with Lipschitz boundary, $\lambda$ and $\gamma$ are real parameters and $2^*:=2n/(n-2s)$ is the fractional critical Sobolev exponent.

As a consequence of this regularity result for $\Phi_{\lambda,\gamma}$ we prove the existence of a nontrivial weak solution for two different nonlocal critical equations driven by the fractional Laplace operator $(-\Delta)^s$ which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy, \quad x\in\mathbb{R}^n.$$

These two existence results were obtained using, respectively, the direct method in the calculus of variations and critical points theory.

Contents



## 1. Introduction

*Key words and phrases.* Integrodifferential operators, fractional Laplacian, variational methods, direct methods in the calculus of variations, lower semicontinuity, Concentration–Compactness Principle, critical exponent.

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1.1. **Motivations.** Elliptic equations can be studied with a variety of methods and techniques. In this paper we are interested in the variational approach, which allows to treat a large classes of problems. The classical example for the problems studied via variational methods is given by the Dirichlet Principle, namely the fact that the solution of a differential equation (of elliptic type) coupled with a boundary condition can be obtained as a minimizer of an appropriate functional.

The fundamental idea behind the Dirichlet Principle is the interpretation of an abstract differential problem $F(u) = 0$ as

$$\mathcal{I}'(u) = 0,$$

where $\mathcal{I}$ is a suitable functional defined on a set of functions, and $\mathcal{I}'$ is its derivative in a suitable sense. In other words, zeros of $F$ are seen as critical points (not necessarily minima) of the functional $\mathcal{I}$. The equation $\mathcal{I}'(u) = 0$ is the Euler-Lagrange equation associated with $\mathcal{I}$. Of course not all the differential problems can be written in the form $\mathcal{I}'(u) = 0$. When this is possible, the problem is called variational.

Many times, it is much easier to find a critical point of $\mathcal{I}$ than to work directly on the equation $F(u) = 0$. Furthermore, in many applications the functional $\mathcal{I}$ has a fundamental physical meaning. Indeed, often $\mathcal{I}$ is an energy of some sort and hence finding a minimum point means not only solving the differential equation, but also finding the solution of minimal energy, which has particular relevance in concrete problems. The interpretation of $\mathcal{I}$ as an energy explains why the functionals associated with differential problems are normally called energy functionals, even when the problem has no direct physical applications.

The methods concerned with the minimization of functionals go under the name of *direct methods of the Calculus of Variations*, while the ones related to finding critical points of functionals give rise to a branch of nonlinear analysis known as *Critical Point Theory.*

The starting point of the so-called direct methods of the Calculus of Variations is the *Weierstrass Theorem* (saying that a weakly lower semicontinuous and coercive functional defined on a reflexive Banach space admits a global minimum), as well as in critical points theory the crucial idea is that the existence of critical points is related to the topological properties of the sublevels of the functional, provided some compactness properties are satisfied.

Of course, when using direct minimization we need that the functional is bounded from below and, in this case, we look for its global minima, which are the most natural critical points. In looking for global minima of a functional the two relevant notions are the weakly lower semicontinuity and the coercivity, as stated in the Weierstrass Theorem (see [1, Remark 1.5.7]). The coercivity of the functional assures that the minimizing sequence is bounded, while the semicontinuity gives the existence of the minimum for the functional.

1.2. **Aims of the paper.** Along the present paper we study the weak lower semicontinuity of the functional

$$\Phi_{\lambda,\gamma}(u) := \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{\gamma}{2}\Big(\int_\Omega |u(x)|^{2^*}\,dx\Big)^{2/2^*}, \tag{1.1}$$

defined on the functional space $X_0$ given by

$$X_0 := \big\{g\in H^s(\mathbb{R}^n) : g = 0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\big\}. \tag{1.2}$$

Here $\Omega$ is an open bounded subset of $\mathbb{R}^n$, $n > 2s$ and $s \in (0,1)$, with Lipschitz boundary, $\lambda$ and $\gamma$ are real parameters, the exponent $2^* := 2n/(n-2s)$ is the fractional critical Sobolev exponent (notice that when $s = 1$ it reduces to the classical critical Sobolev exponent $2_* := 2n/(n-2)$), and the functional space $H^s(\mathbb{R}^n)$ denotes the fractional Sobolev space defined as the linear space of functions $g \in L^2(\mathbb{R}^n)$ such that

$$\text{the map } (x,y)\mapsto \frac{g(x)-g(y)}{|x-y|^{n/2+s}} \text{ is in } L^2(\mathbb{R}^n\times\mathbb{R}^n, dxdy).$$

The space $X_0$ was introduced in [32] in a general nonlocal framework, in order to give a variational formulation for nonlocal integrodifferential equations of fractional type depending on a suitable kernel $K$. When the kernel $K$ is given by the model function $K(x) := |x|^{-(n+2s)}$ we get the fractional Laplace operator. In this setting the space $X_0$ can be characterized as in (1.2): for a proof of this we refer to [35, Lemma 7-$b$)]. Further properties of the space $X_0$ can be found in [29, 30, 33, 34, 35, 36, 37].

The main problem in proving the weak lower semicontinuity of the functional $\Phi_{\lambda,\gamma}$ relies on the fact that the embeddings $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ are continuous but not compact: for this we refer to [33, Lemma 8] and [14, Theorem 6.5]. Thanks to these results we can define the best fractional critical Sobolev constant $S_s$ as follows

$$S_s := \inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} S_s(v)\,, \tag{1.3}$$

where for any $v \in H^s(\mathbb{R}^n) \setminus \{0\}$ the function $S_s(\cdot)$ is given by

$$S_s(v) := \frac{\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}\,dx\Big)^{2/2^*}}\,. \tag{1.4}$$

We stress that $S_s$ does not depend on $\Omega$, because the minimization occurs on the whole of $H^s(\mathbb{R}^n)$, and that $S_s$ is a constant strictly positive.

In order to overcome the difficulties related to the lack of compactness we will perform a Concentration–Compactness Principle for fractional Sobolev spaces (see [25]), which represents the nonlocal counterpart of the famous result of Lions given in [21, 22]. Using this strategy we prove the following result:

**Theorem 1.** *Let $s \in (0,1)$, $n > 2s$, $\Omega$ be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let $2^* = 2n/(n-2s)$.*

*Then, the functional*

$$X_0 \ni u \mapsto \Phi_{\lambda,\gamma}(u)$$

*is weakly lower semicontinuous for any $\lambda \in \mathbb{R}$ and any $\gamma \in [0, S_s]$, where $S_s$ is the best fractional critical Sobolev constant defined as in* (1.3).

When $\lambda = 0$ Theorem 1 represents the fractional counterpart of [24, Theorem 2.1], where the author studied the energy functional associated with quasilinear elliptic equations in presence of critical nonlinearities.

As a consequence of Theorem 1 we have the following result:

**Corollary 2.** *Let $s \in (0,1)$, $n > 2s$, $\Omega$ be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let $2^* = 2n/(n-2s)$. Let $\lambda_{1,s}$ be the first eigenvalue of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary data.*

*Then, for any $h \in L^2(\Omega)$ the functional $\mathcal{I}_h$ defined as*

$$\mathcal{I}_h(u) := \Phi_{\lambda,\gamma}(u) - \int_\Omega h(x)u(x)\,dx$$

*admits a global minimum in $X_0$, provided $\lambda < \lambda_{1,s}$ and $\gamma \in [0,\gamma_\lambda)$, where*

$$\gamma_\lambda := S_s \min\big\{1, \lambda/\lambda_{1,s}\big\},$$

*and $S_s$ is as in* (1.3).

The proof of Corollary 2 relies on direct minimization and it is based on the Weierstass Theorem. For a precise definition of $\lambda_{1,s}$ we refer to Section 2.

1.3. **Applications to nonlocal fractional critical equations.** The functional $\mathcal{I}_h$ introduced in Corollary 2 represents the Euler–Lagrange functional of the nonlocal critical equation

$$\begin{cases} (-\Delta)^s u - \lambda u = \gamma\Big(\displaystyle\int_\Omega |u(x)|^{2^*}\,dx\Big)^{2/2^*-1}|u|^{2^*-2}u + h & \text{in } \Omega\\ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{1.5}$$

where $(-\Delta)^s$ is the fractional Laplace operator which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy, \quad x\in\mathbb{R}^n.$$

Moreover, $\lambda$ and $\gamma$ are real parameters and $h$ is a perturbation. Also, the homogeneous Dirichlet datum in (1.5) is given in $\mathbb{R}^n\setminus\Omega$ and not simply on the boundary $\partial\Omega$, as it happens in the classical case of the Laplacian, consistently with the nonlocal nature of the fractional Laplacian operator $(-\Delta)^s$.

Nonlocal fractional problems appear in many fields such as, among the others, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves. Recently, a lot of interest has been devoted to this kind of problems and to their concrete applications; see, for instance the seminal papers [9, 10, 11] and [2, 3, 8, 12, 15, 16, 17, 23, 31, 40] as well as the references therein.

In the sequel we focus our attention on nonlocal critical equations of fractional type. A great interest has been given to the study of equations with critical nonlinearities, both in the standard Laplace setting (see [6, 26, 27] and, for an overview, [1, 19, 39]) and, more recently, in the nonlocal fractional framework (see, e.g., the recent papers [3, 29, 30, 35, 36, 37, 40]).

As usual, when dealing with the critical setting (in the sense of the Sobolev embeddings) the main difficulties in finding a solution are related to the lack of compactness.

As a byproduct of Corollary 2 we deduce the following existence result for problem (1.5):

**Corollary 3.** *Let $s\in(0,1)$, $n>2s$, $\Omega$ be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let $2^*=2n/(n-2s)$. Let $\lambda_{1,s}$ be the first eigenvalue of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary data and let $h\in L^2(\Omega)$.*

*Then, problem (1.5) admits a weak solution $u\in X_0$, for any $\lambda<\lambda_{1,s}$ and $\gamma\in[0,\gamma_\lambda)$, where $\gamma_\lambda = S_s\min\{1,\lambda/\lambda_{1,s}\}$ and $S_s$ is as in (1.3). Finally, $u$ is not identically zero, provided $h\not\equiv 0$.*

As usual, for a weak solution of problem (1.5), we mean a solution of the following problem

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\ \qquad = \gamma\Big(\displaystyle\int_\Omega |u(x)|^{2^*}\,dx\Big)^{2/2^*-1}\int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)\,dx \\ \qquad\qquad + \displaystyle\int_\Omega h(x)\varphi(x)dx, \quad \forall\,\varphi\in X_0 \\ u\in X_0. \end{cases} \tag{1.6}$$

The proof of Corollary 3 is based on direct minimization of the functional associated with problem (1.6) (i.e. the functional $\mathcal{I}_h$).

Along this paper we are interested, in particular, in equations depending on parameters which, in many cases, come from concrete applications. The existence of solutions for this kind of problems is a relevant topic, as well as the study of how solutions depend on these parameters. A recent result on the existence of solutions for nonlocal equations depending on two real parameters with subcritical growth nonlinearities appeared in [23].

Motivated by this recent work, one of the aims of this paper is to study the existence of solutions for the following general nonlocal critical equation depending on three real parameters

$$\begin{cases} (-\Delta)^s u - \lambda u = \gamma\Big(\displaystyle\int_\Omega |u(x)|^{2^*}\,dx\Big)^{2/2^*-1}|u|^{2^*-2}u + \mu f(x,u) & \text{in } \Omega\\ u = 0 & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{1.7}$$

where $\lambda$, $\gamma$ and $\mu$ are real parameters, while $f$ is a lower order perturbation of the critical power function. Precisely, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that

$$\sup\Big\{|f(x,t)| : \text{a.e. } x\in\Omega\,,\ t\in[0,M]\Big\} < +\infty \ \text{ for any } M>0\,; \tag{1.8}$$

$$\lim_{|t|\to+\infty}\frac{f(x,t)}{|t|^{2^*-1}} = 0, \ \text{ uniformly in } x\in\Omega\,. \tag{1.9}$$

Note that conditions (1.8) and (1.9) are the standard assumptions to be satisfied by the perturbation in presence of critical terms (see, for instance, the seminal paper [6, Section 2]).

One of the aim of this paper is to prove the existence of a non-trivial weak solution for problem (1.7). We stress that the trivial function $u\equiv 0$ in $\mathbb{R}^n$ is a solution of problem (1.7) if and only if $f(\cdot,0)=0$. As a consequence of this, if $f(\cdot,0)\neq 0$ and problem (1.7) admits a solution $u$, then we can immediately deduce that $u\not\equiv 0$. While, in the case when $f(\cdot,0)=0$, we need some extra assumptions on $f$, in order to show the existence of non-trivial solutions. Precisely, when $f(\cdot,0)=0$ we assume the following extra condition:

$$\begin{gathered}\text{there exist a non-empty open set } D\subseteq\Omega \text{ and a set } B\subseteq D\\ \text{of positive Lebesgue measure such that}\\ \limsup_{t\to 0^+}\frac{\operatorname{essinf}_{x\in B}F(x,t)}{t^2} = +\infty \quad\text{and}\quad \liminf_{t\to0^+}\frac{\operatorname{essinf}_{x\in D}F(x,t)}{t^2} > -\infty,\end{gathered} \tag{1.10}$$

where

$$F(x,t) := \int_0^t f(x,\tau)d\tau, \tag{1.11}$$

for a.e. $x\in\Omega$ and any $t\in\mathbb{R}$, that is $F$ is the primitive of the nonlinearity $f$ with respect to the second variable.

As a model for $f$ we can take the functions $f(x,t) := a(x)|t|^{r-2}t + b(x)|t|^{q-2}t + c(x)$, with $1<r<2\leqslant q<2^*$ and $a,b,c\in L^\infty(\Omega)$. If $c\equiv 0$ a.e. in $\Omega$, we assume also that $\operatorname{essinf}_{x\in\Omega}a(x)>0$. We would like to stress that the nonlinearity $f$ may behave like $|t|^\nu$, namely

$$f(\cdot,t)\cong|t|^\nu$$

as $t\to 0$ and $|t|\to+\infty$ for any $\nu\in(0,2^*-1)$, that is $f$ can be both sublinear and superlinear at zero and at infinity.

Assumption (1.10) is a sort of subquadratical growth condition at zero. Note also that condition (1.10) is trivially satisfied if the following stronger assumption holds true:

$$\begin{gathered}\text{there exists a non-empty open set } B\subseteq\Omega\\ \text{of positive Lebesgue measure such that}\\ \lim_{t\to0^+}\frac{\operatorname{essinf}_{x\in B}F(x,t)}{t^2} = +\infty.\end{gathered}$$

In this paper we will prove the existence of non-trivial weak solutions of problem (1.7) using variational and topological methods. By a weak solutions of (1.7) we mean a solution of the following problem

$$
\begin{cases}
\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\
\qquad\displaystyle = \gamma\Big(\int_\Omega |u(x)|^{2^*}\,dx\Big)^{2/2^*-1}\int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)\,dx \\
\qquad\qquad\displaystyle +\mu\int_\Omega f(x,u(x))\varphi(x)dx, \quad \forall\,\varphi\in X_0 \\
u\in X_0.
\end{cases}
\tag{1.12}
$$

Problem (1.12) represents the Euler–Lagrange equation of the functional $\mathcal{J}_{\lambda,\gamma,\mu}: X_0\to\mathbb{R}$ defined as

$$
\begin{aligned}
\mathcal{J}_{\lambda,\gamma,\mu}(u) &:= \Phi_{\lambda,\gamma}(u) - \mu\int_\Omega F(x,u(x))dx \\
&= \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx \\
&\qquad - \frac{\gamma}{2}\Big(\int_\Omega |u(x)|^{2^*}\,dx\Big)^{2/2^*} - \mu\int_\Omega F(x,u(x))dx,
\end{aligned}
\tag{1.13}
$$

where $F$ is the function defined in (1.11).

The main existence result of the present paper can be stated as follows:

**Theorem 4.** *Let $s\in(0,1)$, $n>2s$, $\Omega$ be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let $\lambda_{1,s}$ be the first eigenvalue of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary data and $\gamma_\lambda := S_s\min\{1,\lambda/\lambda_{1,s}\}$, with $S_s$ as in (1.3). Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function verifying (1.8) and (1.9). In addition, if $f(x,0)=0$ for a.e. $x\in\Omega$, assume also (1.10).*

*Then, for any $\lambda<\lambda_{1,s}$ and any $\gamma\in[0,\gamma_\lambda)$ there exists a positive constant $\mu_\lambda$, depending on $\lambda$, such that for any $\mu\in(0,\mu_\lambda)$ problem (1.7) admits a weak solution $u_\mu\in X_0$ which is not identically zero.*

*Moreover,*

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u_\mu(x)-u_\mu(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\to 0$$

*as $\mu\to 0^+$ and the function*

$$\mu\mapsto \mathcal{J}_{\lambda,\gamma,\mu}(u_\mu)$$

*is negative and strictly decreasing in $(0,\mu_\lambda)$.*

Actually, using a truncation argument, we can prove that problem (1.7) admits a non-trivial non-negative weak solution, according to the following result:

**Corollary 5.** *Let all the assumptions of Theorem 4 be satisfied and assume $f(\cdot,0)=0$.*

*Then, for any $\lambda<\lambda_{1,s}$ and any $\gamma\in[0,\gamma_\lambda)$ there exists a positive constant $\mu_\lambda$, depending on $\lambda$, such that, for any $\mu\in(0,\mu_\lambda)$ problem (1.7) admits a non-trivial non-negative weak solution $u_+\in X_0$.*

In general, when $f(\cdot,0)\neq 0$, problem (1.7) admits changing-sign solutions, as it happens if we look at the classical critical case involving the Laplace operator.

Theorem 4 will be proved using variational and topological techniques, in particular performing [4, Theorem 2.1; part *a*)] (see also [28]), which assures the existence of a critical point (actually a minimum) for a functional, under suitable regularity assumptions on it. For more details and related topics we refer to the recent monograph [20].

The main difficulty in proving Theorem 4 is related to the study of the regularity of the functional associated with problem (1.7). This is mainly due to the fact that the equation (1.7) contains a critical nonlinearity and the space $X_0$ is not compactly embedded into $L^{2^*}(\Omega)$. In order to overcome this difficulty we will make use of Theorem 1.

Another difficulty is related to the proof of the non-triviality of the solution. Of course, if $f(\cdot, 0) \neq 0$, there is nothing to prove, since it it obvious that any solution of problem (1.7) is not identically zero. On the other hand, when $f(\cdot, 0) = 0$, the non-triviality of the solution is more complicated to be proved: for this a crucial role will be played by the subquadratical growth assumption (1.10).

Finally, we would like to note that Theorem 4 represents the nonlocal counterpart of [5, Theorem 3], where the authors considered a critical equation driven by the $p$–Laplace operator in a bounded domain of $\mathbb{R}^n$.

The present paper is organized as follows. In Section 2 we give some notations and we recall some properties of the functional space we work in. We also give some tools which will be useful along the paper. Section 3 is devoted to the study of the lower semicontinuity of the functional $\Phi_{\lambda,\gamma}$: here we prove Theorem 1 and we give some applications of it. In Section 4 we study problem (1.7) and we prove Theorem 4 and Corollary 5.

## 2. Some preliminaries

This section is devoted to the notations used along the paper. We also give some preliminary results which will be useful in the sequel.

2.1. **Notations and definitions.** In this subsection we briefly recall some properties of the functional space $X_0$, firstly introduced in [32], and we give some notations. The reader familiar with this topic may skip this section and go directly to the next one.

The space $X_0$ is defined as in (1.2), where $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \Big( \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}. \tag{2.1}$$

For further details on the fractional Sobolev spaces we refer to [14] and to the references therein.

Of course, the space $X_0$ is non-empty, since $C^2_0(\Omega) \subseteq X_0$ by [32, Lemma 11] and it depends on the set $\Omega$. Moreover, by [33, Lemma 6] and the fact that any function $v \in X_0$ is such that $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, in the sequel we can take

$$X_0 \ni v \mapsto \|v\|_{X_0} = \left( \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2} \tag{2.2}$$

as norm on $X_0$. Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space (for this see [33, Lemma 7]), with scalar product

$$\langle u, v\rangle_{X_0} := \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy. \tag{2.3}$$

In the sequel, we will denote by $\lambda_{1,s}$ the first eigenvalue of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary data, namely the first eigenvalue of the problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

For the existence and the basic properties of this eigenvalue we refer to [34, Proposition 9 and Appendix A], where a spectral theory for general integrodifferential nonlocal operators was developed. Further properties can be also found in [29, 36, 38].

When $\lambda < \lambda_{1,s}$ we can take as a norm on $X_0$ the function

$$X_0 \ni v \mapsto \|v\|_{X_0,\lambda} = \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dx \, dy - \lambda \int_\Omega |v(x)|^2 \, dx \right)^{1/2}, \tag{2.4}$$

since for any $v \in X_0$ it holds true (see [34, Lemma 10])

$$m_\lambda \|v\|_{X_0} \leqslant \|v\|_{X_0,\lambda} \leqslant M_\lambda \|v\|_{X_0}, \tag{2.5}$$

where

$$m_\lambda := \min\left\{ \sqrt{1 - \lambda/\lambda_{1,s}}\,,\, 1 \right\} \quad \text{and} \quad M_\lambda := \max\left\{ \sqrt{1 - \lambda/\lambda_{1,s}}\,,\, 1 \right\}.$$

Note that (2.5) is a consequence of the variational characterization of $\lambda_{1,s}$ given in [34, Proposition 9] and of the choice of $\lambda$.

Finally, we recall that in [33, Lemma 8] and in [35, Lemma 9] the authors proved that the embedding $j : X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [1, 2^*]$, while it is compact whenever $\nu \in [1, 2^*)$. In the sequel for any $\nu \in [1, 2^*)$, we will denote by $c_\nu$ the positive constant such that

$$\|v\|_{L^\nu(\mathbb{R}^n)} \leqslant c_\nu \|v\|_{X_0}, \quad \text{for any } v \in X_0. \tag{2.6}$$

Of course, by this and taking into account (2.5), it is easy to see that for any $\nu \in [1, 2^*)$

$$\|v\|_{L^\nu(\mathbb{R}^n)} \leqslant c_\nu m_\lambda^{-1} \|v\|_{X_0,\lambda}, \quad \text{for any } v \in X_0.$$

Also, taking into account the definition of $S_s$ in (1.3) we get

$$\|v\|_{L^{2^*}(\mathbb{R}^n)} \leqslant S_s^{-1/2} \|v\|_{X_0} \leqslant S_s^{-1/2} m_\lambda^{-1} \|v\|_{X_0,\lambda}, \quad \text{for any } v \in X_0. \tag{2.7}$$

2.2. **Some useful tools.** The main tools used along this paper in order to prove the existence results stated in Corollary 3 and in Theorem 4 are given by the Weierstarss Theorem, which is the starting point of the direct minimization, and a result in critical points theory due to Ricceri and stated in [28, Theorem 2.1]. For the sake of clarity, we recall this last result, in the form given in [4], here below:

**Theorem.** ([4, Theorem 2.1; part $a$)]) *Let $Y$ be a reflexive real Banach space, let $\Phi, \Psi : Y \to \mathbb{R}$ be two Gâteaux differentiable functionals such that $\Phi$ is strongly continuous, sequentially weakly lower semicontinuous and coercive in $Y$ and $\Psi$ is sequentially weakly upper semicontinuous in $Y$.*

*Let $J_\mu$ be the functional defined as $J_\mu := \Phi - \mu\Psi$, $\mu \in \mathbb{R}$, and for any $r > \inf_Y \Phi$ let $\varphi$ be the function defined as*

$$\varphi(r) := \inf_{u \in \Phi^{-1}\big((-\infty, r)\big)} \frac{\displaystyle\sup_{v \in \Phi^{-1}\big((-\infty, r)\big)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

*Then, for any $r > \inf_Y \Phi$ and any $\mu \in (0, 1/\varphi(r))$[1], the restriction of the functional $J_\mu$ to $\Phi^{-1}\big((-\infty, r)\big)$ admits a global minimum, which is a critical point (precisely a local minimum) of $J_\mu$ in $Y$.*

In the sequel we also will need the following lemma, whose proof can be found in [35, Lemma 5]:

**Lemma 6.** *Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions (1.8) and (1.9). Then, for any $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$*

$$|f(x,t)| \leqslant 2^* \varepsilon |t|^{2^*-1} + M(\varepsilon) \tag{2.8}$$

[1] Note that, by definition, $\varphi(r) \geqslant 0$ for any $r > \inf_Y \Phi$. Here and in the following, if $\varphi(r) = 0$, by $1/\varphi(r)$ we mean $+\infty$, i.e. we set $1/\varphi(r) = +\infty$.

*and so, as a consequence,*

$$|F(x,t)| \leqslant \varepsilon\,|t|^{2^*} + M(\varepsilon)|t|\,, \tag{2.9}$$

*where $F$ is defined as in* (1.11).

## 3. Lower semicontinuity of functionals of fractional type

In this section we prove a semicontinuity property for the functional $\Phi_{\lambda,\gamma}$. The arguments used along this section are similar to the ones performed in [24, Theorem 2.1], where functionals defined in the classical Sobolev spaces were considered when a critical term occurs.

The difficulty is treating functional $\Phi_{\lambda,\gamma}$ is related to the presence of a critical term and to the fact that the embedding of $X_0$ into $L^{2^*}(\Omega)$ is not compact. In order to overcome this problem our proof relies on a Concentration-Compactness Principle in fractional Sobolev spaces (see [25]).

3.1. **Proof of Theorem 1.** By [18] we known that $C^\infty_0(\Omega)$ is a dense subset of $X_0$ . Hence, using density arguments, in order to prove Theorem 1 it is enough to show that the functional

$$C^\infty_0(\Omega) \ni u \mapsto \Phi_{\lambda,\gamma}(u) \text{ is weakly lower semicontinuous} \tag{3.1}$$

for any $\lambda \in \mathbb{R}$ and any $\gamma \in [0, S_s]$ .

For this let $u_j$ be a sequence in $C^\infty_0(\Omega)$ such that

$$u_j \rightharpoonup u \quad \text{weakly in } X_0 \tag{3.2}$$

as $j \to +\infty$. Then, by [32, Lemma 11] and [35, Lemma 9] we get that

$$u_j \rightharpoonup u \quad \text{weakly in } L^{2^*}(\mathbb{R}^n)$$

as $j \to +\infty$ . Hence,

$$|(-\Delta)^{s/2} u_j|^2\, dx \overset{*}{\rightharpoonup} \bar\mu \tag{3.3}$$

and

$$|u_j|^{2^*}\, dx \overset{*}{\rightharpoonup} \bar\nu \tag{3.4}$$

as $j \to +\infty$ in the weak* convergence of measures.

Thus, by [25, Theorem 1.5] we have that there exist a finite set of distinct points $x_1,\dots,x_k \in \bar\Omega$ and positive numbers $\mu_1,\dots,\mu_k$ and $\nu_1,\dots,\nu_k$ such that

$$\bar\mu \geqslant |(-\Delta)^{s/2} u|\, dx + \sum_{j=1}^{k} \mu_j \delta_{x_j}, \tag{3.5}$$

$$\bar\nu = |u|^{2^*}\, dx + \sum_{j=1}^{k} \nu_j \delta_{x_j}, \tag{3.6}$$

and, finally,

$$\nu_j \leqslant S_s^{-2^*/2} \mu_j^{2^*/2}, \quad \text{for any} \quad j = 1,\dots,k\,. \tag{3.7}$$

Here $\delta_x$ denotes the Dirac delta function at $x$, while $S_s$ is the constant given in (1.3).

The continuity of the embedding $X_0 \hookrightarrow L^2(\Omega)$ gives that

$$u_j \to u \quad \text{in} \quad L^2(\Omega) \tag{3.8}$$

as $j \to +\infty$.

By (3.4)–(3.6) and (3.8) we get

$$
\begin{aligned}
\Phi_{\lambda,\gamma}(u_j) &= \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy-\frac{\lambda}{2}\int_\Omega|u_j(x)|^2\,dx\\
&\qquad -\frac{\gamma}{2}\Big(\int_\Omega|u_j(x)|^{2^*}\,dx\Big)^{2/2^*}\\
&\geqslant \frac{1}{2}\left(\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy+\sum_{j=1}^k\mu_j\right)-\frac{\lambda}{2}\int_\Omega|u(x)|^2\,dx\\
&\qquad -\frac{\gamma}{2}\left(\int_\Omega|u(x)|^{2^*}\,dx+\sum_{j=1}^k\nu_j\right)^{2/2^*}\\
&= \Phi_{\lambda,\gamma}(u)+\frac{1}{2}\sum_{j=1}^k\mu_j\\
&\qquad -\frac{\gamma}{2}\left(\left(\int_\Omega|u(x)|^{2^*}\,dx+\sum_{j=1}^k\nu_j\right)^{2/2^*}-\Big(\int_\Omega|u(x)|^{2^*}\,dx\Big)^{2/2^*}\right).
\end{aligned}
\tag{3.9}
$$

Now, using the inequality

$$(a+b)^r\leqslant a^r+b^r \qquad a,b\geqslant 0,\ r\in(0,1),$$

the non-negativity of $\gamma$, (3.7) and (3.9) we obtain

$$
\begin{aligned}
\Phi_{\lambda,\gamma}(u_j) &\geqslant \Phi_{\lambda,\gamma}(u)+\frac{1}{2}\sum_{j=1}^k\mu_j-\frac{\gamma}{2}\left(\sum_{j=1}^k\nu_j\right)^{2/2^*}\\
&\geqslant \Phi_{\lambda,\gamma}(u)+\frac{1}{2}\sum_{j=1}^k\mu_j-\frac{\gamma}{2}\left(\sum_{j=1}^k\Big(\frac{\mu_j}{S_s}\Big)^{2^*/2}\right)^{2/2^*}.
\end{aligned}
\tag{3.10}
$$

Now, applying the inequality

$$(a+b)^q\geqslant a^q+b^q \qquad a,b\geqslant 0,\ q>1,$$

with $q=2^*/2$ and using again the non-negativity of $\gamma$, by (3.10) we get

$$
\begin{aligned}
\Phi_{\lambda,\gamma}(u_j) &\geqslant \Phi_{\lambda,\gamma}(u)+\frac{1}{2}\sum_{j=1}^k\mu_j-\frac{\gamma}{2}\sum_{j=1}^k\frac{\mu_j}{S_s}\\
&= \Phi_{\lambda,\gamma}(u)+\frac{1}{2}\left(1-\frac{\gamma}{S_s}\right)\sum_{j=1}^k\mu_j\\
&\geqslant \Phi_{\lambda,\gamma}(u),
\end{aligned}
\tag{3.11}
$$

provided $\gamma\leqslant S_s$. Passing to the liminf as $j\to+\infty$ in (3.11) we obtain the assertion stated in (3.1) for any $\lambda\in\mathbb{R}$ and $\gamma\in[0,S_s]$.

Now, we can conclude the proof of Theorem 1 using density arguments. Let $u_j$ be a sequence in $X_0$ such that

$$u_j\rightharpoonup u\quad\text{weakly in } X_0 \tag{3.12}$$

as $j\to+\infty$. Then, since $C_0^\infty(\Omega)$ is a dense subset of $X_0$, for any $j\in\mathbb{N}$ there exists $u_j^k\in C_0^\infty(\Omega)$ such that

$$u_j^k\to u_j\ \text{ in } X_0 \tag{3.13}$$

as $k \to +\infty$.

By (3.12) and (3.13), recalling the definition given in (2.3), we have that for any $\varphi \in X_0$

$$\langle u_j^k - u, \varphi \rangle_{X_0} = \langle u_j^k - u_j, \varphi \rangle_{X_0} + \langle u_j - u, \varphi \rangle_{X_0} \to 0$$

as $j, k \to +\infty$, that is

$$u_j^k \rightharpoonup u \text{ weakly in } X_0$$

as $j, k \to +\infty$.

Since $u_j^k \in C_0^\infty(\Omega)$ and (3.1) holds true, we deduce that

$$\liminf_{j,\,k \to +\infty} \Phi_{\lambda,\,\gamma}(u_j^k) \geqslant \Phi_{\lambda,\,\gamma}(u) \tag{3.14}$$

for any $\lambda \in \mathbb{R}$ and any $\gamma \in [0, S_s]$.

Moreover, by (3.13) and the definition of $\Phi_{\lambda,\,\gamma}$ it is easy to see that for any $j \in \mathbb{N}$

$$\lim_{k \to +\infty} \Phi_{\lambda,\,\gamma}(u_j^k) = \Phi_{\lambda,\,\gamma}(u_j),$$

so that, passing to the liminf as $j \to +\infty$

$$\liminf_{j,\,k \to +\infty} \Phi_{\lambda,\,\gamma}(u_j^k) = \liminf_{j \to +\infty} \lim_{k \to +\infty} \Phi_{\lambda,\,\gamma}(u_j^k) = \liminf_{j \to +\infty} \Phi_{\lambda,\,\gamma}(u_j). \tag{3.15}$$

By (3.14) and (3.15) we get that

$$\liminf_{j \to +\infty} \Phi_{\lambda,\,\gamma}(u_j) \geqslant \Phi_{\lambda,\,\gamma}(u),$$

that is the functional

$$X_0 \ni u \mapsto \Phi_{\lambda,\,\gamma}(u)$$

is weakly lower semicontinuous for any $\lambda \in \mathbb{R}$ and any $\gamma \in [0, S_s]$. This concludes the proof of Theorem 1.

3.2. **Some applications.** As applications of the result stated in Theorem 1 here we prove Corollary 2 and Corollary 3.

*Proof of Corollary 2:* Let $h \in L^2(\Omega)$ be fixed. First of all we show that

$$\text{the map } u \mapsto \int_\Omega h(x)u(x)\,dx \quad \text{is continuous in the weak topology of } X_0. \tag{3.16}$$

For this, let $u_j$ be a sequence in $X_0$ such that $u_j \rightharpoonup u$ weakly in $X_0$ as $j \to +\infty$. Then, by [33, Lemma 8], $u_j \to u$ in $L^2(\Omega)$, and so

$$\int_\Omega h(x)u_j(x)\,dx \to \int_\Omega h(x)u(x)\,dx$$

as $j \to +\infty$. Hence, (3.16) is proved.

By (3.16) and Theorem 1 we get that the functional

$$\mathcal{I}_h(u) = \Phi_{\lambda,\,\gamma}(u) - \int_\Omega h(x)u(x)\,dx$$

is weakly lower semicontinuous in $X_0$ for any $\lambda \in \mathbb{R}$ and any $\gamma \in [0, S_s]$.

Now, let us show that $\mathcal{I}_h$ is coercive in $X_0$. By the definitions of $\mathcal{I}_h$ and $\Phi_{\lambda,\gamma}$, the Hölder inequality, (2.5), (2.6) and (2.7) we get that

$$\begin{aligned}\mathcal{I}_h(u) &= \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx \\ &\qquad - \frac{\gamma}{2}\Big(\int_\Omega |u(x)|^{2^*}\,dx\Big)^{2/2^*} - \int_\Omega h(x)u(x)\,dx \\ &\geqslant \frac{1}{2}\|u\|^2_{X_0,\lambda} - \frac{\gamma}{2}\|u\|^2_{L^{2^*}(\Omega)} - \|h\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\geqslant \frac{m_\lambda^2}{2}\|u\|^2_{X_0} - \frac{\gamma}{2S_s}\|u\|^2_{X_0} - c_2\|h\|_{L^2(\Omega)}\|u\|_{X_0} \\ &= \frac{1}{2}\Big(m_\lambda^2 - \frac{\gamma}{S_s}\Big)\|u\|^2_{X_0} - c_2\|h\|_{L^2(\Omega)}\|u\|_{X_0},\end{aligned} \tag{3.17}$$

provided $\gamma \geqslant 0$ and $\lambda < \lambda_{1,s}$. Hence, choosing $\gamma$ such that $\gamma < m_\lambda^2 S_s$, by (3.17) we deduce that

$$\mathcal{I}_h(u) \to +\infty,$$

as $\|u\|_{X_0} \to +\infty$, namely, the functional $\mathcal{I}_h$ is coercive in $X_0$, provided $\lambda < \lambda_{1,s}$ and $\gamma \in [0,\gamma_\lambda)$, where, taking into account the definition of $m_\lambda$, the constant $\gamma_\lambda$ is given by

$$\gamma_\lambda := S_s \min\{1, 1-\lambda/\lambda_{1,s}\}.$$

Note that $\gamma_\lambda \leqslant S_s$. Thus, the functional $\mathcal{I}_h$ is weakly lower semicontinuous and coercive in $X_0$, for any $\lambda < \lambda_{1,s}$ and $\gamma \in [0,\gamma_\lambda)$.

Also $X_0$ is a Hilbert space (see [33, Lemma 7]) and so it is reflexive. Hence, by the (generalized) Weierstrass Theorem (see [1, Remark 1.5.7]), $\mathcal{I}_h$ admits a global minimum in $X_0$ for any $\lambda < \lambda_{1,s}$ and $\gamma \in [0,\gamma_\lambda)$. This concludes the proof of Corollary 2. □

Now, we can prove our first existence result related to the critical problem (1.5).

*Proof of Corollary 3:* First of all, note that $\mathcal{I}_h$ is the Euler–Lagrange functional associated with problem (1.5). Also, $\mathcal{I}_h$ is Fréchet differentiable in $X_0$ with

$$\begin{aligned}\langle \mathcal{I}_h'(u),\varphi\rangle &= \int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\ &\quad - \gamma\Big(\int_\Omega |u(x)|^{2^*}\,dx\Big)^{2/2^*-1}\int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)\,dx - \int_\Omega h(x)\varphi(x)dx\end{aligned}$$

for any $\varphi \in X_0$.

By Corollary 2 we know that $\mathcal{I}_h$ admits a global minimum $u$ in $X_0$ for any $\lambda < \lambda_{1,s}$ and $\gamma \in [0,\gamma_\lambda)$. It is easy to see that $u$ is a critical point of $\mathcal{I}_h$ (for this see, e.g., [1, Remark 1.5.1]) and so it is a weak solution of problem (1.5).

Of course, $u$ is not identically zero, since the trivial function does not solve equation (1.5) (unless $h \equiv 0$). □

## 4. A critical nonlocal fractional equation

This section is devoted to the study of the critical problem (1.7). In this case we can not use a direct minimization, as for problem (1.5), since in general the functional $\mathcal{J}_{\lambda,\gamma,\mu}$ naturally associated with (1.12) (which is the weak formulation of (1.7)) is unbounded from below in $X_0$. Indeed, if we consider the case when $\mu > 0$ and $f(\cdot,\tau)$ is superlinear (i.e. $F(\cdot,\tau)$ is superquadratic) at infinity, we have that for any $u \in X_0 \setminus \{0\}$

$$\mathcal{J}_{\lambda,\gamma,\mu}(tu) = \frac{t^2}{2}\|u\|^2_{X_0,\lambda} - \frac{\gamma t^2}{2}\|u\|^2_{L^{2^*}(\Omega)} - \mu\int_\Omega F(x,tu(x))\,dx \to -\infty,$$

as $t \to +\infty$.

Due to these reasons, we will study problem (1.12) performing the methods of the critical points theory, i.e. looking for critical points of $\mathcal{J}_{\lambda,\gamma,\mu}$. In particular, along the present paper we will apply [4, Theorem 2.1; part $a$)].

In the sequel, we will be able to overcome the problem related to the lack of compactness thanks to our assumptions (1.8) and (1.9) and to the weak lower semicontinuity of the functional $\Phi_{\lambda,\gamma}$ proved in Theorem 1 . As a consequence of this, we prove the existence of a weak solution $u_\mu$ for problem (1.7) for suitable values of the parameters $\lambda$ , $\gamma$ and $\mu$ .

On the other hand, the subquadratical growth condition (1.10) will be crucial, in the case when $f(\cdot,0)=0$, in order to show that $u_\mu$ is not the trivial function. Notice that, if $f(\cdot,0)\neq 0$, the trivial function does not solve equation (1.7) and so, obviously, $u_\mu\not\equiv 0$.

4.1. **Proof of Theorem 4.** The idea of the proof consists in applying [4, Theorem 2.1; part $a$)]) (recalled in Subsection 2.2) to the functional $\mathcal{J}_{\lambda,\gamma,\mu}$ .

To this purpose, we write the functional $\mathcal{J}_{\lambda,\gamma,\mu}$ as follows:

$$\mathcal{J}_{\lambda,\gamma,\mu}(u)=\Phi_{\lambda,\gamma}(u)-\mu\Psi(u),\quad u\in X_0,$$

where $\Phi_{\lambda,\gamma}$ is the functional given in (1.1), while

$$\Psi(u):=\int_\Omega F(x,u(x))\,dx.$$

First of all, note that $X_0$ is a Hilbert space (see [33, Lemma 7]) and the functionals $\Phi_{\lambda,\gamma}$ and $\Psi$ are Fréchet differentiable in $X_0$.

Also, by Theorem 1 the map

$$u\mapsto\Phi_{\lambda,\gamma}(u)$$

is lower semicontinuous in the weak topology of $X_0$ for any $\lambda\in\mathbb{R}$ and $\gamma\in[0,S_s]$.

Now, we claim that

$$\text{the map }\ u\mapsto\int_\Omega F(x,u(x))dx\ \text{ is continuous in the weak topology of }\ X_0. \tag{4.1}$$

For this, let $u_j$ be a sequence in $X_0$ such that $u_j\rightharpoonup u$ weakly in $X_0$ as $j\to+\infty$. Then, $u_j$ is bounded in $X_0$ and so, as a consequence of (2.7), it is bounded in $L^{2^*}(\mathbb{R}^n)$, that is there exists $\kappa>0$ such that

$$\|u_j\|^{2^*}_{L^{2^*}(\Omega)}\leqslant\kappa,\quad\text{for any } j\in\mathbb{N}. \tag{4.2}$$

Here we used also the fact that $u_j=0$ outside $\Omega$.

Since $L^{2^*}(\mathbb{R}^n)$ is a reflexive space we have that, up to a subsequence

$$u_j\rightharpoonup u_\infty\quad\text{weakly in } L^{2^*}(\mathbb{R}^n) \tag{4.3}$$

as $j\to+\infty$, while by [33, Lemma 8], up to a subsequence,

$$u_j\to u_\infty\quad\text{in } L^{\nu}(\mathbb{R}^n) \tag{4.4}$$

and

$$u_j\to u_\infty\quad\text{a.e. in } \mathbb{R}^n \tag{4.5}$$

as $j\to+\infty$ for any $\nu\in[1,2^*)$ (see, for instance [7, Theorem IV.9]).

Now, let us fix $\delta>0$. By (2.9) (used here with $\varepsilon=\delta/(2\kappa)$, we have that

$$|F(x,u_j(x))|\leqslant\frac{\delta}{2\kappa}\,|u_j(x)|^{2^*}+M(\delta/(2\kappa))|u_j(x)|,\ \text{ a.e. in }\Omega. \tag{4.6}$$

Then, since $X_0$ is embedded in $L^{2^*}(\Omega)$ and in $L^1(\Omega)$, we have that $F(\cdot,u_j(\cdot))\in L^1(\Omega)$ and, putting

$$\eta(\delta):=\left(\frac{\delta}{2\kappa^{1/2^*}M(\delta/(2\kappa))}\right)^{2^*/(2^*-1)},$$

for any measurable subset $\Omega'$ of $\Omega$ such that

$$\text{meas}(\Omega')\leqslant\eta(\delta), \tag{4.7}$$

we get

$$
\begin{aligned}
\int_{\Omega'}|F(x,u_j(x))|\,dx &\leqslant \frac{\delta}{2\kappa}\int_{\Omega'}|u_j(x)|^{2^*}\,dx + M(\delta/(2\kappa))\int_{\Omega'}|u_j(x)|\,dx\\
&\leqslant \frac{\delta}{2\kappa}\,\|u_j\|^{2^*}_{L^{2^*}(\Omega)} + M(\delta/(2\kappa))\|u_j\|_{L^1(\Omega')}\\
&\leqslant \frac{\delta}{2} + M(\delta/(2\kappa))\,\mathrm{meas}(\Omega')^{(2^*-1)/2^*}\,\|u_j\|_{L^{2^*}(\Omega)}\\
&\leqslant \frac{\delta}{2} + M(\delta/(2\kappa))\eta(\delta)^{(2^*-1)/2^*}\kappa^{1/2^*}\\
&= \delta,
\end{aligned}
\tag{4.8}
$$

thanks to (4.2), (4.6), (4.7) and Hölder inequality. Hence, $F(\cdot,u_j(\cdot))$ is uniformly integrable[2] in $\Omega$. Moreover, by (4.5) and the fact that the map $t\mapsto F(\cdot,t)$ is continuous in $t\in\mathbb{R}$, we get

$$
F(\cdot,u_j(\cdot))\to F(\cdot,u(\cdot)),\quad \text{a.e. in}\quad \Omega \tag{4.9}
$$

as $j\to+\infty$. Thus, (4.8) and (4.9) and the Vitali Convergence Theorem yield

$$
\int_\Omega F(x,u_j(x)\,dx \to \int_\Omega F(x,u(x)\,dx,
$$

as $j\to+\infty$. Then, the claim (4.1) is proved.

Hence, the functionals $\Phi_{\lambda,\gamma}$ and $\Psi$ have the regularity required by [4, Theorem 2.1; part $a$)] (see Subsection 2.2). Also, arguing as in the proof of Corollary 2 (see formula (3.17)), we get that

$$
\Phi_{\lambda,\gamma}(u)\geqslant \frac{1}{2}\Big(m_\lambda^2-\frac{\gamma}{S_s}\Big)\|u\|^2_{X_0},
$$

that is $\Phi_{\lambda,\gamma}$ is coercive in $X_0$ and $\inf\limits_{u\in X_0}\Phi_{\lambda,\gamma}(u)=0$, provided $\lambda<\lambda_{1,s}$ and $\gamma\in[0,\gamma_\lambda)$, with $\gamma_\lambda=S_s\min\big\{1,\lambda/\lambda_{1,s}\big\}$.

From now on, let us fix $\lambda<\lambda_{1,s}$ and $\gamma\in[0,\gamma_\lambda)$. Let also $r>0$ and $\varphi_{\lambda,\gamma}$ be the function defined as follows

$$
\varphi_{\lambda,\gamma}(r):=\inf_{u\in\Phi^{-1}_{\lambda,\gamma}\big((-\infty,r)\big)}\frac{\displaystyle\sup_{v\in\Phi^{-1}_{\lambda,\gamma}\big((-\infty,r)\big)}\Psi(v)-\Psi(u)}{r-\Phi_{\lambda,\gamma}(u)}. \tag{4.10}
$$

It is easy to see that $\varphi_{\lambda,\gamma}(r)\geqslant 0$ for any $r>0$.

By [4, Theorem 2.1; part $a$)],

$$
\begin{gathered}
\text{for any } r>0 \text{ and any } \mu\in\Big(0,1/\varphi_{\lambda,\gamma}(r)\Big) \text{ the restriction}\\
\text{of } \mathcal{J}_{\lambda,\gamma,\mu} \text{ to } \Phi^{-1}_{\lambda,\gamma}\big((-\infty,r)\big) \text{ admits a global minimum } u_{\mu,r},\\
\text{which is a critical point (namely a local minimum) of } \mathcal{J}_{\lambda,\gamma,\mu} \text{ in } X_0.
\end{gathered}
\tag{4.11}
$$

Remember that, when $\varphi_{\lambda,\gamma}(r)=0$, by $1/\varphi_{\lambda,\gamma}(r)$ we mean $+\infty$.

Let $\mu_\lambda$ be defined as follows

$$
\mu_\lambda:=\sup_{r>0}\frac{1}{\varphi_{\lambda,\gamma}(r)}.
$$

Note that $\mu_\lambda>0$, since $\varphi_{\lambda,\gamma}(r)\geqslant 0$ for any $r>0$.

Now, let us fix $\bar\mu\in(0,\mu_\lambda)$. First of all, thanks to the definition of $\mu_\lambda$, it is easy to see that

$$
\text{there exists } \bar r_{\bar\mu}>0 \text{ such that } \bar\mu\leqslant 1/\varphi_{\lambda,\gamma}(\bar r_{\bar\mu}). \tag{4.12}
$$

[2]Or, according to the different terminologies, absolutely continuous in $\Omega$, uniformly with respect to $j\in\mathbb{N}$.

Then, by (4.11) applied with $r = \bar{r}_{\bar{\mu}}$, we have that for any $\mu$ such that

$$0 < \mu < \bar{\mu} \leqslant 1/\varphi_{\lambda,\gamma}(\bar{r}_{\bar{\mu}}),$$

the function

$$u_\mu := u_{\mu,\bar{r}_{\bar{\mu}}}$$

is a global minimum of the functional $\mathcal{J}_{\lambda,\gamma,\mu}$ restricted to $\Phi_{\lambda,\gamma}^{-1}\big((-\infty, \bar{r}_{\bar{\mu}})\big)$, i.e.

$$\mathcal{J}_{\lambda,\gamma,\mu}(u_\mu) \leqslant \mathcal{J}_{\lambda,\gamma,\mu}(u), \quad \text{for any } u \in X_0 \text{ such that } \Phi_{\lambda,\gamma}(u) < \bar{r}_{\bar{\mu}} \tag{4.13}$$

and

$$\Phi_{\lambda,\gamma}(u_\mu) < \bar{r}_{\bar{\mu}}\,, \tag{4.14}$$

and also $u_\mu$ is a critical point of $\mathcal{J}_{\lambda,\gamma,\mu}$ in $X_0$ and so it is a weak solution of problem (1.7).

In this way we have shown that for any $\lambda < \lambda_{1,s}$, any $\gamma \in [0, \gamma_\lambda)$ and any $\mu \in (0, \mu_\lambda)$, problem (1.7) admits a weak solution $u_\mu \in X_0$.

Now, it remains to show that $u_\mu$ is not the trivial function. Of course, when $f(\cdot, 0) \neq 0$, it easily follows that $u_\mu \not\equiv 0$ in $X_0$, since the trivial function does not solve problem (1.7).

Let us consider the case when $f(\cdot, 0) = 0$ and let us fix $\bar{\mu} \in (0, \mu_\lambda)$ and $\mu \in (0, \bar{\mu})$ and let $u_\mu$ be as in (4.13) and (4.14). In this setting, in order to prove that $u_\mu \not\equiv 0$ in $X_0$, first we claim that there exists a sequence $w_j$ in $X_0$ such that

$$\limsup_{j \to +\infty} \frac{\Psi(w_j)}{\Phi_{\lambda,\gamma}(w_j)} = +\infty. \tag{4.15}$$

By the assumption on the limsup in (1.10) there exists a sequence $\xi_j$ in $\mathbb{R}^+$ such that $\xi_j \to 0^+$ as $j \to +\infty$ and

$$\lim_{j \to +\infty} \frac{\operatorname{essinf}_{x \in B} F(x, \xi_j)}{\xi_j^2} = +\infty, \tag{4.16}$$

namely, we have that for any $M > 0$ and $j$ sufficiently large

$$\operatorname{essinf}_{x \in B} F(x, \xi_j) > M\xi_j^2. \tag{4.17}$$

Now, let $C$ be a set of positive Lebesgue measure such that $C \subset B$ and let $v \in X_0$ be a function such that

*i)* $v(x) \in [0, 1]$ for every $x \in \mathbb{R}^n$;
*ii)* $v(x) = 1$ for every $x \in C$;
*iii)* $v(x) = 0$ for every $x \in \Omega \setminus D$.

Of course $C$ exists since $B$ has positive Lebesgue measure, while the function $v$ exists thanks to the fact that $C_0^2(\Omega) \subseteq X_0$ (see [32, Lemma 11]).

Finally, let $w_j := \xi_j v$ for any $j \in \mathbb{N}$. It is easily seen that $w_j \in X_0$ for any $j \in \mathbb{N}$ (actually, $w_j \in C_0^2(\Omega)$ if $v$ does). Furthermore, taking into account the properties of $v$

stated in $i$)–$iii$), the fact that $C \subset B \subseteq D \subseteq \Omega$ and $F(\cdot, 0) = 0$, and (4.17) we have

$$
\begin{aligned}
\frac{\Psi(w_j)}{\Phi_{\lambda,\gamma}(w_j)} &= \frac{\displaystyle\int_\Omega F(x, w_j(x))\,dx}{\Phi_{\lambda,\gamma}(w_j)} \\
&= \frac{\displaystyle\int_C F(x, w_j(x))\,dx + \int_{D\setminus C} F(x, w_j(x))\,dx}{\Phi_{\lambda,\gamma}(w_j)} \\
&= \frac{\displaystyle\int_C F(x, \xi_j)\,dx + \int_{D\setminus C} F(x, \xi_j v(x))\,dx}{\Phi_{\lambda,\gamma}(w_j)} \\
&\geqslant \frac{\displaystyle M\,\mathrm{meas}(C)\xi_j^2 + \int_{D\setminus C} F(x, \xi_j v(x))\,dx}{\Phi_{\lambda,\gamma}(w_j)}\,,
\end{aligned} \tag{4.18}
$$

for $j$ sufficiently large.

Moreover, note that by definition of $\Phi_{\lambda,\gamma}$ and the non-negativity of $\gamma$ we get

$$
\Phi_{\lambda,\gamma}(w_j) = \frac{1}{2}\|w_j\|^2_{X_0,\lambda} - \frac{\gamma}{2}\|w_j\|^2_{L^{2^*}(\Omega)} \leqslant \frac{1}{2}\|w_j\|^2_{X_0,\lambda}\,, \tag{4.19}
$$

so that, thanks to this, the definition of $w_j$ and (4.18) give

$$
\frac{\Psi(w_j)}{\Phi_{\lambda,\gamma}(w_j)} \geqslant \frac{\displaystyle 2M\,\mathrm{meas}(C)\xi_j^2 + 2\int_{D\setminus C} F(x, \xi_j v(x))\,dx}{\xi_j^2\|v\|^2_{X_0,\lambda}}. \tag{4.20}
$$

Now we have to distinguish two different cases, i.e. the case when the liminf in (1.10) is $+\infty$ (and so the liminf is actually a limit) and the one in which the liminf in (1.10) is finite.

**Case 1:** $\displaystyle\lim_{t\to 0^+} \frac{\mathrm{essinf}_{x\in D} F(x,t)}{t^2} = +\infty$.

Then, there exists $\rho_M > 0$ such that

$$
\mathrm{essinf}_{x\in D} F(x,t) \geqslant Mt^2, \tag{4.21}
$$

for any $0 < t < \rho_M$.

Since $\xi_j \to 0^+$ and $0 \leqslant v \leqslant 1$ in $\Omega$, then $w_j(x) = \xi_j v(x) \to 0^+$ as $j \to +\infty$ uniformly in $x \in \Omega$. Hence, $0 \leqslant w_j(x) < \rho_M$ for $j$ sufficiently large and for any $x \in \Omega$. Hence, as a consequence of (4.20) and (4.21) (used here with $t = w_j(x)$, $j$ large), we deduce that

$$
\begin{aligned}
\frac{\Psi(w_j)}{\Phi_{\lambda,\gamma}(w_j)} &\geqslant \frac{\displaystyle 2M\,\mathrm{meas}(C)\xi_j^2 + 2\int_{D\setminus C} F(x, \xi_j v(x))\,dx}{\xi_j^2\|v\|^2_{X_0,\lambda}} \\
&\geqslant \frac{\displaystyle 2M\,\mathrm{meas}(C)\xi_j^2 + 2M\xi_j^2\int_{D\setminus C} v^2(x)\,dx}{\xi_j^2\|v\|^2_{X_0,\lambda}} \\
&= \frac{\displaystyle 2M\,\mathrm{meas}(C) + 2M\int_{D\setminus C} v^2(x)\,dx}{\|v\|^2_{X_0,\lambda}},
\end{aligned}
$$

for $j$ sufficiently large. The arbitrariness of $M$ gives (4.15) and so the claim is proved.

**Case 2:** $\displaystyle\liminf_{t\to 0^+} \frac{\mathrm{essinf}_{x\in D} F(x,t)}{t^2} = \ell \in \mathbb{R}$.

Then, for any $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ such that for any $t$ with $0 < t < \rho_\varepsilon$

$$
\mathrm{essinf}_{x\in D} F(x,t) \geqslant (\ell - \varepsilon)t^2. \tag{4.22}
$$

Arguing as above, we can suppose that $0 \leqslant w_j(x) = \xi_j v(x) < \rho_\varepsilon$ for $j$ large enough and any $x \in \Omega$. Thus, by (4.20) and (4.22) (used with $t = \xi_j v(x)$ with $j$ large) we get

$$\begin{aligned} \frac{\Psi(w_j)}{\Phi_{\lambda,\gamma}(w_j)} &\geqslant \frac{2M\,\mathrm{meas}(C)\xi_j^2 + 2\displaystyle\int_{D\setminus C} F(x,\xi_j v(x))\,dx}{\xi_j^2\|v\|^2_{X_0,\lambda}} \\ &\geqslant \frac{2M\,\mathrm{meas}(C)\xi_j^2 + 2(\ell-\varepsilon)\xi_j^2\displaystyle\int_{D\setminus C} v^2(x)\,dx}{\xi_j^2\|v\|^2_{X_0,\lambda}} \\ &= \frac{2M\,\mathrm{meas}(C) + 2(\ell-\varepsilon)\displaystyle\int_{D\setminus C} v^2(x)\,dx}{\|v\|^2_{X_0,\lambda}}, \end{aligned} \tag{4.23}$$

provided $j$ is sufficiently large.

Choosing $M > 0$ large enough, say

$$M\,\mathrm{meas}(C) > \max\Big\{0, -2\ell\int_{D\setminus C} v^2(x)\,dx\Big\},$$

and $\varepsilon > 0$ small enough so that

$$\varepsilon\int_{D\setminus C} v^2(x)\,dx < \frac{M\,\mathrm{meas}(C)}{2} + \ell\int_{D\setminus C} v^2(x)\,dx,$$

by (4.23) we get

$$\begin{aligned} \frac{\Psi(w_j)}{\Phi_{\lambda,\gamma}(w_j)} &\geqslant \frac{2M\,\mathrm{meas}(C) + 2(\ell-\varepsilon)\displaystyle\int_{D\setminus C} v^2(x))\,dx}{\|v\|^2_{X_0,\lambda}} \\ &\geqslant \frac{2}{\|v\|^2_{X_0,\lambda}}\left(M\,\mathrm{meas}(C) + \ell\int_{D\setminus C} v^2(x))\,dx - M\,\mathrm{meas}(C)/2 - \ell\int_{D\setminus C} v^2(x)\,dx\right) \\ &= \frac{M\,\mathrm{meas}(C)}{\|v\|^2_{X_0,\lambda}}, \end{aligned}$$

for $j$ large enough. Also in this case the arbitrariness of $M$ gives assertion (4.15).

Now, note that

$$\|w_j\|_{X_0,\lambda} = |\xi_j|\,\|v\|_{X_0,\lambda} \to 0,$$

as $j \to +\infty$, so that for $j$ large enough

$$\|w_j\|_{X_0,\lambda} < \sqrt{2\bar r_{\bar\mu}},$$

where $\bar r_{\bar\mu}$ is given in (4.12). As a consequence of this and (4.19) we get that $\Phi_{\lambda,\gamma}(w_j) < \bar r_{\bar\mu}$, that is

$$w_j \in \Phi^{-1}_{\lambda,\gamma}\big((-\infty, \bar r_{\bar\mu})\big), \tag{4.24}$$

provided $j$ is large enough. Also, by (4.15) and the fact that $\mu > 0$

$$\mathcal{J}_{\lambda,\gamma,\mu}(w_j) = \Phi_{\lambda,\gamma}(w_j) - \mu\Psi(w_j) < 0, \tag{4.25}$$

for $j$ sufficiently large.

Since $u_\mu$ is a global minimum of the restriction of $\mathcal{J}_{\lambda,\gamma,\mu}$ to $\Phi^{-1}_{\lambda,\gamma}\big((-\infty, \bar r_\mu)\big)$ (see (4.13)), by (4.24) and (4.25) we conclude that

$$\mathcal{J}_{\lambda,\gamma,\mu}(u_\mu) \leqslant \mathcal{J}_{\lambda,\gamma,\mu}(w_j) < 0 = \mathcal{J}_{\lambda,\gamma,\mu}(0), \tag{4.26}$$

so that $u_\mu \not\equiv 0$ in $X_0$. Thus, $u_\mu$ is a non-trivial weak solution of problem (1.7). The arbitrariness of $\mu$ and $\bar\mu$ gives that $u_\mu \not\equiv 0$ for any $\mu \in (0, \mu_\lambda)$.

Moreover, from (4.26) we get that the map

$$(0,\mu_\lambda)\ni\mu\mapsto \mathcal{J}_{\lambda,\gamma,\mu}(u_\mu)\ \text{ is negative.} \tag{4.27}$$

Now, we claim that $\lim\limits_{\mu\to 0^+}\|u_\mu\|_{X_0}=0$. For this, first note that, arguing as in (3.17), we have

$$\frac{1}{2}\left(m_\lambda^2-\frac{\gamma}{S_s}\right)\|u_\mu\|_{X_0}^2\leqslant \Phi_{\lambda,\gamma}(u_\mu), \tag{4.28}$$

so that, using (4.14), we get

$$\frac{1}{2}\left(m_\lambda^2-\frac{\gamma}{S_s}\right)\|u_\mu\|_{X_0}^2< \bar r_{\bar\mu},$$

that is

$$\|u_\mu\|_{X_0}<\kappa_\lambda,$$

where $\kappa_\lambda$ is a suitable positive constant depending on $\lambda$ (note that $\gamma\in[0,\gamma_\lambda)$ and so it depends on $\lambda$), but independent of $\mu$.

As a consequence of this and by using Lemma 6 (see formula (2.8) with $\varepsilon=1$), together with the embedding properties (2.6) and (2.7) and the fact that $u_\mu\in X_0$, it follows that

$$\begin{aligned}\left|\int_\Omega f(x,u_\mu(x))u_\mu(x)dx\right| &\leqslant 2^*\|u_\mu\|_{L^{2^*}(\Omega)}^{2^*}+M(1)\|u_\mu\|_{L^1(\Omega)}\\ &\leqslant \frac{2^*}{S_s^{2^*/2}}\|u_\mu\|_{X_0}^{2^*}+M(1)c_1\|u_\mu\|_{X_0}\\ &< \frac{2^*}{S_s^{2^*/2}}\,\kappa_\lambda^{2^*}+M(1)c_1\,\kappa_\lambda=:M_\lambda.\end{aligned} \tag{4.29}$$

Since $u_\mu$ is a critical point of $\mathcal{J}_{\lambda,\gamma,\mu}$, then $\langle \mathcal{J}'_{\lambda,\gamma,\mu}(u_\mu),\varphi\rangle=0$, for any $\varphi\in X_0$ and every $\mu\in(0,\bar\mu)$. In particular $\langle \mathcal{J}'_{\lambda,\gamma,\mu}(u_\mu),u_\mu\rangle=0$, that is

$$\langle \Phi'_{\lambda,\gamma}(u_\mu),u_\mu\rangle=\mu\int_\Omega f(x,u_\mu(x))u_\mu(x)dx, \tag{4.30}$$

for every $\mu\in(0,\bar\mu)$.

Then, (4.28)–(4.30) yield

$$\begin{aligned}0&<\left(m_\lambda^2-\frac{\gamma}{S_s}\right)\|u_\mu\|_{X_0}^2\leqslant 2\Phi_{\lambda,\gamma}(u_\mu)\\ &=\langle \Phi'_{\lambda,\gamma}(u_\mu),u_\mu\rangle=\mu\int_\Omega f(x,u_\mu(x))u_\mu(x)\,dx<\mu\,M_\lambda,\end{aligned}$$

for any $\mu\in(0,\bar\mu)$. Letting $\mu\to 0^+$, we get $\lim\limits_{\mu\to 0^+}\|u_\mu\|_{X_0}=0$, as claimed.

Finally, we have to show that the map

$$\mu\mapsto \mathcal{J}_{\lambda,\gamma,\mu}(u_\mu)\ \text{ is strictly decreasing in }\ (0,\mu_\lambda).$$

For this we observe that for any $u\in X_0$

$$\mathcal{J}_{\lambda,\gamma,\mu}(u)=\mu\left(\frac{\Phi_{\lambda,\gamma}(u)}{\mu}-\Psi(u)\right). \tag{4.31}$$

Now, let us fix $0<\mu_1<\mu_2<\bar\mu<\mu_\lambda$ and let $u_{\mu_i}$ be the global minimum of the functional $\mathcal{J}_{\lambda,\gamma,\mu_i}$ restricted to $\Phi^{-1}_{\lambda,\gamma}\big((-\infty,\bar r_{\bar\mu})\big)$ for $i=1,2$ (for this see (4.13)). Also, let

$$m_{\mu_i}:=\left(\frac{\Phi_{\lambda,\gamma}(u_{\mu_i})}{\mu_i}-\Psi(u_{\mu_i})\right)=\inf_{v\in\Phi^{-1}_{\lambda,\gamma}\left((-\infty,\bar r_{\bar\mu})\right)}\left(\frac{\Phi_{\lambda,\gamma}(v)}{\mu_i}-\Psi(v)\right)\quad i=1,2\,.$$

Clearly, (4.27), (4.31) and the positivity of $\mu$ imply that

$$m_{\mu_i}<0\quad\text{for}\ \ i=1,2. \tag{4.32}$$

Moreover,

$$m_{\mu_2} \leqslant m_{\mu_1}, \tag{4.33}$$

thanks to the fact that $0 < \mu_1 < \mu_2$ and $\Phi_{\lambda,\gamma} \geqslant 0$ by (3.17). Then, by (4.31)–(4.33) and again by the fact that $0 < \mu_1 < \mu_2$, we get that

$$\mathcal{J}_{\lambda,\gamma,\mu_2}(u_{\mu_2}) = \mu_2 m_{\mu_2} \leqslant \mu_2 m_{\mu_1} < \mu_1 m_{\mu_1} = \mathcal{J}_{\lambda,\gamma,\mu_1}(u_{\mu_1}),$$

so that the map $\mu \mapsto \mathcal{J}_{\lambda,\gamma,\mu}(u_\mu)$ is strictly decreasing in $(0,\bar{\mu})$. The arbitrariness of $\bar{\mu} < \mu_\lambda$ shows that $\mu \mapsto \mathcal{J}_{\lambda,\gamma,\mu}(u_\mu)$ is strictly decreasing in $(0,\mu_\lambda)$. This concludes the proof of Theorem 4.

4.2. **Existence of a non-negative solution.** In this subsection we show that, as in the classical case of the Laplacian, it is possible to prove that the solution of problem (1.7) given by Theorem 4 has constant sign.

*Proof of Corollary 5:* Here we can argue as in [23, Corollary 3] (where the subcritical case was considered), with minor corrections related to the presence of the critical term. We prefer to repeat here all the calculations, just for the sake of clarity and in order to make the paper self contained.

As usual, our proof is based on a truncation argument. Let $F_+$ and $f_+$ be the functions defined as

$$F_+(x,t) := \int_0^t f_+(x,\tau)d\tau,$$

with

$$f_+(x,t) := \begin{cases} f(x,t) & \text{if } t \geqslant 0 \\ 0 & \text{if } t < 0 \end{cases}$$

for a.e. $x \in \Omega$ and $t \in \mathbb{R}$.

First of all, note that both $f_+$ and $F_+$ are well defined a.e. $x \in \Omega$ and $t \in \mathbb{R}$. Furthermore, since $f(\cdot,0) = 0$, then $f_+$ is a Carathéodory function in $\Omega \times \mathbb{R}$ and so $t \mapsto F_+(\cdot,t)$ is differentiable in $\mathbb{R}$. Moreover, it is easily seen that $f_+$ and $F_+$ satisfy conditions (1.8), (1.9) and (1.10), respectively.

Let $\mathcal{J}^+_{\lambda,\gamma,\mu} : X_0 \to \mathbb{R}$ be the functional defined as follows

$$\mathcal{J}^+_{\lambda,\gamma,\mu}(u) := \Phi_{\lambda,\gamma}(u) - \mu\Psi_+(u),$$

with

$$\Psi_+(u) := \int_\Omega F_+(x,u(x))\,dx.$$

It is easy to see that the functional $\Psi_+$ is well defined, is Fréchet differentiable at any $u \in X_0$ (being $F_+$ differentiable in $\mathbb{R}$) and has the regularity properties required by [4, Theorem 2.1; part $a$)] (see Subsection 2.2). Also, for any $\varphi \in X_0$

$$\begin{aligned} \langle (\mathcal{J}^+_{\lambda,\gamma,\mu})'(u),\varphi\rangle = \int_{\mathbb{R}^n\times\mathbb{R}^n} & \frac{\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\ & - \gamma\Big(\int_\Omega |u(x)|^{2^*}\,dx\Big)^{2/2^*-1}\int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)\,dx \\ & - \mu\int_\Omega f_+(x,u(x))\varphi(x)\,dx. \end{aligned} \tag{4.34}$$

Hence, by [4, Theorem 2.1; part $a$)], there exists a critical point $u_+ \in X_0$ of $\mathcal{J}^+_{\lambda,\gamma,\mu}$, provided $\lambda < \lambda_{1,s}$, $\gamma \in [0,\gamma_\lambda)$ and $\mu \in (0,\mu_\lambda)$, for a suitable $\mu_\lambda > 0$.

Also $u_+ \not\equiv 0$ in $X_0$. Indeed, since $f(\cdot,0) = 0$, also $f_+(\cdot,0) = 0$ and so, in order to prove that $u_+ \not\equiv 0$, we can argue exactly as in the proof of Theorem 4, jus replacing $f$ with $f_+$, $F$ with $F_+$ and $\Psi$ with $\Psi_+$ in formulas (4.15)–(4.26).

We claim that $u_+$ is non-negative in $\mathbb{R}^n$. For this we take $\varphi := (u_+)^-$ in (4.34), where $v^-$ is the negative part of $v$, i.e. $v^- := \max\{-v,0\}$. We remark that, since $u_+ \in X_0$, we have that $(u_+)^- \in X_0$, by [32, Lemma 12], and so the choice of such $\varphi$ is admissible. In this way, since $u_+$ is a critical point of $\mathcal{J}^+_{k,\lambda,\mu}$, we get

$$
\begin{aligned}
0 &= \langle (\mathcal{J}^+_{\lambda,\gamma,\mu})'(u_+), (u_+)^- \rangle \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big(u_+(x)-u_+(y)\big)\big((u_+)^-(x)-(u_+)^-(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\
&\qquad - \lambda \int_\Omega u_+(x)(u_+)^-(x)\,dx \\
&\qquad - \gamma\Big(\int_\Omega |u_+(x)|^{2^*}\,dx\Big)^{2/2^*-1}\int_\Omega |u_+(x)|^{2^*-2}u_+(x)(u_+)^-(x)\,dx \\
&\qquad - \mu\int_\Omega f_+(x,u_+(x))(u_+)^-(x)\,dx \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big(u_+(x)-u_+(y)\big)\big((u_+)^-(x)-(u_+)^-(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\
&\qquad - \lambda\int_\Omega \big|(u_+)^-(x)\big|^2\,dx \\
&\qquad - \gamma\Big(\int_\Omega |u_+(x)|^{2^*}\,dx\Big)^{2/2^*-1}\int_\Omega |(u_+)^-(x)|^{2^*}\,dx,
\end{aligned}
\tag{4.35}
$$

thanks to the definition of $f_+$ and of negative part.

Now, we claim that for any $w \in X_0$ the following relation holds true a.e. $x, y \in \mathbb{R}^n$

$$
\big(w(x)-w(y)\big)\big(w^-(x)-w^-(y)\big) \leqslant -\big|w^-(x)-w^-(y)\big|^2. \tag{4.36}
$$

Indeed, writing $w = w^+ - w^-$ and taking into account that

$$
w^+(x)w^-(x) = 0 \ \text{ and } \ w^+(x)w^-(y) \geqslant 0, \quad \text{a.e. } x, y \in \mathbb{R}^n,
$$

we get

$$
\begin{aligned}
(w(x)-w(y))(w^-(x)-w^-(y)) &= (w^+(x)-w^+(y))(w^-(x)-w^-(y)) - (w^-(x)-w^-(y))^2 \\
&= -w^+(x)w^-(y) - w^+(y)w^-(x) - (w^-(x)-w^-(y))^2 \\
&\leqslant -\big|w^-(x)-w^-(y)\big|^2,
\end{aligned}
$$

a.e. $x, y \in \mathbb{R}^n$. Hence, the claim (4.36) is proved.

Thus, by (4.35) and (4.36) applied here with $w = u_+$, and the fact that $\gamma \geqslant 0$, we obtain

$$
\begin{aligned}
0 &= \langle (\mathcal{J}^+_{\lambda,\gamma,\mu})'(u_+), (u_+)^- \rangle \\
&\leqslant -\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big|(u_+)^-(x)-(u_+)^-(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega \big|(u_+)^-(x)\big|^2\,dx \\
&= -\|(u_+)^-\|^2_{X_0} - \lambda\|(u_+)^-\|^2_{L^2(\Omega)} \\
&\leqslant -\kappa_\lambda \|(u_+)^-\|^2_{X_0},
\end{aligned}
$$

where $\kappa_\lambda := \max\big\{1, 1+\lambda/\lambda_{1,s}\big\} > 0$. Hence, $\|(u_+)^-\|_{X_0} = 0$, so that $(u_+)^- \equiv 0$ a.e. in $\mathbb{R}^n$, that is $u_+ \geqslant 0$ a.e. in $\mathbb{R}^n$. The assertion is proved. □

4.3. **Final comments.** As a remark we would like to note that, if we replace condition (1.10) with the following one

$$
\begin{gathered}
\text{there exist a non-empty open set } D \subseteq \Omega \text{ and a set } B \subseteq D \\
\text{of positive Lebesgue measure such that} \\
\limsup_{t\to 0^-} \frac{\operatorname{essinf}_{x\in B} F(x,t)}{t^2} = +\infty \quad \text{and} \quad \liminf_{t\to 0^-} \frac{\operatorname{essinf}_{x\in D} F(x,t)}{t^2} > -\infty,
\end{gathered}
$$

arguing as in Theorem 4, it is to show that problem (1.7) admits a non-trivial weak solution.

In this case this solution is non-positive in $\mathbb{R}^n$, provided $f(\cdot,0)=0$. To this purpose, it is enough to consider the functional

$$\mathcal{J}^-_{\lambda,\gamma,\mu}(u) := \Phi_{\lambda,\gamma}(u) - \mu\Psi_-(u)\,,\ \ u\in X_0$$

with

$$\Psi_-(u) := \int_\Omega F_-(x,u(x))\,dx,$$

and

$$F_-(x,t) := \int_0^t f_-(x,\tau)d\tau\,, \qquad f_-(x,t) := \begin{cases} 0 & \text{if } t>0 \\ f(x,t) & \text{if } t\leqslant 0 \end{cases}$$

a.e. $x\in\Omega$ and $t\in\mathbb{R}$, and argue as in the proof of Corollary 5.

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