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Controlling General Polynomial Networks

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Abstract: We consider networks of massive particles connected by non-linear springs. Some particles interact with heat baths at different temperatures, which are modeled as stochastic driving forces. The structure of the network is arbitrary, but the motion of each particle is 1D. For polynomial interactions, we give sufficient conditions for Hörmander's "bracket condition" to hold, which implies the uniqueness of the steady state (if it exists), as well as the controllability of the associated system in control theory. These conditions are constructive; they are formulated in terms of inequivalence of the forces (modulo translations) and/or conditions on the topology of the connections. We illustrate our results with examples, including "conducting chains" of variable cross-section. This then extends the results for a simple chain obtained in [5] .

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1. Introduction

We consider a network of interacting particles described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a set $\mathcal{V}$ of vertices and a set $\mathcal{E}$ of edges. Each vertex represents a particle, and each edge represents a spring connecting two particles. We single out a set $\mathcal{V}_* \subset \mathcal{V}$

of particles, each of which interacts with a heat bath. We address the question of when such a system has a unique stationary state. This question has been studied for several special cases: Starting from a linear chain [5,4], results have become more refined in terms of the relation between the spring potentials and the pinning potentials which tie the masses to the laboratory frame [11,2]. This problem is very delicate, as is apparent from the extensive study in [8] for the case of only 2 masses.

We provide conditions on the interaction potentials that imply Hörmander's "bracket condition," from which it follows that the semigroup associated to the process has a smoothing effect. This, together with some stability assumptions, implies the *uniqueness* of the stationary state. The *existence* is not discussed in this paper, but seems well understood in the sense that the interaction potentials must be somehow stronger than the pinning potentials. This issue will be explained in a forthcoming paper [3].

Since the problem is known (see for example [7] and [1]) to be tightly related to the control problem where the stochastic driving forces are replaced with deterministic control forces, we shall use the terminology of control theory, and mention the implications of our results from the control-theoretic viewpoint.

We work with unit masses and interaction potentials that are polynomials of degree at least 3, and we say that two such potentials V_1 and V_2 have *equivalent second derivative* if there is a $\delta \in \mathbf{R}$ such that $V_1''(\,\cdot\,) = V_2''(\,\cdot + \delta)$.

We start with the set $\mathcal{V}_*$ of particles that interact with heat baths, and are therefore *controllable*. One of our results (Corollary 5.6) is formulated as a condition for some of the particles in the set of first neighbors $\mathcal{N}(\mathcal{V}_*)$ of $\mathcal{V}_*$ to be also controllable. Basically, the condition is that these particles must be "inequivalent" in a sense that involves both the topology of their connections to $\mathcal{V}_*$ and the corresponding interaction potentials. More precisely, a sufficient condition for a particle $v \in \mathcal{N}(\mathcal{V}_*)$ to be controllable is that for each other particle $w \in \mathcal{N}(\mathcal{V}_*)$ at least one of the two conditions holds:

(a) v and w are connected to $\mathcal{V}_*$ in a topologically different way,
(b) there is a particle c in $\mathcal{V}_*$ such that the interaction potential between c and v and that between c and w have inequivalent second derivative.

It is then possible to use this condition recursively, taking control of more and more masses at each step (Theorem 5.7). If by doing so we can control all the masses in the graph, then Hörmander's bracket condition holds.

In Sect. 6 we give examples of physically relevant networks whose controllability can be established using this method.

Our results imply in particular that connected graphs are controllable for "almost all" choices of the interaction potentials, provided that they are polynomials of degree at least 3 (Corollary 6.3).

2. The system

We define a Hamiltonian for the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows. Each particle $v \in \mathcal{V}$ has position $q_v \in \mathbf{R}$ and momentum $p_v \in \mathbf{R}$ and is "pinned down" by a potential $U_v(q_v)$. Throughout, we assume the masses being 1, for simplicity of notation. See Remark 4.20 on how to adapt the results when the masses are not all equal.

We denote each edge $e \in \mathcal{E}$ by $\{u, v\}$ (or equivalently $\{v, u\}$) where u, v are the vertices adjacent to e.[1] To each edge $e = \{u, v\}$, we associate an interaction potential

[1] Due to the physical nature of the problem, we assume that the graph has no self-edge.

$V_{uv}(q_u - q_v)$, or equivalently $V_{vu}(q_v - q_u)$ with

$$V_{vu}(q_v - q_u) \equiv V_{uv}(q_u - q_v) \; . \tag{2.1}$$

Note that we do not require the potentials to be even functions; the condition (2.1) just makes sure that considering $e = \{u, v\}$ or $e = \{v, u\}$ leads to the same physical interaction, which is consistent with the fact that the edges are not oriented.

With the notation $q = (q_v)_{v \in \mathcal{V}}$ and $p = (p_v)_{v \in \mathcal{V}}$ the Hamiltonian is then

$$H(q, p) = \sum_{v \in \mathcal{V}} \big(p_v^2/2 + U_v(q_v)\big) + \sum_{e \in \mathcal{E}} V_e(\delta q_e) \; ,$$

where it is understood that $V_e(\delta q_e)$ denotes one of the two expressions in (2.1).

Finally, we make the following assumptions:

Assumption 2.1

1. *All functions are smooth.*
2. *The level sets of H are compact, i.e., for each $K > 0$ the set $\{(q, p) \mid H(q, p) \le K\}$ is compact.*
3. *The function* $\exp(-\beta H)$ *is integrable for some $\beta > 0$.*

Each particle $v \in \mathcal{V}_*$ is coupled to a heat bath at temperature $T_v > 0$ with coupling constant $\gamma_v > 0$. For convenience, we set $\gamma_v = 0$ when $v \notin \mathcal{V}_*$. The model is then described by the system of stochastic differential equations

$$\begin{aligned} dq_v &= p_v \, dt \; , \\ dp_v &= -U_v'(q_v) dt - \partial_{q_v} \Big(\sum_{e \in \mathcal{E}} V_e(\delta q_e) \Big) dt - \gamma_v p_v dt + \sqrt{2 T_v \gamma_v} \, dW_v(t) \; , \end{aligned} \tag{2.2}$$

where the W_v are identical independent Wiener processes. The solutions to (2.2) form a Markov process. The generator of the associated semigroup is given by

$$L \equiv X_0 + \sum_{v \in \mathcal{V}_*} \gamma_v T_v \partial_{p_v}^2 \; ,$$

with

$$X_0 \equiv - \sum_{v \in \mathcal{V}_*} \gamma_v p_v \partial_{p_v} + \sum_{v \in \mathcal{V}} \big(p_v \partial_{q_v} - U_v'(q_v) \partial_{p_v} \big) - \sum_{\{u,v\} \in \mathcal{E}} V_{uv}'(q_u - q_v) \cdot (\partial_{p_u} - \partial_{p_v}) \; .$$

From now on, we assume that the interaction potentials V_e, $e \in \mathcal{E}$ are *polynomials of degree at least 3*. The condition on the degree means that we require throughout the presence of non-harmonicities. The fully-harmonic case has been described earlier [6], and the case where some but not all the potentials are harmonic is not covered here. We will show in a counter-example (Example 6.8) that the non-harmonicities are really essential for our results. We make no assumption about the pinning potentials U_v; we do not require them to be polynomials, and some or all of them may be identically zero.

We work in the space $\mathbf{R}^{2|\mathcal{V}|}$ with coordinates $x = (q, p)$. We identify the vector fields over $\mathbf{R}^{2|\mathcal{V}|}$ and the corresponding first-order differential operators in the usual way. This enables us to consider Lie algebras of vector fields over $\mathbf{R}^{2|\mathcal{V}|}$, where the Lie bracket $[\,\cdot\, , \,\cdot\,]$ is the usual commutator of two operators.

Definition 2.2. *We define $\mathcal{M}$ as the smallest Lie algebra that*

(i) contains ∂_{p_v} for all $v \in \mathcal{V}_*$,[2]
(ii) is closed under the operation $[\,\cdot\,, X_0]$,
(iii) is closed under multiplication by smooth scalar functions.

By the definition of a Lie algebra, $\mathcal{M}$ is closed under linear combinations and Lie brackets.

Definition 2.3. *We say that a particle $v \in \mathcal{V}$ is* controllable *if we have $\partial_{q_v}, \partial_{p_v} \in \mathcal{M}$. We say that the network $\mathcal{G}$ is* controllable *if all the particles are controllable, i.e., if*

$$\partial_{q_v}, \partial_{p_v} \in \mathcal{M} \qquad \textit{for all } v \in \mathcal{V}\ . \tag{2.3}$$

The aim of this paper is to give sufficient conditions on $\mathcal{G}$ and the interaction potentials, which guarantee that the network is controllable.

If the network is controllable in the sense (2.3), then Hörmander's condition[3] [9] holds: for all x, the vector fields $F \in \mathcal{M}$ evaluated at x span all of $\mathbf{R}^{2|\mathcal{V}|}$, *i.e.*,

$$\{F(x) \mid F \in \mathcal{M}\} = \mathbf{R}^{2|\mathcal{V}|} \quad \text{for all } x \in \mathbf{R}^{2|\mathcal{V}|}\ . \tag{2.4}$$

Hörmander's condition implies that the transition probabilities of the Markov process (2.2) are smooth, and that so is any invariant measure (see for example [1, Cor. 7.2]). We now briefly mention two implications of these smoothness properties. Proposition 2.4 and Proposition 2.5 below can be deduced from arguments similar to those exposed in [7], and will be discussed in more detail in the forthcoming paper [3].

Proposition 2.4. *Under Assumption 2.1, if (2.4) holds, then the Markov process (2.2) has at most one invariant probability measure.*

The control-theoretic problem corresponding to (2.2) is the system of ordinary differential equations

$$\begin{aligned} \dot q_v &= p_v, \\ \dot p_v &= -U_v'(q_v) - \partial_{q_v}\Big(\sum_{e\in\mathcal{E}} V_e(\delta q_e)\Big) + (u_v(t) - \gamma_v p_v)\cdot \mathbf{1}_{v\in\mathcal{V}_*}\ , \end{aligned} \tag{2.5}$$

where for each $v \in \mathcal{V}_*$, $u_v : \mathbf{R} \to \mathbf{R}$ is a smooth *control function* (*i.e.*, the stochastic driving forces have been replaced with deterministic functions).[4]

Proposition 2.5. *Under the hypotheses of Proposition 2.4, the system (2.5) is* controllable *in the sense that for each $x^{(0)} = (q^{(0)}, p^{(0)})$ and $x^{(f)} = (q^{(f)}, p^{(f)})$, there are a time T and some smooth controls u_v, $v \in \mathcal{V}_*$, such that the solution $x(t)$ of (2.5) with $x(0) = x^{(0)}$ verifies $x(T) = x^{(f)}$.*

In fact, (2.4) is a well-known condition in control theory. See for example [10], which addresses the case of piecewise constant control functions. In particular, (2.4) implies by [10, Thm. 3.3] that for every initial condition $x^{(0)}$ and each time $T > 0$, the set $A(x^{(0)}, T)$ of all points that are accessible at time T (by choosing appropriate controls) is connected and full-dimensional.

[2] Due to the identification mentioned above, we view here ∂_{p_v} as a constant vector field over $\mathbf{R}^{2|\mathcal{V}|}$.

[3] The condition (2.4) is slightly different, but equivalent to the usual statement of Hörmander's criterion. This can be checked easily. In particular, closing $\mathcal{M}$ under multiplication by smooth scalar functions does not alter the set in (2.4), and will be very convenient.

[4] Whether or not we keep the dissipative terms $-\gamma_v p_v$ in (2.5) makes no difference since they can always be absorbed in the control functions.

3. Strategy

We want to show that $\partial_{q_v}, \partial_{p_v} \in \mathcal{M}$ for all $v \in \mathcal{V}$. The next lemma shows that we only need to worry about the ∂_{p_v}.

Lemma 3.1. *Let A be a subset of $\mathcal{V}$.*

$$\text{If} \quad \sum_{v\in A} \partial_{p_v} \in \mathcal{M} \quad \text{then} \quad \sum_{v\in A} \partial_{q_v} \in \mathcal{M} \; .$$

Proof. Assuming $\sum_{v\in A} \partial_{p_v} \in \mathcal{M}$, we find that

$$\big[\sum_{v\in A} \partial_{p_v}, X_0\big] = \sum_{v\in A} \partial_{q_v} - \sum_{v\in \mathcal{V}_*\cap A} \gamma_v \partial_{p_v} \tag{3.1}$$

is in $\mathcal{M}$. But since $\partial_{p_v} \in \mathcal{M}$ for all $v \in \mathcal{V}_*$, the linear structure of $\mathcal{M}$ implies $\sum_{v\in\mathcal{V}_*\cap A} \gamma_v \partial_{p_v} \in \mathcal{M}$. Adding this to the vector field in (3.1) shows that $\sum_{v\in A} \partial_{q_v} \in \mathcal{M}$, as claimed. □

Definition 3.2. *We say that a set $A \subset \mathcal{V}$ is* jointly controllable *if $\sum_{v\in A} \partial_{p_v}$ is in $\mathcal{M}$ (and therefore, also $\sum_{v\in A} \partial_{q_v}$ by Lemma 3.1).*

Requiring all the particles in a set A to be (individually) controllable is stronger than asking the set A to be jointly controllable (indeed, if all the ∂_{p_v}, $v \in A$ are in $\mathcal{M}$, then so is their sum). We will obtain jointly controllable sets and then "refine" them until we control particles individually.

The strategy is as follows. In the next section, we start with a controllable particle c, and show that its first neighbors split into jointly controllable sets. Then, in Sect. 5, we consider several controllable particles, and basically intersect the jointly controllable sets obtained for each of them, in order to control "new" particles individually. Finally, we iterate this procedure, taking control of more particles at each step, until we establish (under some conditions) the controllability of the whole network.

Remark 3.3. Observe in the following that our results neither involve the interaction potentials U_v nor the coupling constants γ_v.

4. The neighbors of one controllable particle

We consider in this section a particle $c \in \mathcal{V}$, and denote by $\mathcal{T}^c$ the set of its first neighbors (the "targets"). The following notion of equivalence among polynomials enables us to split $\mathcal{T}^c$ into equivalence classes.

Definition 4.1. *Two polynomials f and g are called* equivalent *if there is a $\delta \in \mathbf{R}$ such that $f(\,\cdot\,) = g(\,\cdot\, + \delta)$.*

Definition 4.2. *We say that two particles $v, u \in \mathcal{T}^c$ are equivalent (with respect to c) if the two polynomials V''_{cv} and V''_{cu} are equivalent.*

Since this relation is symmetric and transitive, the set $\mathcal{T}^c$ is naturally decomposed into a disjoint union of equivalence classes:

$$\mathcal{T}^c = \cup_i \mathcal{T}^c_i \; .$$

An explanation of why we use the second derivative of the potentials instead of the first one (*i.e.*, the force) will be given in Example 6.7. The main result of this section is

Theorem 4.3. *Assume that c is controllable. Then, each equivalence class $\mathcal{T}_i^c$ is jointly controllable, i.e.,*

$$\sum_{v\in\mathcal{T}_i^c} \partial_{p_v} \in \mathcal{M} \qquad \text{for all } i\ . \tag{4.1}$$

Furthermore, there are constants δ_{cv} such that for all i,

$$\sum_{v\in\mathcal{T}_i^c} (q_c - q_v + \delta_{cv})\partial_{p_v} \in \mathcal{M}\ . \tag{4.2}$$

The second part of the theorem will be used in the next section to intersect the equivalence classes $\mathcal{T}_i^c$ of several controllable particles c. We will now prepare the proof of Theorem 4.3. We assume in the remainder of this section that c is controllable. And since c is fixed, we write $\mathcal{T}$ and $\mathcal{T}_i$ instead of $\mathcal{T}^c$ and $\mathcal{T}_i^c$.

Lemma 4.4. *We have*

$$\sum_{v\in\mathcal{T}} V''_{cv}(q_c - q_v)\partial_{p_v} \in \mathcal{M}\ . \tag{4.3}$$

Proof. From Lemma 3.1 we conclude that $\partial_{q_c} \in \mathcal{M}$. Therefore, we find that

$$[\partial_{q_c}, X_0] = -U''_c(q_c)\partial_{p_c} - \sum_{v\in\mathcal{T}} V''_{cv}(q_c - q_v)\cdot(\partial_{p_c} - \partial_{p_v})$$

is in $\mathcal{M}$. Now, since $\partial_{p_c} \in \mathcal{M}$ and since $\mathcal{M}$ is closed under multiplication by scalar functions, we can subtract all the contributions that are along ∂_{p_c} and obtain (4.3). □

We need a bit of technology to deal with equivalent polynomials.

Definition 4.5. *Let $g(t) = \sum_{i=0}^k a_i t^i/i!$ be a polynomial of degree $k \geq 1$. If $a_{k-1} = 0$, we say that g is* adjusted. *As can be checked, the polynomial $\tilde{g}(\,\cdot\,) \equiv g(\,\cdot\, - a_{k-1}/a_k)$ is always adjusted, and is referred to as the* adjusted representation *of g.*

Observe that a polynomial and its adjusted representation are by construction equivalent and have the same degree and the same leading coefficient. In fact, given a polynomial g of degree $k \geq 1$, $\tilde{g}$ is the only polynomial equivalent to g that is adjusted. This adjusted representation will prove to be very useful thanks to the following obvious

Lemma 4.6. *Two polynomials g and h of degree at least 1 are equivalent iff $\tilde{g} = \tilde{h}$.*

Remark 4.7. If all the interaction potentials are *even*, then all the V''_{cv} are automatically adjusted, and some parts of the following discussion can be simplified.

We shift the argument of each V''_{cv} by a constant δ_{cv} so that they are all adjusted. We let $\tilde{f}_v$ be the adjusted representation of V''_{cv} and use the notation

$$x_v \equiv q_c - q_v + \delta_{cv}$$

so that

$$\tilde{f}_v(x_v) = V''_{cv}(q_c - q_v) \qquad \text{for all} \qquad q_c, q_v \in \mathbf{R}\ .$$

With this notation, (4.3) reads as

$$\sum_{v\in\mathcal{T}} \tilde{f}_v(x_v)\partial_{p_v} \in \mathcal{M}\ . \tag{4.4}$$

We will now mostly deal with "diagonal" vector fields, *i.e.*, vector fields of the kind (4.4), where the component along ∂_{p_v} depends only on x_v. When taking commutators, it is crucial to remember that x_v is only a notation for $q_c - q_v + \delta_{cv}$.

Remark 4.8. By the definition of equivalence and Lemma 4.6, two particles $v, w \in \mathcal{T}$ are equivalent iff $\tilde{f}_v$ and $\tilde{f}_w$ coincide.

Lemma 4.9. *Consider some functions g_v, $v \in \mathcal{T}$.*

$$\text{If} \quad \sum_{v\in\mathcal{T}} g_v(x_v)\partial_{p_v} \in \mathcal{M} \qquad \text{then} \qquad \sum_{v\in\mathcal{T}} g'_v(x_v)\partial_{p_v} \in \mathcal{M} \; . \tag{4.5}$$

Proof. This is immediate by commuting with ∂_{q_c} (which is in $\mathcal{M}$ by Lemma 3.1). □

We now introduce the main tool.

Definition 4.10. *Given two vector fields Y and Z, we define the* double commutator $[\![Y : Z]\!]$ *by*

$$[\![Y : Z]\!] \equiv [[X_0, Y], Z] \; .$$

Obviously, if the vector fields Y and Z are in $\mathcal{M}$, then so is $[\![Y : Z]\!]$.

Lemma 4.11. *Consider some functions g_v, h_v, $v \in \mathcal{T}$. Then*

$$[\![\sum_{v\in\mathcal{T}} g_v(x_v)\partial_{p_v} : \sum_{v'\in\mathcal{T}} h_{v'}(x_{v'})\partial_{p_{v'}}]\!] = \sum_{v\in\mathcal{T}} (g_v h_v)'(x_v)\, \partial_{p_v} \; . \tag{4.6}$$

Proof. Observe first that (omitting the arguments x_v)

$$\big[\, X_0, \sum_{v\in\mathcal{T}} g_v\partial_{p_v} \,\big] = \sum_{v\in\mathcal{T}} (p_c - p_v) g'_v \partial_{p_v} - \sum_{v\in\mathcal{T}} g_v\partial_{q_v} + \sum_{v\in\mathcal{T}\cap\mathcal{V}_*} \gamma_v g_v \partial_{p_v} \; .$$

Commuting with $\sum_{v'\in\mathcal{T}} h_{v'}(x_{v'})\partial_{p_{v'}}$ gives the desired result. □

We will prove Theorem 4.3 starting from (4.4) and using only (4.5) and double commutators of the kind (4.6).

Let d_v be the degree of $\tilde{f}_v$. Note that since the interaction potentials are of degree at least 3, we have $d_v \geq 1$. We define

$$d \equiv \max_{v\in\mathcal{T}} d_v \geq 1$$

as the maximal degree of the adjusted polynomials $\tilde{f}_v$ with $v \in \mathcal{T}$. We can then write

$$\tilde{f}_v(x) = \sum_{j=0}^{d} b_{vj}\, x^j/j! \; ,$$

for some real coefficients b_{vj}, $j = 0, \ldots, d$, with

$$b_{vj} = 0 \quad \text{if} \quad j > d_v \quad \text{and} \quad b_{v,d_v-1} = 0 \; ,$$

for all $v \in \mathcal{T}$.

Definition 4.12. *We define the set of particles $v \in \mathcal{T}$ corresponding to the maximal degree d:*

$$\mathcal{D}^{\,d} \equiv \{v \in \mathcal{T} \mid d_v = d\} \; .$$

For every ℓ, $0 \leq \ell \leq d$, we define the set

$$\mathcal{B}^d_\ell \equiv \{b_{v\ell} \mid v \in \mathcal{D}^{\,d}\}$$

of distinct *values taken by the coefficients of $x_v^\ell/\ell!$ in $\tilde{f}_v$, $v \in \mathcal{D}^{\,d}$.*

We begin with a technical lemma. Observe how it is expressed in terms of the x_v. In a sense, this shows that the x_v are really the "natural" variables for this problem. Thus, in addition to making the notion of equivalence trivial (Remark 4.8), working with adjusted representations will be very convenient from a technical point of view.

Lemma 4.13. *The following hold:*

(i) For each $b \in \mathcal{B}_d^d$, *we have*

$$\sum_{v \in \mathcal{D}^d \,:\, b_{vd}=b} x_v \partial_{p_v} \in \mathcal{M} \quad \textit{and} \quad \sum_{v \in \mathcal{D}^d \,:\, b_{vd}=b} \partial_{p_v} \in \mathcal{M} \,. \tag{4.7}$$

(ii) Furthermore,

$$\sum_{v \in \mathcal{D}^d} x_v \partial_{p_v} \in \mathcal{M} \quad \textit{and} \quad \sum_{v \in \mathcal{D}^d} \partial_{p_v} \in \mathcal{M} \,. \tag{4.8}$$

(iii) Let α_v, β_v, $v \in \mathcal{D}^d$ *be real constants. If* $d \geq 2$, *we have the two implications*

$$\textit{if} \sum_{v \in \mathcal{D}^d} \alpha_v\, \partial_{p_v} \in \mathcal{M}\,, \quad \textit{then} \quad \sum_{v \in \mathcal{D}^d} \alpha_v\, x_v\, \partial_{p_v} \ \in \mathcal{M}\,, \tag{4.9}$$

$$\textit{if} \sum_{v \in \mathcal{D}^d} \alpha_v\, \partial_{p_v}, \ \sum_{v \in \mathcal{D}^d} \beta_v\, \partial_{p_v} \in \mathcal{M}\,, \quad \textit{then} \quad \sum_{v \in \mathcal{D}^d} \alpha_v\, \beta_v\, \partial_{p_v} \ \in \mathcal{M}\,. \tag{4.10}$$

Remark 4.14. Observe that the assumption $d \geq 1$ is crucial in the proof of (i). Requiring the $\tilde{f}_v$ to be non-constant ensures that we can find non-trivial double commutators, which is the crux of our analysis. See Example 6.8 for what goes wrong for harmonic potentials.

Proof. (i). By (4.4) and using (4.5) recursively $d-1$ times, we find that

$$Y \equiv \sum_{v \in \mathcal{T}} \bigl(\partial^{d-1} \tilde{f}_v\bigr)(x_v)\, \partial_{p_v} = \sum_{v \in \mathcal{T}} (b_{v,d-1} + b_{vd} x_v)\, \partial_{p_v}$$

is in $\mathcal{M}$. But now, by (4.6),

$$[\![Y : Y/2]\!] = \sum_{v \in \mathcal{T}} b_{vd} (b_{v,d-1} + b_{vd} x_v) \partial_{p_v} \in \mathcal{M} \,.$$

Taking more double commutators with $Y/2$, we obtain for all $r \geq 1$:

$$\sum_{v \in \mathcal{T}} b_{vd}^r\, (b_{v,d-1} + b_{vd} x_v) \partial_{p_v} \in \mathcal{M} \,.$$

But the sum above is really only over $\mathcal{D}^d$ since $b_{vd} \neq 0$ only if $v \in \mathcal{D}^d$. Moreover, for these v, we have $b_{v,d-1} = 0$ since the polynomials are adjusted, so that for all $i \geq 2$,

$$\sum_{v \in \mathcal{D}^d} b_{vd}^i x_v \partial_{p_v} \in \mathcal{M} \,. \tag{4.11}$$

Let $b \in \mathcal{B}_d^d$. Using Lemma A.1 with $s = 1$ and with the set of distinct and non-zero values $\{b_{vd} \mid v \in \mathcal{D}^d\} = \mathcal{B}_d^d$ we find real numbers $r_1, r_2, \ldots, r_n$ (with $n = |\mathcal{B}_d^d|$) such that $\sum_{i=1}^n r_i \, b_{vd}^{i+1}$ equals 1 if $b_{vd} = b$ and 0 when $b_{vd} \neq b$. Thus,

$$\sum_{i=1}^n r_i \sum_{v \in \mathcal{D}^d} b_{vd}^{i+1} x_v \partial_{p_v} = \sum_{v \in \mathcal{D}^d \,:\, b_{vd}=b} x_v \partial_{p_v}$$

is in $\mathcal{M}$ by (4.11). This together with (4.5) establishes the second inclusion of (4.7), so that we have shown (i).

The statement (ii) follows by summing (i) over all $b \in \mathcal{B}_d^d$.

Proof of (iii). Let us assume that $d \geq 2$ and that $\sum_{v \in \mathcal{D}^d} \alpha_v \, \partial_{p_v} \in \mathcal{M}$. By (4.7), we have for each $b \in \mathcal{B}_d^d$ that

$$\frac{1}{b} \sum_{v \in \mathcal{D}^d \,:\, b_{vd}=b} x_v \partial_{p_v} = \sum_{v \in \mathcal{D}^d \,:\, b_{vd}=b} \frac{x_v}{b_{vd}} \partial_{p_v} \in \mathcal{M} \; .$$

Taking the double commutator with $\sum_{v \in \mathcal{D}^d} \alpha_v \; \partial_{p_v}$ and summing over all $b \in \mathcal{B}_d^d$ shows that

$$U \equiv \sum_{v \in \mathcal{D}^d} \frac{\alpha_v}{b_{vd}} \partial_{p_v} \in \mathcal{M} \; .$$

Since we assume here $d \geq 2$, we have $Z \equiv \sum_{v \in \mathcal{T}} \big(\partial^{d-2} \tilde{f}_v\big)(x_v) \partial_{p_v} \in \mathcal{M}$. But then,

$$[\![U : Z]\!] = \sum_{v \in \mathcal{T}} \frac{\alpha_v}{b_{vd}} \big(\partial^{d-1} \tilde{f}_v\big)(x_v) \partial_{p_v} = \sum_{v \in \mathcal{D}^d} \frac{\alpha_v}{b_{vd}} \left(b_{v,d-1} + b_{vd} \, x_v\right) \partial_{p_v}$$

is also in $\mathcal{M}$. Recalling that $b_{v,d-1} = 0$ for all $v \in \mathcal{D}^d$, we obtain (4.9). Finally (4.10) follows from (4.9) and the double commutator

$$[\![\sum_{v \in \mathcal{D}^d} \alpha_v \; x_v \; \partial_{p_v} : \sum_{v \in \mathcal{D}^d} \beta_v \; \partial_{p_v}]\!] = \sum_{v \in \mathcal{D}^d} \alpha_v \; \beta_v \; \partial_{p_v} \; .$$

This completes the proof. □

With these preparations, we can now prove Theorem 4.3.

Proof (of Theorem 4.3).

We distinguish the cases $d = 1$ and $d \geq 2$.

Case $d = 1$: This case is easy. Since all the $\tilde{f}_v$ have degree 1, we have that $\tilde{f}_v(x_v) = b_{v1} \; x_v$ for all $v \in \mathcal{T}$, with $b_{v1} \neq 0$. Consequently, the sets $\mathcal{T}_i$ consist of those v which have the same b_{v1} (see Remark 4.8). Thus, we have by (4.7) that for each $\mathcal{T}_i$:

$$\sum_{v \in \mathcal{T}_i} \partial_{p_v} \in \mathcal{M} \quad \text{and} \quad \sum_{v \in \mathcal{T}_i} x_v \partial_{p_v} \in \mathcal{M} \qquad (\text{if } d = 1) \; . \tag{4.12}$$

This shows that the conclusion of Theorem 4.3 holds when $d = 1$.

Case $d \geq 2$: In this case, (4.7) is not enough. First, (4.7) says nothing about the masses $v \in \mathcal{T} \setminus \mathcal{D}^d$, for which $b_{vd} = 0$. And second, (4.7) provides us with no way to “split” the ∂_{p_v} corresponding to a common (non-zero) value b of b_{vd}, even though the corresponding v might be inequivalent due to some b_{vk} with $k < d$. To fully make use of these coefficients, we must develop some more advanced machinery.

Definition 4.15. *We denote by $\mathcal{P}_d$ the vector space of real polynomials in one variable of degree at most d. We consider the operator $\mathrm{G} : \mathcal{P}_d \to \mathcal{P}_d$ defined by*

$$(\mathrm{G}v)(x) \equiv (x \cdot v(x))' \ ,$$

and we introduce the set of operators

$$\mathcal{F} \equiv \mathrm{span}\{\mathrm{G}, \mathrm{G}^2, \dots, \mathrm{G}^{d+1}\} \ .$$

Observe that by (4.4) and (4.8) we have

$$[\![\sum_{v\in\mathcal{T}} \tilde{f}_v(x_v)\partial_{p_v} : \sum_{v\in\mathcal{D}^{\,d}} x_v\partial_{p_v}]\!] = \sum_{v\in\mathcal{D}^{\,d}} (\mathrm{G}\tilde{f}_v)(x_v)\partial_{p_v} \in \mathcal{M} \ .$$

Note that we obtain a sum over $\mathcal{D}^{\,d}$ only. By taking more double commutators with $\sum_{v\in\mathcal{D}^{\,d}} x_v\partial_{p_v}$, we find that $\sum_{v\in\mathcal{D}^{\,d}}(\mathrm{G}^k\tilde{f}_v)(x_v)\partial_{p_v}$ is in $\mathcal{M}$ for all $k \geq 1$. Thus, by the linear structure of $\mathcal{M}$, we obtain

Lemma 4.16. *For all $\mathrm{P} \in \mathcal{F}$, we have*

$$\sum_{v\in\mathcal{D}^{\,d}} (\mathrm{P}\tilde{f}_v)(x_v)\partial_{p_v} \in \mathcal{M} \ .$$

It is crucial to understand that it is the *same* operator P that is applied simultaneously to all the components, and that the components in $\mathcal{T} \setminus \mathcal{D}^{\,d}$ are "projected out."

We now show that some very useful operators are in $\mathcal{F}$.

Proposition 4.17. *The following hold:*

(i) The projector

$$\mathrm{S}_\ell : \mathcal{P}_d \to \mathcal{P}_d \ , \quad \sum_{i=0}^{d} b_i x^i/i! \mapsto b_\ell x^\ell/\ell!$$

belongs to $\mathcal{F}$ for all $\ell = 0, \dots, d$.

(ii) The identity operator $\mathbf{1}$ acting on $\mathcal{P}_d$ is in $\mathcal{F}$.

Proof. Consider the basis $B = (e_0, e_1, \dots, e_d)$ of $\mathcal{P}_d$ where $e_j(x) = x^j/j!$. Observe that for all $j \geq 0$ we have $\mathrm{G}e_j = (j+1)e_j$, so that G is diagonal in the basis B. Thus, G^k is represented by the matrix $\mathrm{diag}(1^k, 2^k, \dots, (d+1)^k)$ for all $k \geq 1$. Consequently, for each $\ell \in \{0, \dots, d\}$, there is by Lemma A.1 with $s = 0$ a linear combination of $\mathrm{G}, \mathrm{G}^2, \dots, \mathrm{G}^{d+1}$ that is equal to S_ℓ. This proves (i). Moreover, we have that $\sum_{\ell=0}^{d} \mathrm{S}_\ell = \mathbf{1}$, so that the proof of (ii) is complete. □

Lemma 4.18. *For all $\ell = 0, \dots, d$, and for each $b \in \mathcal{B}_\ell^d = \{b_{v\ell} \mid v \in \mathcal{D}^{\,d}\}$ we have*

$$\sum_{v\in\mathcal{D}^{\,d} \,:\, b_{v\ell}=b} \partial_{p_v} \in \mathcal{M} \ . \tag{4.13}$$

Proof. Let $\ell \in \{0, 1, \ldots, d\}$. Using Lemma 4.16 and Proposition 4.17(i) we find that $\sum_{v\in\mathcal{D}^d}(b_{v\ell}x^\ell/\ell!)\partial_{p_v}$ is in $\mathcal{M}$. Using (4.5) repeatedly, we find that $\sum_{v\in\mathcal{D}^d} b_{v\ell}\partial_{p_v}$ is in $\mathcal{M}$. Thus, by (4.10),

$$\sum_{v\in\mathcal{D}^d} b^i_{v\ell}\partial_{p_v} \in \mathcal{M} \qquad \text{for all } i \geq 1.$$

Then, applying Lemma A.1 to the set $\mathcal{B}^d_\ell \setminus \{0\}$ and with $s = 0$, we conclude that

$$\sum_{v\in\mathcal{D}^d\,:\,b_{v\ell}=b} \partial_{p_v} \in \mathcal{M} \quad \text{for all} \quad b \in \mathcal{B}^d_\ell \setminus \{0\}\ . \tag{4.14}$$

If $0 \notin \mathcal{B}^d_\ell$, we are done. Else, we obtain that (4.13) holds also for $b = 0$ by summing the vector field (4.14) over all $b \in \mathcal{B}^d_\ell \setminus \{0\}$ and subtracting the result from $\sum_{v\in\mathcal{D}^d}\partial_{p_v}$ (which is in $\mathcal{M}$ by (4.8)). This completes the proof. □

Remember that by Remark 4.8, a given equivalence class $\mathcal{T}_i$ is either a subset of $\mathcal{D}^d$ or completely disjoint from it.

Lemma 4.19. *Let $\mathcal{T}_i$ be an equivalence class such that $\mathcal{T}_i \subset \mathcal{D}^d$. Then*

$$\sum_{v\in\mathcal{T}_i}\partial_{p_v} \in \mathcal{M}\ , \quad \textit{and} \quad \sum_{v\in\mathcal{T}_i} x_v\,\partial_{p_v} \in \mathcal{M}\ . \tag{4.15}$$

Proof. All the polynomials $\tilde{f}_v$, $v \in \mathcal{T}_i$ are equal. Thus, there are coefficients $c_\ell \in \mathcal{B}^d_\ell$, $\ell = 0, 1, \ldots, d$ such that

$$\mathcal{T}_i = \bigcap_{\ell=0}^{d}\{v \in \mathcal{D}^d \mid b_{v\ell} = c_\ell\}\ . \tag{4.16}$$

By Lemma 4.18, we have for all $\ell = 0, \ldots, d$ that

$$\sum_{v\in\mathcal{D}^d\,:\,b_{v\ell}=c_\ell}\partial_{p_v} \in \mathcal{M}\ . \tag{4.17}$$

Now observe that whenever two sets $B, B' \subset \mathcal{D}^d$ are such that $\sum_{v\in B}\partial_{p_v} \in \mathcal{M}$ and $\sum_{v\in B'}\partial_{p_v} \in \mathcal{M}$, we have by (4.10) that $\sum_{v\in B\cap B'}\partial_{p_v} \in \mathcal{M}$. Applying this recursively to the intersection in (4.16) and using (4.17) shows that $\sum_{v\in\mathcal{T}_i}\partial_{p_v} \in \mathcal{M}$. Using now (4.9) implies that $\sum_{v\in\mathcal{T}_i} x_v\partial_{p_v} \in \mathcal{M}$, which completes the proof. □

With these tools, we are now ready to complete the proof of Theorem 4.3 (for the case $d \geq 2$). By Lemma 4.19, we are done if $\mathcal{D}^d = \mathcal{T}$ (*i.e.*, if all the $\tilde{f}_v$, $v \in \mathcal{T}$ have degree d). If this is not the case, we proceed as follows.

Observe that Lemma 4.16 and Proposition 4.17(ii) imply that $\sum_{v\in\mathcal{D}^d}\tilde{f}_v(x_v)\,\partial_{p_v}$ is in $\mathcal{M}$. Subtracting this from (4.4) shows that

$$\sum_{v\in\mathcal{T}\setminus\mathcal{D}^d}\tilde{f}_v(x_v)\partial_{p_v} \in \mathcal{M}\ .$$

Thus, we can start the above procedure again with this new "smaller" vector field, each component being a polynomial of degree at most

$$d' \equiv \max_{v\in\mathcal{T}\setminus\mathcal{D}^{\,d}} d_v \ ,$$

with obviously $d' < d$. Defining then $\mathcal{D}^{\,d'} = \{v \in \mathcal{T} \mid d_v = d'\}$, we get as in Lemma 4.19 that (4.15) holds for all $\mathcal{T}_i \subset \mathcal{D}^{\,d'}$. We then proceed like this inductively, dealing at each step with the components of highest degree and "removing" them, until all the remaining components have the same degree d^- (which is equal to $\min_{v\in\mathcal{T}} d_v$). If $d^- \geq 2$ we obtain again as in Lemma 4.19 that (4.15) holds for all $\mathcal{T}_i \subset \mathcal{D}^{\,d^-}$. And if $d^- = 1$, the conclusion follows from (4.12). Thus, (4.15) holds for every equivalence class $\mathcal{T}_i$, regardless of the degree of the polynomials involved. The proof of Theorem 4.3 is complete. □

Remark 4.20. Our method also covers the case where each particle $v \in \mathcal{V}$ can have an arbitrary positive mass m_v. The proofs work the same way, if we replace the functions $\tilde{f}_v$ with $\tilde{f}_v = V''_{cv}(x_v)/(m_c m_v)$. Thus, if for example all the V''_{cv}, $v \in \mathcal{T}$ are the same, but all the particles in $\mathcal{T}$ have distinct masses, then all the new $\tilde{f}_v$ are different, and the particles in $\mathcal{T}$ belong each to a separate $\mathcal{T}_i$.

5. Controlling a network

We now show how Theorem 4.3 can be used recursively to control a large class of networks. The idea is very simple: at each step of the recursion, we apply Theorem 4.3 to a controllable particle (or a set of such) in order to show that some neighboring vertices are also controllable. Starting this procedure with the particles in $\mathcal{V}_*$ (which are controllable by the definition of $\mathcal{M}$), we obtain under certain conditions that the whole network is controllable.

In order to make the distinction clear, we will say that a particle c is a *controller* if it is controllable and if we intend to use it as a starting point to control other particles.

Definition 5.1. *Let $\mathcal{J}$ be the collection of jointly controllable sets (i.e., of sets $A \subset \mathcal{V}$ such that $\sum_{v\in A} \partial_{p_v} \in \mathcal{M}$, and therefore also $\sum_{v\in A} \partial_{q_v}$ by Lemma 3.1).*

Obviously, a particle v is controllable iff $\{v\} \in \mathcal{J}$. The next lemma shows what we "gain" in $\mathcal{J}$ when we apply Theorem 4.3 to a controller c. Remember that the set $\mathcal{T}^c$ of first neighbors of c is partitioned into equivalence classes $\mathcal{T}_i^c$, as discussed in Sect. 4.

Lemma 5.2. *Let $c \in \mathcal{V}$ be a controller. Then,*

(i) for all i,

$$\mathcal{T}_i^c \in \mathcal{J} \ ,$$

(ii) for all i and for all $A \in \mathcal{J}$ the sets

$$A \cap \mathcal{T}_i^c, \qquad A \setminus \mathcal{T}_i^c \quad \textit{and} \quad \mathcal{T}_i^c \setminus A \tag{5.1}$$

are in $\mathcal{J}$.

We illustrate some possibilities in Fig. 5.1.

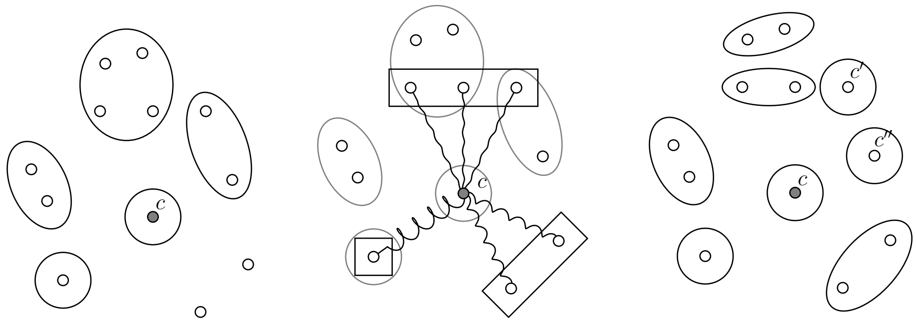


Fig. 5.1: a: A controller c and a few sets in $\mathcal{J}$, shown as ovals. b: The equivalence classes $\mathcal{T}_i^c$ are shown as rectangles. (Only the edges incident to c are shown.) c: New sets "appear" in $\mathcal{J}$. In particular, c' and c'' are controllable.

Proof. (i). This is an immediate consequence of (4.1) and the definition of $\mathcal{J}$.
(ii). We consider an equivalence class $\mathcal{T}_i^c$ and a set $A \in \mathcal{J}$. By (4.2) we find that

$$[\sum_{v\in\mathcal{T}_i^c}(q_c - q_v + \delta_{cv})\partial_{p_v}, \sum_{v\in A}\partial_{q_v}] = \sum_{v\in A\cap\mathcal{T}_i^c}\partial_{p_v} - \mathbf{1}_{c\in A}\cdot\sum_{v\in\mathcal{T}_i^c}\partial_{p_v}$$

is in $\mathcal{M}$. By the linear structure of $\mathcal{M}$ and since $\sum_{v\in\mathcal{T}_i^c}\partial_{p_v}$ is in $\mathcal{M}$ by (4.1), we can discard the second term and we find $\sum_{v\in A\cap\mathcal{T}_i^c}\partial_{p_v} \in \mathcal{M}$. This proves that $A\cap\mathcal{T}_i^c$ is in $\mathcal{J}$. Then, subtracting $\sum_{v\in A\cap\mathcal{T}_i^c}\partial_{p_v}$ from $\sum_{v\in A}\partial_{p_v}$ (resp. from $\sum_{v\in\mathcal{T}_i^c}\partial_{p_v}$) shows that $\sum_{v\in A\setminus\mathcal{T}_i^c}\partial_{p_v}$ (resp. $\sum_{v\in\mathcal{T}_i^c\setminus A}\partial_{p_v}$) is in $\mathcal{M}$, which completes the proof of (ii). □

We can now give an algorithm that applies Lemma 5.2 recursively, and that can be used to show that a large variety of networks is controllable.

Proposition 5.3. *Consider the following algorithm that builds step by step a collection of sets $\mathcal{W}\subset\mathcal{J}$ and a list of controllable particles K.*
Start with $\mathcal{W} = \{\{v\} \mid v\in\mathcal{V}_*\}$ and put (in any order) the vertices of $\mathcal{V}_*$ in K.

1. *Take the first unused controller $c\in K$.*
2. *Add each equivalence class $\mathcal{T}_i^c$ to $\mathcal{W}$.*
3. *For each $\mathcal{T}_i^c$ and each $A\in\mathcal{W}$ add the sets of (5.1) to $\mathcal{W}$.*
4. *If in 2. or 3. new singletons appear in $\mathcal{W}$, add the corresponding vertices (in any order) at the end of K.*
5. *Consider c as used. If there is an unused controller in K, start again at 1. Else, stop.*

We have the following result: if in the end K contains all the vertices of $\mathcal{V}$, then the network is controllable.

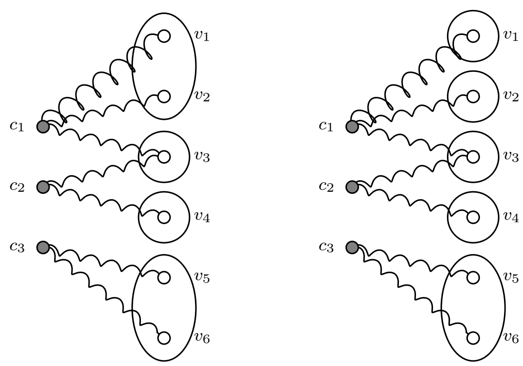
Proof. By Lemma 5.2, the collection $\mathcal{W}$ remains at each step a subset of $\mathcal{J}$, and K contains only controllers. Thus, the result holds by construction. □

The algorithm stops after at most $|\mathcal{V}|$ iterations, and one can show that the result does not depend on the order in which we use the controllers. This algorithm is probably the easiest to implement, but does not give much insight into what really makes a network controllable in our criteria. For this reason, we now formulate a similar result in terms of equivalence with respect to a *set* of controllers, which underlines the role of the "cooperation" of several controllers.

Definition 5.4. *We consider a set C of controllers and denote by $\mathcal{N}(C)$ the set of first neighbors of C that are not themselves in C. We say that two particles $v, w \in \mathcal{N}(C)$ are C-siblings if v and w are connected to C in exactly the same way, i.e., if for every $c \in C$ the edges $\{c, v\}$ and $\{c, w\}$ are either both present or both absent.*

Moreover, we say that v and w are C-equivalent if they are C-siblings, and if in addition, for each $c \in C$ that is linked to v and w, we have that v and w are equivalent with respect to c (i.e., there is a $\delta \in \mathbf{R}$ such that $V''_{cv}(\,\cdot\,) = V''_{cw}(\,\cdot\, + \delta)$).

The C-equivalence classes form a refinement of the sets of C-siblings, see Fig. 5.2.



(a) C-siblings (b) C-equivalence classes

Fig. 5.2: Illustration of Definition 5.4. We assume that all the springs are identical, except for the edge $\{c_1, v_1\}$. The particles $v_1, \dots, v_6$ form 4 sets of C-siblings, with $C = \{c_1, c_2, c_3\}$. The one containing v_1 and v_2 is further split into two C-equivalence classes, since v_1 and v_2 are by assumption inequivalent with respect to c_1.

Proposition 5.5. *Let C be a set of controllers. Then, for each C-equivalence class $U \subset \mathcal{N}(C)$, we have $U \in \mathcal{J}$.*

Proof. See Fig. 5.3. Let $U = \{v_1, \dots, v_n\} \subset \mathcal{N}(C)$ be a C-equivalence class. We denote by $c_1, \dots, c_k$ the controllers in C that are linked to v_1, and therefore also to $v_2, \dots, v_n$, since the elements of U are C-siblings. For each $j \in \{1, \dots, k\}$, there is a $\mathcal{T}_i^{c_j}$ with $v_1, \dots, v_n \in \mathcal{T}_i^{c_j}$, and we define $\mathcal{S}_j \equiv \mathcal{T}_i^{c_j} \setminus C$. We consider the set

$$\widehat{U} \equiv \bigcap_{j=1}^{k} \mathcal{S}_j \ .$$

Clearly, $U \subset \widehat{U}$, and $\widehat{U} \in \mathcal{J}$ by Lemma 5.2. We have $\widehat{U} = \{v_1, \dots, v_n, v_1^*, \dots, v_r^*\}$, where the v_j^* are those particles that are equivalent to $v_1, \dots, v_n$ from the point of view of $c_1, \dots, c_k$, but that are also connected to some controller(s) in $C \setminus \{c_1, \dots, c_k\}$. In particular, for each $j \in \{1, \ldots, r\}$, there is a $c_j^* \in C \setminus \{c_1, \ldots, c_k\}$ and an i such that v_j^* is in $\mathcal{S}_j^* \equiv \mathcal{T}_i^{c_j^*}$. By construction, $\mathcal{S}_j^* \cap U = \phi$. Thus,

$$\widehat{U} \setminus \bigcup_{j=1}^{r} \mathcal{S}_j^* = U .$$

Starting from $\widehat{U} \in \mathcal{J}$ and removing one by one the $\mathcal{S}_j^*$, we find by Lemma 5.2(ii) that U is indeed in $\mathcal{J}$, as we claim. □

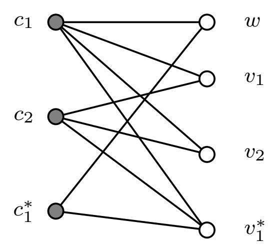


Fig. 5.3: Illustration of the proof of Proposition 5.5 for identical springs. We consider the C-equivalence class $U = \{v_1, v_2\}$, where C contains c_1, c_2, c_1^* and possibly other particles (not shown) that are not linked to v_1, v_2. With the notation of the proof, we have $\mathcal{S}_1 = \{w, v_1, v_2, v_1^*\}$ and $\mathcal{S}_2 = \{v_1, v_2, v_1^*\}$ so that $\widehat{U} = \mathcal{S}_1 \cap \mathcal{S}_2 = \{v_1, v_2, v_1^*\}$. Since v_1^* belongs to $\mathcal{S}_1^* = \{w, v_1^*\}$, we find $\widehat{U} \setminus \mathcal{S}_1^* = U$.

An immediate consequence is

Corollary 5.6. *Let C be a set of controllers. If a vertex $v \in \mathcal{N}(C)$ is alone in its C-equivalence class, then it is controllable.*

Applying this recursively, we obtain

Theorem 5.7. *We start with $C_0 \equiv \mathcal{V}_*$. For each $k \geq 0$, let*

$$C_{k+1} \equiv C_k \cup \{v \in \mathcal{N}(C_k) \mid \textit{ no other vertex in } \mathcal{N}(C_k) \textit{ is } C_k\textit{-equivalent to } v\} .$$

Then, if $C_k = \mathcal{V}$ for some $k \geq 0$, the network is controllable.

Proof. By Corollary 5.6 we have that each C_k contains only controllers (remember also that $\mathcal{V}_*$ contains only controllers by the definition of $\mathcal{M}$). Thus if $C_k = \mathcal{V}$ for some $k \geq 0$ we find that all vertices are controllers, which is what we claim. □

6. Examples

In this section we illustrate by several examples the possibilities and the limits of our controllability criteria.

Example 6.1. A single controller c can control *several* particles if the interaction potentials between c and its neighbors have pairwise inequivalent second derivative. See Fig. 6.1.

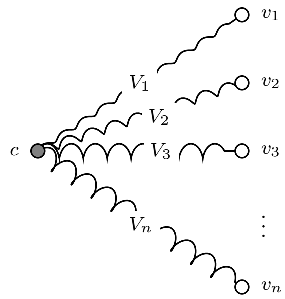


Fig. 6.1: If no two springs are equivalent, the v_i are controllable. Springs from the v_i to other particles or from one v_i to another may exist but are not shown. They do not change the conclusion.

The example above does not use the topology of the network (*i.e.*, the notion of siblings), but only the inequivalence due to the second derivative of the potentials. We have the following immediate generalization, which we formulate as

Theorem 6.2. *Assume that $\mathcal{G}$ is connected, that $\mathcal{V}_*$ is not empty, and that for each $v \in \mathcal{V}$, the first neighbors of v are all pairwise inequivalent with respect to v (i.e., no two distinct neighbors u, w of v are such that $V''_{vu}(\,\cdot\,) = V''_{vw}(\,\cdot\, + \delta)$ for some constant $\delta \in \mathbf{R}$). Then, the network is controllable.*

Proof. We use Theorem 5.7. Observe that under these assumptions, we have at each step $C_{k+1} = C_k \cup \mathcal{N}(C_k)$. Thus, since the network is connected, there is indeed a $k \geq 0$ such that $C_k = \mathcal{V}$. □

One can restate Theorem 6.2 as a genericity condition:

Corollary 6.3. *Assume that $\mathcal{G}$ is connected, that $\mathcal{V}_*$ is not empty, and that for each $e \in \mathcal{E}$ the degree of the polynomial V_e is fixed (and is at least 3). Then, $\mathcal{G}$ is almost surely controllable if we pick the coefficients of each V_e at random according to a probability law that is absolutely continuous w.r.t. Lebesgue.*

Example 6.4. The 1D chain (shown in Fig. 6.2) is controllable. Our theory applies when the interactions are polynomials of degree at least 3; for a somewhat different variant, see [5]. To apply our criteria, we start with $C = \{c\}$. Clearly, v_1 is alone in its C-equivalence class, and is therefore controllable by Corollary 5.6. We then take $C' = \{c, v_1\}$. Since v_2 is alone in its C'-equivalence class, it is also controllable. Continuing like this, we find that the whole chain is controllable.

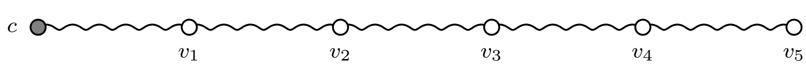


Fig. 6.2: A one-dimensional chain.

Observe that the chain described in the example above is controllable whether some pairs of springs are equivalent or not. There are in fact many networks that are controllable thanks to their topology alone, regardless of the potentials. We have in particular

Example 6.5. We consider the network in Fig. 6.3(a), with the set $C = \{c_1, \dots, c_4\}$ of controllers. Since no two v_i are C-siblings, they are each in a distinct C-equivalence class (regardless of the potentials). By Corollary 5.6 they are controllable. We use this argument recursively to show that the network in Fig. 6.3(b) is controllable. Starting with $C = \{c_1, \dots, c_4\}$, we obtain that the particles in the second column are also controllable. Then, doing the same with the set C' that contains these particles and the c_i, we find that the third column is controllable. Continuing like this, we gain control of the whole network.

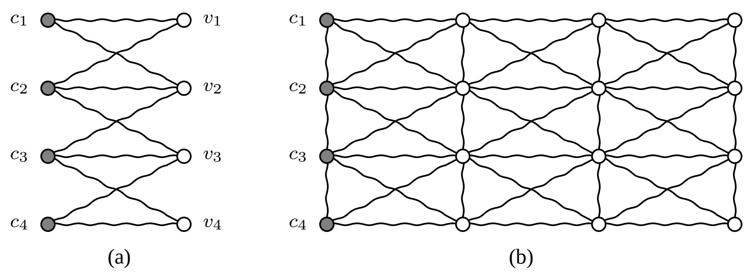


Fig. 6.3: Two networks that are controllable by their topology alone, regardless of the potentials (as long as they are polynomials of degree at least 3).

Our theory is local in the sense that the central tool (Theorem 4.3) involves only a controller and its first neighbors. When we "walk through the graph," starting from $\mathcal{V}_*$ and taking at each step control of more particles, we only look at the interaction potentials that involve the particles we already control and their first neighbors. We never look "farther" in the graph. This makes our criteria quite easy to apply, but this is also the main limitation of our theory, as illustrated in

Example 6.6. We consider the network shown in Fig. 6.4, where c is a controller. If V''_{cv_1} and V''_{cv_3} are equivalent, then our theory fails to say anything about the controllability of the network. In order to draw any conclusion, one has to look at "what comes next" in the network. Of course, if the lower branch is an exact copy of the upper one (*i.e.*, if the interaction and pinning potentials are the same), then the network is truly uncontrollable, and this is obvious for symmetry reasons. However, without such an "unfortunate" symmetry, the network may still be controllable. Indeed, by the study above, we know that the vector field $Y \equiv \partial_{q_{v_1}} + \partial_{q_{v_3}}$ is in $\mathcal{M}$. By commuting with X_0 and subtracting some contributions already in $\mathcal{M}$, one easily obtains that the vector field

$$U''_{v_1}\partial_{p_{v_1}} + U''_{v_3}\partial_{p_{v_3}} + V''_{v_1v_2} \cdot (\partial_{p_{v_1}} - \partial_{p_{v_2}}) + V''_{v_3v_4} \cdot (\partial_{p_{v_3}} - \partial_{p_{v_4}})$$

is in $\mathcal{M}$. Observe that now the pinning potentials U_{v_1} and U_{v_3} as well as the interaction potentials $V_{v_1v_2}$ and $V_{v_3v_4}$ come into play. Taking first commutators with Y and then taking double commutators among the obtained vector fields, one obtains further vector fields of the form $\sum_{i=1}^4 g_i(q_{v_1}, q_{v_2}, q_{v_3}, q_{v_4})\partial_{p_{v_i}}$, where the g_i involve derivatives and products of the potentials mentioned above. In many cases, these are enough to prove that the network in Fig. 6.4 is controllable, even though our theory fails to say so.

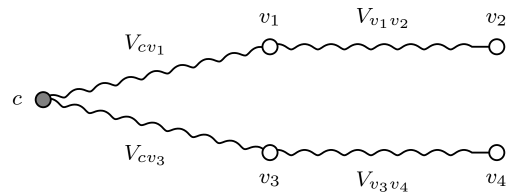


Fig. 6.4: The network used in Example 6.6. If V_{cv_3} is equivalent to V_{cv_1} our theory does not allow to conclude, but the network might still be controllable.

One question that might arise is: why does only the *second* derivative of the interaction potentials enter the theory? The next example shows that this issue is related to the notion of locality mentioned above.

Example 6.7. We consider the network in Fig. 6.5, where c is a controller. We study the case where

$$\begin{aligned} V_{cv}(q_c - q_v) &= (q_c - q_v)^4, & U_v(q_v) &= q_v^6 \,, \\ V_{cw}(q_c - q_w) &= (q_c - q_w)^4 + a \cdot (q_c - q_w), & U_w(q_w) &= q_w^6 + b \cdot q_w \,, \end{aligned}$$

for some constants a and b. The terms in a and b act as constant forces on c and w. Since $V''_{cv} \sim V''_{cw}$, the particles v and w are equivalent with respect to c by our definition. Thus, our theory fails to say anything. We seem to be missing the fact that when $a \neq 0$, the particles v and w can be told apart due to the first derivative of the potentials. However, having $a \neq 0$ is not enough; the controllability of the network also depends on b. Indeed, if $a = b$, the vector field X_0 is symmetric in v and w, and therefore the network is genuinely uncontrollable. If now $a \neq b$, we have checked, by following a different strategy of taking commutators, that the network is controllable. Consequently, when two potentials have equivalent second derivative, but inequivalent first derivative, no conclusion can be drawn in general without knowing more about the network (here, it is one of the pinning potentials, but in more complex situations, it can be some subsequent springs).

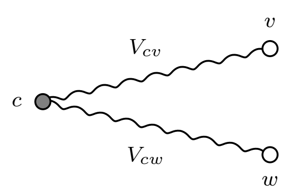


Fig. 6.5: The network discussed in Example 6.7.

Our theory applies only to strictly anharmonic systems, since we assume that the interaction potentials have degree at least 3. The next example shows what can go wrong if we drop this assumption. Again, this is related to the locality of our criteria.

Example 6.8. We consider the harmonic system shown in Fig. 6.6. The vertex c is a controller, and all the pinning potentials are equal and harmonic, *i.e.*, of the form $\lambda x^2/2$. The interaction potentials are also harmonic. The spring $\{c, v_1\}$ has coupling constant 2, the springs $\{c, v_2\}$ and $\{v_2, v_3\}$ have coupling constant 1 and the spring $\{v_3, v_4\}$ has coupling $k > 0$. Since $V''_{cv_1} \equiv 2$ and $V''_{cv_2} \equiv 1$, the particles v_1 and v_2 are inequivalent with respect to c. Yet, this is not enough to obtain that they are controllable (unlike in the strictly anharmonic case covered by our theory). With standard methods for harmonic systems, it can be shown that the network is controllable iff $k \neq 2$. When $k = 2$, one of the eigenmodes decouples from the controller c, and no particle except c is controllable. Thus, one cannot obtain that v_1 and v_2 are controllable without knowing more about the potentials that come farther in the graph.

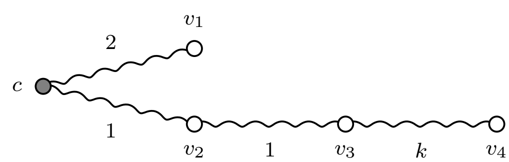


Fig. 6.6: A harmonic network that may or may not be controllable depending on the value of the coupling constant k.

7. Comparison with other commutator techniques

It is perhaps useful to compare the techniques used in this paper to those used elsewhere: To unify notation, we consider the hypoellipticity problem in the classical form

$$L = X_0 + \sum_{i>0} X_i^2 \ .$$

In [5], the authors considered a chain, so that $\mathcal{V}_*$ is just the first and the last particle in the chain. Starting with ∂_{p_1} (the left end of the chain) one then forms (with simplified notation, which glosses over details which can be found in that paper)

$$\partial_{q_1} = [\partial_{p_1}, X_0] \ , \quad \partial_{p_2} = (M_{1,2})^{-1} [\partial_{q_1}, X_0] \ , \quad \partial_{q_2} = [\partial_{p_2}, X_0] \ ,$$

and so on, going through the chain. Here, the particles are allowed to move in several dimensions, and $M_{j,j+1}$ is basically the Hessian matrix of $V_{j,j+1}$. This technique requires that $M_{j,j+1}$ be invertible, which implies some restrictions on the potentials.

Villani [12] uses another sequence of commutators:

$$C_0 = \{X_i\}_{i>0} \ , \quad C_{j+1} = [C_j, X_0] + \text{ remainder}_j \ .$$

With this superficial notation, the current paper uses again a walk through the network, but the basic step involves double commutators of the form

$$[\![Z_1 : Z_2]\!]$$

with Z_i typically of the form $\sum g_v(x_v) \partial_{p_v}$, where we use abundantly that the V_e are polynomials. This allows for the "fanning out" of Fig. 6.1 and is at the basis of our ability to control very general networks. In particular, this shows that networks with variable cross-section can be controlled.

A. Vandermonde determinants

Lemma A.1. *Let* $c_1, \ldots, c_n \in \mathbf{R}$ *be distinct and non-zero, and let* $s \geq 0$. *Then, for all* $k \in \{1, \ldots, n\}$ *there are constants* $r_1, \ldots, r_n \in \mathbf{R}$ *such that for all* $j = 1, \ldots, n$,

$$\sum_{i=1}^{n} r_i \, c_j^{i+s} = \delta_{jk} \ .$$

Proof. We have that the Vandermonde determinant

$$\begin{vmatrix} c_1^{s+1} & c_1^{s+2} & \cdots & c_1^{s+n} \\ c_2^{s+1} & c_2^{s+2} & \cdots & c_2^{s+n} \\ \vdots & & & \vdots \\ c_n^{s+1} & c_n^{s+2} & \cdots & c_n^{s+n} \end{vmatrix} = \left(\prod_{i=1}^{n} c_i^{s+1} \right) \begin{vmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{n-1} \\ 1 & c_2 & c_2^2 & \cdots & c_2^{n-1} \\ \vdots & \vdots & & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^{n-1} \end{vmatrix} = \prod_{i=1}^{n} c_i^{s+1} \prod_{j=i+1}^{n} (c_j - c_i)$$

is non-zero under our assumptions. Thus, the columns of this matrix form a basis of $\mathbf{R}^n$, which proves the lemma. □

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