Phaseless inverse scattering problems in 3-d

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1 Introduction

Consider the Schrödinger equation in $\mathbb{R}^3$ with the compactly supported potential $q(x), x \in \mathbb{R}^3$. The problem of the reconstruction of the function $q(x)$ from measurements of the solution of that equation on a certain set is called "inverse scattering problem". In this paper we prove uniqueness theorems for some 3-d inverse scattering problems in the case when only the modulus of the complex valued wave field is measured, while the phase is unknown. This is the phaseless case. In the past, phaseless inverse scattering problems were studied only in the 1-d case (section 1.2). As to the 3-d inverse scattering problems in the frequency domain, it was assumed in all studies so far that both the modulus and the phase of the complex valued wave field are measured, see, e.g. [2] for uniqueness results in the case of a piecewise analytic potential and [26, 27] for global uniqueness results and reconstruction methods.

Below $C^{s+\alpha}$ are Hölder spaces, where $s \geq 0$ is an integer and $\alpha \in (0,1)$. Let $\Omega, G \subset \mathbb{R}^3$ be two bounded domains, $\Omega \subset G$. For an arbitrary point $y \in \mathbb{R}^3$ and for an arbitrary number $\omega \in (0,1)$ denote $B_\omega(y) = \{x : |x-y| < \omega\}$ and $P_\omega(y) = \mathbb{R}^3 \diagdown B_\omega(y)$. For any two sets $M, N \subset \mathbb{R}^3$ let $dist(M,N)$ be the Hausdorff distance between them. Let $G_1 \subset \mathbb{R}^3$ be a convex bounded domain with its boundary $S \in C^1$. Let $\varepsilon \in (0,1)$ be a number. We assume that $\Omega \subset G_1 \subset G$, $dist(S, \partial G) > 2\varepsilon$ and $dist(S, \partial\Omega) > 2\varepsilon$. Hence,

$$dist(\partial B_\varepsilon(y), \partial\Omega) > \varepsilon, \forall y \in S, \quad (1)$$

$$dist(\partial B_\varepsilon(y), \partial G) > \varepsilon, \forall y \in S. \quad (2)$$

Below either $m = 2$ or $m = 4$, and we will specify this later. We impose the following conditions on the potential $q(x)$

$$q(x) \in C^m\left(\mathbb{R}^3\right), q(x) = 0 \text{ for } x \in \mathbb{R}^3 \diagdown G, \quad (3)$$

$$q(x) \geq 0. \quad (4)$$

As a rule, the minimal smoothness of unknown coefficients is not the first priority of proofs of uniqueness theorems of multidimensional coefficient inverse problems, see, e.g. [26, 27] and Theorem 4.1 in [29]. Since our proofs require either C^2 or C^4 smoothness of solutions of Cauchy problems for some hyperbolic equations, we are not concerned below with minimal smoothness assumptions. In particular, the reason of imposing C^4 (rather than C^2) smoothness conditions in Theorems 3 and 4 is rooted in smoothness requirements of uniqueness theorems of [4, 5, 6, 19, 20, 21], which we use here.

1.1 One of main results

We now formulate one of our four main theorems. Three other theorems are formulated in section 2. Let $x_0 = (x_{0,1}, x_{0,2}, x_{0,3})$ be the source position. Consider the following problem

$$\Delta_x u + k^2 u - q(x) u = -\delta(x - x_0), x \in \mathbb{R}^3, \tag{5}$$

$$u(x, x_0, k) = O\left(\frac{1}{|x - x_0|}\right), |x| \to \infty, \tag{6}$$

$$\sum_{j=1}^{3} \frac{x_j - x_{j,0}}{|x - x_0|} \partial_{x_j} u(x, x_0, k) - iku(x, x_0, k) = o\left(\frac{1}{|x - x_0|}\right), |x| \to \infty. \tag{7}$$

Here the radiation conditions (6), (7) are valid for every fixed source position x_0. To establish existence and uniqueness of the solution of the problem (5)-(7), we refer to Theorem 6 of Chapter 9 of the book [32] as well as to Theorem 3.3 of the paper [31]. As to the smoothness of the solution of the problem (5)-(7), we refer to Theorem 6.17 of the book [12]. Thus, combining these results, we obtain that for each pair $(k, x_0) \in \mathbb{R} \times \mathbb{R}^3$ there exists unique solution $u(x, x_0, k)$ of the problem (5), (6), (7) such that

$$u(x, x_0, k) = u_0(x, x_0, k) + u_s(x, x_0, k), \tag{8}$$

$$u_0 = \frac{\exp(ik|x - x_0|)}{4\pi |x - x_0|}, \; u_s \in C^{m+1+\alpha}(P_\omega(x_0)), \forall \alpha, \omega \in (0, 1). \tag{9}$$

In (8), (9) $u_0(x, x_0, k)$ is the incident spherical wave and $u_s(x, x_0, k)$ is the scattered wave.

Inverse Problem 1 (IP1). Let $m = 2$ in (3). Suppose that the function $q(x)$ satisfying (3), (4) is unknown for $x \in \Omega$ and known for $x \in \mathbb{R}^3 \diagdown \Omega$. Also, assume that the following function $f_1(x, x_0, k)$ is known

$$f_1(x, x_0, k) = |u(x, x_0, k)|^2, \forall x_0 \in S, \forall x \in B_\varepsilon(x_0), x \neq x_0, \forall k \in (a, b),$$

where $(a, b) \subset \mathbb{R}$ is an arbitrary interval. Determine the function $q(x)$ for $x \in \Omega$.

Theorem 1 is one of four main results of this paper.

Theorem 1. *Consider IP1. Let two potentials $q_1(x)$ and $q_2(x)$ satisfying conditions (3), (4) be such that $q_1(x) = q_2(x) = q(x)$ for $x \in \mathbb{R}^3 \diagdown \Omega$. Let $u_1(x, x_0, k)$ and $u_2(x, x_0, k)$ be corresponding solutions of the problem (5)-(7) satisfying conditions (8), (9). Assume that*

$$|u_1(x, x_0, k)|^2 = |u_2(x, x_0, k)|^2, \forall x_0 \in S, \forall x \in B_\varepsilon(x_0), x \neq x_0, \forall k \in (a, b). \tag{10}$$

Then $q_1(x) \equiv q_2(x)$.

Corollary 1. *Fix two arbitrary points $y_0 \in S$ and $y \in B_\varepsilon(y_0)$ such that $y \neq y_0$. Suppose that all conditions of Theorem 1 are in place, except that (10) is replaced with*

$$|u_1(y, y_0, k)|^2 = |u_2(y, y_0, k)|^2, \forall k \in (a, b). \tag{11}$$

Then $u_1(y, y_0, k) = u_2(y, y_0, k)$ for all $k \in \mathbb{R}$.

Remark 1. The proof of Corollary 1 can be immediately derived from the proof of Theorem 1. Completely analogous corollaries are valid for each of Theorems 2-4 of this paper. Their proofs can also be immediately derived from proofs of corresponding theorems. We omit formulations of those corollaries for brevity.

We now outline the main difficulty, which did not allow to prove uniqueness results for phaseless 3-d inverse scattering problems so far. As an example we consider IP1. Analogous difficulties take place for three other inverse problems formulated in section 2. In IP1 one should work with a complex valued function $r(k), k \in \mathbb{R}$ such that its modulus $|r(k)|$ is known for all $k \in (a,b)$. The function $r(k)$ admits the analytic continuation from the real line $\mathbb{R}$ in the half-plane $\{k \in \mathbb{C} : \operatorname{Im} k > -\gamma\}$ for a certain number $\gamma > 0$. Since $|r(k)|^2 = r(k)\overline{r}(k)$, then the function $|r(k)|^2$ is analytic for $k \in \mathbb{R}$ as the function of the real variable k. Here $\overline{r}(k)$ is the complex conjugate of $r(k)$. Hence, the modulus $|r(k)|$ is known for all $k \in \mathbb{R}$. Denote $\mathbb{C}_+ = \{k \in \mathbb{C} : \operatorname{Im} k \geq 0\}$. Proposition 4.2 of [17] implies that if $r(k)$ would not have zeros in $\mathbb{C}_+$, then this function would be uniquely reconstructed for all $k \in \mathbb{R}$ from the values of $|r(k)|$ for $k \in \mathbb{R}$, also see Lemma 4 in subsection 3.2. However, the *main difficulty* is to properly account for zeros of $r(k)$ in the upper half-plane $\mathbb{C}_+ \diagdown \mathbb{R}$. Indeed, let $z_1, ..., z_n \in \mathbb{C}_+ \diagdown \mathbb{R}$ be some of such zeros of $r(k)$. Consider the function $\widehat{r}(k)$ defined as

$$\widehat{r}(k) = \left(\prod_{j=1}^{n} \frac{k - \overline{z}_j}{k - z_j} \right) r(k).$$

Hence, $|\widehat{r}(k)| = |r(k)|, \forall k \in \mathbb{R}$. Furthermore, the function $\widehat{r}(k)$ is analytic in the half-plane $\{k \in \mathbb{C} : \operatorname{Im} k > -\gamma\}$. Therefore, in order to prove uniqueness, one needs to figure out how to combine the knowledge of $|r(k)|$ for $k \in \mathbb{R}$ with a linkage between the function $r(k)$ and the originating differential operator.

This difficulty was handled in [16] in 1-d, using the fact that the function $r(k)$ depends only on one variable k in this case. Unlike [16], the function $r(x, x_0, k) = u(x, x_0, k)$ depends on x, x_0, k in the 3-d case. Hence, the above zeros depend now on both x and x_0, i.e. $z_j = z_j(x, x_0)$. Thus, compared with the 1-d problem, the main difficulty of the 3-d case is that it is necessary to figure out how to take into account the dependence of zeros $z_j(x, x_0)$ from x and x_0. To do this, we essentially use here properties of the solution of the Cauchy problem for an associated hyperbolic PDE.

1.2 Published results

The phaseless inverse scattering problem is of central importance in some applications, where only the amplitude of the scattered signal can be measured. An example is neutron specular reflection, see, e.g. [3]. Uniqueness of the phaseless inverse scattering problem in the 1-d case was first proved in [16]. Next, the result of [16] was extended to the discontinuous impedance case in [25]. Also, see [1] for a relevant result. A survey can be found in [17]. Uniqueness theorem for a 1-d phaseless inverse problem arising in crystallography was proven in [15]. This problem is essentially different from the one considered in [16].

Inverse problems without the phase information are well known in optics, since it is often impossible to measure the phase of the optical signal, unlike its amplitude. In optics, such a problem is usually formulated as the problem about the recovery of a compactly supported complex valued function from the modulus of its Fourier transform. The latter is called the "phase retrieval problem" [9]. This problem arises in x-ray crystallography [22], astronomical imaging [10] and other subfields of optics [9]. Some numerical methods for this problem can be found in, e.g. [8, 9, 10, 13, 30]. Recently regularization algorithms were developed in the 1-d case for a similar, the so-called "autocorrelation problem" [7, 11]. Uniqueness theorems for the phase retrieval problem can be found in [14, 18].

In section 2 we formulate three more phaseless inverse scattering problems as well uniqueness theorems 2-4 for them. In section 3 we prove Theorem 1. Theorem 2 is proved in section 4. Finally, Theorems 3 and 4 are proved in section 5.

2 Other problems and results

In IP1 the modulus of the total wave field $u = u_0 + u_s$ is known on a certain set. We now consider the case when the modulus of the scattered wave is known.

Inverse Problem 2 (IP2). Let $m = 2$ in (3). Suppose that the function $q(x)$ satisfying (3), (4) is unknown for $x \in \Omega$ and known for $x \in \mathbb{R}^3 \diagdown \Omega$. Also, assume that the following function $f_2(x, x_0, k)$ is known

$$f_2(x, x_0, k) = |u_s(x, x_0, k)|^2, \forall x_0 \in S, \forall x \in B_\varepsilon(x_0), x \neq x_0, \forall k \in (a, b).$$

Determine the function $q(x)$ for $x \in \Omega$.

Theorem 2. *Consider IP2. Assume that all conditions of Theorem 1 hold, except that (10) is replaced with*

$$|u_{s,1}(x, x_0, k)|^2 = |u_{s,2}(x, x_0, k)|^2, \forall x_0 \in S, \forall x \in B_\varepsilon(x_0), x \neq x_0, \forall k \in (a, b), \tag{12}$$

where $u_{s,j} = u_j - u_0, j = 1, 2$. *In addition, assume that* $q(x) \neq 0$, $\forall x \in S$. *Then* $q_1(x) \equiv q_2(x)$.

Theorems 1 and 2 are formulated only for the over-determined data. Indeed, in both IP1 and IP2 the number of free variables in the data exceeds the number of free variables in the unknown coefficient. The reason of this is that even if the phase is known, still all current uniqueness results for 3-d inverse scattering problems in the case when the $\delta-$function is the source function are valid only if the data are over-determined ones, see, e.g. [2, 26, 27] for the frequency domain and §1 of chapter 7 of [24] for an inverse scattering problem in the time domain. Suppose now that the function $\delta(x - x_0)$ in (5) is replaced with such a function $p(x)$ that $p(x) \neq 0$ in $\overline{\Omega}$. And consider the inverse problem of the reconstruction of the potential $q(x)$ from values of the function $u(x, k)$ for all $x \in S, k \in \mathbb{R}$. Then uniqueness theorem for this problem can be proved for the non-overdetermined case. This proof can be handled by the method, which was introduced in the originating paper [5]. Also, see, e.g.

[6, 19, 20] and sections 1.10, 1.11 of [4] for some follow up works of authors of [5] on this method; a survey can be found in [21]. This technique is based on Carleman estimates.

Consider the function $\chi(x) \in C^{\infty}(\mathbb{R}^3)$ such that $\chi(x) = 1$ in G_1 and $\chi(x) = 0$ for $x \notin G$. Let $x_0 \in S$. For a number $\sigma > 0$ consider the function $\delta_\sigma(x - x_0)$,

$$\delta_\sigma(x - x_0) = C \frac{\chi(x)}{(2\sqrt{\pi\sigma})^3} \exp\left(-\frac{|x - x_0|^2}{4\sigma}\right),$$

where the number $C > 0$ is such that

$$\int_G \delta_\sigma(x - x_0)\, dx = 1.$$

The function $\delta_\sigma(x - x_0)$ approximates the function $\delta(x - x_0)$ in the distribution sense for sufficiently small values of σ. The function $\delta_\sigma(x - x_0)$ is acceptable in Physics as a proper replacement of $\delta(x - x_0)$, since there is no "true" delta-function in the reality. On the other hand, the above mentioned method of [5] is applicable to the case when $\delta(x - x_0)$ is replaced with $\delta_\sigma(x)$. Therefore, it seems to be worthy from the Physics standpoint to consider Inverse Problems 3,4 below.

Let in (3) $m = 4$. To apply results, which follow from the method of [5], consider the function $g(x)$ such that

$$g \in C^4\left(\mathbb{R}^3\right), g(x) = 0 \text{ in } \mathbb{R}^3 \diagdown G, \tag{13}$$

$$g(x) \neq 0 \text{ in } \overline{G}_1. \tag{14}$$

Consider the following problem

$$\Delta v + k^2 v - q(x) v = -g(x), x \in \mathbb{R}^3, \tag{15}$$

$$v(x,k) = O\left(\frac{1}{|x|}\right), |x| \to \infty, \tag{16}$$

$$\sum_{j=1}^{3} \frac{x_j}{|x|} \partial_{x_j} v(x,k) - ikv(x,k) = o\left(\frac{1}{|x|}\right), |x| \to \infty. \tag{17}$$

The same results of [12, 31, 32] as ones in subsection 1.1 guarantee that for each $k \in \mathbb{R}$ there exists unique solution $v(x,k) \in C^{5+\alpha}(\mathbb{R}^3), \forall \alpha \in (0,1)$ of the problem (15), (16), (17).

Inverse Problem 3 (IP3). Let $m = 4$ in (3). Suppose that the function $q(x)$ satisfying conditions (3), (4) is unknown for $x \in \Omega$ and known for $x \in \mathbb{R}^3 \diagdown \Omega$. Assume that the following function $f_3(x,k)$ is known

$$f_3(x,k) = |v(x,k)|^2, \forall x \in S, \forall k \in (a,b). \tag{18}$$

Determine the function $q(x)$ for $x \in \Omega$.

Theorem 3. *Consider IP3. Let the function* $g(x)$ *satisfies conditions (13), (14). Consider two functions* $q_1(x), q_2(x)$ *satisfying conditions (3), (4) and such that* $q_1(x) = q_2(x) = q(x)$ *for* $x \in \mathbb{R}^3 \diagdown \Omega$. *For* $j = 1, 2$ *let* $v_j(x,k) \in C^{5+\alpha}(\mathbb{R}^3)$ *be the solution of the problem (15)-(17) with* $q(x) = q_j(x)$. *Assume that*

$$|v_1(x,k)|^2 = |v_2(x,k)|^2, \forall x \in S, \forall k \in (a,b). \tag{19}$$

Then $q_1(x) \equiv q_2(x)$.

We now pose an analog of IP2. Let $v_0(x,k)$ be the solution of the problem (15)-(17) for the case $q(x) \equiv 0$,

$$v_0(x,k) = \int\limits_G \frac{\exp(ik|x-\xi|)}{4\pi|x-\xi|} g(\xi)\, d\xi.$$

Hence, one can interpret the function $v_0(x,k)$ as the solution of the problem (15)-(17) for case of the background medium.

Inverse Problem 4 (IP4). Let $m = 4$ in (3). Suppose that the function $q(x)$ satisfying (3), (4) is unknown for $x \in \Omega$ and known for $x \in \mathbb{R}^3 \diagdown \Omega$. Let $v_s(x,k) = v(x,k) - v_0(x,k)$. Assume that the following function $f_4(x,k)$ is known

$$f_4(x,k) = |v_s(x,k)|^2, \forall x \in S, \forall k \in (a,b).$$

Determine the function $q(x)$ for $x \in \Omega$.

Theorem 4. *Consider IP4. Let all conditions of Theorem 3 hold, except that (19) is replaced with*

$$|v_{s,1}(x,k)|^2 = |v_{s,2}(x,k)|^2, \forall x \in S, \forall k \in (a,b), \tag{20}$$

where $v_{s,j}(x,k) = v_j(x,k) - v_0(x,k), j = 1, 2$. *Assume that* $q(x) \neq 0, \forall x \in S$. *Then* $q_1(x) \equiv q_2(x)$.

3 Proof of Theorem 1

3.1 Functions U and u

Consider the solution $U(x, x_0, t)$ of the following Cauchy problem

$$U_{tt} = \Delta_x U - q(x) U, (x,t) \in \mathbb{R}^3 \times (0, \infty), \tag{21}$$

$$U(x,0) = 0, U_t(x,0) = \delta(x - x_0). \tag{22}$$

It was shown in §1 of Chapter 7 of the book [24] that the function U has the form

$$U(x, x_0, t) = \frac{\delta(t - |x - x_0|)}{4\pi|x - x_0|} + \widetilde{U}(x, x_0, t), \tag{23}$$

where

$$\widetilde{U}(x,x_0,t) = -\int\limits_{D(x,x_0,t)} \frac{q(\xi)\,U(\xi,x_0,t-|x-\xi|)}{4\pi|x-\xi|} d\xi, \tag{24}$$

$$D(x,x_0,t) = \{\xi : |x-\xi| + |x_0-\xi| < t\}, \tag{25}$$

$$\widetilde{U}(x,x_0,t) = 0 \text{ for } t \in (0, |x-x_0|), \tag{26}$$

$$\widetilde{U}(x,x_0,t) = -\frac{1}{16\pi|x-x_0|} \int\limits_{L(x,x_0)} q(\xi)\,ds + O(t-|x-x_0|), t \to |x-x_0|^+. \tag{27}$$

Here $L(x,x_0)$ is the interval of the straight line connecting points x, x_0.

Let $\Phi \subset \mathbb{R}^3$ be an arbitrary bounded domain. Choose the number $\omega_0 = \omega_0(x_0,\Phi) \in (0,1)$ so small that $\Phi \cap P_\omega(x_0) \neq \varnothing$. Let $T > \max_{x\in\overline{\Phi}} |x-x_0|$ be an arbitrary number. Denote

$$\Psi(\Phi,x_0,\omega,T) = \left\{(x,t) : x \in \overline{\Phi} \cap P_\omega(x_0), t \in [|x-x_0|, T]\right\}.$$

It was shown in §3 of chapter 2 of the book [28] that the function U can be represented as the following series

$$U(x,x_0,t) = \sum_{n=0}^{\infty} U_n(x,x_0,t), \tag{28}$$

where

$$U_0(x,x_0,t) = \frac{\delta(t-|x-x_0|)}{4\pi|x-x_0|}, \tag{29}$$

$$U_n(x,x_0,t) = -\int\limits_{D(x,x_0,t)} \frac{q(\xi)\,U_{n-1}(\xi,x_0,t-|x-\xi|)}{4\pi|x-\xi|} d\xi, n \geq 1. \tag{30}$$

Convergence estimates of §3 of chapter 2 of [28] imply that series (28) converges in the norm of the space $C^2(\Psi(\Phi,x_0,\omega,T))$. Hence, for any fixed source position $x_0 \in \mathbb{R}^3$ and for $|\beta| = 0,1,2$

$$D^\beta_{x,t}\widetilde{U}(x,x_0,t) \in C(\Psi(\Phi,x_0,\omega,T)), \forall \omega \in (0,\omega_0(x_0,\Phi)), \forall T > \max_{\overline{\Phi}} |x-x_0|. \tag{31}$$

We now refer to some results about the asymptotic behavior of solutions of hyperbolic equations as $t \to \infty$. More precisely, we refer to Lemma 6 of Chapter 10 of the book [32] as well as to Remark 3 after that lemma. It follows from these results that there exist numbers $C_1 = C_1(q,\Phi,x_0,\omega) > 0, c_1 = c_1(q,\Phi,x_0,\omega) > 0$ depending only on the function q, the domain Φ, the source position x_0 and the number $\omega \in (0,\omega_0(x_0,\Phi))$ such that

$$\left|D^\beta_{x,t}\widetilde{U}(x,x_0,t)\right| \leq C_1 e^{-c_1 t} \text{ in } \{(x,t) : x \in \Phi \cap P_\omega(x_0), t \geq |x-x_0|\}, |\beta| = 0,1,2. \tag{32}$$

It follows from (23), (31) and (32) that one can apply the operator $\mathcal{F}$ of the Fourier transform with respect to t to functions $D_{x,t}^{\beta}\widetilde{U}\left(x,x_0,t\right),|\beta|=0,1,2$. Let

$$\mathcal{F}\left(U\right)\left(x,x_0,k\right)=\int_0^{\infty}U\left(x,x_0,t\right)e^{ikt}dt,\ \forall x,x_0\in\mathbb{R}^3,x\neq x_0,\forall k\in\mathbb{R}.$$

Using again the same results of references [12, 31, 32] as ones cited in section 1.1, we obtain that

$$u\left(x,x_0,k\right)=\mathcal{F}\left(U\right)\left(x,x_0,k\right),\forall x,x_0\in\mathbb{R}^3,x\neq x_0,\forall k\in\mathbb{R}. \tag{33}$$

In particular, (32) and (33) imply that for each pair $x,x_0\in\mathbb{R}^3$ such that $x\neq x_0$ there exists a number $\gamma=\gamma\left(x,x_0,q\right)>0$ such that the function $u\left(x,x_0,k\right)$ admits the analytic continuation with respect to k from the real line $\mathbb{R}$ in the half-plane $\{k\in\mathbb{C}:\operatorname{Im}k>-\gamma\}$.

Suppose that $G\cap P_{\omega}\left(x_0\right)\neq\varnothing$. Then, using (23)-(27), (32) and (33), we obtain the following asymptotic behavior of functions $u\left(x,x_0,k\right)$ and $u_s\left(x,x_0,k\right)$ for every fixed source position $x_0\in\mathbb{R}^3$ and for every fixed value of $\omega\in\left(0,\omega_0\left(x_0,\Phi\right)\right)$

$$u\left(x,x_0,k\right)=\frac{\exp\left(ik\left|x-x_0\right|\right)}{4\pi\left|x-x_0\right|}\left[1+O\left(\frac{1}{k}\right)\right],|k|\to\infty,k\in\mathbb{C}_+, \tag{34}$$

$$u_s\left(x,x_0,k\right)=-\frac{i\exp\left(ik\left|x-x_0\right|\right)}{16\pi\left|x-x_0\right|k}\left[\int_{L(x,x_0)}q\left(\xi\right)ds+O\left(\frac{1}{k}\right)\right],|k|\to\infty,k\in\mathbb{C}_+, \tag{35}$$

uniformly for $x\in\overline{G}\cap P_{\omega}\left(x_0\right)$. Therefore, we have proven Lemma 1.

Lemma 1. *The solution $U\left(x,x_0,t\right)$ of the problem (21), (22) can be represented in the form (23), where the function $\widetilde{U}\left(x,x_0,t\right)$ satisfies conditions (24)-(32). Furthermore, (33) holds, where the function $u\left(x,x_0,k\right)$ is the unique solution of the problem (5), (6), (7) satisfying conditions (8), (9). In addition, for every pair of points $x,x_0\in\mathbb{R}^3$ such that $x\neq x_0$ the function $u\left(x,x_0,k\right)$ admits the analytic continuation with respect to k from the real line $\mathbb{R}$ in the half-plane $\{k\in\mathbb{C}:\operatorname{Im}k>-c_1\}$, where $c_1=c_1\left(q,\Phi,x_0,\omega\right)>0$ is the number in (32). Finally, if $G\cap P_{\omega}\left(x_0\right)\neq\varnothing$, then asymptotic formulas (34) and (35) hold uniformly for $x\in\overline{G}\cap P_{\omega}\left(x_0\right)$.*

3.2 Lemmata 2-5

Lemma 2. *Let $x_0\in\mathbb{R}^3$ and $x\in G,x\neq x_0$ be two arbitrary points. Then the function $u\left(x,x_0,k\right)$ has at most finite number of zeros in $\mathbb{C}_+$.*

Lemma 3. *Let $x_0\in\mathbb{R}^3$ and $x\in G$ be two arbitrary points. Assume that*

$$\int_{L(x,x_0)}q\left(\xi\right)ds\neq 0.$$

Then the function $u_s(x, x_0, k)$ has at most finite number of zeros in $\mathbb{C}_+$.

Lemmata 2 and 3 follow immediately from (34) and (35) respectively.

Lemma 4. *Let $\gamma > 0$ be a number. Let the function $d(k)$ be analytic in $\{k \in \mathbb{C} : \operatorname{Im} k > -\gamma\}$ and does not have zeros in $\mathbb{C}_+$. Assume that*

$$d(k) = \frac{C}{k^n}[1 + o(1)] \exp(ikL), |k| \to \infty, k \in \mathbb{C}_+,$$

where $C \in \mathbb{C}$ and n,$L \in \mathbb{R}$ are some numbers and also $n \geq 0$. Then the function $d(k)$ can be uniquely determined for $k \in \{k \in \mathbb{C} : \operatorname{Im} k > -\gamma\}$ by the values of $|d(k)|$ for $k \in \mathbb{R}$. Furthermore, for $k \in \mathbb{R}$

$$\arg d(k) = \frac{1}{\pi} \lim_{R\to\infty} \lim_{\varepsilon\to 0^+} \left[\int_{-R}^{k-\varepsilon} \frac{\ln|d(\xi)|}{k-\xi} d\xi + \int_{k+\varepsilon}^{R} \frac{\ln|d(\xi)|}{k-\xi} d\xi \right] + Lk - \arg C + \frac{n\pi}{2}. \tag{36}$$

The right hand side of (36) can be any of branches of the function arg. However, this does not make any difference for us, since we are interested in the function $d(k) = |d(k)| \exp[i \arg d(k)]$. Since the function $d(k)$ is uniquely determined on the real line, then the analyticity of this function implies that it is uniquely determined in $\{k \in \mathbb{C} : \operatorname{Im} k > -\gamma\}$. Lemma 4 follows immediately from Proposition 4.2 of [17]. Hence, we omit the proof.

Lemma 5. *Let the function $d(k)$ be analytic for all $k \in \mathbb{R}$. Then the function $|d(k)|$ can be uniquely for all $k \in \mathbb{R}$ by the values of $|d(k)|$ for $k \in (a, b)$.*

This lemma was actually proven in subsection 1.1 for the function $r(k)$.

3.3 Proof of Theorem 1

Choose an arbitrary point $x_0 \in S$ and an arbitrary point $x \in B_\varepsilon(x_0)$ such that $x \neq x_0$. Denote

$$h_1(k) = u_1(x, x_0, k), h_2(k) = u_2(x, x_0, k). \tag{37}$$

It follows from Lemma 1 that we can regard $h_1(k)$ and $h_2(k)$ as analytic functions in the half-plane $\{k \in \mathbb{C} : \operatorname{Im} k > -\theta_1\}$, where $\theta_1 > 0$ is a certain number. Next, since $|h_1(k)| = |h_2(k)|$ for $k \in (a, b)$, then Lemma 5 implies that

$$|h_1(k)| = |h_2(k)|, \forall k \in \mathbb{R}. \tag{38}$$

By Lemma 2 each of functions $h_1(k), h_2(k)$ has at most finite number of zeros in $\mathbb{C}_+$. Let $\{a_1, ..., a_n\} \subset (\mathbb{C}_+ \diagdown \mathbb{R})$ and $\{b_1, ..., b_m\} \subset (\mathbb{C}_+ \diagdown \mathbb{R})$ be sets of all zeros of functions $h_1(k)$ and $h_2(k)$ respectively in the upper half-plane. Here and below each zero is counted as many times as its order is. Let $\{a'_1, ..., a'_{r_1}\} \subset \mathbb{R}$ and $\{b'_1, ..., b'_{r_2}\} \subset \mathbb{R}$ be sets of all real zeros of functions $h_1(k)$ and $h_2(k)$ respectively.

First, we prove that

$$\{a'_1, ..., a'_{r_1}\} = \{b'_1, ..., b'_{r_2}\}. \tag{39}$$

Let, for example a_1' be the zero of the order $n_1 \geq 1$ of the function $h_1(k)$ as well as the zero of the order $m_1 \geq 0$ of the function $h_2(k)$. Then $h_1(k) = (k - a_1')^{n_1} \widehat{h}_1(k)$ and $h_2(k) = (k - a_1')^{m_1} \widehat{h}_1(k)$, where

$$\widehat{h}_1(a_1') \widehat{h}_2(a_1') \neq 0. \tag{40}$$

Using (38), we obtain

$$\left|(k - a_1')^{n_1}\right| \cdot \left|\widehat{h}_1(k)\right| = \left|(k - a_1')^{m_1}\right| \cdot \left|\widehat{h}_2(k)\right|, \ \forall k \in \mathbb{R}. \tag{41}$$

Assume, for example that $m_1 < n_1$. Dividing both sides of (41) by $|(k - a_1')^{m_1}|$ and setting the limit at $k \to a_1'$, we obtain $\left|\widehat{h}_2(a_1')\right| = 0$. This contradicts to (40). Hence, $m_1 = n_1$. Since a_1' is an arbitrary element of the set $\left\{a_1', ..., a_{r_1}'\right\}$, then (39) follows.

Let $\{c_1, ..., c_r\} \subset \mathbb{R}$ be the set of all real zeros of both functions $h_1(k)$ and $h_2(k)$. Define functions $\widetilde{h}_1(k), \widetilde{h}_2(k)$ as

$$\widetilde{h}_1(k) = h_1(k) \left(\prod_{j=1}^{n} \frac{k - \overline{a}_j}{k - a_j} \right) \left(\prod_{s=1}^{r} \frac{1}{k - c_s} \right), \tag{42}$$

$$\widetilde{h}_2(k) = h_2(k) \left(\prod_{j=1}^{m} \frac{k - \overline{b}_j}{k - b_j} \right) \left(\prod_{s=1}^{r} \frac{1}{k - c_s} \right). \tag{43}$$

Hence, $\widetilde{h}_1(k)$ and $\widetilde{h}_2(k)$ are analytic functions in $\{k \in \mathbb{C} : \operatorname{Im} k > -\theta_1\}$. Also, these functions do not have zeros in $\mathbb{C}_+$. Furthermore, it follows from (38), (42) and (43) and $\left|\widetilde{h}_1(k)\right| = \left|\widetilde{h}_2(k)\right|, \forall k \in \mathbb{R}$. Also, using (34), we obtain the following asymptotic behavior of both functions $\widetilde{h}_1(k)$ and $\widetilde{h}_2(k)$

$$\widetilde{h}_j(k) = \frac{\exp(ik|x - x_0|)}{4\pi |x - x_0| k^r} \left[1 + O\left(\frac{1}{k}\right)\right], |k| \to \infty, k \in \mathbb{C}_+, j = 1, 2.$$

Hence, Lemma 4 implies that $\widetilde{h}_1(k) = \widetilde{h}_2(k), \forall k \in \mathbb{R}$. Hence, (42) and (43) lead to

$$h_1(k) \prod_{j=1}^{n} \frac{k - \overline{a}_j}{k - a_j} = h_2(k) \prod_{j=1}^{m} \frac{k - \overline{b}_j}{k - b_j}, \ \forall k \in \mathbb{R}.$$

Hence,

$$h_1(k) \prod_{j=1}^{m} \frac{k - b_j}{k - \overline{b}_j} = h_2(k) \prod_{j=1}^{n} \frac{k - a_j}{k - \overline{a}_j}, \ \forall k \in \mathbb{R}. \tag{44}$$

Rewrite (44) as

$$h_1(k) + h_1(k) \left(\prod_{j=1}^{m} \frac{k - b_j}{k - \overline{b}_j} - 1 \right) = h_2(k) + h_2(k) \left(\prod_{j=1}^{n} \frac{k - a_j}{k - \overline{a}_j} - 1 \right). \tag{45}$$

Consider the function $w_1(k)$,

$$w_1(k) = \prod_{j=1}^{n} \frac{k-a_j}{k-\overline{a}_j} - 1.$$

This function can be rewritten in the form

$$w_1(k) = Q(k) \prod_{j=1}^{n} \frac{1}{k-\overline{a}_j},$$

where $Q(k)$ is a polynomial of the degree less than n. By a partial fraction expansion, $A(k)$ can be written in the form

$$w_1(k) = \sum_{j=1}^{n'} \frac{C_j}{(k-\overline{a}_j)^{s_j}},$$

where $C_j \in \mathbb{C}$ are some numbers and $s_j, n' \geq 1$ are integers. Direct calculations verify that the inverse Fourier transform $\mathcal{F}^{-1}$ of the function $(k-\overline{a}_j)^{-s_j}$ is

$$\mathcal{F}^{-1}\left(\frac{1}{(k-\overline{a}_j)^{s_j}}\right) = H(t)\,\widehat{C}_j t^{s_j-1} \exp(-i\overline{a}_j t)$$

with a certain constant $\widehat{C}_j \in \mathbb{C}$. Hence,

$$\mathcal{F}^{-1}(w_1(k)) := \lambda_1(t) = H(t) \sum_{j=1}^{n'} \widetilde{C}_j t^{s_j-1} \exp(-i\overline{a}_j t), \tag{46}$$

where constants $\widetilde{C}_j = C_j \widehat{C}_j \in \mathbb{C}$. Similarly, denoting

$$w_2(k) = \prod_{j=1}^{m} \frac{k-b_j}{k-\overline{b}_j} - 1,$$

we obtain

$$\mathcal{F}^{-1}(w_2(k)) := \lambda_2(t) = H(t) \sum_{j=1}^{m'} B_j t^{s_j-1} \exp\left(-i\overline{b}_j t\right), \tag{47}$$

where constants $B_j \in \mathbb{C}$. For $j = 1, 2$ let $\mathcal{F}^{-1}(h_j) = \widehat{h}_j(t)$. By (33) and (37)

$$\widehat{h}_j(t) = U_j(x, x_0, t), \tag{48}$$

where $U_j(x, x_0, t)$ is the solution of the problem (21), (22) with $q(x) = q_j(x)$. We now apply the operator $\mathcal{F}^{-1}$ to both sides of (45). Using (46), (47) and the convolution theorem, we obtain

$$\widehat{h}_1(t) + \int_0^t \widehat{h}_1(t-s)\,\lambda_2(s)\,ds = \widehat{h}_2(t) + \int_0^t \widehat{h}_2(t-s)\,\lambda_1(s)\,ds. \tag{49}$$

Since $x_0 \in S, x \in B_\varepsilon(x_0)$, then (1), (2) and (25) imply that

$$D(x,x_0,t) \subset \left(G \diagdown \overline{\Omega}\right), \forall t \in (|x-x_0|, \varepsilon). \tag{50}$$

Since $q_1(x) = q_2(x)$ for $x \in \mathbb{R}^3 \diagdown \Omega$, then (28), (29), (30) and (50) imply that

$$U_1(x,x_0,t) = U_2(x,x_0,t) = U(x,x_0,t), \forall t \in (|x-x_0|, \varepsilon). \tag{51}$$

By (48) and (51)

$$\widehat{h}_1(t) = \widehat{h}_2(t) = \widehat{h}(t) = U(x,x_0,t), \forall t \in (|x-x_0|, \varepsilon). \tag{52}$$

Hence, (49) implies that

$$\int_0^t \widehat{h}(t-s)\lambda(s)\,ds = 0, \ \forall t \in (|x-x_0|, \varepsilon), \tag{53}$$

where

$$\lambda(t) = \lambda_1(t) - \lambda_2(t). \tag{54}$$

Using (23), (26), (27), (31) and (52), we obtain

$$\widehat{h}(t) = \frac{\delta(t - |x-x_0|)}{4\pi|x-x_0|} + p(t), \tag{55}$$

$$p(t) = 0, t \in (0, |x-x_0|), \tag{56}$$

$$\lim_{t \to |x-x_0|^+} p(t) = -\frac{1}{16\pi|x-x_0|} \int_{L(x,x_0)} q(\xi)\,ds, \tag{57}$$

$$p \in C^2(t \geq |x-x_0|). \tag{58}$$

Introduce a new variable $\tau \Leftrightarrow t$, where $\tau = t - |x-x_0|$. Then (53) and (55)-(58) lead to the following integral equation of the Volterra type with the continuous kernel $p(\tau + |y-y_0| - s)$ and with respect to the function $\lambda(\tau)$

$$\lambda(\tau) + 4\pi|x-x_0| \int_0^\tau p(\tau + |x-x_0| - s)\lambda(s)\,ds = 0, \ \forall \tau \in (0, \varepsilon - |x-x_0|). \tag{59}$$

Hence, $\lambda(\tau) = 0$ for all $\tau \in (0, \varepsilon - |x-x_0|)$. On the other hand, (46), (47) and (54) imply that $\lambda(t)$ is analytic function of the real variable $t > 0$. Hence, $\lambda(t) = 0, \forall t \geq 0$. This implies that $\{a_1, ..., a_n\} = \{b_1, ..., b_m\}$. Thus, (37) and (44) lead to $u_1(x,x_0,k) = u_2(x,x_0,k), \forall k \in \mathbb{R}$.

Since $x_0 \in S$ and $x \in B_\varepsilon(x_0)$ are two arbitrary points such that $x \neq x_0$, then we have established that

$$u_1(x, x_0, k) = u_2(x, x_0, k), \forall x_0 \in S, \forall x \in B_\varepsilon(x_0), x \neq x_0, \forall k \in \mathbb{R}. \tag{60}$$

Consider an arbitrary point $y_0 \in S$. Since $q_1(x) = q_2(x) = q(x)$ in $\mathbb{R}^3 \diagdown \Omega$, then, using (5), we obtain

$$\Delta_x u_j + k^2 u_j - q(x) u_j = -\delta(x - y_0), x \in \mathbb{R}^3 \diagdown \Omega, j = 1, 2.$$

Hence, (60) and the well known result about uniqueness of the continuation problem for elliptic equations (see, e.g. §1 of chapter 4 of [24]) imply that $u_1(x, y_0, k) = u_2(x, y_0, k), \forall x \in \mathbb{R}^3 \diagdown \Omega, \forall k \in \mathbb{R}$. Hence,

$$u_1(x, x_0, k) = u_2(x, x_0, k), \forall x, x_0 \in S, x \neq x_0, \forall k \in \mathbb{R}. \tag{61}$$

Using (8), (9), (34) and (35), we obtain for $j = 1, 2$

$$\lim_{k \to \infty} \left[4ik \left(\frac{u_j}{u_0} - 1 \right)(x, x_0, k) \right] = \int\limits_{L(x,x_0)} q_j(\xi)\, ds, \ \forall x, x_0 \in \mathbb{R}^3, x \neq x_0. \tag{62}$$

Hence, (61) and (62) lead to

$$\int\limits_{L(x,x_0)} (q_1 - q_2)(\xi)\, ds = 0, \ \forall x, x_0 \in S. \tag{63}$$

Finally, (63) and the classical uniqueness theorem for the Radon transform implies that $q_1(x) \equiv q_2(x)$. $\square$

The idea of using (63) for the proof of the uniqueness of an inverse scattering problem in the time domain can be found in §1 of chapter 7 of [24]

4 Proof of Theorem 2

Consider an arbitrary point $x_0 \in S$. Since $q_1(x) = q_2(x) = q(x)$ for $x \in \mathbb{R}^3 \diagdown \Omega$ and $q(x) \neq 0, \forall x \in S$, then (1) and (2) imply that one can choose $\varepsilon > 0$ so small that

$$\int\limits_{L(y,x_0)} q_1(\xi)\, ds = \int\limits_{L(y,x_0)} q_2(\xi)\, ds = \int\limits_{L(y,x_0)} q(\xi)\, ds \neq 0, \forall y \in B_\varepsilon(x_0). \tag{64}$$

Choose an arbitrary point $x \in B_\varepsilon(x_0)$ such that $x \neq x_0$. Denote $h_{s,j}(k) = u_{s,j}(x, x_0, k)$. It follows from (64) and Lemmata 1, 3, 4 and 5 that we can apply to functions $h_{s,j}(k)$ the same procedure as the one described in section 3.3. For brevity we keep notations (46), (47) and

(54). For $j=1,2$ let $\widehat{h}_{s,j}(t)=\mathcal{F}^{-1}(h_{s,j})$. Let $\widetilde{U}(x,x_0,t)$ be the function defined in (23), (24). Then, using (55), (56) and (58), we obtain analogously with (52)

$$\widehat{h}_{s,j}(t)=\widetilde{U}(x,x_0,t)=p(t),\forall t\in(|x-x_0|,\varepsilon),j=1,2. \tag{65}$$

Hence, using (65), we obtain similarly with (59)

$$\int_0^\tau p(\tau-s+|x-x_0|)\lambda(s)\,ds=0,\forall\tau\in(0,\varepsilon-|x-x_0|), \tag{66}$$

where $\tau=t-|x-x_0|$. Differentiating both sides of (66) with respect to τ and using (57), (58) and (64), we obtain

$$\lambda(\tau)-m(x,x_0)\int_0^\tau p'(\tau+|x-x_0|-s)\lambda(s)\,ds=0,\ \forall\tau\in(0,\varepsilon-|x-x_0|),$$

$$m(x,x_0)=\frac{16\pi}{|x-x_0|}\left(\int_{L(x,x_0)}q(\xi)\,ds\right)^{-1}.$$

Hence, similarly with the proof of Theorem 1, we conclude that $\lambda(t)=0,\forall t\geq 0$. This leads to

$$u_{s,1}(x,x_0,k)=u_{s,2}(x,x_0,k),\forall k\in\mathbb{R}. \tag{67}$$

Next, since $u_j(x,x_0,k)=u_0(x,x_0,k)+u_{s,j}(x,x_0,k),j=1,2$ and since again $x_0\in S$ and $x\in B_\varepsilon(x_0)$ are two arbitrary points such that $x\neq x_0$, then (67) implies (60). The rest of the proof is the same as the proof of Theorem 1 after (60). □

5 Proofs of Theorems 3 and 4

5.1 Functions V and v

Consider the following Cauchy problem

$$V_{tt}=\Delta V-q(x)V,\ \ (x,t)\in\mathbb{R}^3\times(0,\infty), \tag{68}$$

$$V(x,0)=0,V_t(x,0)=g(x). \tag{69}$$

Corollary 4.2 of Chapter 4 of the book [23] implies that there exists unique solution $V\in H^2(\mathbb{R}^3\times(0,T)),\forall T>0$ of the problem (68), (69). By the Kirchhoff formula this function $V(x,t)$ is the solution of the following integral equation

$$V(x,t)=\frac{1}{4\pi t}\int_{|x-\xi|=t}g(\xi)\,dS_\xi-\int_{|x-\xi|<t}\frac{q(\xi)V(\xi,t-|x-\xi|)}{4\pi|x-\xi|}d\xi. \tag{70}$$

Construct functions $V_n(x,t)$ as

$$V_0(x,t) = \frac{1}{4\pi} \int\limits_{|x-\xi|=t} g(\xi)\, dS_\xi, \tag{71}$$

$$V_n(x,t) = -\int\limits_{|x-\xi|<t} \frac{q(\xi) V_{n-1}(\xi, t-|x-\xi|)}{4\pi |x-\xi|} d\xi,\ n = 1, 2, ..., \tag{72}$$

The above mentioned technique of §3 of chapter 2 of the book [28] implies that the function $V(x,t)$ can be represented as

$$V(x,t) = \sum_{n=0}^{\infty} V_n(x,t), \tag{73}$$

and this series converges in the norm of the space $C^4\left(\overline{\Phi}\times[0,T]\right)$ for any bounded domain $\Phi \subset \mathbb{R}^3$ and for any number $T > 0$. Hence,

$$V \in C^4\left(\overline{\Phi}\times[0,T]\right). \tag{74}$$

Using again Lemma 6 in Chapter 10 of the book [32] as well as Remark 3 after that lemma, we obtain that for any bounded domain $\Phi \subset \mathbb{R}^3$ there exist constants $C_2 = C_2(q,g,\Phi) > 0, c_2 = c_2(q,g,\Phi) > 0$ depending only on functions q, g and the domain Φ such that

$$\left|D^\beta_{x,t} V(x,t)\right| \le C_2 e^{-c_2 t},\ \forall x \in \Phi, \forall t \ge 0; |\beta| = 0, 1, ..., 4. \tag{75}$$

Hence, using again Theorem 6 of Chapter 9 of [32], Theorem 3.3 of [31] and Theorem 6.17 of [12], we obtain that the Fourier transform of the function $V(x,t)$ is the unique solution $v(x,k) \in C^{5+\alpha}(\mathbb{R}^3), \forall \alpha \in (0,1)$ of the problem (15)-(17), i.e.

$$v(x,k) = \mathcal{F}(V), \forall x \in \mathbb{R}^3, \forall k \in \mathbb{R}. \tag{76}$$

Furthermore, it follows from (75) and (76) that for every point $x \in G$ the function $v(x,k)$ admits the analytic continuation with respect to k from the real line $\mathbb{R}$ in the half-plane $\{k \in \mathbb{C} : \operatorname{Im} k > -c_2(q,g,G)\}$.

Hence, using the integration by parts in integral (76) of the Fourier transform as well as (69), (74) and (75), we obtain the following asymptotic behavior of the function $v(x,k)$, uniformly for $x \in \overline{G}$

$$v(x,k) = \frac{1}{k^2}\left[-g(x) + O\left(\frac{1}{k}\right)\right], |k| \to \infty, k \in \mathbb{C}_+. \tag{77}$$

Let

$$V_s(x,t) = V(x,t) - V_0(x,t). \tag{78}$$

Then

$$v_s(x,k) = \mathcal{F}(V_s). \tag{79}$$

Next, using (68) and (69), we obtain

$$\partial_t^2 V_s = \Delta V_s - q(x)(V_s + V_0), (x,t) \in \mathbb{R}^3 \times (0,\infty), \tag{80}$$

$$V_s(x,0) = \partial_t V_s(x,0) = 0. \tag{81}$$

Hence, (69), (80) and (81) imply that

$$\partial_t^r V_s(x,0) = 0 \text{ for } r = 0,1,2 \text{ and } \partial_t^3 V_s(x,0) = -(qg)(x). \tag{82}$$

Hence, using (82) and the integration by parts in the right hand side of (79), we obtain the following asymptotic behavior of the function $v_s(x,k)$, uniformly for $x \in \overline{G}$

$$v_s(x,k) = \frac{1}{k^4}\left[-(qg)(x) + o(1)\right], |k| \to \infty, k \in \mathbb{C}_+. \tag{83}$$

Thus, we have proven Lemma 6.

Lemma 6. *There exists unique solution $V(x,t)$ of the problem (68), (69) such that (74) is valid for every bounded domain $\Phi \subset \mathbb{R}^3$ and for every $T > 0$. Estimate (75) is valid for this function $V(x,t)$. In addition, the Fourier transform (76) of the function $V(x,t)$ is the unique solution $v(x,k) \in C^{5+\alpha}(\mathbb{R}^3), \forall \alpha \in (0,1)$ of the problem (15)-(17). Also, for every $x \in \Phi$ the function $v(x,k)$ admits the analytic continuation with respect to k from the real line $\mathbb{R}$ in the half-plane $\{k \in C : \operatorname{Im} k > -c_2\}$. Finally, asymptotic formulas (77) and (83) hold uniformly for $x \in \overline{G}$.*

Lemma 7 follows immediately from Lemma 6, (14), (77) and (83).

Lemma 7. *For every point $x \in \overline{G}_1$ there exists at most finite number of zeros of the function $v(x,k)$ in $\mathbb{C}_+$. Next, assume that there exists a point $x' \in \overline{G}_1$ such that $q(x') \neq 0$. Then the function $v_s(x',k)$ has at most finite number of zeros in $\mathbb{C}_+$.*

5.2 Proof of Theorem 3

Consider an arbitrary point $\widetilde{x} \in S$ and denote similarly with (37) $h_j(k) = v_j(\widetilde{x},k), j = 1,2$. By Lemma 6 there exists a number $\theta_2 > 0$ such that each of functions $h_1(k)$, $h_2(k)$ admits the analytic continuation from the real line $\mathbb{R}$ in the half-plane $\{k \in \mathbb{C} : \operatorname{Im} k > -\theta_2\}$. By Lemma 7 each function $h_1(k), h_2(k)$ has at most finite number of zeros in $\mathbb{C}_+$. Hence, the asymptotic behavior (77) enables us to apply the technique of section 3.3. For $j = 1,2$ let $\widehat{h}_j(t) = \mathcal{F}^{-1}(h_j)$. Let the function $V_j(x,t)$ satisfying (74) be the solution of the problem (68), (69) with $q(x) := q_j(x)$. Then (76) implies that

$$\widehat{h}_j(t) = V_j(\widetilde{x},t), j = 1,2. \tag{84}$$

Since $\widetilde{x} \in S$, then it follows from (1) and (70)-(73) that

$$V_1(\widetilde{x},t) = V_2(\widetilde{x},t) = V(\widetilde{x},t), \ \forall t \in (0,\varepsilon). \tag{85}$$

We briefly note that another way of establishing (85) is via the energy estimate. Using (84) and (85), we obtain

$$\widehat{h}_1(t) = \widehat{h}_2(t) = V(\widetilde{x}, t), \ \forall t \in (0, \varepsilon). \tag{86}$$

Using arguments, which are completely analogous with those of section 3.3, keeping the same notations (46), (47), (54) and using (86), we obtain the following analog of (66)

$$\int\limits_0^t V(\widetilde{x}, t-\tau)\lambda(\tau)\,d\tau = 0, \ \forall t \in (0, \varepsilon). \tag{87}$$

Differentiating both sides of (87) twice, using initial conditions (69) as well as (74), we obtain the following Volterra integral equation of the second kind with respect to the function $c(t)$

$$\lambda(t) + \frac{1}{g(\widetilde{x})}\int\limits_0^t V_{tt}(\widetilde{x}, t-\tau)\lambda(\tau)\,d\tau = 0, \ \forall t \in (0, \varepsilon). \tag{88}$$

Hence, analogously with subsection 3.3, we conclude that $\lambda(t) = 0$, $\forall t \geq 0$ and

$$v_1(\widetilde{x}, k) = v_2(\widetilde{x}, k), \forall k \in \mathbb{R}. \tag{89}$$

Since $\widetilde{x} \in S$ is an arbitrary point, then (89) implies that

$$v_1(x, k) = v_2(x, k), \forall x \in S, \forall k \in \mathbb{R}. \tag{90}$$

Since $V_j(x,t) = \mathcal{F}^{-1}(v_j), j = 1, 2$, then (90) leads to

$$V_1(x,t)\mid_{S_\infty} = V_2(x,t)\mid_{S_\infty} := \eta(x,t), \tag{91}$$

where $S_\infty = S \times (0, \infty)$. Let $\widehat{V}(x,t)$ be any of two functions $V_1(x,t), V_2(x,t)$. Since

$$q_1(x) = q_2(x) = q(x) \ \text{ for } x \in \mathbb{R}^3 \diagdown \Omega, \tag{92}$$

then (68), (69) and (91) imply that the function $\widehat{V}(x,t)$ satisfies the following conditions

$$\widehat{V}_{tt} = \Delta\widehat{V} - q(x)\widehat{V}, \ \ (x,t) \in \left(\mathbb{R}^3 \diagdown G_1\right) \times (0, \infty), \tag{93}$$

$$\widehat{V}(x, 0) = 0, \partial_t \widehat{V}(x, 0) = g(x), x \in \mathbb{R}^3 \diagdown G_1, \tag{94}$$

$$\widehat{V}(x,t)\mid_{S_\infty} = \eta(x,t). \tag{95}$$

Recall that both functions $V_j \in H^2(\mathbb{R}^3 \times (0,T))$, $\forall T > 0$. On the other hand, the standard energy estimate tells us that the problem (93)-(95) has at most one solution $\widehat{V} \in H^1((\mathbb{R}^3 \diagdown G_1) \times (0,T))$, $\forall T > 0$. Hence, $V_1(x,t) = V_2(x,t) = V(x,t)$ for $(x,t) \in (\mathbb{R}^3 \diagdown G_1) \times (0, \infty)$.

Denote $\varphi(x,t) = \partial_\nu V(x,t) \mid_{S_\infty}$, where ν is the unit normal vector at S pointing outside of the domain G_1. For $j = 1,2$ consider functions $W_j(x,t) = \partial_t V_j(x,t)$. Then

$$\partial_t^2 W_j = \Delta W_j - q_j(x) W_j, (x,t) \in G_1 \times (0,\infty), \tag{96}$$

$$W_j(x,0) = g(x), \partial_t W_j(x,0) = 0, \tag{97}$$

$$W_j(x,t) \mid_{S_\infty} = \eta_t(x,t), \tag{98}$$

$$\partial_\nu W_j(x,t) \mid_{S_\infty} = \varphi_t(x,t). \tag{99}$$

In addition, (74) implies that

$$W_j \in C^3\left(\overline{\Omega} \times [0,T]\right), \forall T > 0. \tag{100}$$

Finally, using Theorem 4.7 of [19], we obtain that relations (96)-(100) imply that $q_1(x) = q_2(x)$ in Ω. To finish the proof, we refer to (92). □

Remark 2. Let the number $T > diam(G_1)/2$. Then one can replace in (96) the time interval $(0,\infty)$ with $(0,T)$ and also replace in (98), (99) S_∞ with $S_T = S \times (0,T)$. These new conditions (96), (98) and (99) together with (97) and (100) still imply that $q_1(x) = q_2(x)$ in Ω, see either Theorem 1.10.5.2 of [4], or Theorem 3.2.2 of [20], or Theorem 3.2 of [21].

5.3 Proof of Theorem 4

Consider again an arbitrary point $\widetilde{x} \in S$. For $j = 1,2$ let $h_j(k) = v_{s,j}(\widetilde{x},k)$ and $\widehat{h}_j(t) = \mathcal{F}^{-1}(h_j)$. Using (71) and (78), we obtain similarly with (85)

$$\widehat{h}_1(t) = \widehat{h}_2(t) = V_s(\widetilde{x},t), \ \forall t \in (0,\varepsilon). \tag{101}$$

We again apply arguments, which are completely analogous to those of section 3.3. Also, we keep the same notations (46), (47) and (54). Hence, we obtain from (101) the following analog of (87)

$$\int_0^t V_s(\widetilde{x}, t-\tau)\lambda(\tau)\,d\tau = 0, \ \forall t \in (0,\varepsilon). \tag{102}$$

Differentiate (102) four times and use (74) and (82). We obtain the following analog of the Volterra integral equation (88)

$$\lambda(t) - \frac{1}{(qg)(\widetilde{x})}\int_0^t \partial_t^4 V_s(\widetilde{x}, t-\tau)\lambda(\tau)\,d\tau = 0, \ \forall t \in (0,\varepsilon).$$

Hence, similarly with subsection 3.3 we conclude that $\lambda(t) = 0$, $\forall t \geq 0$, which implies that $v_{s,1}(\widetilde{x},k) = v_{s,2}(\widetilde{x},k), \forall k \in \mathbb{R}$. Since $\widetilde{x} \in S$ is an arbitrary point, then

$$v_{s,1}(x,k) = v_{s,2}(x,k), \ \forall x \in S, \forall k \in \mathbb{R}. \tag{103}$$

Since $v_j(x,k) = v_{s,j}(x,k) + v_0(x,k)$, then (103) implies that (90) is valid again. The rest of the proof is the same as the proof of Theorem 3 after (90). □

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