# Singularity theory for non-twist KAM tori

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# Contents

# Abstract

We introduce a new method to study bifurcations of KAM tori with fixed Diophantine frequency. This is based on Singularity Theory of critical points of a real-valued function that we call *potential*. The potential is constructed in such a way that nondegenerate critical points of the potential correspond to twist invariant tori (i.e. with nondegenerate torsion) and degenerate critical points of the potential correspond to non-twist invariant tori. Hence bifurcating points correspond to non-twist tori. Invariant tori are classified using the classification of critical points of the potential as provided by Singularity Theory. We show that, under general conditions, this classification is robust. Our construction is developed for general Hamiltonian systems and general exact symplectic forms. It is applicable to both the close-to-integrable case and the *'far from integrable'* case where a bifurcation of invariant tori has been detected (e.g. numerically). In the close-to-integrable case, our method applies to *any* finitely determinate singularity of the frequency map for the integrable system.


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CHAPTER 1

# Introduction

## 1.1. Main contribution

The celebrated Kolmogorov-Arnold-Moser (KAM) Theory [**3, 37, 47**] establishes the persistence under perturbation of quasi-periodic solutions, with sufficiently irrational frequency, of Hamiltonian systems. Geometrically, quasi-periodic motions for Hamiltonian systems are invariant tori. Maximal dimensional invariant tori bearing quasi-periodic motion are known as KAM tori. We refer the reader to [**19**] for a tutorial on KAM Theory and to [**17**] for a very clear exposition of Kolmogorov's Theorem [**37**]. KAM results require a nondegeneracy condition. In its simplest form the nondegeneracy condition is known as *twist condition.* In action-angle variables, this requires the frequency map (as function of the action) to be a local diffeomorphism. Persistence of a *Cantor set* of KAM tori has been proved under the very weak *Rüssmann nondegenerate* condition [**16, 56, 58, 60, 72**].

Recently, there has been considerable interest in studying the persistence of tori with fixed frequency, where twist condition is violated but nevertheless, the system depends on sufficient parameters that control the frequencies. The parameters of the Hamiltonian correspond to characteristics of the system that can be tuned by the designer to obtain the desired effect of the system. For example [**2, 23, 24**], in the design of plasma confinement devices, it has been heuristically argued that non-twist invariant tori are very efficient barriers for the undesired effect of *transport.* Similar considerations have appeared in mixing of fluids [**20, 21, 59**]. An important assumption is that the persistent tori must have a prescribed, *fixed* frequency. KAM results under the Rüssmann condition do not give information of the persistence of a torus with *prescribed frequency.* Indeed, it is known [**63**] that under the Rüssmann condition, the set of the persistent frequencies is, in general, different from the set of the unperturbed frequencies. This happens, for example, when the given frequency lies in the boundary of the image of the frequency map or when the perturbed tori become nondegenerate.

In this monograph we give a new methodology for the study of bifurcations of KAM tori with fixed Diophantine frequency vector. It turns out that bifurcating tori are non-twist. Our method is developed for general Hamiltonian systems, hence the notion of non-twist has an extended meaning. Our formulation generalizes the usual one which is given for integrable systems or by means of the *Birkhoff Normal Form* (BNF). In both the integrable and the BNF cases a torus is twist if and only if its *torsion* is non-degenerate. In the integrable case, the torsion is the derivative of the frequency map at the corresponding action. In the BNF case, the torsion is provided by the first order terms of the BNF. We formulate a concept of torsion of a torus which is intrinsic, in the sense that it only depends on the geometric properties of the torus and of the system. Then, by a twist torus we mean a torus with nondegenerate torsion.

The method presented here is based on the construction of a real-valued function, that we call the *potential*, in such a way that critical points of the potential correspond to invariant tori with fixed frequency. This leads to a natural classification of invariant tori based the application of *Singularity Theory* to the potential. An important feature of this classification is that, under rather general conditions, a torus is *degenerate* if and only if it is non-twist. Hence, a torus that is degenerate in our terminology has a degenerate BNF. Moreover, the classification obtained with our method is persistent under small perturbations of the system.

We also prove that, given a parametric family of Hamiltonian systems, there is a (local) bijection between the bifurcation diagram of the critical points of the parametric potential and the bifurcation diagram of invariant tori for the parametric family.

Our methodology does not requires either the computation of the BNF or the existence of an invariant torus. This is useful on predicting the breakdown of tori in a fold bifurcation.

Our methodology is stated in an *a-posteriori* context: the close-to-integrability assumption has been replaced by the assumption of the existence of an approximate solution of the invariance equation, with sufficient accuracy, satisfying some transversality conditions. Hence, our methodology is suitable for validating numerical computations in cases in which bifurcations of invariant tori are observed numerically, but the system is *far from integrable*. Moreover, our methodology leads to efficient numerical algorithms to compute *both* the potential and invariant tori. It is applicable to models where the symplectic structure is not the standard one. It is also suitable to compute tori that are not necessarily graphs of the angle variable e.g. *meandering tori* [**64**]. We plan to discuss the implementation of these algorithms in a future work.

In the close-to-integrable case, our method enables us to obtain persistence of an invariant torus, with fixed frequency, in both twist and non-twist cases (including small-twist and Rüssmann). Moreover, if the frequency map of the integrable system has a *finitely determined singularity*, then the perturbed torus is of the same class as the unperturbed one (under the classification obtained by applying Singularity Theory to the potential). Finite-determined singularities for the frequency include the classical twist case and the seven *elementary catastrophes* (see Table 1 in Appendix B).

The simplest situation in which one has a non-twist torus is the integrable area preserving map with frequency map $\hat{\omega}(y) = \omega + y^2$:

$$f_0(x, y) = \begin{pmatrix} x + \hat{\omega}(y) \\ y \end{pmatrix} .$$

The first simple observation is that the $f_0$-invariant non-twist torus

$$Z_0 = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$$

satisfies the Rüssmann condition (see [**58, 60**]). Indeed, $\frac{\partial^2 \hat{\omega}(y)}{\partial y^2}$ generates $\mathbb{R}$. In this case, $\omega$ is in the boundary of the image of the frequency map $\hat{\omega}(y)$, and the non-twist torus with frequency $\omega$ can be destroyed easily. Indeed, there is a smooth function $\delta(\varepsilon)$, with $\delta(0) = 0$, such that $\hat{\omega}(y) + \delta(\varepsilon)$ does not lie in the image of the frequency map for any $y$ in the domain of $\hat{\omega}$ and $\varepsilon \neq 0$. Then, for any $\varepsilon \neq 0$, the integrable map with frequency map $\hat{\omega}(y) + \delta(\varepsilon)$ does not have an invariant torus with frequency $\omega$. Hence, to study the bifurcations of $Z_0$ we need unfolding parameters.

The second observation is that $Z_0$ corresponds to the degenerate critical point $p = 0$ of the scalar function $A(p) = \frac{p^3}{3}$ and that the unfolding of this singularity is the one-dimensional parameter family of functions: $A(\mu, p) = \mu p + \frac{p^3}{3}$.

Then, a natural way to do the bifurcation analysis of $Z_0$ is to embed $f_0$ in the parametric family $f_\mu$ of integrable symplectomorphisms, with frequency map $\hat{\omega}(\mu, y) = \omega + \nabla_y A(\mu, y)$. It is clear that there are either two, one or zero $f_\mu$-invariant tori with frequency $\omega$, depending on whether $\mu < 0$, $\mu = 0$ or $\mu > 0$. Hence, the non-twist torus $Z_0$ is a fold torus.

The method introduced here generalizes the above bifurcation analysis for KAM invariant tori and enables us to show that under small perturbations degenerate tori persist and the persistent torus has the same degeneracy type. In the present example, this means that, for sufficiently small perturbations of $f_\mu$, the perturbed system will have a bifurcation diagram similar to that in Figure 1.

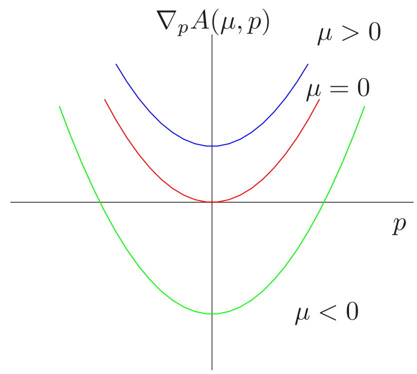


FIGURE 1. The graph shows $\nabla_p A(\mu, p) = \mu + p^2$. The values of $(\mu, p)$ at which $\nabla_p A(\mu, p) = 0$ correspond to invariant tori for the integrable symplectomorphism with frequency map $\hat{\omega}(\mu, y) = \omega + y^2 + \mu$.

Our results for the close-to-integrable case generalize the results in [**22, 23**]. In [**22**] an approach similar to that taken here is used to study the fold singularity in the standard non-twist family. In [**23**], using different techniques, the fold and the cusp were studied for the case of two degrees of freedom.

## 1.2. Methodology

Our strategy follows three main stages. The first one consists on a KAM result based on the *automatic reducibility* of KAM invariant tori. We add external parameters in such a way that the infinite-dimensional problem of finding invariant tori, with fixed Diophantine frequency, is reduced to the finite dimensional problem of finding zeros of a smooth function (similar ideas can be found in [**48, 51, 62, 73**]). In the second stage, the latter problem is transformed to a problem of finding critical points of a smooth real-valued function that we call *potential*. This is achieved by using the geometrical properties of symplectomorphisms and Lagrangian tori. The third stage consists on applying Singularity Theory to the potential.

Our methodology is developed by using classical and new techniques in Symplectic Geometry and parametric KAM theory. An important ingredient in our geometric constructions is the *primitive function* of a given Hamiltonian system [**29**]. The primitive function is a geometric object that depends only on the system and on the symplectic structure. It should be noticed that the primitive functions and the *generating functions* are different objects. One difference is that the domain of the primitive function is the domain of the given system (not *mixed* variables). Hence, the use of the primitive function does not requires the application of the Implicit Function Theorem. This, among other things, makes the primitive function suitable for the study of non-twist tori where generating functions may fail to exist. More properties of the primitive function will be discussed throughout the paper. Another important geometrical ingredient is a generalization of *symplectic deformations* and *moment maps*, also developed here. Roughly, a symplectic deformation is a parametric family of Hamiltonian systems and its moment map is a vector of 'Hamiltonians' corresponding to the vector field given by the variation of the family with respect to the parameter.

For the KAM part, we prove a *parametric KAM Theorem* that may be of independent interest. It is known that KAM procedures involve several non-geometrical steps (e.g. solving small divisor equations, Fourier series). It has been found convenient [**51, 57, 54**] to add extra parameters so that the KAM step can proceed always by adjusting the parameters. Then, at the end, one just solves the *finite dimensional problem of setting the artificial parameters to zero*. As noted in [**62, 63, 73**] this has the advantage that it separates the analytic steps from the nondegeneracy condition. The final nondegeneracy condition amounts to the nondegeneracy conditions of a finite dimensional version of the Implicit Function Theorem. This is especially useful in degenerate problems. The procedure of adding parameters at the iterative step is stable from the analytic point of view and leads to very efficient proofs of KAM results under the Rüssmann non-degeneracy conditions [**61**]. These considerations on the parameter are important when we want to validate numerical computations and when the considered families are far-from-integrable. In our applications of our parametric KAM Theorem, we use different types of parameters playing different roles with the geometry and with the analysis of the problem:

- For technical reasons, we find it convenient to introduce some parameters which change the exactness properties of the symplectomorphisms. We call these parameters *dummy* because at the end of the KAM procedure – but not in the intermediate steps – they will vanish. The reason for this is that if a symplectomorphism has an invariant torus, then it has to be exact. The use of dummy parameters yields a generalization of the *Translated Curve Theorem* in [**57, 54**].
- To obtain a nondegeneracy condition that is weaker than the twist condition, we introduce some parameters that we call *modifying* parameters because they generalize the Moser's modifying terms in [**51**]. The modifying parameters are introduced in such a way that invariant tori are obtained by setting the modifying parameters equal to zero. We show that the modifying parameter, as a function of the momentum of the torus, is

the gradient of a real-valued function, which we call *the potential.* Hence, invariant tori correspond to critical points of the potential.

- In addition to the above parameters, there are other parameters on which the systems may depend, and we call them the *intrinsic* parameters. Intrinsic parameters include *unfolding* and *perturbation* parameters. We show that the potential depends smoothly on the intrinsic parameters.
  We introduce unfolding parameters in such a way that properties, similar to those for degenerate critical points of functions [**5, 25**], are satisfied. Namely, degenerate (non-twist) tori, with fixed frequency, are persistent and the persistent torus has the same type of degeneracy as the unperturbed one.

We consider both the maps and the vector field cases. The tori considered here are of maximal dimension. The KAM results in this paper are formulated and proved for functions depending analytically on the phase space variable and smoothly on the parameter. These KAM results also hold for functions with smooth dependence on both the phase space variable and the parameter (with some loss of derivatives). However, because this involves many technical details, this case is not included here.

## 1.3. Outline of the monograph

For ease of reading, here we briefly describe the content of this monograph. Figure 2 contains a map of the main sections.

Chapter 2 contains the setting of the monograph. Most of the notation used throughout the monograph is established in Section 2.1. Basic concepts on Symplectic Geometry are reviewed in Section 2.2. In Section 2.3, the concept of *symplectic deformation* and *moment map* are introduced. These will play a key role in the definition of the potential and are not standard in the literature. Section 2.4 reviews some basic definitions of spaces of analytic functions and the solvability of the *'one-bite'* small divisors equations.

This monograph contains three main parts. Part 1 contains the geometrical properties of KAM invariant tori. Part 2 contains parametric KAM results based on reducibility of Lagrangian invariant tori. The KAM results given here are stated and proved in the analytic category. In Part 3 our geometric and KAM results are combined with Singularity Theory to develop a new method to study bifurcations of KAM invariant tori.

**Part 1.** Chapter 3 contains geometrical properties of a KAM torus. In Section 3.1, it is proved that any torus which is invariant for a symplectomorphism, with dynamics an ergodic rotation, is *Lagrangian* and moreover the linear dynamics around the torus has a block-triangular form, with the identity on the diagonal. This property is known as *automatic reducibility.* These results are available in the literature, in a less general setting [**34**]. In Section 3.2 we give a geometric definition of the *torsion* of a torus with respect to a symplectomorphism and a given frequency vector. This leads to an intrinsic definition of twist tori.

Chapter 4 contains our main geometrical construction. First, we introduce the definition of an invariant *fibered Lagrangian deformation* (FLD) with frequency $\omega$. Roughly, given a smooth family of symplectomorphism $g_\lambda$, with $s$-dimensional parameter $\lambda$, a $g$-invariant FLD with frequency $\omega$ is a smooth family $\{\mathbf{K}(p) = (\lambda(p), K_p)\}_{p\in \mathrm{D}}$ of parameters $\lambda(p)$ and tori $K_p$ that satisfy the invariance equation:

$$g_{\lambda(p)} \circ K_p = K_p \circ \mathrm{R}_\omega,$$

where $\omega$ is fixed and $\mathrm{R}_\omega : \mathbb{T}^n \to \mathbb{T}^n$ is the rigid rotation: $\mathrm{R}_\omega(\theta) = \theta + \omega$. Then, using the primitive function and the moment map, we define the *potential* and the *momentum* of $\mathbf{K}$ with respect to $g$ and $\omega$. We show that both the momentum and the potential are invariant under canonical changes of coordinates. In Theorem 4.7, under certain general conditions, it is shown that $\lambda(p)$ is the gradient of the potential. Theorem 4.8 gives a relation between the Hessian of the potential of $\mathbf{K}$ and the torsion of $K_p$.

**Part 2.** In Chapter 5 we discuss the nondegeneracy conditions involved in KAM procedures based on the automatic reducibility and with fix frequency. First, in Section 5.1 it is proved that, in the analytic category, approximately invariant tori, with Diophantine frequency, are approximately Lagrangian and approximately reducible. Quantitative estimates are provided. Then, in Section 5.2 we describe and motivate the use of external parameters (dummy and modifying) on performing a step of the iterative procedure. It is shown that it is always possible to introduce a modifying parameter in such a way that an iterative KAM procedure can be performed (even in the non-twist case).

Chapter 6 contains our most general parametric KAM result, Theorem 6.2. Given a $\Lambda$-parametric family of real-analytic Hamiltonian systems, we provide sufficient conditions for the *existence and local uniqueness* of Lagrangian tori $K(\zeta)$ and parameters $\Lambda(\zeta)$, depending smoothly on an external parameter $\zeta$, in such a way that for each value of the parameter $\zeta$, $K(\zeta)$ is invariant for the system in the family corresponding to $\Lambda(\xi)$, the frequency is fixed and Diophantine. This result and the method used to prove it are different from the parametric KAM results in [**11, 10**], where a Whitney smooth Diophantine family of persistent tori is proved for systems that are close-to-integrable. The close-to-integrable hypothesis has been replaced by the assumption of the existence of a torus that solves approximately the invariance equation with sufficiently small error as satisfying some transversality conditions. Our KAM procedure computes the Fourier coefficients of tori that better approximate solutions of the invariance equation. Hence, the original coordinate system is not changed in the procedure.

Chapter 7 contains a Transformed Torus Theorem, Theorem 7.4. This is a result on the existence and local uniqueness of invariant FLD with fixed Diophantine frequency. Briefly, given a smooth family of symplectomorphism $f_\mu$, depending on an intrinsic parameter $\mu$, *dummy* and *modifying* parameters are introduced by embedding the family $f_\mu$ into a smooth family $\mathbf{f}_{(\mu,\sigma,\lambda)}$ in such a way that $\mathbf{f}_{(\mu,\sigma,\lambda)}$ is exact if and only if $\sigma = 0$ and $\mathbf{f}_{(\mu,0,0)} = f_\mu$. The dummy parameter is $\sigma \in \Sigma \subset \mathbb{R}^n$ and the modifying parameter is $\lambda \in \Lambda \subset \mathbb{R}^s$, with $0 \le s \le n$. Theorem 7.4 gives sufficient conditions for the existence and the local uniqueness of a $\mathbf{f}$-invariant parametric FLD with fixed Diophantine frequency $\omega$. Theorem 7.4 is stated in *a-posteriori context*: the main hypothesis is the existence of parameters $\mu_0, \lambda_0$ and a torus $K_0$ such that $K_0$ is approximately $\mathbf{f}_{(\mu_0,0,\lambda_0)}$-invariant, with frequency $\omega$, and satisfies a nondegeneracy condition. This nondegeneracy condition depends on the way the modifying parameters are introduced. For close-to-integrable systems, it is always possible to introduce modifying parameters in such a way that any torus, invariant for the integrable system, satisfies the our non-degeneracy condition. We show that Theorem 7.4 yields a generalization of the *Translated Curve Theorem* in [**57, 54**]. Under suitable choice of the parameters, Theorem 7.4 is a consequence of Theorem 6.2.

**Part 3.** In Chapter 8 we develop the Singularity Theory for KAM tori. Section 8.1 contains a method to characterize invariant tori as critical points of the potential. Briefly, given $f_\mu$, a smooth family of symplectomorphism, and $K_*$ an $f_{\mu_*}$-invariant torus with Diophantine frequency $\omega$, we show that it is always possible to introduce modifying parameters, $\lambda$, in such a way that the nondegeneracy condition in the Transformed Tori Theorem (Theorem 7.4) holds. The dimension of the modifying parameter is determined by the dimension of the kernel of the torsion of $K_*$. Then, the Transformed Tori Theorem guarantees the existence and local uniqueness of an $f$-invariant FLD with frequency $\omega$, $\mathbf{K}$. We combine the results in Section 4.1 with Singularity Theory to classify invariant tori in terms of the classification of critical points of the potential of $\mathbf{K}$, as provided by Singularity Theory. We prove the robustness of this classification. We also show that there is a local one-to-one correspondence between the bifurcation diagram obtained using the critical points of the potential and the bifurcation diagram obtained using invariant tori.

In Chapter 9 we apply the results of Chapter 8 to close-to-integrable systems. For completeness in Section 9.1 we consider the integrable case. In Section 9.2 we obtain persistence of an invariant torus with fixed frequency in both the twist and the non-twist cases, including *small twist*. The bifurcation theory of non-twist tori is described in Section 9.3. In Section 9.4 we show that it is always possible to introduce modifying parameters in such a way that the potential of the BNF and the potential are close.

**Appendices.** Our results and methods are developed in full detail for symplectic maps. The corresponding results for Hamiltonian vector fields are discussed briefly in Appendix A. A brief summary on Singularity Theory is included in Appendix B.

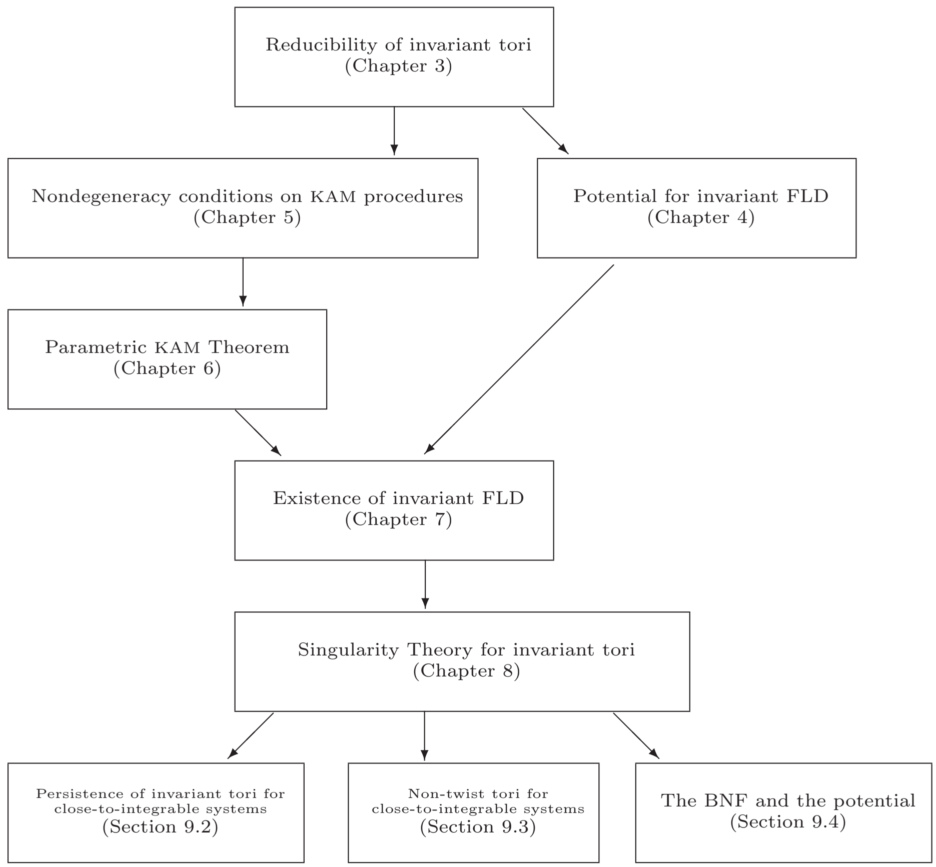


Figure 2. Map of the main sections of the monograph

CHAPTER 2

# Preliminaries

Here we review some standard definitions and results in Symplectic Geometry and KAM theory. The concepts of *symplectic deformation* and *moment map* are introduced. These will be important in our geometric constructions.

## 2.1. Elementary notations

This section contains some notation and definitions that are standard in the literature. As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of non-negative integer, integer, real and complex numbers, respectively. $\mathbb{R}^{m\times d}$ and $\mathbb{C}^{m\times d}$ denote the spaces of $m\times d$ matrices with components in $\mathbb{R}$ and $\mathbb{C}$, respectively. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the $n$-dimensional *torus* with covering space $\mathbb{R}^n$. The $n$-dimensional complex torus is defined by $\mathbb{T}^n_{\mathbb{C}} = \mathbb{C}^n/\mathbb{Z}^n$, with covering space $\mathbb{C}^n$. The standard annulus $\mathbb{T}^n\times\mathbb{R}^n$ is denoted by $\mathbb{A}^n$.

An *annulus* is a subset $\mathcal{A} \subset \mathbb{A}^n$ that is diffeomorphic to $\mathbb{T}^n\times U$, with $U\subset\mathbb{R}^n$ a connected open set. The covering of an annulus $\mathcal{A}$ is denoted by $\tilde{\mathcal{A}}\subset\mathbb{R}^n\times\mathbb{R}^n$ and the coordinates on $\mathcal{A}$ (and $\tilde{\mathcal{A}}$) are denoted by $z = (z_1,\ldots,z_{2n}) = (x, y)$, with $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$.

We will use the notations $\pi_i(z) = z_i$, $\pi_x(z) = z^x = x$ and $\pi_y(z) = z^y = y$ for the projections. For a matrix $P\in\mathbb{C}^{2n\times d}$, we use the notation $P^x = \pi_x P = (I_n\ O_n)P$ and $P^y = \pi_y P = (O_n\ I_n)P$, where $I_n$ and $O_n$ are the $n\times n$ identity and zero matrices, respectively. The $n\times d$ zero matrix is represented by $O_{n\times d}$.

By a diffeomorphism between two manifolds we mean a diffeomorphic immersion. By embedding we mean a smooth injective immersion. Following standard practice, *smooth* means $C^\ell$, with $\ell$ sufficiently large. The precise value of $\ell$ can be ascertained by looking at the arguments in detail.

A function $u:\mathbb{R}^n\to\mathbb{R}$ is 1-periodic if $u(\theta+e) = u(\theta)$ for all $\theta\in\mathbb{R}^n$ and $e\in\mathbb{Z}^n$. A function $u:\mathbb{T}^n\to\mathbb{R}$ is a 1-periodic function $u:\mathbb{R}^n\to\mathbb{R}$. Similarly, a function $g:\tilde{\mathcal{A}}\to\mathbb{R}$ is 1-periodic in $x$ if $g(x+e,y) = g(x,y)$ for all $x\in\mathbb{R}^n$ and $e\in\mathbb{Z}^n$. A function $g:\mathcal{A}\to\mathbb{R}$ is viewed as a function $g:\tilde{\mathcal{A}}\to\mathbb{R}$ that is 1-periodic in $x$.

The average of a continuous function $u:\mathbb{T}^n\to\mathbb{R}$ is $\langle u\rangle = \int_{\mathbb{T}^n} u(\theta)\,\mathrm{d}\theta$. The notation $\langle\cdot\rangle$ extends component-wise to vector and matrix valued continuous functions.

A rigid rotation on $\mathbb{T}^n$ with rotation vector $\omega\in\mathbb{R}^n$ is the function $\mathrm{R}_\omega:\mathbb{T}^n\to\mathbb{T}^n$ given by $\mathrm{R}_\omega(\theta) = \theta+\omega$. The rotation $\mathrm{R}_\omega$ is *ergodic* if $\omega$ is rationally independent, i.e. $k^\top\omega\notin\mathbb{Z}$ for all $k\in\mathbb{Z}^n\setminus\{0\}$.

We use the following notation for derivation. A sub-index in the derivatives is included indicating the variables with respect to which one is taking derivatives.

For example, if $f$ is defined on $(t,z)\in\mathbb{R}^d\times\mathcal{A}$

$$\begin{aligned}\mathrm{D}_t f(t,z) &= \left(\frac{\partial f}{\partial t_1}(t,z) \ \cdots \ \frac{\partial f}{\partial t_d}(t,z)\right),\\ \mathrm{D}_z f(t,z) &= \left(\frac{\partial f}{\partial z_1}(t,z) \ \cdots \ \frac{\partial f}{\partial z_{2n}}(t,z)\right).\end{aligned}$$

## 2.2. Geometric preliminaries

Here we review basic concepts in Symplectic Geometry [**8, 15, 35, 46, 69**]. From now on, $\mathcal{A}\subset\mathbb{A}^n$ is assumed to be an annulus.

### 2.2.1. Symplectic structures.

DEFINITION 2.1. A *symplectic form* on $\mathcal{A}$ is a closed, nondegenerate differential 2-form $\boldsymbol{\omega}$ on $\mathcal{A}$. If $\boldsymbol{\omega}$ is exact, i.e. $\boldsymbol{\omega}=\mathrm{d}\boldsymbol{\alpha}$, the 1-form $\boldsymbol{\alpha}$ is called an *action form* of $\boldsymbol{\omega}$.

DEFINITION 2.2. Let $\mathcal{A}$ be endowed with the symplectic form $\boldsymbol{\omega}$. An *almost complex structure* on $\mathcal{A}$, *compatible* with $\boldsymbol{\omega}$, is a linear symplectomorphism $\boldsymbol{J}:\mathrm{T}\mathcal{A}\to\mathrm{T}\mathcal{A}$ which is anti-involutive (i.e. $\boldsymbol{J}^2=-\boldsymbol{I}$), and such that the 2-form $\boldsymbol{g}(u,v)=-\boldsymbol{\omega}(u,\boldsymbol{J}v)$ induces a Riemannian metric on $\mathcal{A}$. $(\boldsymbol{\omega},\boldsymbol{J},\boldsymbol{g})$ is called a *compatible triple* on $\mathcal{A}$.

Using local coordinates, if $\boldsymbol{\alpha}(z)=\sum_{i=1}^{2n}a_i(z)\,dz_i$ denotes the 1-form $\boldsymbol{\alpha}_z$, where $a:\mathcal{A}\to\mathbb{R}^{2n}$ is 1-periodic in $x$. Then, the matrix representation of the 2-form $\boldsymbol{\omega}_z=\mathrm{d}\boldsymbol{\alpha}_z$ is

$$\Omega(z)=\mathrm{D}_z a(z)^\top-\mathrm{D}_z a(z). \tag{2.1}$$

The nondegeneracy of $\boldsymbol{\omega}_z$ is equivalent to $\det\Omega(z)\neq 0$. If $J(z)$ is the matrix representation of $\boldsymbol{J}_z$ and $G(z)$ is the positive-definite symmetric matrix giving $\boldsymbol{g}_z$ on $\mathrm{T}_z\mathcal{A}\simeq\mathbb{R}^{2n}$, then $\boldsymbol{\omega}$, $\boldsymbol{J}$ and $\boldsymbol{g}$ are related as follows:

$$\Omega^\top=-\Omega,\ J^2=-I_{2n},\ G^\top=G,$$

$$\Omega=J^\top\Omega J=GJ=-J^\top G,\ G=J^\top GJ=-\Omega J=J^\top\Omega\ .$$

With the above notation we say that $(\Omega=\mathrm{D}a^\top-\mathrm{D}a,J,G)$ is the coordinate representation of the compatible triple $(\boldsymbol{\omega}=\mathrm{d}\boldsymbol{\alpha},\boldsymbol{J},\boldsymbol{g})$.

The prototype example of a compatible tripe is the following. Let $\boldsymbol{\omega}_0$ be the *standard symplectic structure* on the standard annulus $\mathbb{A}^n$: $\boldsymbol{\omega}_0=\sum_{i=1}^n dy_i\wedge dx_i$. An action form for $\boldsymbol{\omega}_0$ is $\boldsymbol{\alpha}_0=\sum_{i=1}^d y_i\ dx_i$. The Euclidean metric $\boldsymbol{g}_0$ induces a compatible almost complex structure $\boldsymbol{J}_0$. The matrix representations of $\boldsymbol{\alpha}_0$, $\boldsymbol{\omega}_0$, $\boldsymbol{J}_0$ and $\boldsymbol{g}_0$ are, respectively,

$$a_0(z)=\begin{pmatrix}y\\0\end{pmatrix},\ \Omega_0=J_0=\begin{pmatrix}O_n & -I_n\\ I_n & O_n\end{pmatrix},\ G_0=\begin{pmatrix}I_n & O_n\\ O_n & I_n\end{pmatrix}. \tag{2.2}$$

REMARK 2.3. Throughout this paper we assume that $\mathcal{A}$ is an annulus endowed with a compatible triple $(\boldsymbol{\omega}=\boldsymbol{\alpha},\boldsymbol{J},\boldsymbol{g})$. This can always be obtained so that this is not loss of generality. However, it simplifies the exposition of the results in this paper.

**2.2.2. Symplectomorphisms and local primitive functions.** From now on, $\mathcal{A}, \mathcal{A}' \subset \mathbb{A}^n$ are assumed to be annuli endowed with the compatible triples ($\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}, \boldsymbol{J}, \boldsymbol{g}$) and ($\boldsymbol{\omega}' = \mathrm{d}\boldsymbol{\alpha}', \boldsymbol{J}', \boldsymbol{g}'$), respectively. Let $(\Omega, a, J, G)$ and $(\Omega', a', J', G')$ be the corresponding coordinate representation.

Definition 2.4. A diffeomorphism $f : \mathcal{A}_0 \to \mathcal{A}'$ is a *symplectomorphism* if $f^*\boldsymbol{\omega}' = \boldsymbol{\omega}$. Let $\mathrm{Symp}(\mathcal{A}_0, \mathcal{A}')$ denote the set of symplectomorphisms $f : \mathcal{A}_0 \to \mathcal{A}'$ that are *homotopic to the identity*, i.e. $f(x, y) - (x, 0)$ is 1-periodic in $x$.

Definition 2.5. A symplectomorphism $f : \mathcal{A}_0 \to \mathcal{A}'$ is *exact* if there is a smooth function $S^f : \mathcal{A}_0 \to \mathbb{R}$, called *primitive function of* $f$, such that $f^*\boldsymbol{\alpha}' - \boldsymbol{\alpha} = \mathrm{d}S^f$. Let $\mathrm{Symp}_e(\mathcal{A}_0, \mathcal{A}')$ denote the set of exact symplectomorphisms that are homotopic to the identity.

In coordinates, the symplectic and the exact symplectic properties of a diffeomorphism $f$ are equivalent to, respectively:

$$\mathrm{D}_z f(z)^\top\, \Omega'(f(z))\, \mathrm{D}_z f(z) = \Omega(z), \quad \forall\, z \in \mathcal{A}_0, \tag{2.3}$$

$$\mathrm{D}_z S^f(z) = a'(f(z))^\top \mathrm{D}_z f(z) - a(z)^\top, \quad \forall\, z \in \mathcal{A}_0. \tag{2.4}$$

Remark 2.6. The primitive function $S^f$ of an exact symplectomorphism $f$ is unique up to addition of constants. Moreover, $S^f$ is the primitive function of $g \circ f$, for any diffeomorphism, $g : \mathcal{A}' \to \mathcal{A}'$, preserving the action form $\boldsymbol{\alpha}'$, i.e. $g^*\boldsymbol{\alpha}' = \boldsymbol{\alpha}'$. See [**29, 32, 33**] for more properties of primitive functions.

Remark 2.7. Primitive functions play a key role in our geometric constructions and they should be not confused with generating functions. One important difference is that primitive functions are intrinsic geometric objects, whereas generating functions are defined through local coordinates. Moreover, primitive functions are not defined in *mixed variables*. These, among other reasons, make the primitive functions suitable to study non-twist tori. Consider for example, the integrable symplectomorphism on $\mathbb{A}^n$

$$f(x, y) = \begin{pmatrix} x + y^2 \\ y \end{pmatrix}.$$

Then, $S^f(x, y) = \frac{2}{3} y^3$ and a Lagrangian generating function in the variables $(x, \bar{x})$ is given by $\frac{2}{3}(x - \bar{x})^{3/2}$, which is defined for $\bar{x} > x$.

The objects introduced above lift to corresponding objects on the covering manifolds $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$. If it does not lead to confusion, we will abuse notation and use the same letters to denote the objects in the covering manifold. From Poincaré Lemma, the lift of any symplectomorphism $f : \mathcal{A}_0 \to \mathcal{A}'$ is exact (in the covering spaces $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$). We will refer to the primitive function $\tilde{S}^f : \tilde{\mathcal{A}}_0 \to \mathbb{R}$ of the lift of $f$ as the *local primitive function* of $f$. Since $\tilde{S}^f$ satisfies

$$\mathrm{D}_z \tilde{S}^f(z) = a'(f(z))^\top \mathrm{D}_z f(z) - a(z)^\top, \quad \forall\, z \in \tilde{\mathcal{A}}_0 \tag{2.5}$$

and $f(x, y) - (x, 0)$ is 1-periodic in $x$, then all the partial derivatives of $\tilde{S}^f$ are 1-periodic in $x$. Then, $\tilde{S}^f$ can be written as follows:

$$\tilde{S}^f(z) = x^\top \mathrm{C}^f + S^f(z), \tag{2.6}$$

where $S^f : \mathcal{A}_0 \to \mathbb{R}$ is 1-periodic in $x$ and $\mathrm{C}^f \in \mathbb{R}^n$ is a constant vector. $\mathrm{C}^f \in \mathbb{R}^n$ is known as the *Calabi invariant* of $f$ [**6, 14**] (see also §3 in [**40**]). It is clear that $f$ is exact symplectic if and only if $\mathrm{C}^f = 0$.

REMARK 2.8. The local primitive function and the Calabi invariant satisfy

$$\tilde{S}^{f\circ g} = \tilde{S}^f \circ g + \tilde{S}^g, \tag{2.7}$$

$$\mathrm{C}^{f\circ g} = \mathrm{C}^f + \mathrm{C}^g. \tag{2.8}$$

**2.2.3. Embedding of tori.** We consider $n$-dimensional tori embedded in $\mathcal{A}_0 \subset \mathcal{A}$. We do not assume that the tori are graphs of functions over the angle.

DEFINITION 2.9. Let $\mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$ denote the set of embeddings $K : \mathbb{T}^n \to \mathcal{A}_0$ that are *homotopic to the zero section*, i.e. the components of $K(\theta) - (\theta, 0)$ are 1-periodic.

DEFINITION 2.10. The *averaged action* of $K \in \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$ is given by

$$C^K = \left\langle a(K(\theta))^\top \mathrm{D}_\theta K(\theta) \right\rangle^\top \in \mathbb{R}^n. \tag{2.9}$$

REMARK 2.11. Given $f \in \mathrm{Symp}(\mathcal{A}_0, \mathcal{A})$, $C^{f\circ K} - C^K$ is the *net flux* of $f$ through $K \in \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$. It is easy to verify that

$$C^{f\circ K} - C^K = \mathrm{C}^f.$$

For this reason, the Calabi invariant of $f$ is often referred to as the *net flux* [**6, 19**].

Given $K \in \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$, the geometric structures $\boldsymbol{\omega}$ and $\boldsymbol{g}$ on $\mathcal{A}$ pull-back to the corresponding structures on $\mathcal{K} = K(\mathbb{T}^n)$, via $K^*\boldsymbol{\omega}$ and $K^*\boldsymbol{g}$, whose matrix representations at a point $K(\theta) \in \mathcal{K}$ are, respectively,

$$\Omega_K(\theta) = \mathrm{D}_\theta K(\theta)^\top\ \Omega(K(\theta))\ \mathrm{D}_\theta K(\theta), \tag{2.10}$$

$$G_K(\theta) = \mathrm{D}_\theta K(\theta)^\top\ G(K(\theta))\ \mathrm{D}_\theta K(\theta). \tag{2.11}$$

$K^*\boldsymbol{g}$ is a Riemannian structure on $\mathcal{K}$, and hence $G_K(\theta)$ is symmetric and positive-definite.

DEFINITION 2.12. $K \in \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$ is *Lagrangian* if $K^*\boldsymbol{\omega} = 0$. Let $\mathrm{Lag}(\mathbb{T}^n, \mathcal{A}_0) \subset \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathrm{A}_0)$ denote the set of *Lagrangian* tori.

From Poincaré Lemma, the lift of $K \in \mathrm{Lag}(\mathbb{T}^n, \mathcal{A}_0)$ pull-backs the one form $\boldsymbol{\alpha}$ into an exact form (in $\mathbb{R}^n$):

$$K^*\boldsymbol{\alpha} = \mathrm{d}\,\tilde{S}^K. \tag{2.12}$$

The function $\tilde{S}^K : \mathbb{R}^n \to \mathbb{R}$ will be called *local primitive function* of the Lagrangian torus $K$. Using that $K$ is homotopic to the zero section, one shows that $\tilde{S}^K$ can be written as follows:

$$\tilde{S}^K(\theta) = \theta^\top C^K + S^K(\theta), \tag{2.13}$$

where $S^K : \mathbb{T}^n \to \mathbb{R}$ is 1-periodic and $C^K$ is the averaged action of $K$, given by (2.9). The local primitive function of a Lagrangian torus is unique up to addition of constants.

REMARK 2.13. Let $K \in \mathrm{Lag}(\mathbb{T}^n, \mathcal{A}_0)$ and let $\psi : \mathbb{T}^n \to \mathbb{T}^n$ be a homotopic to the identity diffeomorphism, then

$$C^{K\circ\psi} = C^K.$$

In particular, the averaged action $C^K$ is independent of the parameterization of the Lagrangian torus $\mathcal{K} = K(\mathbb{T}^n)$. Moreover, $C^K$ depends only on $\mathcal{K}$ and the cohomology class of the action form $\boldsymbol{\alpha}$.

**2.2.4. Hamiltonian vector fields.** The notion of vector field preserving the symplectic form is reviewed here.

DEFINITION 2.14. A *local Hamiltonian function* on $\mathcal{A}$ is a function $\tilde{h} : \tilde{\mathcal{A}} \to \mathbb{R}$ of the form

$$\tilde{h}(z) = -x^\top \mathrm{C}^{\tilde{h}} + h(z),$$

where $h : \mathcal{A} \to \mathbb{R}$ is 1-periodic in $x$ and $\mathrm{C}^{\tilde{h}} \in \mathbb{R}^n$ is a constant, which we call *infinitesimal Calabi invariant* of $\tilde{h}$.

A local Hamiltonian function induces a *local Hamiltonian vector field* (or *local symplectic vector field*) $\mathcal{X}_{\tilde{h}}$ on $\mathcal{A}$ such that

$$\mathrm{i}_{\mathcal{X}_{\tilde{h}}} \boldsymbol{\omega} = -\,\mathrm{d}\tilde{h},$$

where i denotes the contraction of a form with a vector. In coordinates:

$$\mathcal{X}_{\tilde{h}}(z) = \Omega(z)^{-1} \nabla_z \tilde{h}(z) \ .$$

Notice that $\mathcal{X}_{\tilde{h}}(z)$ is 1-periodic in $x$. If $\mathrm{C}^{\tilde{h}} = 0$, we say that $\tilde{h} = h$ is a *(global) Hamiltonian function*, and that $\mathcal{X}_h(z)$ is a *(global) Hamiltonian vector field*.

The *infinitesimal primitive function*, corresponding to a local Hamiltonian function $\tilde{h}$ is defined as follows:

$$S^{\tilde{h}}(z) = a(z)^\top \mathcal{X}_{\tilde{h}}(z) - h(z).$$

Definition 2.14 extends in a natural way to the non-autonomous case (time-dependent). The evolution operator $\Phi_{t,t_0}(z)$ of a time-dependent (local) Hamiltonian function $\tilde{h}_t(z)$, that satisfies

$$\frac{\partial \Phi_{t,t_0}}{\partial t}(z) = X_{\tilde{h}_t} \circ \Phi_{t,t_0}(z)$$

and $\Phi_{t_0,t_0}(z) = z$ is symplectic. The local primitive function of $\Phi_{t,t_0}$ is

$$\tilde{S}_{t,t_0}(z) = \int_{t_0}^{t} (\boldsymbol{\alpha}(X_{\tilde{h}_s}) - \tilde{h}_s) \circ \Phi_{s,t_0}(z)\ ds \ .$$

The Hamiltonian is global if and only if $\Phi_{t,t_0}(z)$ is exact symplectic, for all $t$.

## 2.3. Symplectic deformations and moment maps

Here we establish the definition and main properties of the *generator* and the *moment map* of a smooth family of symplectomorphisms. These form the basis for our geometric constructions. Roughly, the generator gives the variation of the family as the parameter moves and the moment map is the *'Hamiltonian'* corresponding to this variation.

Let us start by making precise the meaning of symplectic deformation.

DEFINITION 2.15. Let $\Xi \subset \mathbb{R}^m$ be open. A *symplectic deformation* with base $\Xi$ is a smooth function $g : \Xi \times \mathcal{A}_0 \to \mathcal{A}'$ inducing a family of symplectomorphisms:

$$\begin{array}{rccc} g: & \Xi & \longrightarrow & \mathrm{Symp}(\mathcal{A}_0, \mathcal{A}') \\ & \mathbf{t} & \longrightarrow & g_{\mathbf{t}}. \end{array}$$

where $g_{\mathbf{t}}(z) = g(\mathbf{t}, z)$.

A *Hamiltonian deformation* is a symplectic deformation $g$ such that $g_{\mathbf{t}}$ is exact, for all $\mathbf{t} \in \Xi$.

In what follows we define some geometric objects, naturally related to symplectic deformations.

DEFINITION 2.16. Given a symplectic deformation $g : \Xi \times \mathcal{A}_0 \to \mathcal{A}'$, the *local primitive function of* $g$ is a smooth function $\tilde{S}^g : \Xi \times \tilde{\mathcal{A}}_0, \to \mathbb{R}$ such that, for each $\mathbf{t} \in \Xi$, the function $\tilde{S}^g_{\mathbf{t}}(z) = \tilde{S}^g(\mathbf{t}, z)$, is the local primitive function of $g_{\mathbf{t}}$: $\tilde{S}^g_{\mathbf{t}} = \tilde{S}^{g_{\mathbf{t}}}$.

The *Calabi invariant of* $g$ is a smooth function $\mathrm{C}^g : \Xi \to \mathbb{R}$ such that $\mathrm{C}^g(\mathbf{t})$ is the Calabi invariant of $g_{\mathbf{t}}$: $\mathrm{C}^g(\mathbf{t}) = \mathrm{C}^{g_{\mathbf{t}}}$.

For each $\mathbf{t} \in \Xi$, write $\tilde{S}^g_{\mathbf{t}}$ as follows:

$$\tilde{S}^g_{\mathbf{t}}(z) = \mathrm{C}^g(\mathbf{t})^\top x + S^g_{\mathbf{t}}(z), \tag{2.14}$$

where $S^g : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}$ is 1-periodic in $x$. If $g$ is a Hamiltonian deformation $S^g$ is called *primitive function of* $g$.

Let us now define the generator and the moment map of a symplectic deformation.

DEFINITION 2.17. Let $g : \Xi \times \mathcal{A}_0 \to \mathcal{A}'$ be a symplectic deformation with base $\Xi$ and let $\tilde{S}^g : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}$ the local primitive function of $g$. Assume that $\mathcal{A}'_0 \subseteq g_{\mathbf{t}}(\mathcal{A}_0)$, for all $\mathbf{t} \in \Xi$.

i) The *generator* of $g$ is the function $\mathcal{G}^{\mathrm{g}} : \Xi \times \tilde{\mathcal{A}}'_0 \to \mathbb{R}^{2n \times m}$ defined by
$$\mathcal{G}^{\mathrm{g}}(\mathbf{t}, z) = \mathrm{D}_{\mathbf{t}} g(\mathbf{t}, g^{-1}_{\mathbf{t}}(z)).$$

ii) The *local moment map* of $g$ is the function $\tilde{\mathcal{M}}^g : \Xi \times \tilde{\mathcal{A}}'_0 \to \mathbb{R}^m$ defined by
$$\tilde{\mathcal{M}}^g(\mathbf{t}, z)^\top = a'(z)^\top \mathcal{G}^{\mathrm{g}}(\mathbf{t}, z) - \mathrm{D}_{\mathbf{t}} \tilde{S}^g(\mathbf{t}, g^{-1}_{\mathbf{t}}(z)).$$

For each $\mathbf{t} \in \Xi$, $\mathcal{G}^{\mathrm{g}}_{\mathbf{t}} : \tilde{\mathcal{A}}'_0 \to \mathbb{R}^{2n \times m}$ and $\tilde{\mathcal{M}}^g_{\mathbf{t}} : \tilde{\mathcal{A}}_0 \to \mathbb{R}^m$ are defined by $\tilde{\mathcal{M}}^g_{\mathbf{t}}(z) = \tilde{\mathcal{M}}^g(\mathbf{t}, z)$ and $\mathcal{G}^{\mathrm{g}}_{\mathbf{t}}(z) = \mathcal{G}^{\mathrm{g}}(\mathbf{t}, z)$, respectively.

If $g_{\mathbf{t}}$ is exact for all $\mathbf{t} \in \Xi$, then $\tilde{\mathcal{M}}^g$ is denoted by $\mathcal{M}^g$ and called the *moment map* of $g$.

REMARK 2.18. Let $g$, $\tilde{S}^g$ and $\tilde{\mathcal{M}}^g$ be as in Definition 2.17 and write $\tilde{S}^g$ as in (2.14). Then,
$$\tilde{\mathcal{M}}^g(\mathbf{t}, z) = \mathcal{M}^g(\mathbf{t}, z) - \mathrm{D}_{\mathbf{t}} \mathrm{C}^g(\mathbf{t})^\top x,$$
where $\mathcal{M}^g(\mathbf{t}, z)^\top = a'(z)^\top \mathcal{G}^{\mathrm{g}}(\mathbf{t}, z) - \mathrm{D}_{\mathbf{t}} S^g(\mathbf{t}, g^{-1}_{\mathbf{t}}(z))$ is 1-periodic in $x$.

In the symplectic geometry literature [**15, 28, 27**] the definition of moment map is slightly different from that given in Definition 2.17. If $\mathcal{A} = \mathcal{A}'$, $\boldsymbol{\omega} = \boldsymbol{\omega}'$ and the symplectic deformation is also a Hamiltonian deformation such that its generator is independent of $\mathbf{t}$ (autonomous case), then our definition coincides with that given in [**15**]. Our definition is motivated by the following result.

LEMMA 2.19. *Let* $g$ *and* $\tilde{\mathcal{M}}^g$ *be as in Definition 2.17. Then the following equality holds:*
$$\mathcal{G}^{\mathrm{g}}(\mathbf{t}, z) = \Omega'(z)^{-1} \mathrm{D}_z \tilde{\mathcal{M}}^g(\mathbf{t}, z)^\top.$$

PROOF. From Definition 2.16 we have, for all $\mathbf{t} \in \Xi$ and all $z \in \mathcal{A}_0$,
$$(g^*_{\mathbf{t}} \boldsymbol{\alpha}')(z) = \boldsymbol{\alpha}(z) + \mathrm{d}\, \tilde{S}^g_{\mathbf{t}}(z).$$
This implies, for $i = 1, \ldots, m$,
$$\begin{aligned} \mathrm{D}_{\mathbf{t}_i} \mathrm{D}_z S^g(\mathbf{t}, z) = {} & \left(\mathrm{D}_z a'(g_{\mathbf{t}}(z)) \mathrm{D}_{\mathbf{t}_i} g(\mathbf{t}, z)\right)^\top \mathrm{D}_z g(\mathbf{t}, z) \\ & + a'(g_{\mathbf{t}}(z))^\top \mathrm{D}_{\mathbf{t}_i} \mathrm{D}_z g(\mathbf{t}, z). \end{aligned}$$

This and the definition of $\tilde{\mathcal{M}}^g$ (see Definition 2.17) imply for $i = 1, \dots, m$ and $j = 1, \dots 2n$

$$\mathrm{D}_{z_j} \tilde{\mathcal{M}}_i^g(\mathbf{t}, z) = \mathrm{D}_{\mathbf{t}_i} g(\mathbf{t}, g_{\mathbf{t}}^{-1}(z))^\top \left( -\mathrm{D}_{z_j} a'(z)^\top + \mathrm{D}_{z_j} a'(z) \right).$$

Performing straightforward computations and using equality (2.1) one obtains:

$$\mathrm{D}_{\mathbf{t}} g(\mathbf{t}, z)^\top \Omega'(g_{\mathbf{t}}(z)) = -\mathrm{D}_z \tilde{\mathcal{M}}^g(\mathbf{t}, g_{\mathbf{t}}(z)).$$

□

REMARK 2.20. If $\boldsymbol{\omega}' = \boldsymbol{\omega}$, the functions $\mathcal{G}^{\mathrm{g}}$ and $\tilde{\mathcal{M}}^g$ in Definition 2.17 have a natural geometrical meaning. For $i = 1, \dots, m$, let $(\mathcal{G}_{\mathbf{t}}^{\mathrm{g}})_i$ and $(\tilde{\mathcal{M}}_{\mathbf{t}}^g)_i$ be the $i$-th column of $\mathcal{G}_{\mathbf{t}}^{\mathrm{g}}$ and the $i$-th coordinate of $\tilde{\mathcal{M}}_{\mathbf{t}}^g$, respectively. Lemma 2.19 says that $(\tilde{\mathcal{M}}_{\mathbf{t}}^g)_i$ is a local Hamiltonian of the vector field $(\mathcal{G}_{\mathbf{t}}^{\mathrm{g}})_i$:

$$\mathrm{i}_{(\mathcal{G}_{\mathbf{t}}^{\mathrm{g}})_i} \boldsymbol{\omega} = -\mathrm{d}(\tilde{\mathcal{M}}_{\mathbf{t}}^g)_i.$$

LEMMA 2.21. *Let $g : \Xi \times \mathcal{A}_0' \to \mathcal{A}$ and $f : \Xi \times \mathcal{A}_0 \to \mathcal{A}$ be two smooth symplectic deformations, with base $\Xi \subset \mathbb{R}^m$, such that $g_{\mathbf{t}}(\mathcal{A}_0') = \mathcal{A}_0$, for all $\mathbf{t} \in \Xi$. Then the following holds:*

**a)** *If $h : \Xi \times \mathcal{A}_0' \to \mathcal{A}$ is defined by $h_{\mathbf{t}} = f_{\mathbf{t}} \circ g_{\mathbf{t}}$ then*

$$\tilde{\mathcal{M}}_{\mathbf{t}}^h = \tilde{\mathcal{M}}_{\mathbf{t}}^f + \tilde{\mathcal{M}}_{\mathbf{t}}^g \circ f_{\mathbf{t}}^{-1}.$$

**b)** *If $h : \Xi \times \mathcal{A}_0 \to \mathcal{A}'$ is defined by $h_{\mathbf{t}} = g_{\mathbf{t}}^{-1}$ then*

$$\tilde{\mathcal{M}}_{\mathbf{t}}^h = -\tilde{\mathcal{M}}_{\mathbf{t}}^g \circ g_{\mathbf{t}}.$$

**c)** *If $h : \Xi \times \mathcal{A}_0' \to \mathcal{A}'$ is defined by $h_{\mathbf{t}} = g_{\mathbf{t}}^{-1} \circ f_{\mathbf{t}} \circ g_{\mathbf{t}}$ then*

$$\tilde{\mathcal{M}}_{\mathbf{t}}^h = \tilde{\mathcal{M}}_{\mathbf{t}}^g \circ f_{\mathbf{t}}^{-1} \circ g_{\mathbf{t}} + \tilde{\mathcal{M}}_{\mathbf{t}}^f \circ g_{\mathbf{t}} - \tilde{\mathcal{M}}_{\mathbf{t}}^g \circ g_{\mathbf{t}}.$$

PROOF. From Definition 2.17, Part a) is a re-phrasing of (2.7) and the other ones follow from a). □

## 2.4. Analytic preliminaries

To deal with small divisors equations, we work with Banach spaces of real-analytic functions with control on derivatives in complex neighborhoods of the real domains. Here we introduce these spaces.

**2.4.1. Spaces of real-analytic functions.** A complex strip of $\mathbb{T}^n$ of width $\rho > 0$ is defined by:

$$\mathbb{T}_\rho^n = \{\theta \in \mathbb{T}_{\mathbb{C}}^n \ : \ |\mathrm{Im}\ \theta_i| < \rho, \, i = 1, \dots, n\}.$$

A function defined on $\mathbb{T}^n$ is *real-analytic* if that can be holomorphically extended to a complex strip $\mathbb{T}_\rho^n$.

DEFINITION 2.22. Given $r \in \mathbb{N}$ and $\rho > 0$, let $\mathrm{A}(\mathbb{T}_\rho^n, C^r)$ denote the set of holomorphic functions $u : \mathbb{T}_\rho^n \to \mathbb{C}$ such that $u(\mathbb{T}^n) \subset \mathbb{R}$ and such that all its partial derivatives up to order $r$ can be continuously extended on the boundary of $\mathbb{T}_\rho^n$. Endow $\mathrm{A}(\mathbb{T}_\rho^n, C^r)$ with the norm

$$\|u\|_{\rho, C^r} = \sup_{k \le r} \sup_{\theta \in \mathbb{T}_\rho^n} \left| \mathrm{D}^k u(\theta) \right|,$$

where $\left|\mathrm{D}^k u(\theta)\right|$ denotes the supremum norm of the components of $\mathrm{D}^k u(\theta)$. We also use the notation $\|u\|_{\rho, C^0} = \|u\|_\rho$.

A complex strip of an annulus $\mathcal{A}$ is a complex connected open neighborhood $\mathcal{B} \subset \mathbb{T}^n_{\mathbb{C}} \times \mathbb{C}^n$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{B} \cap \mathbb{A}^n$. A function defined on an annulus is *real-analytic* if it can be holomorphically extended to a complex strip of the annulus.

Definition 2.23. Given $r \in \mathbb{N}$ and $\mathcal{B}$, a complex strip of $\mathcal{A}$, let $\mathrm{A}(\mathcal{B}, C^r)$ denote the set of bounded holomorphic functions $u : \mathcal{B} \to \mathbb{C}$ such that $u(\mathcal{A}) \subset \mathbb{R}$ and such that all the partial derivatives up to order $r$ can be continuously extended to $\bar{\mathcal{B}}$. Endow $\mathrm{A}(\mathcal{B}, C^r)$ with the norm

$$\|u\|_{\mathcal{B},C^r} = \sup_{k \leq r} \sup_{z \in \mathcal{B}} \left|\mathrm{D}^k u(z)\right| ,$$

where $\left|\mathrm{D}^k u(z)\right|$ denotes the supremum norm of the components of $\mathrm{D}^k u(z)$. We also use the notation $\|u\|_{\mathcal{B}} = \|u\|_{\mathcal{B},C^0}$.

The sets $\mathrm{A}(\mathbb{T}^n_\rho, C^r)$, $\mathrm{A}(\mathcal{B}, C^r)$, endowed with the corresponding norms, are Banach spaces. Definitions 2.22 and 2.23 extend component-wise to vector or matrix-valued functions, or in general for tensor functions, the norm is defined by taking the maximum of the norms of the components.

### 2.4.2. Real-analytic symplectomorphisms and real-analytic parameterizations of tori.

Assume that the exact symplectic form $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$ has an holomorphic extension to the complex strip $\mathcal{B}$ of $\mathcal{A}$.

Definition 2.24. Given $r \in \mathbb{N}$, $\rho > 0$ and $\mathcal{B}_0 \subset \mathcal{B}$, a complex strip of an annulus $\mathcal{A}_0 \subset \mathcal{A}$, let $\mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^r)$ denote the set of holomorphic embeddings $K : \mathbb{T}^n_\rho \to \mathcal{B}_0$ such that:

a) $\mathrm{closure}(K(\mathbb{T}^n_\rho)) \subset \mathcal{B}_0$;
b) The components of $K(\theta) - (\theta, 0)$ are in $\mathrm{A}(\mathbb{T}^n_\rho, C^r)$;
c) If $r \geq 1$, the components of the inverse of $G_K$ (given in (2.11)) are in $\mathrm{A}(\mathbb{T}^n_\rho, C^{r-1})$.

$\mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^r)$ is endowed with the distance $\|K_1 - K_2\|_{\rho,C^r}$.

Definition 2.25. Given $r \in \mathbb{N}$ and $\mathcal{B}_0 \subset \mathcal{B}$, a complex strip of $\mathcal{A}_0 \subset \mathcal{A}$, let $\mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^r)$ denote the set of holomorphic maps $f : \mathcal{B}_0 \to \mathcal{B}$ such that

a) The components of $f(x, y) - (x, 0)$ are in $\mathrm{A}(\mathcal{B}_0, C^r)$;
b) $f^*\omega = \omega$.

$\mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^r)$ is endowed with the distance $\|f_1 - f_2\|_{\mathcal{B}_0,C^r}$.

## 2.5. One-bite small divisors equations

Here we review the standard results for the 'one-bite' small divisors equations. Given $\omega \in \mathbb{R}^n$, define the linear operator $\mathcal{L}_\omega$ as follows:

$$\mathcal{L}_\omega \, u = u - u \circ \mathrm{R}_\omega \; . \tag{2.15}$$

The analytic core of KAM techniques is the following *small divisors equation*:

$$\mathcal{L}_\omega u = v - \langle v \rangle \, , \tag{2.16}$$

where $v$ is smooth and known and $u$ has to be determined. Note that, given $v$, if $u(0)$ is fixed then (2.16) determines the dense set $\{u(t\omega) : t \in \mathbb{Z}\}$.

REMARK 2.26. Let $v : \mathbb{T}^n \to \mathbb{R}$ be a continuous function and assume that $\mathrm{R}_\omega$ is an ergodic rotation. Then, if there exists a continuous zero-average solution of equation (2.16), this is unique and will be denoted by $\mathcal{R}_\omega v$. All the solutions of (2.16) differ by a constant.

It is well-known that a sufficient condition for the solvability of the small divisors equation (2.16) is that $\omega$ satisfies a Diophantine property defined below.

DEFINITION 2.27. Given $\gamma > 0$ and $\tau \geq n$, $\omega \in \mathbb{R}^n$ is a Diophantine frequency vector of type $(\gamma, \tau)$ if and only if

$$\left|k^\top \omega - m\right| \geq \gamma \,|k|_1^{-\tau}, \qquad \forall k \in \mathbb{Z}^n \setminus \{0\},\, m \in \mathbb{Z},$$

where $|k|_1 = \sum_{i=1}^n |k_i|$. The set of $n$-dimensional Diophantine frequencies of type $(\gamma, \tau)$ is denoted by $\mathcal{D}_n(\gamma, \tau)$.

Equation (2.16) can be solved in the smooth category, as stated in the following result. For a proof see e.g [**19, 55**].

LEMMA 2.28. *Let $\omega \in \mathcal{D}_n(\gamma, \tau)$, for some $\gamma > 0$ and $\tau \geq n$. Let $\ell \in \mathbb{R}$ be not an integer be such that $\ell - \tau > 0$ is not an integer. Then, for any $C^\ell$-function $v : \mathbb{T}^n \to \mathbb{R}$, there exists a unique $C^{\ell-\tau}$-function, $u : \mathbb{T}^n \to \mathbb{R}$ with zero-average satisfying* (2.16)*.*

If $\ell$ and $\ell - \tau$ are integers, then Lemma 2.28 holds in the so-called $\Lambda_\ell$ spaces [**65, 74**]. In the analytic category, we have the following well-known result, for a proof see [**19, 55**].

LEMMA 2.29 (Rüssmann estimates). *Let $\omega \in \mathcal{D}_n(\gamma, \tau)$, for some $\gamma > 0$ and $\tau \geq n$. There exists a positive constant $c_R$, depending only on $n$ and $\tau$, such that for any $v \in \mathrm{A}(\mathbb{T}^n_\rho, C^0)$, with $\rho > 0$, there exists a unique zero-average solution $u$ of* (2.16)*, denoted by $u = \mathcal{R}_\omega v$. Moreover, $u \in \mathrm{A}(\mathbb{T}^n_{\rho-\delta}, C^0)$ for any $0 < \delta < \rho$, and*

$$\|u\|_{\rho-\delta} \leq c_R \gamma^{-1} \delta^{-\tau} \|v\|_\rho \,. \tag{2.17}$$

From Lemma 2.29, it is clear that $\mathcal{R}_\omega : \mathrm{A}(\mathbb{T}^n_\rho, C^0) \to \mathrm{A}(\mathbb{T}^n_{\rho-\delta}, C^0)$ is a continuous linear operator. Moreover, the following equalities hold:

$$\mathcal{R}_\omega \, \mathcal{L}_\omega \, u \;=\; u - \langle u \rangle \,, \tag{2.18}$$

$$\mathcal{L}_\omega \, \mathcal{R}_\omega \, v \;=\; v - \langle v \rangle \,. \tag{2.19}$$

Furthermore, performing some computations and using (2.15) one shows

$$\mathcal{L}_\omega(u\, v)(\theta) \;= u(\theta + \omega)\, \mathcal{L}_\omega v(\theta) + \mathcal{L}_\omega u(\theta)\, v(\theta). \tag{2.20}$$

Equations (2.19) and (2.20) imply

$$\mathcal{L}_\omega(\mathcal{R}_\omega u\, \mathcal{R}_\omega v)(\theta) = \mathcal{R}_\omega u(\theta + \omega)(v(\theta) - \langle v \rangle) + (u(\theta) - \langle u \rangle)\mathcal{R}_\omega v,$$

from which we have

$$\langle\, \mathcal{R}_\omega u(\theta + \omega)\, v(\theta) + u(\theta)\, \mathcal{R}_\omega v(\theta) \,\rangle \;=\; 0. \tag{2.21}$$

The above definitions for $\mathcal{L}_\omega$ and $\mathcal{R}_\omega$ extend component-wise to vector and matrix-valued functions. These extensions also satisfy Lemma 2.28, Lemma 2.29 and equalities (2.18), (2.19) and (2.21). We will refer to $\mathcal{L}_\omega$ as the *one-bite cohomology operator* and to as the *one-bite solver operator*

# Part 1

# Geometrical properties of KAM invariant tori

CHAPTER 3

# Geometric properties of an invariant torus

Let $\mathcal{A}_0 \subset \mathcal{A}$ be annuli in $\mathbb{A}^n$, let $(\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}, \boldsymbol{J}, \boldsymbol{g})$ be a compatible triple on $\mathcal{A}$, with coordinate representation $(\Omega, \alpha, J, G)$.

## 3.1. Automatic reducibility

Let $f \in \mathrm{Symp}(\mathcal{A}_0,\mathcal{A})$ be given. It is well-known that the existence of an $f$-invariant torus, $K \in \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$, such that the internal dynamics of $K(\mathbb{T}^n)$ is an ergodic rotation, has important geometrical and dynamical consequences. These are stated in Lemma 3.1. In particular, $f$ is exact symplectic and $K$ Lagrangian [**34**] (see also §8 in [**40**]). Moreover, the linearized dynamics around the torus is upper-triangular. This is sometimes referred to as *'automatic reducibility'*. These geometrical arguments are depicted in Figure 1 (from [**18**]) for the two dimensional case.

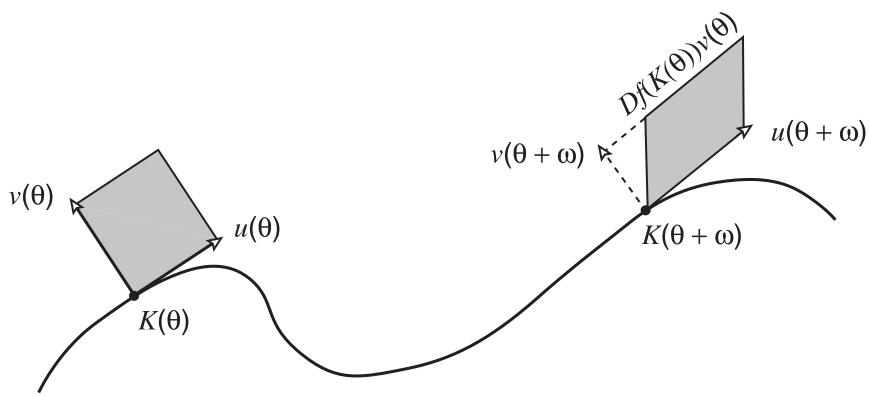


FIGURE 1. If $K$ is $f$-invariant, then $u(\theta) = \mathrm{D}K(\theta)$ spans the tangent space of $K(\mathbb{T}^n)$ at $K(\theta)$, and the symplectic conjugate $v(\theta)$ spans a complementary space.

LEMMA 3.1. *Let* $f \in \mathrm{Symp}(\mathcal{A}_0, \mathcal{A})$. *Assume that* $K \in \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$ *is* $f$-*invariant and that the dynamics in* $\mathcal{K} = K(\mathbb{T}^n)$ *is the ergodic rotation* $\mathrm{R}_\omega(\theta) = \theta + \omega$*:*

$$f \circ K - K \circ \mathrm{R}_\omega = 0. \tag{3.1}$$

*Then the following hold.*

**a)** $f$ *is exact symplectic.*
**b)** $K$ *is Lagrangian.*

**c)** *Define* $L_K, N_K : \mathbb{T}^n \to \mathbb{R}^{2n\times n}$ *by*

$$L_K(\theta) = \mathrm{D}_\theta K(\theta), \tag{3.2}$$

$$N_K(\theta) = J(K(\theta))\ \mathrm{D}_\theta K(\theta)\ G_K(\theta)^{-1}, \tag{3.3}$$

*where* $G_K$ *is given in* (2.11). *Let* $M_K : \mathbb{T}^n \to \mathbb{R}^{2n\times 2n}$ *be obtained by juxtaposing the* $2n\times n$ *matrices in* (3.2) *and* (3.3)*:*

$$M_K(\theta) = \begin{pmatrix} L_K(\theta) & N_K(\theta) \end{pmatrix}. \tag{3.4}$$

*Then, the vector bundle morphism induced by* $M_K$*:*

$$\begin{array}{rccc} \boldsymbol{M}_K : & \mathbb{T}^n\times\mathbb{R}^{2n} & \longrightarrow & \mathrm{T}_{\mathcal{K}}\mathcal{A}_0 \\ & (\theta,\xi) & \longrightarrow & (K(\theta), M_K(\theta)\xi) \end{array}$$

*is an isomorphism such that* $\boldsymbol{M}_K^*\boldsymbol{\omega} = \boldsymbol{\omega}_0$*. In particular,*

$$M_K(\theta)^{-1} = -\Omega_0\ M_K(\theta)^\top\ \Omega(K(\theta)). \tag{3.5}$$

**d)** *Transformation by* $\boldsymbol{M}_K$ *reduces the linearized dynamics* $\mathrm{D}_z f\circ K$ *to a block-triangular matrix:*

$$M_K(\theta+\omega)^{-1}\mathrm{D}_z f(K(\theta))M_K(\theta) = \begin{pmatrix} I_n & T_{(f,K)}(\theta) \\ O_n & I_n \end{pmatrix} \tag{3.6}$$

*where* $T_{(f,K)}$ *is the symmetric* $n\times n$ *matrix defined by*

$$T_{(f,K)}(\theta) = N_K(\theta+\omega)^\top\ \Omega(K(\theta+\omega))\ \mathrm{D}_z f(K(\theta))\ N_K(\theta). \tag{3.7}$$

Proof. Taking derivatives of both sides of the invariance equation (3.1) we have:

$$\mathrm{D}_z f(K(\theta))\ \mathrm{D}_\theta K(\theta) = \mathrm{D}_\theta K(\theta+\omega)\ . \tag{3.8}$$

Let $\tilde{S}^f(z) = x^\top \mathrm{C}^f + S^f(x)$ be the local primitive function of $f$. To prove Part a) it is sufficient to prove that the Calabi invariant of $f$, $\mathrm{C}^f$, is zero. Using that $K^x(\theta)-\theta$ is 1-periodic, and equalities (2.5), (2.6),(3.1) and (3.8) we have

$$\begin{aligned} (\mathrm{C}^f)^\top &= \left\langle \mathrm{D}_\theta((\mathrm{C}^f)^\top K^x + S^f\circ K)(\theta)\right\rangle \\ &= \left\langle \mathrm{D}_\theta(\tilde{S}^f\circ K)(\theta)\right\rangle \\ &= \left\langle \left(a(f(K(\theta)))^\top \mathrm{D}_z f(K(\theta)) - a(K(\theta))^\top\right)\ \mathrm{D}_\theta K(\theta)\right\rangle \\ &= \left\langle a(K(\theta+\omega))^\top \mathrm{D}_\theta K(\theta+\omega) - a(K(\theta))^\top \mathrm{D}_\theta K(\theta)\right\rangle, \\ &= 0. \end{aligned}$$

Let $\Omega_K(\theta)$ be given by (2.10). Using (2.3), (3.1) and (3.8), it is easy to show that $\Omega_K(\theta+\omega) = \Omega_K(\theta)$, for all $\theta\in\mathbb{T}^n$. The ergodicity of $\mathrm{R}_\omega$ implies that $\Omega_K(\theta) = \langle\Omega_K\rangle$ is constant. Moreover, since $K^*\boldsymbol{\omega} = \mathrm{d}(K^*\boldsymbol{\alpha})$, we have that $\langle\Omega_K\rangle = 0$. In more algebraic terms, the components of $\Omega_K(\theta)$ are sums of derivatives of periodic functions:

$$(\Omega_K(\theta))_{i,j} = \sum_{m=1}^{2n} \left(\partial_{\theta_i}\left(a_m(K(\theta))\right)\partial_{\theta_j} K_m(\theta) - \partial_{\theta_j}\left(a_m(K(\theta))\right)\partial_{\theta_i} K_m(\theta)\right).$$

From which one obtains $0 = \langle\Omega_K\rangle = \Omega_K$. This proves Part b).

Let $\mathcal{N}$ be the bundle generated by the column vectors of $N_K$. Then, the Lagrangianity of $\mathcal{K} = K(\mathbb{T}^n)$ implies that the sub-bundles $\mathrm{T}\mathcal{K}$ and $\mathcal{N}$ are $\boldsymbol{g}$-orthogonal: $L_K(\theta)^\top G(K(\theta))N_K(\theta) \equiv 0$. Moreover, it is clear that the column vectors of $L_K$

span the tangent bundle $\mathrm{T}\mathcal{K}$. Hence, $\mathcal{N}$ is a normal bundle complementary to $\mathrm{T}\mathcal{K}$ and $\boldsymbol{M}$ is a bundle isomorphism such that $\boldsymbol{M}^*\boldsymbol{\omega} = \boldsymbol{\omega}_0$. This proves c).

From the $f_*$-invariance of the tangent bundle $\mathrm{T}\mathcal{K}$ (see (3.8)) we obtain that

$$M_K(\theta+\omega)^{-1}\mathrm{D}_z f(K(\theta))M_K(\theta) \tag{3.9}$$

is a block-triangular matrix with the identity $I_n$ in the top-left corner. Since $\boldsymbol{M}_K^*\boldsymbol{\omega} = \boldsymbol{\omega}_0$, the cocycle (3.9) is $\boldsymbol{\omega}_0$-symplectic, obtaining the special form (3.6) and the symmetry of $T_{(f,K)}(\theta)$. Formula (3.7) follows easily. □

Several observations are in order concerning Lemma 3.1.

Remark 3.2. If in Lemma 3.1 the ergodicity condition does not hold, the $f$-invariant torus $K$ may be not Lagrangian. For an example see [**34**].

Remark 3.3. Part c) in Lemma 3.1 is an infinitesimal version of the celebrated Weinstein Lagrangian Neighborhood Theorem [**69**] (see also Chapter 8 in [**15**]).

Remark 3.4. Given $K \in \mathrm{Lag}(\mathbb{T}^n, \mathcal{A}_0)$, there are several ways one can construct a normal bundle $\mathcal{N}$, complementary to the tangent $\mathrm{T}K(\mathbb{T}^n)$ one. In this paper, we choose the normal bundle spanned by the column vector of $N_K$, given in (3.3). When $f$ depends on parameters, this choice of $\mathcal{N}$ yields to smooth dependence of $\mathcal{N}$ with respect to parameters.

Remark 3.5. A statement similar to Lemma 3.1, concerning the automatic reducibility, can be found in [**18**] with a different $M_K$. The main difference is that the transformation $M_K$ defined here is symplectic, and consequently the matrix $T_{(f,K)}$ is symmetric. This is not necessarily the case with the transformation $M_K$ taken in [**18**]. The formulation adopted here simplifies some computations and provides insight on the geometric properties of invariant tori.

Remark 3.6. In Lemma 5.2 it is shown that, if $f$ is real-analytic, $\omega$ is Diophantine and $K$ is a real-analytic approximately $f$-invariant torus, with error $e = f\circ K - K\circ \mathrm{R}_\omega$, then equalities (3.5) and (3.6) hold up to some terms that can be bounded by the norm of the error $e$.

## 3.2. Geometric definition of non-twist tori

Here we define *non-twist tori* using the geometric properties of embedding of tori. The definition given here coincides with the usual one in the close-to-integrable case or that using the BNF. To motivate our definition, we notice that the components of matrix $T_{(f,K)}(\theta)$, given in (3.7), are the symplectic areas between the column vectors of $N_K(\theta+\omega)$ and the column vectors of $\mathrm{D}_z f(K(\theta))N_K(\theta)$. If $\mathcal{N}$ is the normal bundle generated by the column vectors of $N_K$, then, $N_K(\theta+\omega)$ spans $\mathcal{N}_{K(\theta+\omega)}$ and $\mathrm{D}_z f(K(\theta))N_K(\theta)$ is the push-forward of the vectors spanning the normal bundle $\mathcal{N}_{K(\theta)}$. Hence, the matrix $\langle T_{(f,K)}\rangle$ measures how much the normal bundle $\mathcal{N}$ is *twisted* in average when applying the push-forward $f_*$.

Definition 3.7. Let $f \in \mathrm{Symp}(\mathcal{A}_0, \mathcal{A})$, $\omega \in \mathbb{R}^n$ and $K : \mathbb{T}^n \to \mathcal{A}_0$. The *torsion* of $K$ with respect to $f$ and $\omega$ is the $n\times n$-matrix given by

$$\bar{\mathrm{T}}_{(f,K)} = \langle T_{(f,K)}\rangle\,,$$

where $T_{(f,K)}$ is as in (3.7).

If $\bar{\mathrm{T}}_{(f,K)}$ is invertible, we say that $K$ is *twist* with respect to $f$ and $\omega$. Otherwise we say that $K$ is *non-twist* with respect to $f$ and $\omega$.

EXAMPLE 3.8. Assume that $\mathbb{A}^n$ is endowed with the standard symplectic form $\boldsymbol{\omega}_0$. Let $U \subset \mathbb{R}^n$ be open and simply connected. Let $f_0 : \mathbb{T}^n \times U \to \mathbb{A}^n$ be the integrable symplectomorphism with frequency map $\hat{\omega}(y) = \omega + \nabla_y A(y)$:

$$f_0(x, y) = \begin{pmatrix} x + \hat{\omega}(y) \\ y \end{pmatrix}.$$

Then, a direct computation shows that, for any $p \in U$, the torsion of the torus $Z_p(\theta) = \begin{pmatrix} \theta \\ p \end{pmatrix}$, with respect to $f_0$ and $\omega$, is $\bar{\mathrm{T}}_{(f,Z)}(p) = \mathrm{D}_y\hat{\omega}(y)$ Hence, Definition 3.7 extends the usual definition of torsion for integrable systems.

Definition 3.7 will be used for both invariant and approximately invariant tori.

## 3.3. Intrinsic character of the reducibility and of the torsion

Let $f \in \mathrm{Symp}(\mathcal{A}_0, \mathcal{A})$, $\omega \in \mathbb{R}^n$ and $K \in \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$. Assume that $K$ is $f$-invariant with frequency $\omega$ and that $\mathrm{R}_\omega$ is an ergodic rotation. The automatic reducibility of $\mathcal{K} = K(\mathbb{T}^n)$ is provided by the following two intrinsic geometric properties:

R1) The tangent bundle of $\mathcal{K}$, $\mathrm{T}\mathcal{K}$, is $f_*$-invariant and there exists a smooth function $L'_K : \mathbb{T}^n \to \mathbb{R}^{2n \times n}$ such that:
  a) the column vectors of $L'_K(\theta)$ span $\mathrm{T}_{K(\theta)}\mathcal{K}$;
  b) $\mathrm{D}f(K(\theta))\, L'_K(\theta) = L'_K(\theta + \omega)$.

R2) There exist a Lagrangian bundle $\mathcal{N}'$, complementary to $\mathrm{T}\mathcal{K}$, and a smooth function $N'_K : \mathbb{T}^n \to \mathbb{R}^{2n \times n}$ such that:
  a) the columns vectors of $N'_K(\theta)$ span $\mathcal{N}_{K(\theta)}$;
  b) the transformation $\boldsymbol{M}'_K : \mathbb{T}^n \times \mathbb{R}^{2n} \to \mathrm{T}_{\mathcal{K}}\mathcal{A}$ induced by the matrix

$$M'_K(\theta) = \begin{pmatrix} L'_K(\theta) & N'_K(\theta) \end{pmatrix}$$

  takes the symplectic form $\boldsymbol{\omega}$ into the standard form $\boldsymbol{\omega}_0$.

Indeed, Property R1 guarantees the block-triangular form, with the identity $I_n$ in the top-left corner, of

$$M'_K(\theta + \omega)^{-1} \mathrm{D}_z f(K(\theta)) M'_K(\theta)$$

and Property R2 gives:

$$M'_K(\theta + \omega)^{-1} \mathrm{D}_z f(K(\theta)) M'_K(\theta) = \begin{pmatrix} I_n & T'_{(f,K)}(\theta) \\ O_n & I_n \end{pmatrix}, \tag{3.10}$$

where

$$T'_{(f,K)}(\theta) = N'_K(\theta + \omega)^\top\ \Omega(K(\theta + \omega))\ \mathrm{D}_z f(K(\theta))\ N'_K(\theta)\,.$$

Part c) in Lemma 3.1 shows that R1 and R2 hold if one chooses $L'_K = L_K$ and $N'_K = N_K$, given in (3.2) and (3.3), respectively. Even if other choices would also work, this simplifies some computations and some quantitative estimates in our constructions.

The following result gives a characterization of all possible matrices $L'_K$ and $N'_K$ satisfying R1 and R2 in terms of the matrices $L_K$ and $N_K$ in (3.2) and (3.3).

PROPOSITION 3.9. *Let $f \in \mathrm{Symp}(\mathcal{A}_0, \mathcal{A})$ be given. Assume that $K \in \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$ is $f$-invariant with internal dynamics the ergodic rotation $\mathrm{R}_\omega$. The smooth functions $L'_K : \mathbb{T}^n \to \mathbb{R}^{2n \times n}$ and $N'_K : \mathbb{T}^n \to \mathbb{R}^{2n \times n}$ satisfy* R1 *and* R2 *if and only if* *there is an invertible matrix* $A \in \mathbb{R}^{n\times n}$ *and a function* $U : \mathbb{T}^n \to \mathbb{R}^{n\times n}$ *such that, for all* $\theta \in \mathbb{T}^n$, $U(\theta)$ *is symmetric and*

$$\begin{aligned} L'_K(\theta) &= L_K(\theta)A, \\ N'_K(\theta) &= L_K(\theta)U(\theta)\,A^{-\top} + N_K(\theta)\,A^{-\top}. \end{aligned}$$

*Moreover, if*

$$M'_K(\theta) = M_K(\theta)\begin{pmatrix} I_n & U(\theta) \\ O_n & I_n \end{pmatrix}\begin{pmatrix} A & O_n \\ O_n & A^{-\top} \end{pmatrix}, \tag{3.11}$$

*then*

$$M'_K(\theta+\omega)^{-1}\mathrm{D}_z f(K(\theta))M'_K(\theta) = \begin{pmatrix} I_n & T'_{(f,K)}(\theta) \\ O_n & I_n \end{pmatrix},$$

*where*

$$T'_{(f,K)}(\theta) = A^{-1}\left(U(\theta) - U(\theta+\omega) + T_{(f,K)}(\theta)\right)A^{-\top}. \tag{3.12}$$

PROOF. Straightforward. □

A consequence of Proposition 3.9 is that, under certain regularity conditions, the dynamics around an invariant torus can be reduced to an block-triangular matrix with constant coefficients.

PROPOSITION 3.10. *Let* $f \in \mathrm{Symp}(\mathcal{A}_0, \mathcal{A})$ *be given. Assume that* $K \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A}_0)$ *is* $f$*-invariant with internal dynamics the rotation* $\mathrm{R}_\omega$*. Also assume that* $\omega$ *is Diophantine and* $T_{(f,K)}$ *is sufficiently smooth in such a way that* $\mathcal{R}_\omega T_{(f,K)}$ *is smooth, being* $\mathcal{R}_\omega$ *the one-bite solver operator (see Section 2.5). Let* $M'_K$ *be given by* (3.11)*, with* $U(\theta) = -\mathcal{R}_\omega T_{(f,K)}(\theta)$*. Then,*

$$M'_K(\theta+\omega)^{-1}\mathrm{D}_z f(K(\theta))M'_K(\theta) = \begin{pmatrix} I_n & A^{-1}\left\langle T_{(f,K)}\right\rangle A^{-\top} \\ O_n & I_n \end{pmatrix}.$$

*In particular, since* $T_{(f,K)}$ *is symmetric, there is* $A \in \mathbb{R}^{n\times n}$ *is such that:*

$$A^{-1}\left\langle T_{(f,K)}\right\rangle A^{-\top} = \mathrm{diag}(t_1, \ldots, t_n).$$

*Then, the dynamics of* $\mathrm{D}_z f \circ K$ *is reducible to*

$$\begin{pmatrix} I_n & \mathrm{diag}(t_1, \ldots, t_n) \\ O_n & I_n \end{pmatrix}.$$

PROOF. Straightforward. □

The torsion of an invariant torus with rationally independent frequency (see Definition 3.7) is intrinsic in the following sense.

PROPOSITION 3.11. *Let* $f \in \mathrm{Symp}(\mathcal{A}_0, \mathcal{A})$*,* $K \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A}_0)$ *and* $\omega \in \mathbb{R}^n$ *be as in Proposition 3.10. If* $L'_K$ *and* $N'_K$ *satisfy* R1 *and* R2*, then the symmetric matrices* $T_{(f,K)}$ *and* $T'_{(f,K)}$ *defined in* (3.7) *and* (3.3)*, respectively, satisfy*

$$\left\langle T'_{(f,K)}\right\rangle = A^{-1}\left\langle T_{(f,K)}\right\rangle A^{-\top},$$

*where* $A \in \mathbb{R}^{n\times n}$ *is invertible. Hence, the torsion of* $K$ *with respect to* $f$ *and* $\omega$ *is defined up to congruence of matrices.*

PROOF. This follows from equality (3.12). □

REMARK 3.12. Given $f$, $K$ and $\omega$ as in Proposition 3.11, Sylvester's law of inertia and Proposition 3.11 imply that the number of positive/negative/zero eigenvalues of $\bar{\mathrm{T}}_{(f,K)}$ are invariants of $K$, $f$ and $\omega$.

CHAPTER 4

# Geometric properties of fibered Lagrangian deformations

In this chapter we introduce the potential for families of parameters and tori that satisfy certain conditions (specified in Definition 4.1). We show the properties of the potential that will enables us to connect Singularity and KAM theories.

## 4.1. The potential of a fibered Lagrangian deformation

Let us first introduce the objects for which we will associate a *potential*.

DEFINITION 4.1. Let $\mathrm{D}, \Lambda \subset \mathbb{R}^s$ be open.

i) A *Lagrangian deformation* with base D is a smooth function $K : \mathrm{D}\times\mathbb{T}^n \longrightarrow \mathcal{A}_0$ such that, for each $p \in \mathrm{D}$, $K_p \equiv K(p, \cdot) \in \mathrm{Lag}(\mathbb{T}^n, \mathcal{A}_0)$.
ii) A *fibered Lagrangian deformation* (FLD) with base sets D and $\Lambda$ is a smooth bundle map

$$\begin{array}{rccc} \mathbf{K}: & \mathrm{D}\times\mathbb{T}^n & \longrightarrow & \Lambda\times\mathcal{A}_0 \\ & (p,\theta) & \longrightarrow & (\lambda(p), K_p(\theta)), \end{array}$$

such that, for each $p \in \mathrm{D}$, $K_p \in \mathrm{Lag}(\mathbb{T}^n, \mathcal{A}_0)$ and $\lambda : \mathrm{D} \to \Lambda$ is the base parameter map.
iii) Let $\omega \in \mathbb{R}^n$ and let $g : \Lambda \times \mathcal{A}_0 \to \mathcal{A}$ be a Hamiltonian deformation with base $\Lambda$. A FLD $\mathbf{K} = (\lambda, K) : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ is *g-invariant with frequency* $\omega$ if for any $p \in \mathrm{D}$, $K_p$ is $g_{\lambda(p)}$-invariant with frequency $\omega$:

$$g_{\lambda(p)} \circ K_p - K_p \circ \mathrm{R}_\omega = 0.$$

We remark that in the Definition 4.1 the dimensions of D and $\Lambda$ are the same.

REMARK 4.2. A FLD, $\mathbf{K} = (\lambda, K) : \mathrm{D} \times \mathbb{T}^n \to \Lambda \times \mathcal{A}_0$, induces a family of parameter-torus couples as follows:

$$\begin{array}{ccc} \mathrm{D} & \longrightarrow & \Lambda\times\mathrm{Lag}(\mathbb{T}^n, \mathcal{A}_0) \\ p & \longrightarrow & (\lambda(p), K_p). \end{array} \tag{4.1}$$

Since there will not be risk of confusion, the function in (4.1) will also be denoted by $\mathbf{K}$.

DEFINITION 4.3. Let $\mathrm{D}, \Lambda \subset \mathbb{R}^s$ open. Let $g : \Lambda\times\mathcal{A}_0 \to \mathcal{A}$ be a Hamiltonian deformation, and let $\mathscr{M}^g$ and $S^g$ be, respectively, the moment map and the primitive function of $g$. Let $\mathbf{K} : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ be a FLD. The *momentum* and the *potential* of $\mathbf{K}$ with respect to $g$ are the functions $\mathscr{M}^{g,\mathbf{K}} : \mathrm{D} \to \mathbb{R}^s$ and $V^{g,\mathbf{K}} : \mathrm{D} \to \mathbb{R}$ defined respectively by:

$$\begin{array}{rcl} \mathscr{M}^{g,\mathbf{K}}(p) & = & \left\langle \mathscr{M}^g(\lambda(p), K(p,\theta)) \right\rangle, \\ V^{g,\mathbf{K}}(p) & = & -\mathscr{M}^{g,\mathbf{K}}(p)^\top \lambda(p) - \left\langle S^g\left(\lambda(p), g^{-1}_{\lambda(p)}(K(p,\theta))\right)\right\rangle. \end{array}$$

We say that $\mathbf{K}$ is parameterized by the *momentum parameter* if, for any $p \in \mathrm{D}$, the following holds:

$$\mathcal{M}^{g,\mathbf{K}}(p) = p.$$

If $g, \mathbf{K}$ are fixed, we will not include them in the notation.

REMARK 4.4. If the momentum of $\mathbf{K}$ with respect to $g$ is a diffeomorphism, then $\mathbf{K}$ can be re-parameterized by the momentum parameter. Indeed if $p' = \mathcal{M}^{g,\mathbf{K}}(p)$ and $\mathbf{K}'(p',\theta) = \mathbf{K}\left((\mathcal{M}^{g,\mathbf{K}})^{-1}(p'),\theta\right)$, then

$$\mathcal{M}^{g,\mathbf{K}'}(p') = p'.$$

REMARK 4.5. If $\mathbf{K} = (K,\lambda) : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ is a $g$-invariant FLD, with frequency $\omega$, then the potential of $\mathbf{K}$ satisfies

$$V^{g,\mathbf{K}}(p) = -\mathcal{M}^{g,\mathbf{K}}(p)^\top \lambda(p) - \langle S^g\left(\lambda(p), K(p,\theta)\right)\rangle \ .$$

The momentum and the potential of invariant FLD do not change under canonical changes of the phase space variable. This is the content of the following result.

PROPOSITION 4.6. *Let $\mathrm{D},\Lambda \subset \mathbb{R}^s$ be open and let $\omega \in \mathbb{R}^n$ be fixed. Let $g : \Lambda\times\mathcal{A}_0 \to \mathcal{A}$ and $\mathbf{K} = (\lambda, K) : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ be, respectively, a Hamiltonian deformation and a $g$-invariant FLD, with frequency $\omega$. Let $\varphi : \Lambda\times\mathcal{A}' \to \mathcal{A}$ be a Hamiltonian deformation such that $\varphi_\lambda(\mathcal{A}_0') = \mathcal{A}_0$, for all $\lambda \in \Lambda$. Define $g' : \Lambda\times\mathcal{A}_0' \to \mathcal{A}'$ and $\mathbf{K}' : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0'$ by*

$$g'(\lambda,z) = \varphi_\lambda^{-1}(g_\lambda(\varphi_\lambda(z))) \qquad \text{and} \qquad \mathbf{K}'(p,\theta) = (\lambda(p), \varphi_{\lambda(p)}^{-1}(K_p(\theta))).$$

*Then,*

$$\mathcal{M}^{g',\mathbf{K}'} = \mathcal{M}^{g,\mathbf{K}}, \qquad \text{and} \qquad V^{g',\mathbf{K}'} = V^{g,\mathbf{K}}.$$

PROOF. Consider the Lagrangian deformation $K' : \mathrm{D}\times\mathbb{T}^n \to \mathcal{A}_0'$, given by $K'(p,\theta)) = \varphi_{\lambda(p)}^{-1}(K(p,\theta))$. Performing direct computations and using equality (2.7) and Lemma 2.21 one verifies that the following equalities hold for any $p \in \mathrm{D}$:

$$\left\langle \mathcal{M}^{g'}(\lambda(p), K'(p,\theta)))\right\rangle = \langle \mathcal{M}^g(\lambda(p), K(p,\theta))\rangle \,,$$

$$\left\langle S^{g'}(\lambda(p), K'(p,\theta)))\right\rangle = \langle S^g(\lambda(p), K(p,\theta))\rangle \,.$$

Proposition 4.6 follows from Definition 4.3 and Remark 4.5. □

The following result motivates the name *potential* for $V^{g,\mathbf{K}}$. It will enable us to reduce the infinite-dimensional problem of finding invariant tori for a symplectomorphism in the finite-dimensional problem of finding critical points of the potential.

THEOREM 4.7. *Let $\mathrm{D},\Lambda \subset \mathbb{R}^s$ be open. Let $g : \Lambda\times\mathcal{A}_0 \to \mathcal{A}$ be a Hamiltonian deformation and $\omega \in \mathbb{R}^n$. Let $\mathbf{K} = (\lambda, K) : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ be a FLD. Assume that $\mathbf{K}$ is $g$-invariant with frequency $\omega$ and that it is parameterized by the momentum parameter $p$. Then, the following equality holds:*

$$\lambda(p) = -\nabla_p V^{g,\mathbf{K}}(p) \ . \tag{4.2}$$

*Hence, for any $\lambda_* \in \Lambda$ fixed and any $p_* \in \mathrm{D}$, the torus $K_{p*}$ is $g_{\lambda_*}$-invariant with frequency $\omega$ if and only if $p_*$ is a critical point of the real-valued function given by $V^{g,\mathbf{K}}(p) + p^\top\lambda_*$.*

PROOF. Taking derivatives with respect to $p$ of both sides of the invariance equation:

$$g(\lambda(p), K(p,\theta)) = K(p,\theta+\omega), \tag{4.3}$$

we have:

$$\mathrm{D}_z g(\lambda(p), K(p,\theta))\mathrm{D}_p K(p,\theta) + \mathrm{D}_\lambda g(\lambda(p), K(p,\theta))\mathrm{D}_p\lambda(p) = \mathrm{D}_p K(p,\theta+\omega). \tag{4.4}$$

Performing direct computations and using equalities (4.3) and (4.4) and $g_\lambda^*\boldsymbol{\alpha} = \boldsymbol{\alpha} + \mathrm{d}S_\lambda^g$ one obtains:

$$\begin{aligned}
&\mathrm{D}_p \,\langle S^g(\lambda(p), K(p,\theta))\,\rangle = \\
&= \langle\, \mathrm{D}_z S^g(\lambda(p), K(p,\theta))\, \mathrm{D}_p K(p,\theta) + \mathrm{D}_\lambda S^g(\lambda(p), K(p,\theta)))\ \mathrm{D}_p\lambda(p)\,\rangle \\
&= \left\langle \mathrm{D}_\lambda S^g(\lambda(p), K(p,\theta)) - a(g(\lambda(p), K(p,\theta)))^\top \mathrm{D}_\lambda g(\lambda(p), K(p,\theta))\right\rangle \mathrm{D}_p\lambda(p) \\
&= -\,\langle \mathcal{M}^g(\lambda(p), K(p,\theta))\,\rangle^\top\ \mathrm{D}_p\lambda(p),
\end{aligned}$$

where $\mathcal{M}^g$ is the moment map of $g$. Then, using Definition 4.3 and Remark 4.5, we obtain

$$\nabla_p V^{g,\mathbf{K}}(p) = -\mathrm{D}_p\mathcal{M}^{g,\mathbf{K}}(p)^\top\lambda(p) \;=-\lambda(p),$$

where we have used that $p$ is the momentum parameter. □

The following result relates the torsion of $K_p$ with the Hessian of the potential of $\mathbf{K}$. From now on we use the following convention: if $R$ is defined on $\mathrm{D}\times\mathbb{T}^n$, with $\mathrm{D}\subset\mathbb{R}^s$, then $R_p$ denotes the function defined on $\mathbb{T}^n$ by $R_p(\theta) = R(p,\theta)$.

THEOREM 4.8. *Let* $\mathrm{D}, \Lambda \subset \mathbb{R}^s$, $\omega \in \mathbb{R}^n$, $g : \Lambda\times\mathcal{A}_0 \to \mathcal{A}$ *and* $\mathbf{K} = (\lambda, K) : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ *be as in Theorem 4.7. Let* $C^K$, $L_K$, $N_K$, $M_K$ *and* $T_{(f,K)}$ *be given by* (2.9), (3.2), (3.3), (3.4) *and* (3.7), *respectively. For* $(p,\theta)\in\mathrm{D}\times\mathbb{T}^n$, *take* $C(p) = C^{K_p}$, $L(p,\theta) = L_{K_p}(\theta)$, $N(p,\theta) = N_{K_p}(\theta)$, $M(p,\theta) = M_{K_p}(\theta)$, $T(p,\theta) = T_{(g_{\lambda(p)},K_p)}$ *and* $\bar{\mathrm{T}}(p) = \langle T(p,\theta)\rangle$.

*Let* $\mathcal{M}^g$ *be the moment map of* $g$. *Define* $B^g : \mathrm{D}\times\mathbb{T}^n \to \mathbb{R}^{2n\times s}$ *by*

$$B^g(p,\theta) = \left(\mathrm{D}_z\mathcal{M}^g(\lambda(p), K(p,\theta+\omega))\, M(p,\theta+\omega)J_0\right)^\top. \tag{4.5}$$

*Assume that* $\omega\in\mathcal{D}_n(\gamma,\tau)$, *for some* $\gamma>0$ *and* $\tau\geq n$ *and that, for any* $p\in\mathrm{D}$, $T_p(\theta) = T(p,\theta)$ *and* $B_p^g(\theta) = B^g(p,\theta)$ *are sufficiently smooth so that* $\mathcal{R}_\omega T_p$ *and* $\mathcal{R}_\omega B_p^g$ *are smooth, being* $\mathcal{R}_\omega$ *the one-bite solver operator (see Section 2.5). Define*

$$W(p) = \langle\, \mathrm{D}_z\mathcal{M}^g(\lambda(p), K(p,\theta))\ (N(p,\theta) - L(p,\theta)\,\mathcal{R}_\omega T(p,\theta))\,\rangle^\top .$$

*Then, the following equality holds:*

$$\bar{\mathrm{T}}(p)\,\mathrm{D}_p C(p) \;=\; W(p)\,\mathrm{Hess}_p\, V^{g,\mathbf{K}}(p). \tag{4.6}$$

*In particular, for any* $\lambda_*\in\Lambda$ *fixed, if* $K_{p_*}$ *is a* $g_{\lambda_*}$*-invariant torus with frequency* $\omega$, *with* $p_*\in\mathrm{D}$, *and the matrices* $\mathrm{D}_p C^K(p_*)$ *and* $W(p_*)$ *are invertible, then the co-rank of* $\bar{\mathrm{T}}(p_*)$ *equals the co-rank of* $p_*$ *as a critical point of* $V^{g,\mathbf{K}}(p) + p^\top\lambda_*$. *That is,*

$$\dim\ker\bar{\mathrm{T}}(p_*) = \dim\ker\mathrm{Hess}_p V^{g,\mathbf{K}}(p_*).$$

PROOF. Lemma 3.1 implies

$$M(p,\theta+\omega)^{-1}\mathrm{D}_z g(\lambda(p), K(p,\theta))M(p,\theta) = \begin{pmatrix} I_n & T(p,\theta)\\ O_n & I_n\end{pmatrix}. \tag{4.7}$$

From Lemma 2.19 and equalities (3.5) and (4.3) we obtain

$$M(p,\theta+\omega)^{-1}\mathrm{D}_\lambda g(\lambda(p),K(p,\theta)) = B^g(p,\theta), \tag{4.8}$$

Define $\xi : \mathrm{D}\times\mathbb{T}^n \to \mathbb{R}^{2n\times s}$ by

$$\xi(p,\theta) = M(p,\theta)^{-1}\mathrm{D}_p K(p,\theta)$$

and let $\mathcal{L}_\omega$ and $\mathcal{R}_\omega$ be as in Section 2.5. Then, using equalities (4.4), (4.7) and (4.8) we have that the following equality holds:

$$(\mathcal{L}_\omega + \hat{T}_p)\xi_p + B^g_p\,\mathrm{D}_p\lambda(p) = O_{2n\times s}, \tag{4.9}$$

where

$$\hat{T}_p(\theta) = \begin{pmatrix} O_n & T_p(\theta) \\ O_n & O_n \end{pmatrix}.$$

Applying $\mathcal{R}_\omega$ on both sides of (4.9) and use (2.18) to get

$$\xi_p = (I_{2n} - \mathcal{R}_\omega \hat{T}_p)\,\langle \xi_p\rangle - \mathcal{R}_\omega(\,B^g_p - \hat{T}_p\,\mathcal{R}_\omega B^g_p\,)\mathrm{D}_p\lambda(p). \tag{4.10}$$

Equality (4.9) also implies

$$\left\langle\, \hat{T}_p\,\xi_p + B^g_p\,\mathrm{D}_p\lambda(p)\,\right\rangle = O_{2n\times s}. \tag{4.11}$$

Combining (4.10) and (4.11) one obtains

$$\left\langle \hat{T}_p \right\rangle \langle \xi_p\rangle = -\left\langle\, B^g_p - \hat{T}_p\,\mathcal{R}_\omega B^g_p\,\right\rangle\,\mathrm{D}_p\lambda(p),$$

which is equivalent to

$$\bar{\mathrm{T}}\,(p)\,\langle \xi^y_p\rangle \;=\; \left\langle\, B^{g,x}_{\;p} \;-\, T_p\,\mathcal{R}_\omega B^{g,y}_{\;p}\right\rangle \mathrm{Hess}_p V^{g,\mathbf{K}}(p), \tag{4.12}$$

$$O_{n\times s} \;=\; \left\langle B^{g,y}_{\;p}\right\rangle\,\mathrm{Hess}_p V^{g,\mathbf{K}}(p), \tag{4.13}$$

where we have used (4.2) and $P^x = \pi_x P = \begin{pmatrix} I_n & O_n\end{pmatrix} P$ and $P^y = \pi_y P = \begin{pmatrix} O_n & I_n\end{pmatrix} P$, for $P \in \mathbb{R}^{2n\times m}$. Note that equation (4.13) can also be obtained using that $\mathcal{M}^g$ is 1-periodic in $x$:

$$\left\langle B^{g,y}_{\;p}\right\rangle = -\left\langle \mathrm{D}_\theta\left(\mathcal{M}^g(\lambda(p),K(p,\theta))\right)\right\rangle^\top = O_{n\times s}. \tag{4.14}$$

Moreover, since $K_p$ is $g_{\lambda(p)}$-invariant with frequency $\omega$, we have that the torsion matrix $\bar{\mathrm{T}}\,(p)$ is symmetric. Then, using (2.21), (4.5) and (4.12) we have

$$\begin{aligned}\bar{\mathrm{T}}\,(p)\;\langle \xi^y_p\rangle &= \left\langle B^{g,x}_{\;p} + (\mathcal{R}_\omega T_p \circ \mathrm{R}_\omega)\,B^{g,y}_{\;p}\right\rangle \mathrm{Hess}_p V^{g,\mathbf{K}}(p)\\ &= \langle\,(\mathrm{D}_z\mathcal{M}^g\circ\mathbf{K})(N_p - L_p\mathcal{R}_\omega T_p\,)\rangle^\top\,\mathrm{Hess}_p V^{g,\mathbf{K}}(p).\end{aligned}$$

Finally, using (2.1), (2.12) and (2.13) one obtains

$$\begin{aligned}\mathrm{D}_p C(p) =&\quad \mathrm{D}_p\,\left\langle \mathrm{D}_z a(K(p,\theta))^\top \mathrm{D}_\theta K(\theta)\right\rangle^\top\\ =&\; -\left\langle \mathrm{D}_\theta K(p,\theta)^\top\Omega(K(p,\theta))\mathrm{D}_p K(p,\theta)\right\rangle\\ =&\quad \left\langle \xi^y_p\right\rangle.\end{aligned}$$

This finishes the proof of Theorem 4.8. □

Remark 4.9. The matrices $\bar{\mathrm{T}}_p$ and $\mathrm{Hess}_p V^{g,\Phi}(p)$ in (4.6) are symmetric. Moreover, the terms involved in the equality (4.6) have the following meaning.

i) $\mathrm{Hess}_p V^{g,\Phi}(p)$ is the variation of $\lambda(p)$ as $p$ moves.
ii) $\mathrm{D}_p\,C(p)$ is the variation of the averaged action of $K_p$ as $p$ moves.

iii) Let $\mathcal{N}'_p$ the normal bundle spanned by the column vectors of $N'_p = N_p - L_p \mathcal{R}_\omega T_p$. Then, $\bar{\mathrm{T}}(p)$ measures how much $\mathcal{N}'{}_p$ is *twisted* in average when applying the push-forward $(g_{\lambda(p)})_*$.

iv) $W(p)$ is the averaged action of $\mathrm{D}_z G \circ \mathbf{K}$ on $\mathcal{N}'_p$.

The invertibility of $\mathrm{D}_p C(p)$ means that the averaged action of $K_p$ changes when $p$ changes. This depends only on the Lagrangian deformation $K : \mathrm{D}\times\mathbb{T}^n \to \mathcal{A}_0$ and on the symplectic form $\boldsymbol{\omega}$ (it is independent of the action form $\boldsymbol{\alpha}$). The invertibility of $W(p)$ is a transversality condition of the family $K_p$ with respect to the deformation $g$. Indeed, from Lemma 2.19 we have

$$\begin{aligned} W(p) &= \left\langle \mathrm{D}_z G(\lambda(p), K(p,\theta)) N'_p(\theta) \right\rangle^\top \\ &= \left\langle N'_p(\theta+\omega)^\top \Omega(K(p,\theta+\omega)) \mathrm{D}_\lambda g(\lambda(p), K(p,\theta)) \right\rangle \end{aligned}$$

Then, $W(p)$ is invertible if $\mathcal{N}'_p$ is *'transversal'* to $g$.

**4.1.1. Intrinsic character the relation between the torsion and the potential.** Let $\Lambda, \mathrm{D} \subset \mathbb{R}^s$ be open, let $g : \Lambda\times\mathcal{A}_0 \to \mathcal{A}$ be a Hamiltonian deformation and let $\mathbf{K} = (\lambda, K) : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ be a $g$-invariant FLD with frequency $\omega \in \mathbb{R}^n$. Assume that the rotation $\mathrm{R}_\omega$ is ergodic and that $\mathbf{K}$ is parameterized by the moment parameter. Then, the relation between the torsion of $K_p$, with respect to $g_{\lambda(p)}$ and $\omega$, and the Hessian of the potential, given in equality (4.6), is an intrinsic property of $g$ and $\mathbf{K}$, in the following sense.

PROPOSITION 4.10. *Assume the hypotheses of Theorem 4.8 hold. Then, equality* (4.6) *does not depend on the choice of the symplectic vector bundle map that reduces the linearized dynamics* $\mathrm{D}_z g_{\lambda(p)} \circ K_p$ *into an block-triangular form, with the identity on the diagonal.*

PROOF. From Proposition 3.9 we have that any symplectic vector bundle map that reduces $\mathrm{D}_z g_{\lambda(p)} \circ K_p$ has the following form:

$$M'_p(\theta) = M_p(\theta) \begin{pmatrix} I_n & U_p(\theta) \\ O_n & I_n \end{pmatrix} \begin{pmatrix} A_p & O_n \\ O_n & A_p^{-\top} \end{pmatrix},$$

where $A_p$ is an $n{\times}n$ invertible matrix which is independent of $\theta$, and $U_p : \mathbb{T}^n \to \mathbb{R}^{n\times n}$ is smooth and such that $U_p(\theta)$ is symmetric, for all $\theta \in \mathbb{T}^n$. Define

$$\begin{aligned} B'^{g}_p(\theta) &= \left( \mathrm{D}_z \mathcal{M}^g(\lambda(p), K(p,\theta+\omega))\, M'_p(\theta+\omega) J_0 \right)^\top, \\ T'_p(\theta) &= A_p^{-1} (\mathcal{L}_\omega U_p + T_p)\, A_p^{-\top}, \\ \hat{T}'_p(\theta) &= \begin{pmatrix} O_n & T'_p(\theta) \\ O_n & O_n \end{pmatrix}, \\ \xi'_p(\theta) &= M'_p(\theta)^{-1} \mathrm{D}_p K(p,\theta), \end{aligned}$$

where $\mathcal{L}_\omega$ is given in (2.15). Following the same steps we did in the proof of Theorem 4.8 one shows that the following equality holds:

$$\left\langle \hat{T}'_p \right\rangle \left\langle \xi'_p \right\rangle = - \left\langle B'^{g}_p - \hat{T}'_p \mathcal{R}_\omega B'^{g}_p \right\rangle \mathrm{D}_p \lambda(p). \tag{4.15}$$

Now, performing some computations and using (2.21) and (4.14) one shows that the following equality holds:

$$\left\langle B'^{g}_p - \hat{T}'_p \mathcal{R}_\omega B'^{g}_p \right\rangle = \begin{pmatrix} A_p^{-1} & O_n \\ O_n & A_p^\top \end{pmatrix} \left\langle B^g_p - \hat{T}_p \mathcal{R}_\omega B^g_p \right\rangle,$$

Then, equality (4.15) is equivalent to:

$$\left\langle \hat{T}_p \right\rangle \left\langle \begin{pmatrix} I_n & -U_p \\ O_n & I_n \end{pmatrix} \xi_p \right\rangle = -\left\langle B_p^g - \hat{T}_p \, \mathcal{R}_\omega B_p^g \right\rangle \mathrm{D}_p\lambda(p),$$

which is equivalent to (4.6). □

### 4.2. A parametric version of the potential

For future reference, we establish here Theorem 4.11, a parametric version of theorems 4.7 and 4.8. This will enable us to apply Singularity Theory to study non-twist tori. Theorem 4.11 can be proved in a similar way theorems 4.7 and 4.8 were proved.

Let $\mathcal{U}\times\Lambda \subset \mathbb{R}^k\times\mathbb{R}^s$ be open and let $g : \mathcal{U}\times\Lambda\times\mathcal{A}_0 \to \mathcal{A}$ be a Hamiltonian deformation, with base $\mathcal{U}\times\Lambda$. We say that $g_\mu(\lambda, z) = g(\mu, \lambda, z)$ is a $\mu$-parametric family of Hamiltonian deformations with base $\Lambda$. Let $\mathcal{M}^g$ be the moment map of $g$, then for each $\mu \in \mathcal{U}$, the moment map of $g_\mu$ is given by

$$\begin{aligned} \pi_\lambda \mathcal{M}^g(\mu, \lambda, z) &= \begin{pmatrix} O_{s\times k} & I_s \end{pmatrix} \mathcal{M}^g(\mu, \lambda, z) \\ &= a(z)^\top \mathrm{D}_\lambda g(\mu, \lambda, g^{-1}_{(\mu,\lambda)}(z)) - \mathrm{D}_\lambda \tilde{S}^g(\mu, \lambda, g^{-1}_{(\mu,\lambda)}(z)). \end{aligned}$$

We call $\pi_\lambda\mathcal{M}^g(\mu, \cdot, \cdot)$ the $\mu$-parametric family of moment maps of the $\mu$-parametric family of Hamiltonian deformations $g_\mu$.

Let $\mathcal{U}_0 \subset \mathcal{U}$, $\mathrm{D} \subset \mathbb{R}^s$ be open. A *parametric FLD with base sets* $\mathrm{D}$ *and* $\Lambda$ *and parameter* $\mu \in \mathcal{U}_0$ is a smooth function

$$\begin{array}{rccc} \mathbf{K}: & \mathcal{U}_0\times\mathrm{D}\times\mathbb{T}^n & \longrightarrow & \Lambda\times\mathcal{A}_0 \\ & (\mu, p, \theta) & \longrightarrow & (\lambda(\mu, p), K(\mu, p, \theta)), \end{array}$$

such that for any $\mu \in \mathcal{U}_0$ fixed, $\mathbf{K}_\mu(p, \theta) = \mathbf{K}(\mu, p, \theta)$ is a FLD with base sets $\mathrm{D}$ and $\Lambda$.

THEOREM 4.11. *Let* $\mathrm{D}, \Lambda \subset \mathbb{R}^s$ *and* $\mathcal{U} \subset \mathbb{R}^k$ *be open and let* $\omega \in \mathbb{R}^n$ *Let* $g : \mathcal{U}\times\Lambda\times\mathcal{A}_0 \to \mathcal{A}$ *be a Hamiltonian deformation, with base* $\mathcal{U}\times\Lambda$ *and moment map* $\mathcal{M}^g$*. Assume that* $\mathbf{K} = (\lambda, K) : \mathcal{U}_0\times\mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ *is a parametric family of FLD, with base sets* $\mathrm{D}$ *and* $\Lambda$ *and parameter* $\mu \in \mathcal{U}_0 \subset \mathcal{U}$*, such that:*

$$g_{(\mu,\lambda(\mu,p))} \circ K_{(\mu,p)} - K_{(\mu,p)} \circ \mathrm{R}_\omega = 0, \tag{4.16}$$

$$\mathcal{M}^{g,\mathbf{K},\lambda}(\mu, p) = p, \tag{4.17}$$

*where* $\mathcal{M}^{g,\mathbf{K},\lambda} : \mathcal{U}_0\times\mathrm{D} \to \mathbb{R}^s$ *is the* $\mu$*-parametric family of momenta with respect to* $\lambda$*:*

$$\mathcal{M}^{g,\mathbf{K},\lambda}(\mu, p) = \pi_\lambda \left\langle \mathcal{M}^g(\mu, \lambda(\mu, p), K(\mu, p, \theta)) \right\rangle . \tag{4.18}$$

*Then the following hold.*

**a)** *The* parametric potential $V^{g,\mathbf{K}} : \mathcal{U}_0\times\mathrm{D} \to \mathbb{R}$*, defined by*

$$V^{g,\mathbf{K}}(\mu, p) = -\,p^\top\lambda(\mu, p) - \left\langle S^g\left(\mu, \lambda(\mu, p), K(\mu, p, \theta)\right) \right\rangle , \tag{4.19}$$

*satisfies*

$$\lambda(\mu, p) = -\nabla_p V^{g,\mathbf{K}}(\mu, p) .$$

**b)** *For any* $\lambda_* \in \Lambda$*,* $\mu_* \in \mathcal{U}_0$ *and* $p_* \in \mathrm{D}$*, the torus* $K_{(\mu_*,p_*)}$ *is* $g_{(\mu_*,\lambda_*)}$*-invariant with frequency* $\omega$ *if and only if* $p_*$ *is a critical point of* $V^{g,\mathbf{K}}(\mu_*, p) + p^\top\lambda_*$*.*

**c)** *Let* $C^{K}$, $L_{K}$, $N_{K}$, $M_{K}$ *and* $T_{(f,K)}$ *be given by* (2.9), (3.2), (3.3), (3.4) *and* (3.7), *respectively. For* $(\mu,p,\theta)\in\mathcal{U}_0\times\mathrm{D}\times\mathbb{T}^n$, *take* $C(\mu,p)=C^{K_{(\mu,p)}}$, $L(\mu,p,\theta)=L_{K_{(\mu,p)}}(\theta)$, $N(\mu,p,\theta)=N_{K_{(\mu,p)}}(\theta)$, $M(\mu,p,\theta)=M_{K_{(\mu,p)}}$, $T(\mu,p,\theta)=T_{(g_{(\mu,\lambda(\mu,p))},K_{(\mu,p)})}$, $\bar{\mathrm{T}}(\mu,p)=\langle T(\mu,p,\theta)\rangle$. *Let* $\mathcal{M}^g$ *be the moment map of* $g$. *Define*

$$B^{g,\lambda}(\mu,p,\theta)=\left(\mathrm{D}_z\pi_\lambda\mathcal{M}^g(\mu,\lambda(\mu,p),K(\mu,p,\theta+\omega))\,M(\mu,p,\theta+\omega)J_0\right)^{\top},$$

*Assume that* $\omega\in\mathcal{D}_n(\gamma,\tau)$, *for some* $\gamma>0$ *and* $\tau\geq n$ *and let* $\mathcal{R}_\omega$ *be the one-bite solver operator (see Section 2.5). Assume that for any* $p\in\mathrm{D}$, $T_{(\mu,p)}(\theta)=T(\mu,p,\theta)$ *and* $B^{g,\lambda}_{(\mu,p)}(\theta)=B^{g,\lambda}(\mu,p,\theta)$ *are sufficiently smooth so that* $\mathcal{R}_\omega T_{(\mu,p)}$ *and* $\mathcal{R}_\omega B^{g,\lambda}_{(\mu,p)}$ *are smooth, being* $\mathcal{R}_\omega$ *the one-bite solver operator (see Section 2.5). Define*

$$\begin{aligned} W(\mu,p)= &\ \left\langle \mathrm{D}_z\pi_\lambda\mathcal{M}^g(\mu,\lambda(\mu,p),K(\mu,p,\theta))\,N(\mu,p,\theta)\right\rangle^{\top} \\ &-\left\langle \mathrm{D}_z\pi_\lambda\mathcal{M}^g(\mu,\lambda(\mu,p),K(\mu,p,\theta))\,L(\mu,p,\theta)\,\mathcal{R}_\omega T(\mu,p,\theta)\right\rangle^{\top}. \end{aligned} \tag{4.20}$$

*Then, the following equality holds:*

$$\bar{\mathrm{T}}(\mu,p)\,\mathrm{D}_pC(\mu,p)\;=\;W(\mu,p)\,\mathrm{Hess}_p\,V^{g,\mathbf{K}}(\mu,p)\,.$$

**d)** *For any* $\lambda_*\in\Lambda$, $\mu_*\in\mathcal{U}_0$ *and* $p_*\in\mathrm{D}$, *if* $K_{(\mu_*,p_*)}$ *is a* $g_{(\mu_*,\lambda_*)}$*-invariant torus with frequency* $\omega$ *and* $\mathrm{D}_pC^K(\mu_*,p_*)$ *and* $W(\mu_*,p_*)$ *are invertible, then the co-rank of* $\bar{\mathrm{T}}(\mu_*,p_*)$ *equals the co-rank of* $p_*$ *as a critical point of the function* $V^{g,\mathbf{K}}(\mu_*,p)+p^{\top}\lambda_*$, *i.e.*

$$\dim\ker\bar{\mathrm{T}}(\mu_*,p_*)=\dim\ker\mathrm{Hess}_pV^{g,\mathbf{K}}(\mu_*,p_*).$$

With the obvious modifications, a parametric version of Proposition 4.10 holds.

# Part 2

# Parametric KAM results

CHAPTER 5

# Nondegeneracy on a KAM procedure with fixed frequency

Here we formulate a nondegeneracy condition that guarantees the construction of an iterative procedure to find invariant tori with fixed Diophantine frequency. This nondegeneracy condition is more general than the usual *twist* condition. In the standard KAM results, the nondegeneracy condition is chosen in such a way that, in an iterative procedure, the small divisors equations can be solved by adjusting the average of the tori. In parametric KAM results, the averages can be adjusted either by adjusting the parameters or by adjusting the average of the tori. If one requires to keep fix the average of the tori, the nondegeneracy condition relies on adjusting the parameters. Similar, but somewhat different ideas appear in [**62, 73**].

Throughout this chapter, $\mathcal{A}_0 \subset \mathcal{A}$ are assumed to be annuli endowed with the compatible triple $(\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}, \boldsymbol{J}, \boldsymbol{g})$ with coordinate representation $(\Omega = \mathrm{D}a^\top - \mathrm{D}a, \alpha, J, G)$. $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$.

## 5.1. Approximate reducibility of approximately invariant tori

Here we show that the properties of invariant tori for symplectomorphisms stated in Lemma 3.1 are slightly modified when the torus is only approximately invariant. We also provide explicit estimates.

The following well-known Cauchy estimate (see e.g. [**74**]) will be used.

LEMMA 5.1 (Cauchy estimate). *There exists a positive constant $c$, depending only on $n$, such that for all $\rho > 0$ and $0 < \delta < \rho$, if $\eta \in C^\omega(\mathbb{T}^n_\rho)$ then $\mathrm{D}_\theta \eta \in C^\omega(\mathbb{T}^n_{\rho-\delta})$, and*

$$\|\mathrm{D}_\theta\, \eta\|_{\rho-\delta} \leq c\ \delta^{-1}\, \|\eta\|_\rho\ .$$

Let $\mathcal{A} \subset \mathbb{A}$ be an annulus endowed with the compatible triple $(\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}, \boldsymbol{J}, \boldsymbol{g})$, which is assumed to be real-analytic, i.e. holomorphic in a complex strip $\mathcal{B}$ of $\mathcal{A}$. Assume that the $C^1$-norms on $\mathcal{B}$ of the components of $a$, $\Omega$, $\Omega^{-1}$, $J$, $G$, $G^{-1}$ are bounded by a positive constant denoted by $c_{\mathrm{symp}}$. $\mathcal{A}_0 \subset \mathcal{A}$ is an annulus and $\mathcal{B}_0 \subset \mathcal{B}$ is a complex strip of $\mathcal{A}_0$. The following is the main result of this section.

LEMMA 5.2. *Let $f \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^2)$ and $K \in \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^1)$, for some $\rho > 0$ such that $\gamma\rho^\tau < 1$. Assume that $K$ is approximately $f$-invariant with frequency $\omega$ and error given by:*

$$e = f \circ K - K \circ \mathrm{R}_\omega\ . \tag{5.1}$$

*Then, there is a constant $\kappa$, depending on $n$, $\tau$, $c_{\mathrm{symp}}$ and polynomially on $\|\mathrm{D}_z f(K(\theta))\|_\rho$, $\|\mathrm{D}_\theta K\|_\rho$ and $\left\|(G_K)^{-1}\right\|_\rho$, such that for any $0 < \delta < \rho/2$ fixed, the following properties hold.*

**a)** *The Calabi invariant of $f$, $\mathrm{C}^f$, satisfies*

$$|\mathrm{C}^f| \leq \kappa\, \rho^{-1} \left\| e \right\|_\rho .$$

**b)** *$\mathcal{K} = K(\mathbb{T}^n)$ is approximately Lagrangian in the sense that the matrix $\Omega_{_K} = \mathrm{D}_\theta K(\theta)^\top \Omega(K(\theta)) \mathrm{D}_\theta K(\theta)$ satisfies*

$$\left\| \Omega_{_K} \right\|_{\rho-2\delta} \leq \kappa\, \gamma^{-1} \delta^{-(\tau+1)} \left\| e \right\|_\rho .$$

**c)** *The matrix $M_{_K}$, defined by (3.4), is approximately symplectic in the sense that the matrix defined by*

$$R^s_{_K}(\theta) = M_{_K}(\theta)^\top \Omega(K(\theta)) M_{_K}(\theta) - \Omega_0$$

*satisfies*

$$\left\| \, R^s_{_K} \right\|_{\rho-2\delta} \leq \kappa\, \gamma^{-1}\, \delta^{-(\tau+1)} \left\| e \right\|_\rho .$$

**d)** *The linearized dynamics $\mathrm{D}_z f \circ K$ is approximately reducible, in the sense that the matrix*

$$\begin{aligned} R^r_{_{(f,K)}}(\theta) = & - \Omega_0\, M_{_K}(\theta+\omega)^\top \Omega(K(\theta+\omega))\, \mathrm{D}_z f(K(\theta)) M_{_K}(\theta) \\ & - \begin{pmatrix} I_n & T_{_{(f,K)}}(\theta) \\ O_n & I_n \end{pmatrix}, \end{aligned}$$

*with $T_{_{(f,K)}}$ defined in (3.7), satisfies*

$$\left\| R^r_{_{(f,K)}} \right\|_{\rho-2\delta} \leq \kappa\, \gamma^{-1} \delta^{-(\tau+1)} \, \left\| e \right\|_\rho .$$

**e)** *If the error $e$ is sufficiently small such that*

$$\kappa\, \gamma^{-1} \delta^{-(\tau+1)} \left\| e \right\|_\rho \; < 1 ,$$

*then for any $\theta \in \mathbb{T}^n_{\rho-2\delta}$, $M(\theta)$ is invertible and*

$$R^i_{_K}(\theta) = M_{_K}(\theta)^{-1} + \Omega_0 M_{_K}(\theta)^\top \Omega(K(\theta))$$

*with*

$$\left\| R^i_{_K} \right\|_{\rho-2\delta} \leq \kappa\, \gamma^{-1} \delta^{-(\tau+1)} \, \left\| e \right\|_\rho .$$

PROOF. Taking derivatives in (5.1) we have:

$$\mathrm{D}_\theta e(\theta) = \mathrm{D}_z f(K(\theta)) \mathrm{D}_\theta K(\theta) - \mathrm{D}_\theta K(\theta+\omega) . \tag{5.2}$$

Let $\tilde{S}^f(z) = x^\top \mathrm{C}^f + S^f(x)$ be the local primitive function of $f$. Part a) follows by using Cauchy estimates (Lemma 5.1) and the following equality:

$$\begin{aligned} (\mathrm{C}^f)^\top = & \left\langle \left( a(f(K(\theta))) - a(K(\theta+\omega)) \right)^\top \mathrm{D}_\theta K(\theta+\omega) ) \right\rangle + \\ & + \left\langle a(f(K(\theta)))^\top \mathrm{D}_\theta e(\theta) \right\rangle , \end{aligned}$$

which is obtained using (5.2).

To prove Part b), first notice that $\langle \Omega_K \rangle = 0$ (see the proof of Lemma 3.1). Next, let $\mathcal{L}_\omega$ be as in Lemma 2.29, then we have

$$\begin{aligned} \mathcal{L}_\omega \Omega_{_K}(\theta) = \; & \mathrm{D}_\theta K(\theta+\omega)^\top \Delta\Omega(\theta)\, \mathrm{D}_\theta K(\theta+\omega) \\ & + \mathrm{D}_\theta K(\theta+\omega)^\top \Omega(f(K(\theta)))\, \mathrm{D}_\theta e(\theta) \\ & + \mathrm{D}_\theta e(\theta)^\top \Omega(f(K(\theta))) \mathrm{D}_z f(K(\theta)) \mathrm{D}_\theta K(\theta) , \end{aligned}$$

where

$$\Delta\Omega(\theta) = \Omega(f(K(\theta))) - \Omega(K(\theta+\omega)) .$$

Moreover, using the Mean Value Theorem we have

$$\|\Delta\Omega(\theta)\|_\rho \leq c\,\|\mathrm{D}_z\Omega\|_{\mathcal{B}}\,\|e\|_\rho\ , \tag{5.3}$$

where $c$ depends only on $n$. Par b) of Lemma 5.2 follows from the above observations, Lemma 2.29 and Cauchy estimates (Lemma 5.1).

Performing straightforward computations one shows that

$$R^s_{K}(\theta) = \begin{pmatrix} \Omega_{K}(\theta) & O_n \\ O_n & G_{K}(\theta)^{-1}\Omega_{K}(\theta)G_{K}(\theta)^{-1} \end{pmatrix}\ ,$$

from which Part c) follows.

Part d) is proved by computing the four $(n\times n)$-block components of

$$-J_0 M_{K}(\theta+\omega)^\top \Omega(K(\theta+\omega))\mathrm{D}_z f(K(\theta))M_{K}(\theta)\,,$$

which are:

$$\begin{aligned} N_{K}(\theta+\omega)^\top\Omega(K(\theta+\omega))\mathrm{D}_z f(K(\theta))L_{K}(\theta) = I_n + \\ + N_{K}(\theta+\omega)^\top\Omega(K(\theta+\omega))\mathrm{D}_\theta e(\theta), \end{aligned}$$

$$N_{K}(\theta+\omega)^\top\Omega(K(\theta+\omega))\mathrm{D}_z f(K(\theta))N_{K}(\theta) = T_{K}(\theta)\,,$$

$$\begin{aligned} -L_{K}(\theta+\omega)^\top\Omega(K(\theta+\omega))\mathrm{D}_z f(K(\theta))L_{K}(\theta) = O_n + \\ -\Omega_{K}(\theta+\omega) - L_{K}(\theta+\omega)^\top\Omega(K(\theta+\omega))\mathrm{D}_\theta e(\theta)\,, \end{aligned}$$

$$\begin{aligned} -L_{K}(\theta+\omega)^\top\Omega(K(\theta+\omega))\mathrm{D}_z f(K(\theta))N_{K}(\theta) = I_n + \\ + L_{K}(\theta+\omega)^\top\Delta\Omega(\theta)\mathrm{D}_z f(K(\theta))N_{K}(\theta) + \\ - \mathrm{D}_\theta e(\theta)^\top\Omega(f(K(\theta)))\,\mathrm{D}_z f(K(\theta))N_{K}(\theta)\,, \end{aligned}$$

where $\Delta\Omega$ satisfies (5.3) and $L_{K}$ and $N_{K}$ are defined by (3.2) and (3.3), respectively.

Now, if $\|e\|_\rho$ is sufficiently small such that $\left\|\Omega_0\, R^s_{K}\right\|_{\rho-2\delta} < 1/(2n)$, then for any $\theta \in \mathbb{T}^n_{\rho-2\delta}$, $\Omega_0 + R^s_{K}(\theta) = M_{K}(\theta)^\top\Omega(K(\theta))M_{K}(\theta)$ is invertible and moreover

$$\left\|(\Omega_0 + R^s_{K})^{-1}\right\|_{\rho-2\delta} < 2\,.$$

Hence, $M_{K}(\theta)$ is also invertible. Moreover, simple computations involving the Neumann series show that

$$R^i_{K}(\theta) = \Omega_0 R^s_{K}(\theta)(\Omega_0 + R^s_{K}(\theta))^{-1}M_{K}(\theta)^\top\Omega(K(\theta))\ .$$

□

## 5.2. Dummy and modifying parameters

Let $f \in \mathrm{Symp}(\mathcal{A}_0, \mathcal{A})$ and $\omega \in \mathcal{D}_n(\gamma,\tau)$ be fixed. A torus $K \in \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$ is $f$-invariant with frequency $\omega$ if and only if the following functional equation is satisfied:

$$f\circ K = K\circ \mathrm{R}_\omega\,. \tag{5.4}$$

Under the hypotheses of Lemma 5.2, an iterative procedure to find solutions (5.4) can be constructed if the linear equation

$$(\mathcal{L}_\omega + \hat{T}_{(f,K)})\,\xi = \eta \tag{5.5}$$

can be solved, where $\mathcal{L}_\omega u = u - u \circ \mathrm{R}_\omega$ and

$$\hat{T}_{(f,K)}(\theta) = \begin{pmatrix} O_n & T_{(f,K)} \\ O_n & O_n \end{pmatrix},$$

with $T_{(f,K)}$ defined in (3.7).

For shake of clearness and to reduce the amount of technical details, all the functions considered here are assumed to be sufficiently smooth so that $\mathcal{R}_\omega$, the one-bite solver operator (see Section 2.5), applies as many times as needed. Under this assumption, the equation (5.5) has a solution if and only if there is a $\xi_0^y \in \mathbb{R}^n$ such that the following hold:

$$\bar{\mathrm{T}}_{(f,K)}\, \xi_0^y = \left\langle \eta^x + (\mathcal{R}_\omega\, T_{(f,K)} \circ \mathrm{R}_\omega)\, \eta^y \right\rangle, \tag{5.6}$$

$$0 = \langle \eta^y \rangle, \tag{5.7}$$

where we have used (2.21) and (5.5) to obtain (5.6). Moreover, if $\eta$ and $\xi_0^y$ satisfy (5.6) and (5.7), then for *any* $\xi_0^x \in \mathbb{R}^n$, the function $\xi : \mathbb{T}^n \to \mathbb{R}^{2n}$ defined by

$$\xi = (I_{2n} - \mathcal{R}_\omega \hat{T}_{(f,K)})\, \xi_0 + \mathcal{R}_\omega(\eta - \hat{T}_{(f,K)}\, \mathcal{R}_\omega\, \eta), \tag{5.8}$$

with $\xi_0 = \begin{pmatrix} \xi_0^x \\ \xi_0^y \end{pmatrix} \in \mathbb{R}^{2n}$, satisfies equality (5.5). Conversely, any solution of (5.5) can be written as in (5.8), for some $\xi_0 \in \mathbb{R}^{2n}$ and $\xi_0^y$ a solution of (5.6).

The uniqueness of the solutions of (5.5) depends on the kernel of $\mathcal{L}_\omega + \hat{T}_{(f,K)}$. It is clear that a smooth function $\xi : \mathbb{T}^n \to \mathbb{R}^{2n}$ is in the kernel of $\mathcal{L}_\omega + \hat{T}_{(f,K)}$ if and only if there are $\xi_0^x \in \mathbb{R}^n$ and $\xi_0^y \in \ker \bar{\mathrm{T}}_{(f,K)}$ such that

$$\xi = (I_{2n} - \mathcal{R}_\omega \hat{T}_{(f,K)})\, \xi_0.$$

Hence, the dimension of the kernel of $\mathcal{L}_\omega + \hat{T}_{(f,K)}$ is $n + \dim \ker \bar{\mathrm{T}}_{(f,K)}$, where $\bar{\mathrm{T}}_{(f,K)} = \left\langle T_{(f,K)} \right\rangle$ is the torsion of $K$ with respect to $f$ and $\omega$.

The $n$-dimensional part of the kernel of $\mathcal{L}_\omega + \hat{T}_{(f,K)}$ is due to the non-uniqueness of the parameterizations of invariant tori (if $K$ is a solution of (5.4), then for any $\theta_0 \in \mathbb{R}^n$, $K \circ \mathrm{R}_{\theta_0}$ is also a solution of (5.4)). The $n$-dimensional part of the kernel of $\mathcal{L}_\omega + \hat{T}_{(f,K)}$ can be determined by fixing the initial phase of the torus. That is, look for solutions of (5.4) satisfying the following $n$-dimensional constraint:

$$\langle K^x(\theta) - \theta \rangle = 0. \tag{5.9}$$

Under the hypotheses of Lemma 5.2, an iterative procedure to solve equations (5.4) and (5.9) depends on the solvability properties of (5.5) with the following $n$-dimensional constraint

$$\pi_x \langle M_K(\theta)\, \xi(\theta) \rangle = 0, \tag{5.10}$$

where $M_K$ is defined in (3.4) and $\pi_x P = \begin{pmatrix} I_n & O_n \end{pmatrix} P$, for $P \in \mathbb{R}^{2n \times m}$.

Hence if $\eta : \mathbb{T}^n \to \mathbb{R}^{2n}$ is sufficiently smooth, the linear equations (5.5) and (5.10) have a solution if and only if the following $3n \times 2n$-dimensional linear system has a solution $\xi_0 \in \mathbb{R}^{2n}$:

$$\begin{pmatrix} O_n & \bar{\mathrm{T}}_{(f,K)} \\ O_n & O_n \\ I_n & \pi_x \left\langle N_K - L_K \mathcal{R}_\omega T_{(f,K)} \right\rangle \end{pmatrix} \xi_0 = \mathbf{b}, \tag{5.11}$$

where

$$\mathbf{b} = \begin{pmatrix} \left\langle\, \eta^x + (\mathcal{R}_\omega\, T_{(f,K)} \circ \mathrm{R}_\omega)\, \eta^y \,\right\rangle \\ \langle \eta^y \rangle \\ -\,\pi_x \left\langle M_K \mathcal{R}_\omega \left(\eta - \hat{T}_{(f,K)} \mathcal{R}_\omega \eta\right)\right\rangle \end{pmatrix}, \tag{5.12}$$

and $L_K$ and $N_K$ are given in (3.2) and (3.3), respectively. Moreover, the solutions are of the form (5.8).

Summarizing, the existence of solutions of (5.5) and (5.10) is guaranteed by the smoothness of $\eta$ and $T_{(f,K)}$ and the existence of a solution of (5.11). Moreover, the uniqueness of the solutions of (5.5) and (5.10) depends only on the uniqueness of the solutions of (5.11), i.e. on the kernel of the torsion $\bar{\mathrm{T}}_{(f,K)}$.

In an iterative procedure to solve the non-linear equations (5.4) and (5.9), the right hand part of (5.5) takes the following form:

$$\eta(\theta) = \; J_0 M_K(\theta+\omega)^\top \Omega(K(\theta+\omega)) e(\theta), \tag{5.13}$$

where $e(\theta) = f(K(\theta)) - K(\theta+\omega)$. A sufficient condition to perform a step of the iterative procedure is:

$$\pi_y \left\langle J_0 M_K(\theta+\omega)^\top \Omega(K(\theta+\omega)) e(\theta) \right\rangle = 0. \tag{5.14}$$

In [**18**] it was proved that if $f$ is exact symplectic, then the left hand side of (5.14) has norm of quadratic order with respect to the error $e(\theta)$. Then, using the twist condition, an iterative procedure to solve (5.4) was constructed solving at each step a modified linear equation of the following form:

$$(\mathcal{L}_\omega + \hat{T}_{(f,K)})\, \xi = \eta - \begin{pmatrix} 0 \\ \langle \eta^y \rangle \end{pmatrix}.$$

Here we adopt a different technique. First, we embed the non-linear equation (5.4) into a family of equations by introducing a parameter $\sigma \in \mathbb{R}^n$ (*dummy* parameter) such that, at the iterative procedure, $\sigma$ adjusts the average of the term $\eta^x$ and at the end (but not at the intermediate steps) it is equal to zero. To obtain the later property, we use the fact that the existence of an $f$-invariant torus, with dynamics an ergodic rotation, implies the exactness of $f$ (Lemma 3.1). Hence, the parameter $\sigma$ is introduced in such a way that the Calabi invariant of $f$ is changed. Second, rather than assuming that the torsion $\bar{\mathrm{T}}_{(f,K)}$ is invertible (twist condition), we fix $s \in \mathbb{N}$ with $0 \leq s \leq n$ and introduce a parameter $\lambda \in \mathbb{R}^s$ (*modifying* parameter) in such a way that, at the iterative procedure, $\lambda$ controls the possible degeneracies of $\bar{\mathrm{T}}_{(f,K)}$ and at the end $f$-invariant tori with frequency $\omega$ are obtained making $\lambda = 0$. Let us make precise these definitions.

Definition 5.3. Let $\mathcal{A}_1 \subseteq \mathcal{A}$ be an annulus and let $\Lambda \subset \mathbb{R}^s$ be an open neighborhood of 0, with $0 \leq s \leq n$. A *modifying deformation* with base $\Lambda$ is a symplectic deformation $h : \Lambda \times \mathcal{A}_1 \to \mathcal{A}$, such that:

i) $\mathrm{C}^h(\lambda) = 0$, for all $\lambda \in \Lambda$;
ii) $h_0 = \mathrm{id}$ and $S^h_0 = 0$;
iii) $h_\lambda \circ K = K$ for some $K \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A}_0)$ if and only if $\lambda = 0$.

Definition 5.4. Let $\mathcal{A}_2 \subset \mathcal{A}_0$ be an annulus and let $\Sigma \subset \mathbb{R}^n$ be an open neighborhood of 0. A *dummy deformation* with base $\Sigma$ is a symplectic deformation $d : \Sigma \times \mathcal{A}_2 \to \mathcal{A}$, such that:

i) $\mathrm{C}^d(\sigma) = \sigma$, for all $\sigma \in \Sigma$;
ii) $d_0 = \mathrm{id}$ and $S_0^d = 0$.

EXAMPLE 5.5. An example of a modifying deformation, $h_\lambda$, is the time-1 flow of the Hamiltonian vector field $\Omega(z)^{-1}\nabla_z\left(\lambda^\top y\right)$. It is not difficult to check that the moment map of $h$ satisfies $\mathcal{M}^h(0,z) = y$.

If the symplectic structure is the standard one, the symplectomorphisms $h_\lambda$ are *translations in the $x$-direction*. In this case, the moment map is autonomous (i.e. independent of $\lambda$) and $\mathcal{M}^h(\lambda, z) = y$.

EXAMPLE 5.6. An example of a dummy deformation, $d_\sigma$, is the time-1 flow of the symplectic vector field $\Omega(z)^{-1}\nabla_z\left(-\sigma^\top x\right)$. Notice that $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ are the generators of the first cohomology group of the annulus, whose elements are of the form $\sigma_1 \mathrm{d}x_1 + \cdots + \sigma_n \mathrm{d}x_n$ (see Proposition 2.5a in [**40**]). If the symplectic structure is the standard one, the symplectomorphisms $d_\sigma$ are *translations in the $y$-direction*. Then, the parameter $\sigma$ can be viewed as a generalization of the translation in the translated curve technique [**9, 57, 73**].

REMARK 5.7. A dummy parameter changes the Calabi invariant of $(h_\lambda \circ f)$. For $\mathrm{C}^{d_\sigma}(\sigma) = \sigma$, $\mathrm{C}^h(\lambda) = 0$ imply

$$\mathrm{C}^{(d_\sigma \circ h_\lambda \circ f)} = \sigma + \mathrm{C}^f.$$

If $f$ is assumed to be exact, then $d_\sigma \circ h_\lambda \circ f$ is exact if and only if $\sigma = 0$. In such a case, from Lemma 3.1, we have that the only elements of the family $d_\sigma \circ h_\lambda \circ f$ that might have invariant tori are those for which $\sigma = 0$. This is the reason for which $\sigma$ is called *'dummy parameter'*.

REMARK 5.8. The torus $K \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A}_0)$ is $(h_{\lambda_*} \circ f)$-invariant with frequency $\omega$, if and only if

$$f \circ K = h_{\lambda_*}^{-1} \circ K \circ \mathrm{R}_\omega\,. \tag{5.15}$$

Hence, $K$ can be viewed as an $h_{\lambda_*}^{-1}$*-transformed torus for $f$*. Moreover, $K$ is $f$-invariant with frequency $\omega$ if and only if $\lambda_* = 0$. This property is achieved by condition (iii) in Definition 5.3.

LEMMA 5.9. *Let $\mathcal{U} \subset \mathbb{R}^k$ and $\Lambda \subset \mathbb{R}^s$ be open, with $0 \in \Lambda$ and $0 \leq s \leq n$. Let $f : \mathcal{U} \times \mathcal{A}_0 \to \mathcal{A}$ be a Hamiltonian deformation such that $f_\mu(\mathcal{A}_0) \subset \mathcal{A}_1$ for all $\mu \in \mathcal{U}$ and let $h : \Lambda \times \mathcal{A}_1 \to \mathcal{A}$ be a modifying family. Then, the moment map of $g_{(\mu,\lambda)} = h_\lambda \circ f_\mu$, satisfies*

$$\mathcal{M}^g(\mu, \lambda, z) = \begin{pmatrix} \mathcal{M}^f(\mu, h_\lambda^{-1}(z)) \\ \mathcal{M}^h(\lambda, z) \end{pmatrix}, \tag{5.16}$$

*where $\mathcal{M}^f$ is the moment map of $f$ and $\mathcal{M}^h$ is the moment map of $h$.*

PROOF. This follows from Definition 2.17 and Lemma 2.21. □

Let $h : \Lambda \times \mathcal{A} \to \mathcal{A}$ and $d : \Sigma \times \mathcal{A} \to \mathcal{A}$ be, respectively, a modifying deformation and dummy deformation. To find solutions of (5.4) and (5.9), we find sufficient conditions guaranteeing the existence of solutions of the following *modified* non-linear equations:

$$d_\sigma \circ h_\lambda \circ f \circ K - K \circ \mathrm{R}_\omega = 0\,, \tag{5.17}$$

$$\langle K^x(\theta) - \theta \rangle = 0\,, \tag{5.18}$$

where the unknown are the parameters $\sigma$ and $\lambda$ and the embedding $K : \mathbb{T}^n \to \mathcal{A}_0$. To simplify notation let

$$\mathbf{f}(\sigma, \lambda, z) = (d_\sigma \circ h_\lambda \circ f)(z),$$

Let $\tilde{\mathcal{M}}^{\mathbf{f}}$, $\mathcal{M}^h$, $\tilde{\mathcal{M}}^d$ be the (local) moment maps of, respectively, $\mathbf{f}$, $h$ and $d$. It is easy to verify that the following equality holds:

$$\tilde{\mathcal{M}}^{\mathbf{f}}(\sigma, \lambda, z) = \begin{pmatrix} \tilde{\mathcal{M}}^d_\sigma \\ \mathcal{M}^h_\lambda \circ d_\sigma^{-1} \end{pmatrix} . \tag{5.19}$$

We now find sufficient conditions that guarantee an iterative procedure to solve (5.17)-(5.18) can be constructed. We study the solvability of the linearized equations around an approximate solution. Let $K : \mathbb{T}^n \to \mathcal{A}_0$ be smooth and let $\sigma \in \Sigma$ and $\lambda \in \Lambda$ be such that $(\sigma, \lambda, K)$ is an approximate solution of (5.17)-(5.18).

Let $T_{(\sigma,\lambda,K)} : \mathbb{T}^n \to \mathbb{R}^{n\times n}$, $\hat{T}_{(\sigma,\lambda,K)} : \mathbb{T}^n \to \mathbb{R}^{2n\times 2n}$, $B_{(\sigma,\lambda,K)} : \mathbb{T}^n \to \mathbb{R}^{2n\times(n+s)}$ be given by, respectively:

$$T_{(\sigma,\lambda,K)}(\theta) = N_K(\theta+\omega)^\top \ \Omega(K(\theta+\omega)) \ \mathrm{D}_z\mathbf{f}(\sigma, \lambda, K(\theta)) \ N_K(\theta),$$

$$\hat{T}_{(\sigma,\lambda,K)}(\theta) = \begin{pmatrix} O_n & T_{(\sigma,\lambda,K)}(\theta) \\ O_n & O_n \end{pmatrix},$$

$$B^{\mathbf{f}}_{(\sigma,\lambda,K)}(\theta) = \left(\mathrm{D}_z\tilde{\mathcal{M}}^{\mathbf{f}}(\sigma, \lambda, K(\theta+\omega)) M_K(\theta+\omega) J_0\right)^\top . \tag{5.20}$$

Under the hypotheses of Lemma 5.2 we have that the change of variables $\Delta_K = M_K\,\xi$ transforms, up to quadratic errors, $\mathrm{D}_z\mathbf{f}(\sigma, \lambda, K(\theta))$ into an block-triangular system. That is the norm of

$$M_K(\theta+\omega)^{-1}\mathrm{D}_z\mathbf{f}(\sigma, \lambda, K(\theta)) M_K(\theta) - (I_n + \hat{T}_{(\sigma,\lambda}(\theta))$$

is quadratic with respect to the norm of the error

$$e_{(\sigma,\lambda,K)}(\theta) = \mathbf{f}(\sigma, \lambda, K(\theta)) - K(\theta+\omega). \tag{5.21}$$

Moreover, using Lemma 5.2 and Lemma 2.19 one obtains that the norm of

$$M_K(\theta+\omega)^{-1}\mathrm{D}_{(\sigma,\lambda)}\mathbf{f}(\sigma, \lambda, K) - B^{\mathbf{f}}_{(\sigma,\lambda,K)}(\theta)$$

is also quadratic with respect to the norm of the error $e_{(\sigma,\lambda,K)}$ in (5.21). Explicit estimates can be easily derived from the estimates given in Lemma 5.2. Hence, an iterative procedure to solve (5.17)-(5.18) depends on the solvability properties of the following linear equations:

$$(\mathcal{L}_\omega + \hat{T}_{(\sigma,\lambda,K)})\,\xi + B^{\mathbf{f}}_{(\sigma,\lambda,K)}\,\delta = \eta, \tag{5.22}$$

$$\pi_x \left\langle M_K\,\xi \right\rangle = 0, \tag{5.23}$$

where $\delta = (\delta^\sigma, \delta^\lambda) \in \mathbb{R}^{n+s}$.

Applying $\mathcal{R}_\omega$ in (5.22) and using equality (2.18) we have:

$$\begin{aligned} \xi &= \xi_0 - \mathcal{R}_\omega(\hat{T}\xi) - \mathcal{R}_\omega B^{\mathbf{f}}\,\delta + \mathcal{R}_\omega\eta \\ &= \xi_0 - \mathcal{R}_\omega(\hat{T}(\xi_0 - \mathcal{R}_\omega(\hat{T}\xi) - \mathcal{R}_\omega B^{\mathbf{f}}\delta + \mathcal{R}_\omega\eta\,)) - \mathcal{R}_\omega B^{\mathbf{f}}\ \delta + \mathcal{R}_\omega\eta \\ &= (I - \mathcal{R}_\omega\hat{T})\xi_0 - \mathcal{R}_\omega(B^{\mathbf{f}} - \hat{T}\mathcal{R}_\omega B^{\mathbf{f}})\,\delta + \mathcal{R}_\omega(\eta - \hat{T}\mathcal{R}_\omega\eta), \end{aligned}$$

where $\xi_0 = \langle\xi\rangle$. For typographic simplicity we do not include in the notation the dependence on $(\sigma, \lambda, K)$. Hence, any solution of (5.22) is of the following form:

$$\xi = (I - \mathcal{R}_\omega\hat{T})\xi_0 - \mathcal{R}_\omega(B^{\mathbf{f}} - \hat{T}\mathcal{R}_\omega B^{\mathbf{f}})\,\delta + \mathcal{R}_\omega(\eta - \hat{T}\mathcal{R}_\omega\eta). \tag{5.24}$$

Moreover, a direct computation shows that $(\xi, \delta)$ in (5.24) is a solution of (5.22)-(5.23) if and only if the following linear system has a solution (compare with (5.11)):

$$\begin{pmatrix} O_n & \bar{\mathrm{T}} & Q_{13} & Q_{14} \\ O_n & O_n & Q_{23} & Q_{24} \\ I_n & Q_{32} & Q_{33} & Q_{34} \end{pmatrix} \begin{pmatrix} \xi_0 \\ \delta \end{pmatrix} = \mathbf{b}, \tag{5.25}$$

where $\mathbf{b}$ is given in (5.12),

$$\bar{\mathrm{T}} = \langle T \rangle , \tag{5.26}$$

$$Q_{32} = \pi_x \langle N - L\mathcal{R}_\omega T \rangle , \tag{5.27}$$

$$\begin{pmatrix} Q_{13} & Q_{14} \end{pmatrix} = \left\langle B^{\mathbf{f},x} - T\,\mathcal{R}_\omega\, B^{\mathbf{f},y} \right\rangle , \tag{5.28}$$

$$\begin{pmatrix} Q_{23} & Q_{24} \end{pmatrix} = \left\langle B^{\mathbf{f},y} \right\rangle , \tag{5.29}$$

$$\begin{pmatrix} Q_{33} & Q_{34} \end{pmatrix} = -\pi_x \left\langle M_{_K} \mathcal{R}_\omega (B^{\mathbf{f}} - \hat{T}\mathcal{R}_\omega B^{\mathbf{f}}) \right\rangle . \tag{5.30}$$

Let us now show that indeed equation (5.25) determines the increment of the parameter, $\delta = (\delta^\sigma, \delta^\lambda) \in \mathbb{R}^{n+s}$, in such a way that $\delta^\sigma$ controls $\langle \eta^y \rangle$ and $\delta^\lambda$ deals with the possible degeneracies of $\bar{\mathrm{T}}(\sigma, \lambda, K)$. Write the local primitive function of $f_{(\sigma,\lambda)}$ as follows (see Remark 2.18):

$$\tilde{\mathcal{M}}^{\mathbf{f}}(\sigma, \lambda, z) = \mathcal{M}^{\mathbf{f}}(\sigma, \lambda, z) \, - \mathrm{D}_{(\sigma,\lambda)} \mathrm{C}^{\mathbf{f}}(\sigma, \lambda)^\top x,$$

with $\mathcal{M}^{\mathbf{f}}(\sigma, \lambda, z)$ 1-periodic in $x$. Then using (5.20) and (5.29) obtains

$$\begin{aligned} \begin{pmatrix} Q_{23}(\sigma, \lambda, K) & Q_{24}(\sigma, \lambda, K) \end{pmatrix} &= -\left\langle \mathrm{D}_z \tilde{\mathcal{M}}^{\mathbf{f}}(\sigma, \lambda, K(\theta))\, L_{_K}(\theta) \right\rangle \\ &= -\left\langle \mathrm{D}_\theta \tilde{\mathcal{M}}^{\mathbf{f}}(\sigma, \lambda, K(\theta)) \right\rangle \\ &= -\left\langle \mathrm{D}_\theta \mathcal{M}^{\mathbf{f}}(\sigma, \lambda, K(\theta)) \right\rangle + \mathrm{D}_{(\sigma,\lambda)} \mathrm{C}^{\mathbf{f}}(\sigma, \lambda) \\ &= \begin{pmatrix} I_n & O_{n\times s} \end{pmatrix} . \end{aligned}$$

Hence, the linear system (5.25) is equivalent to

$$\begin{pmatrix} O_n & \bar{\mathrm{T}} & Q_{13} & Q_{14} \\ O_n & O_n & I_n & O_{n\times s} \\ I_n & Q_{32} & Q_{33} & Q_{34} \end{pmatrix} \begin{pmatrix} \xi_0 \\ \delta \end{pmatrix} = \mathbf{b}\,, \tag{5.31}$$

which determines $\delta^\sigma = \langle \eta^y \rangle \in \mathbb{R}^n$. Moreover, thanks to $\delta^\lambda$, the solvability of (5.31) does not relies only on the invertibility of the torsion $\bar{\mathrm{T}}_{_{(\sigma,\lambda,K)}}$ (twist condition).

To overcome the under-determination of (5.31) we impose to the solutions $(\sigma, \lambda, K)$ of (5.17)-(5.18) an $s$-dimensional constraint. The constraint that is useful for our proposals is the following

$$\langle \mathcal{M}^h(\lambda, K) \rangle = p\,, \tag{5.32}$$

where $p \in \mathbb{R}^s$ is given and $\mathcal{M}^h$ is the moment map of $h$. The constraint (5.32) enables us to obtain parametric families of FLD, parameterized by the momentum (see Section 4.1). Equation (5.32) adds the following $s$-dimensional equation to (5.22)-(5.23)

$$\left\langle \mathrm{D}_z \mathcal{M}^h(\lambda, K) M_{_K}\, \xi \, + \, \mathrm{D}_\lambda \mathcal{M}^h(\lambda, K)\, \delta^\lambda \right\rangle = p - \langle \mathcal{M}^h(\lambda, K) \rangle\, . \tag{5.33}$$

Define

$$U_{_{(\sigma,\lambda,K)}} = \mathrm{D}_z \mathcal{M}^h(\lambda, K) M_{_K} \mathcal{R}_\omega (B^{\mathbf{f}}_{_{(\sigma,\lambda,K)}} - \hat{T}_{(\sigma,\lambda,K)} \mathcal{R}_\omega B^{\mathbf{f}}_{_{(\sigma,\lambda,K)}})\, . \tag{5.34}$$

Using (5.24), (5.31) and (5.34) one verifies that the linear equations (5.22), (5.23) and (5.33) has a solution if and only if the following linear system has a solution

$$\begin{pmatrix} O_n & \bar{\mathrm{T}} & Q_{13} & Q_{14} \\ O_n & O_n & I_n & O_{n\times s} \\ I_n & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{pmatrix} \begin{pmatrix} \xi_0^x \\ \xi_0^y \\ \delta^\sigma \\ \delta^\lambda \end{pmatrix} = \mathbf{c} \tag{5.35}$$

where

$$Q_{41} = \langle \mathrm{D}_z \mathcal{M}^h(\lambda, K(\theta)) L_K(\theta) \rangle , \tag{5.36}$$

$$Q_{42} = \left\langle \mathrm{D}_z \mathcal{M}^h(\lambda, K(\theta))(N_K(\theta) - L_K(\theta)\mathcal{R}_\omega T_{(\sigma,\lambda,K)}(\theta)) \right\rangle , \tag{5.37}$$

$$Q_{43} = -\left\langle U_{(\sigma,\lambda,K)}(\theta) \right\rangle \begin{pmatrix} I_n \\ O_{s\times n} \end{pmatrix} , \tag{5.38}$$

$$Q_{44} = \langle \mathrm{D}_\lambda \mathcal{M}^h(\lambda, K(\theta)) \rangle - \left\langle U_{(\sigma,\lambda,K)}(\theta) \right\rangle \begin{pmatrix} O_{n\times s} \\ I_s \end{pmatrix} , \tag{5.39}$$

$$\mathbf{c} = \begin{pmatrix} \mathbf{b} \\ p - \langle \mathcal{M}^h(\lambda, K) \rangle - \left\langle \mathrm{D}_z \mathcal{M}^h(\lambda, K) M_K(\eta - \hat{T}_{(\sigma,\lambda,K)} \mathcal{R}_\omega \eta) \right\rangle \end{pmatrix} ,$$

with $\mathbf{b}$ given by (5.12).

Notice that, since $\mathcal{M}^h(\lambda, z)$ is 1-periodic in $x$ we have:

$$Q_{41} = \langle \mathrm{D}_\theta (\mathcal{M}_\lambda^h \circ K) \rangle = O_{s\times n} . \tag{5.40}$$

Hence, the existence and uniqueness of solutions of the linear equation (5.35) is guaranteed by the invertibility of the following $(n+s)\times(n+s)$-dimensional matrix.

$$\begin{pmatrix} \bar{\mathrm{T}}_{(\sigma,\lambda,K)} & Q_{14}(\sigma, \lambda, K) \\ Q_{42}(\sigma, \lambda, K) & Q_{44}(\sigma, \lambda, K) \end{pmatrix} , \tag{5.41}$$

with $\bar{\mathrm{T}}$, $Q_{14}$, $Q_{42}$ and $Q_{44}$ given in (5.26), (5.28), (5.37) and (5.39), respectively.

To find a solution of (5.4) we find $p$-parametric family $(\lambda(p), K_p)$ of solutions of (5.17), (5.18) and (5.32). Then, solve the finite dimensional equation $\lambda(p) = 0$. In several cases (e.g. the close-to-integrable case), it is possible to start with a torus $K$ that is approximately invariant for $h_\lambda \circ f$, for some $\lambda \in \Lambda$. In this case, an iterative procedure can be performed if the the matrix in (5.41) is invertible at $(0, \lambda, K)$. We now analyze the form of the matrix in (5.41) in the case that $\sigma = 0$. First, from (5.19) and (5.20) we have:

$$\begin{aligned} B^{\mathbf{f}}_{(\sigma,\lambda,K)}(\theta) \begin{pmatrix} O_{n\times s} \\ I_s \end{pmatrix} &= \\ &(\mathrm{D}_z \mathcal{M}^h(\lambda, d_\sigma^{-1}(K(\theta+\omega)))\, \mathrm{D}_z d_\sigma^{-1}(K(\theta+\omega))\, M_K(\theta+\omega) J_0 )^\top , \end{aligned} \tag{5.42}$$

Define

$$B^h_{(\lambda,K)}(\theta) = (\mathrm{D}_z \mathcal{M}^h(\lambda, K(\theta+\omega)) M_K(\theta+\omega)\, J_0)^\top . \tag{5.43}$$

Then, since $d_0$ is the identity we have:

$$B^{\mathbf{f}}_{(0,\lambda,K)}(\theta) \begin{pmatrix} O_{n\times s} \\ I_s \end{pmatrix} = B^h_{(\lambda,K)}(\theta). \tag{5.44}$$

From (5.34), (5.39), (5.43) and (5.44) we have

$$\begin{aligned} Q_{44}(0,\lambda,K) &= \langle \mathrm{D}_\lambda \mathcal{M}^h(\lambda, K(\theta))\rangle - \left\langle (B^{h,x}_{(\lambda,K)} \circ \mathrm{R}_{-\omega})^\top \mathcal{R}_\omega B^{h,y}_{(\lambda,K)} \right\rangle \\ &+ \left\langle (B^{h,y}_{(\lambda,K)} \circ \mathrm{R}_{-\omega})^\top \mathcal{R}_\omega (B^{h,x}_{(\lambda,K)} - T_{(0,\lambda,K)}\, \mathcal{R}_\omega\, B^{h,y}_{(\lambda,K)}) \right\rangle . \end{aligned}$$

Moreover, using (2.21), (5.28), (5.43) and (5.44) one obtains

$$\begin{aligned} Q_{14}(0,\lambda,K) &= \left\langle B^{h,x}_{(\lambda,K)} - T_{(0,\lambda,K)}\, \mathcal{R}_\omega B^{h,y}_{(\lambda,K)} \right\rangle \\ &= \left\langle B^{h,x}_{(\lambda,K)} \circ \mathrm{R}_{-\omega} + (\mathcal{R}_\omega T_{(0,\lambda,K)})\, B^{h,y}_{(\lambda,K)} \circ \mathrm{R}_{-\omega} \right\rangle \\ &= \left\langle \mathrm{D}_z \mathcal{M}^h(\lambda, K(\theta))( N_K(\theta) - L_K(\theta) \mathcal{R}_\omega T_{(0,\lambda,K)}(\theta)^\top ) \right\rangle^\top . \end{aligned} \tag{5.45}$$

Remark 5.10. In particular, if $(0,\lambda,K)$ is a solution of (5.17), (5.18) and (5.32), then $T_{(0,\lambda,K)}$ is symmetric and hence

$$Q_{14}(0,\lambda,K) = Q_{42}(0,\lambda,K)^\top .$$

From the above discussion, we have that the invertibility of the matrix in (5.41) at $(0,\lambda,K)$ depends on the torsion of $K$ and on the choice of the modifying family $h$. Concretely, it depends on the properties of $\bar{\mathrm{T}}_{(0,\lambda,K)}$, $\mathrm{D}_\lambda \mathcal{M}^h(\lambda, K(\theta))$, $\mathrm{D}_z \mathcal{M}^h(\lambda, K(\theta)) N_K(\theta)$ and $\mathrm{D}_z \mathcal{M}^h(\lambda, K(\theta)) L_K(\theta)$.

Remark 5.11. If the modifying deformation $h$ is such that its moment map $\mathcal{M}^h$ is autonomous (i.e. independent of $\lambda$), and

$$B^{h,y}_{(\lambda,K)}(\theta) = \mathrm{D}_z \mathcal{M}^h\,(K(\theta))\, L_K(\theta) = O_{s\times n} .$$

Then the matrix in (5.41) takes the following form at $(0,\lambda,K)$:

$$\begin{pmatrix} \bar{\mathrm{T}}_{(0,\lambda,K)}(\theta) & \langle\, \mathrm{D}_z \mathcal{M}^h(K(\theta))\, N_K(\theta)\rangle^\top \\ \langle\, \mathrm{D}_z \mathcal{M}^h(K(\theta))\, N_K(\theta)\rangle & O_{s\times s} \end{pmatrix} .$$

Hence, if $s = n$ and $\langle\, \mathrm{D}_z \mathcal{M}^h(K(\theta))\, N_K(\theta)\rangle$ is invertible, then matrix in (5.41) is also invertible.

Given an approximate solution of (5.4), it is natural to ask whether it is possible to find a modifying deformation $h$ in such a way that the matrix in (5.41) is invertible at $(0,0,K)$. Another question is about the minimum dimension of the modifying parameter $\lambda$. In the following result we show that given a solution of (5.4), $K$, it is possible to find a modifying deformation $h$ in such a way that the matrix in (5.41) is invertible at $(0,0,K)$. Moreover, minimum dimension of $\lambda$ is the dimension of the kernel of the torsion of $K$, with respect to $f$ and $\omega$.

Proposition 5.12. *Let $f \in \mathrm{Symp}(\mathcal{A}_0, \mathcal{A})$, $\omega \in \mathcal{D}_n(\gamma,\tau)$ and $K \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A}_0)$. Let $\bar{\mathrm{T}}_{f,K}$ be torsion of $K$ with respect to $f$ and $\omega$. Fix $s \in \mathbb{N}$ with $\dim\ker \bar{\mathrm{T}}_{(f,K)} \le s \le n$. Assume that $K$ is $f$-invariant with frequency $\omega$. Then, there is an open set $\Lambda \subset \mathbb{R}^s$ with $0 \in \Lambda$ and a modifying deformation $h$ with base $\Lambda \subset \mathbb{R}^s$, such that the matrix in (5.41) is invertible at $(0,0,K)$.*

Proof. Since $K$ is $f$-invariant with frequency $\omega$, from Lemma 3.1 we have that $K(\mathbb{T}^n)$ is Lagrangian and $\bar{\mathrm{T}}_{(0,0,K)}$ is symmetric . Then, there is an invertible matrix $A \in \mathbb{R}^{n\times n}$, such that

$$\bar{\mathrm{T}}_{(0,0,K)} = A \,\mathrm{diag}\begin{pmatrix} t_1 & \dots & t_n \end{pmatrix} A^\top$$

where $t_{s+1}, \dots, t_n$ are different from zero.

Since the column vectors of $N_K(\theta)$ generates a Lagrangian bundle, complementary to the tangent bundle $\mathrm{T}K(\mathbb{T}^n)$, Weinstein's Theorem [**41, 69**] guarantees the existence of a tubular neighborhood $\mathcal{A}_K$ of $K(\mathbb{T}^n)$ and a symplectomorphism $\varphi : \mathcal{A}_K \to \mathbb{A}^n$ satisfying the following properties:

i) $\varphi^* \boldsymbol{\omega}_0 = \boldsymbol{\omega}$;

ii) $\varphi(K(\theta)) = K^0(\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$;

iii) $\mathrm{D}_z\varphi(K(\theta))\, L_K(\theta) = L^0(\theta) = \begin{pmatrix} I_n \\ O_n \end{pmatrix}$;

iv) $\mathrm{D}_z\varphi(K(\theta))\, N_K(\theta) = N^0(\theta) = \begin{pmatrix} O_n \\ I_n \end{pmatrix}$.

Define the modifying deformation $h^0 : \mathbb{R}^s \times \mathbb{A}^n \to \mathbb{A}^n$ as follows:

$$h^0(\lambda, z)^\top = \begin{pmatrix} x + A\, I_{n\times s}\, \lambda \\ y \end{pmatrix}, \tag{5.46}$$

where $I_{n\times s}$ is the $(n \times s)$ matrix such that $(I_{n\times s})_{ij} = 0$ if $i \neq j$ and $(I_{n\times s})_{ii} = 1$. It is easy to check that the moment map of $h^0$ is $\mathcal{M}^{h^0}(z) = I_{s\times n}\, A^\top y$ and that the following equalities hold:

$$\mathrm{D}_z\mathcal{M}^{h^0}(K_0(\theta))\, N^0(\theta) \;=\; I_{s\times n}\, A^\top \tag{5.47}$$

$$\mathrm{D}_z\mathcal{M}^{h^0}(K_0(\theta))\, L^0(\theta) \;=\; O_{s\times n}\,. \tag{5.48}$$

Define a modifying deformation on the (possibly shrunken) tubular neighborhood $\mathcal{A}_K \subset \mathcal{A}_0$ with base a (small) neighborhood of $0 \in \mathbb{R}^s$, $\Lambda$, $h : \Lambda \times \mathcal{A}_K \to \mathcal{A}$, as follows:

$$h_\lambda = \varphi^{-1} \circ h^0_\lambda \circ \varphi\,. \tag{5.49}$$

Then from Part c) of Lemma 2.21, the moment map of $h$ is $\mathcal{M}^h = \mathcal{M}^{h^0} \circ \varphi$. In particular, $\mathcal{M}^h$ is independent of $\lambda$ and moreover from (5.47), (5.48) and the properties of $\varphi$ one obtains:

$$\begin{aligned} \mathrm{D}_z\mathcal{M}^h(K(\theta))\, N_K(\theta) &= \mathrm{D}_z\mathcal{M}^{h^0}(\varphi(K(\theta)))\, \mathrm{D}_z\varphi(K(\theta)) N_K(\theta) &&= I_{s\times n}\, A^\top, \\ \mathrm{D}_z\mathcal{M}^h(K(\theta))\, L_K(\theta) &= \mathrm{D}_z\mathcal{M}^{h^0}(\varphi(K(\theta)))\, \mathrm{D}_z\varphi(K(\theta)) L_K(\theta) &&= O_{s\times n}. \end{aligned}$$

Then, it is easy to verify that the matrices in (5.45), (5.36) and (5.39) take the following equalities hold:

$$\begin{aligned} Q_{14}(0,0,K) &= A\, I_{n\times s}\,, \\ Q_{42}(0,0,K) &= I_{s\times n}\, A^\top \\ Q_{44}(0,0,K) &= O_{s\times s}\,. \end{aligned}$$

Hence, the matrix in (5.41) at $(0,0,K)$ takes the following form:

$$\begin{pmatrix} A & O_{n\times s} \\ O_{n\times s} & I_s \end{pmatrix} \begin{pmatrix} \mathrm{diag}(t_1, \dots t_n) & I_{n\times s} \\ I_{s\times n} & O_{s\times s} \end{pmatrix} \begin{pmatrix} A^\top & O_{n\times s} \\ O_{n\times s} & I_s \end{pmatrix}.$$

This finishes the proof of Proposition 5.12. □

CHAPTER 6

# A parametric KAM theorem

The main contribution of this chapter is Theorem 6.2. This is a KAM result stated and proved in a general setting. It will be used in Chapter 7 to prove existence of FLD, where the parameters $\Lambda$ and $\zeta$ used here will have different components playing different roles.

Throughout this chapter, we assume that $\mathcal{A} \subset \mathbb{A}$ is an annulus endowed with the compatible triple $(\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}, \boldsymbol{J}, \boldsymbol{g})$, which is assumed to be real-analytic, i.e. holomorphic in a complex strip $\mathcal{B}$ of $\mathcal{A}$. More concretely, we assume that the $C^1$-norms on $\mathcal{B}$ of the components of $a$, $\Omega$, $\Omega^{-1}$, $J$, $G$, $G^{-1}$ are bounded by a positive constant denoted by $c_{\mathrm{symp}}$. $\mathcal{A}_0 \subset \mathcal{A}$ is an annulus and $\mathcal{B}_0 \subset \mathcal{B}$ is a complex strip of $\mathcal{A}_0$.

Let $\Xi \subset \mathbb{R}^m$ be open, with $m \geq n$. Let $C^r(\Xi, \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^2))$ denote the set of symplectic deformations with base $\Xi$, $f : \Xi \times \mathcal{A}_0 \to \mathcal{A}$, such that, $f$ is $C^r$ with respect to $\mathbf{t} \in \Xi$ and, $f_{\mathbf{t}} \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^2)$.

Given $f \in C^r\left(\Xi, \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^2)\right)$ and $\omega \in \mathcal{D}_n(\gamma, \tau)$, we aim to find for $\zeta \in D \subset \mathbb{R}^m$ a parameter $\mathbf{t}(\zeta) \in \mathbb{R}^m$ and an embedding $K_\zeta \in \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^1)$ such that $K_\zeta$ is $f_{\mathbf{t}(\zeta)}$-invariant with dynamics the rigid rotation $\mathrm{R}_\omega$. Since we are interested in obtaining solutions $(\mathbf{t}(\zeta), K_\zeta)$ that are smooth in $\zeta$, we impose extra-conditions (as many as the dimension of $\mathbf{t}$) to obtain local uniqueness.

## 6.1. Functional equations and nondegeneracy condition

Let $Z \in C^r(\Xi \times \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^0), \mathbb{R}^m)$ be a $C^r$ functional. Define the *error function* operator

$$\mathcal{F} : \mathbb{R}^m \times \Xi \times \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^0) \longrightarrow (\mathrm{A}(\mathbb{T}^n_\rho, C^0))^{2n} \times \mathbb{R}^m$$

as follows

$$\mathcal{F}(\zeta; \mathbf{t}, K) = \begin{pmatrix} e(\mathbf{t}; K) \\ \nu(\zeta; \mathbf{t}, K) \end{pmatrix} = \begin{pmatrix} f_{\mathbf{t}} \circ K - K \circ \mathrm{R}_\omega \\ Z(\mathbf{t}, K) - \zeta \end{pmatrix} . \tag{6.1}$$

Theorem 6.2 can be viewed as *Generalized Implicit Function Theorem* for the following non-linear functional equation:

$$\mathcal{F}(\zeta; \mathbf{t}, K) = 0 , \tag{6.2}$$

where $\zeta \in D$ is given and the unknown is $(\mathbf{t}, K)$. The main assumption is the existence of an approximate solution $(\zeta_0; \mathbf{t}_0, K_0)$, with sufficiently small error, which satisfies a nondegeneracy condition. To rigorously state our parametric KAM theorem, we first make explicit such a nondegeneracy condition.

Let $(\mathbf{t}, K) \in \Xi \times \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^1)$ be given and let $L_K$ and $N_K$ be defined by (3.2) and (3.3), respectively. Let

$$T_{(\mathbf{t},K)}(\theta) = N_K(\theta + \omega)^\top \; \Omega(K(\theta + \omega)) \; \mathrm{D}_z f(\mathbf{t}, K(\theta)) \; N_K(\theta) , \tag{6.3}$$

$$\hat{T}_{(\mathbf{t},K)}(\theta) = \begin{pmatrix} O_n & T_{(\mathbf{t},K)}(\theta) \\ O_n & O_n \end{pmatrix}, \tag{6.4}$$

and

$$B_{(\mathbf{t},K)}(\theta) = \left(\mathrm{D}_z \mathscr{M}^f(\mathbf{t}, K(\theta+\omega)) M_K(\theta+\omega) J_0\right)^\top, \tag{6.5}$$

where $\mathscr{M}^f$ is the local moment map of $f$. Motivated by Section 5.2 we introduce the following general nondegeneracy condition.

Definition 6.1. Let $\mathcal{R}_\omega$ be the one-bite solver operator (see Section 2.5). A pair $(\mathbf{t}, K) \in \Xi \times \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^1)$ is *nondegenerate* with respect to $\mathcal{F}$, given in (6.1), if the following $(2n+m)\times(2n+m)$ matrix is invertible:

$$Q_{(\mathbf{t},K)} = \begin{pmatrix} \left\langle \hat{T} \right\rangle & \left\langle B - \hat{T}\mathcal{R}_\omega B \right\rangle \\ \mathrm{D}_K Z\left[M_K(I - \mathcal{R}_\omega \hat{T})\right] & \mathrm{D}_{\mathbf{t}} Z - \mathrm{D}_K Z\left[M_K \mathcal{R}_\omega (B - \hat{T}\mathcal{R}_\omega B)\right] \end{pmatrix}, \tag{6.6}$$

where for typographical simplicity we have not written the dependence on $(\mathbf{t}, K)$ of and $\mathrm{D}_K Z$ and $\mathrm{D}_{\mathbf{t}} Z$ are evaluated at $(\mathbf{t}, K)$.

## 6.2. Statement of the parametric KAM theorem

Many of our estimates will involve some quantities that depend in a polynomial way on specific positive numbers, associated to a given $(\mathbf{t}, K) \in \Xi\times\mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^1)$. To reduce the length our statements we will collect them in a vector as follows:

$$\mathbf{n}_\rho(\mathbf{t}, K) = \left( \|\mathrm{D}_\theta K\|_\rho, \left\|G_K^{-1}\right\|_\rho, \left|Q^{-1}_{(\mathbf{t},K)}\right|, \left\|\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\mathbf{t}, K)\right\|_{\mathbb{R}^m\times(\mathrm{A}(\mathbb{T}^n_\rho, C^0))^{2n}} \right), \tag{6.7}$$

where $G_K$, $\mathcal{F}$ and $Q_{(\mathbf{t},K)}$ are defined in (2.11), (6.1) and (6.6), respectively.

Theorem 6.2. *Let $\omega \in \mathcal{D}_n(\gamma, \tau)$, for some $\gamma > 0$ and $\tau \geq n$. Let $m \geq n > 0$, $r \geq 2$ be fixed integer numbers. Let $\Xi \subset \mathbb{R}^m$ be open and let $f \in C^r(\Xi, \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^2))$. Let $\rho_0 > 0$ be fixed and such that $\gamma\,\rho_0^\tau < 1$. Let $\mathcal{F}$ be defined by (6.1), with $Z \in C^r(\Xi\times\mathrm{Emb}(\mathbb{T}^n_{\rho_0/2}, \mathcal{B}_0, C^0), \mathbb{R}^m)$. Assume that there exists positive constants $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_r$ such that for $\zeta \in \Xi$ fixed and any $\rho \in [\rho_0/2, \rho_0]$ and for $k = 0, \ldots, r$*

$$\sup_{(\mathbf{t},K)\in\Xi\times\mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^0)} \left| \mathrm{D}^k_{(\mathbf{t},K)}\mathcal{F}(\zeta; \mathbf{t}, K)\right|_{\mathbb{R}^m\times(\mathrm{A}(\mathbb{T}^n_\rho, C^0))^{2n}} \leq \beta_k\,. \tag{6.8}$$

*Let $(\zeta_0; \mathbf{t}_0, K_0) \in \mathbb{R}^m\times\Xi\times\mathrm{Emb}(\mathbb{T}^n_{\rho_0}, \mathcal{B}_0, C^1)$ be given and such that the matrix $Q_{(\mathbf{t}_0,K_0)}$, defined in (6.6), is invertible. Then, the following statements hold.*

**a) Existence of a solution:** *There exists a positive constant $\kappa$, depending on $n$, $\tau$, $m$, $c_{\mathrm{symp}}$, $\beta_2$, and polynomially on the components of $\mathbf{n}_{\rho_0}(\mathbf{t}_0, K_0)$ (see (6.7)), such that if*

$$\kappa\,\gamma^{-4}\,\delta_0^{-4\tau}\left(1 + \gamma^{-2}\rho_0^{-2\tau}\right)^2 \|\mathcal{F}(\zeta_0; \mathbf{t}_0, K_0)\|_{\rho_0} < c_0,$$

*where*

$$c_0 = \min\left[\mathrm{dist}\,(\mathbf{t}_0, \partial\,\Xi)\,,\ \mathrm{dist}\left(K_0\left(\mathbb{T}^n_{\rho_0}\right), \partial\mathcal{B}_0\right),\ 1\right],$$

*and $0 < \delta_0 < \rho_0/12$ is fixed, then there exists $(\mathbf{t}_{\zeta_0}, K_{\zeta_0}) \in \Xi\times\mathrm{Emb}(\mathbb{T}^n_{\rho_0 - 6\,\delta_0}, \mathcal{B}_0, C^1)$ such that $Q_{(\mathbf{t}_{\zeta_0},K_{\zeta_0})}$ is invertible,*

$$|\mathbf{t}_{\zeta_0} - \mathbf{t}_0| \leq \kappa\gamma^{-2}\,\rho_0^{-2\tau}\,\|\mathcal{F}(\zeta_0; \mathbf{t}_0, K_0)\|_{\rho_0}\,,$$

$$\|K_{\zeta_0} - K_0\|_{\rho_0 - 6\delta_0} \leq \kappa\gamma^{-2}\,\delta_0^{-2\tau}\left(1 + \gamma^{-2}\rho_0^{-2\tau}\right)\|\mathcal{F}(\zeta_0; \mathbf{t}_0, K_0)\|_{\rho_0}\,,$$

*and*

$$\mathcal{F}(\zeta_0; \mathbf{t}_{\zeta_0}, K_{\zeta_0}) = 0\,,$$

*that is* $K_{\zeta_0}$ *is an* $f_{\zeta_0}$*-invariant torus with frequency* $\omega$*.*

**b) Local uniqueness:** *Let* $(\mathbf{t}_1, K_1) \in \Xi \times \mathrm{Emb}(\mathbb{T}^n_{\rho_0}, \mathcal{B}_0, C^1)$*. Assume that*

$$\mathcal{F}(\zeta_0; \mathbf{t}_i, K_i) = 0, \quad \textit{for} \quad i = 0, 1.$$

*Then there exists a constant* $\hat{\kappa}$*, depending on* $n$*,* $\tau$*,* $m$*,* $c_{\mathrm{symp}}$*,* $\beta_2$ *and polynomially on the components of* $\mathbf{n}_{\rho_0}(\mathbf{t}_0, K_0)$ *(see (6.7)), such that if*

$$\hat{\kappa}\, \gamma^{-2} \delta_0^{-2\tau} \left(1 + \gamma^{-2} \rho_0{}^{-2\tau}\right) \left\| (\mathbf{t}_0, K_0) - (\mathbf{t}_1, K_1) \right\|_{\rho_0} < 1,$$

*with* $0 < \delta_0 < \rho_0/8$ *fixed, then* $(\mathbf{t}_1, K_1) = (\mathbf{t}_0, K_0)$*.*

**c) Smooth dependency on $\zeta$ of the solutions:** *Assume that*

$$\mathcal{F}(\zeta_0; \mathbf{t}_0, K_0) = 0\,.$$

*Then, there exist* $\mathrm{D} \subset \mathbb{R}^m$ *a small open neighborhood of* $\zeta_0$*, and a locally unique* $C^r$*-function*

$$\begin{array}{rccl} \Phi: & \mathrm{D} & \longrightarrow & \Xi \times \mathrm{Emb}(\mathbb{T}^n_{\rho_0 - 22\delta_0}, \mathcal{B}_0, C^1) \\ & \zeta & \longrightarrow & \Phi(\zeta) = (\mathbf{t}(\zeta), K_\zeta)\,, \end{array}$$

*with* $0 < \delta_0 < \rho_0/44$*, such that* $\Phi(\zeta_0) = (\mathbf{t}_0, K_0)$ *and, for all* $\zeta \in D$*,* $Q(\zeta) = Q_{\left(\mathbf{t}(\zeta), K_\zeta\right)}$ *is invertible and*

$$\mathcal{F}(\zeta; \mathbf{t}(\zeta), K_\zeta) = 0\,.$$

## 6.3. Proof of the parametric KAM Theorem

Theorem 6.2 is proved following a standard KAM scheme. The main ingredient is the fact that real-analytic tori, that are approximately invariant for a real-analytic symplectomorphism and Diophantine frequency, are also approximately Lagrangian. Therefore, the linear dynamics around them is approximately reducible. A proof of this property and explicit estimates are given in Lemma 5.2.

It is known [**50, 57, 74**] that to construct a *rapidly convergent method* to find solutions of a non-linear problem, it is sufficient to guarantee the approximate solvability of the corresponding linearized equations, with tame bounds. In Section 6.3.1 we show that approximate solvability of the linearized equations, corresponding to (6.2), is guaranteed by the approximate reducibility and the change of the parameters. In Section 6.3.2 we show that the sufficient conditions needed to find an approximate solution of the linearized equations are open on $\Xi \times \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^1)$, for all $\rho_0/2 < \rho \leq \rho_0$. Parts a), b) and c) of Theorem 6.2 are proved in sections 6.3.3, 6.3.4, and 6.3.5, respectively.

Our estimates depend on certain quantities that will be specified in the statement of the results. To reduce the amount of notation we will use the same letter $\kappa$ to denote a generic constant, whose definition will vary from statement to statement.

**6.3.1. Linearized equations.** Throughout this section $\zeta \in \mathbb{R}^m$ and $\rho \in (\rho_0/2, \rho_0)$ are assumed to be fixed. The pair $(\mathbf{t}, K) \in \Xi \times \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^1)$ is assumed to be a fixed approximate solution of (6.2). The linear operator corresponding to (6.1), at $(\mathbf{t}, K)$, is

$$\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta;\mathbf{t},K)\begin{pmatrix}\Delta_t\\ \Delta_K\end{pmatrix} = \begin{pmatrix}(\mathrm{D}_z f_{\mathbf{t}} \circ K)\Delta_K - \Delta_K \circ \mathrm{R}_\omega + (\mathrm{D}_{\mathbf{t}} f_{\mathbf{t}} \circ K)\Delta_t \\ \mathrm{D}_K Z(\mathbf{t},K)\Delta_K + \mathrm{D}_{\mathbf{t}} Z(\mathbf{t},K)\Delta_t\end{pmatrix}. \tag{6.9}$$

Lemma 6.3 (Approximate solution). *Let $\omega \in \mathcal{D}_n(\gamma, \tau)$, for some $\gamma > 0$ and $\tau \geq n$. Let $G_K$, $M_K$, $T_{(\mathbf{t},K)}$, $\hat{T}_{(\mathbf{t},K)}$ and $B_{(\mathbf{t},K)}$ be defined in (2.11), (3.4), (6.3), (6.4), (6.5), respectively. Let $e_{(\mathbf{t},K)}$ be given by*

$$e_{(\mathbf{t},K)} = f_{\mathbf{t}} \circ K - K \circ \mathrm{R}_\omega \,.$$

*Let $\nu \in \mathbb{R}^m$ and $\varphi \in (\mathrm{A}(\mathbb{T}^n_\rho, C^0))^{2n}$ be given, define*

$$\hat{\varphi}(\theta) = -J_0 M_K(\theta+\omega)^\top \Omega(K(\theta+\omega))\,\varphi(\theta)\,. \tag{6.10}$$

*Assume that $Q_{(\mathbf{t},K)}$, defined in (6.6), is invertible, and let $(\xi_0, \Delta_t) \in \mathbb{R}^{2n} \times \mathbb{R}^m$ be the unique solution of the following linear equation:*

$$Q_{(\mathbf{t},K)}\begin{pmatrix}\xi_0\\ \Delta_t\end{pmatrix} = -\begin{pmatrix}\left\langle \hat{\varphi} - \hat{T}_{(\mathbf{t},K)}\mathcal{R}_\omega\,\hat{\varphi}\right\rangle \\ \nu - \mathrm{D}_K Z(\mathbf{t},K)\left[M_K \mathcal{R}_\omega(\,\hat{\varphi} - \hat{T}_{(\mathbf{t},K)}\mathcal{R}_\omega\,\hat{\varphi})\right]\end{pmatrix}. \tag{6.11}$$

*Define $\Delta_K = M_K\,\xi$, where*

$$\xi = (I_{2n} - \mathcal{R}_\omega \hat{T}_{(\mathbf{t},K)})\xi_0 - \mathcal{R}_\omega(B_{(\mathbf{t},K)} - \hat{T}_{(\mathbf{t},K)}\mathcal{R}_\omega B_{(\mathbf{t},K)})\Delta_t - \mathcal{R}_\omega(\hat{\varphi} - \hat{T}_{(\mathbf{t},K)}\mathcal{R}_\omega\,\hat{\varphi})\,. \tag{6.12}$$

*Then, there exists a constant, depending on $n$, $\tau$, $m$, $c_{\mathrm{symp}}$, $\beta_2$ and polynomially on the components of $\mathbf{n}_\rho(\mathbf{t},K)$ (see (6.7)) such that the following hold.*

**a)** *$\Delta_K \in (\mathrm{A}(\mathbb{T}^n_{\rho-2\delta}, C^0))^{2n}$, for any $0 < \delta < \rho/2$. Moreover,*

$$|\Delta_t| \;\leq\; \kappa\,\gamma^{-2}\rho^{-2\tau}\,\|(\varphi,\nu)\|_\rho\,, \tag{6.13}$$

$$\|\Delta_K\|_{\rho-2\delta} \;\leq\; \kappa\,\gamma^{-2}\delta^{-2\tau}\left(1+\gamma^{-2}\rho^{-2\tau}\right)\|(\varphi,\nu)\|_\rho\,. \tag{6.14}$$

**b)** *If $\|e_{(\mathbf{t},K)}\|_\rho$ is sufficiently small, such that matrix $M_K$ is invertible on $\mathbb{T}^n_{\rho-2\delta}$, then the following estimate holds:*

$$\left\|\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta;\mathbf{t},K)\begin{pmatrix}\Delta_t\\ \Delta_K\end{pmatrix} + \begin{pmatrix}\varphi\\ \nu\end{pmatrix}\right\|_{\rho-2\delta} \leq$$
$$\leq \kappa\,\gamma^{-3}\delta^{-(3\tau+1)}\left(1+\gamma^{-2}\rho^{-2\tau}\right)\|e_{(\mathbf{t},K)}\|_\rho\,\|(\varphi,\nu)\|_\rho\,.$$

Proof. First notice that if $\varphi \in (\mathrm{A}(\mathbb{T}^n_\rho, C^0))^{2n}$, then from (6.10), we have that $\hat{\varphi} \in (\mathrm{A}(\mathbb{T}^n_\rho, C^0))^{2n}$. Then Lemma 2.29 and equality (6.12) imply $\xi \in (\mathrm{A}(\mathbb{T}^n_{\rho-2\delta}, C^0))^{2n}$. From Lemma 2.29 and equality (6.10) we have that there is a constant $\kappa$, depending on $n$, $\tau$, $c_{\mathrm{symp}}$ and polynomially on $\|\mathrm{D}_\theta K\|_\rho$, $\|\mathrm{D}_z f_{\mathbf{t}}(K(\theta))\|_\rho$ and $\left\|(G_K)^{-1}\right\|_\rho$, such that for any $0 < \delta < \rho$ the following estimates hold:

$$\begin{aligned}\|\mathcal{R}_\omega\,\hat{\varphi}\|_{\rho-\delta} &\leq \kappa\,\gamma^{-1}\,\delta^{-\tau}\,\|\varphi\|_\rho\,,\\ \left\|\mathcal{R}_\omega\left(\hat{\varphi} - \hat{T}_{(\mathbf{t},K)}\mathcal{R}_\omega\hat{\varphi}\right)\right\|_{\rho-2\delta} &\leq \kappa\,\gamma^{-2}\,\delta^{-2\tau}\,\|\varphi\|_\rho\,.\end{aligned} \tag{6.15}$$

Equality (6.11) and estimates in (6.15) imply the existence of a constant $\kappa$, depending on $n$, $m$, $\tau$, $c_{\rm symp}$ and polynomially on the components of $\mathbf{n}_\rho(\mathbf{t},K)$ (see (6.7)) such that the following estimate holds:

$$|(\xi_0,\Delta_t)| \leq \kappa\,(1+\gamma^{-2}\,\rho^{-2\tau})\ \|(\varphi,\nu)\|_\rho\,. \tag{6.16}$$

From which (6.13) follows. Moreover, using again Lemma 2.29, we have the following estimates

$$\left\| I_{2n} - \mathcal{R}_\omega \hat{T}_{(\mathbf{t},K)} \right\|_{\rho-\delta} \leq \kappa\,\gamma^{-1}\delta^{-\tau}, \tag{6.17}$$

and

$$\left\| \mathcal{R}_\omega ( B_{(\mathbf{t},K)} - \hat{T}_{(\mathbf{t},K)} \mathcal{R}_\omega B_{(\mathbf{t},K)}) \right\|_{\rho-2\delta} \leq \kappa\,\gamma^{-2}\delta^{-2\tau}\,. \tag{6.18}$$

Equality (6.12) and estimates (6.16), (6.17) and (6.18) yield:

$$\|\xi\|_{\rho-2\delta} \leq \kappa\,\gamma^{-2}\delta^{-2\tau}\left(1+\gamma^{-2}\,\rho^{-2\tau}\right)\ \|(\varphi,\nu)\|_\rho\ . \tag{6.19}$$

This proves estimate (6.14). This finishes the proof of Part a). We now prove Part b). Performing some computations, applying $\mathcal{L}_\omega$ to (6.11) and using (2.19) and (6.12) one verifies that the following equalities hold:

$$\begin{aligned} (I_{2n} + \hat{T}_{(\mathbf{t},K)})\,\xi - \xi\circ \mathrm{R}_\omega + B_{(\mathbf{t},K)}\ \Delta_t + \hat{\varphi} &= 0, \\ \mathrm{D}_K Z(\mathbf{t},K)\ [M_K \xi] + \mathrm{D}_{\mathbf{t}} Z(\mathbf{t},K)\ \Delta_t + \nu &= 0\,, \end{aligned}$$

where $\hat{T}_{(\mathbf{t},K)}$ and $B_{(\mathbf{t},K)}$ are given in (6.3) and (6.5), respectively. Next, from parts d) and e) of Lemma 5.2 we have

$$\begin{aligned} M_K(\theta+\omega)^{-1}\mathrm{D}_z f(\mathbf{t},K(\theta)) M_K(\theta) = {}& (I_{2n} + \hat{T}_{(\mathbf{t},K)}(\theta)) \\ &+ R^r_{(f_{\mathbf{t}},K)}(\theta) \\ &+ R^i_K(\theta+\omega)\mathrm{D}_z f(\mathbf{t},K(\theta)) M_K(\theta). \end{aligned}$$

From Lemma 2.19 and Part e) of Lemma 5.2 we have

$$\begin{aligned} M_K(\theta+\omega)^{-1}\mathrm{D}_{\mathbf{t}} f(\mathbf{t},K(\theta)) = {}& B_{(\mathbf{t},K)}(\theta) + R^i_K(\theta+\omega)\,\mathrm{D}_{\mathbf{t}} f(\mathbf{t},K(\theta)) \\ &+ J_0 M_K(\theta+\omega)^\top \left(\mathrm{D}_z \tilde{\mathscr{M}}^f(\mathbf{t},K(\theta+\omega)) - \mathrm{D}_z \tilde{\mathscr{M}}^f(\mathbf{t},f_{\mathbf{t}}(K(\theta))\right)^\top \\ &- J_0 M_K(\theta+\omega)^\top \left(\Omega(K(\theta+\omega)) - \Omega(f_{\mathbf{t}}(K(\theta))\right) \mathrm{D}_{\mathbf{t}} f(\mathbf{t},K(\theta)). \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{D}_{(\mathbf{t},K)} e(\mathbf{t},K) \begin{pmatrix} \Delta_t \\ \Delta_K \end{pmatrix} + \varphi = {}& M_K(\theta+\omega) R^i_K(\theta+\omega)\,\varphi \\ &+ M_K(\theta+\omega)\left[ M_K(\theta+\omega)^{-1}\mathrm{D}_z f(\mathbf{t},K(\theta)) M_K(\theta) - (I_{2n} + \hat{T}_{(\mathbf{t},K)})\right]\xi \\ &+ M_K(\theta+\omega)\left[ M_K(\theta+\omega)^{-1}\mathrm{D}_{\mathbf{t}} f(\mathbf{t},K(\theta)) - B_{(\mathbf{t},K)}(\theta)\right]\Delta_t. \end{aligned}$$

The proof is finished using the estimates in Lemma 5.2. □

REMARK 6.4. If, in Lemma 6.3, $e_{(\mathbf{t},K)} \equiv 0$, then, from the uniqueness of the solutions of (6.11), we have that the linear equation

$$\mathrm{D}_{(\mathbf{t},K)} \mathcal{F}(\zeta;\mathbf{t},K) \begin{pmatrix} \Delta_t \\ \Delta_K \end{pmatrix} + \begin{pmatrix} \varphi \\ \nu \end{pmatrix} = 0\,, \tag{6.20}$$

has a unique solution $(\Delta_K = M_K \xi, \Delta_t)$ where $\xi$ is given in (6.12).

REMARK 6.5. In the terminology of [**68, 74**], Part b) of Lemma 6.3 means that $\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta;\mathbf{t},K)$ has an *approximate right inverse* and Remark 6.4 says that $\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta;\mathbf{t},K)$ has an *approximate left inverse*.

**6.3.2. Sufficient conditions for iteration of the iterative step.** Given $(\mathbf{t},K)\in\Xi\times\mathrm{Emb}(\mathbb{T}^n_\rho,\mathcal{B}_0,C^1)$, an approximate solution of (6.2), with $\rho_0/2<\rho\le\rho_0$, Lemma 6.3 provides a method to find a new one $(\mathbf{t}+\Delta_t, K+\Delta_K)$. To to iterate this procedure, we use the following result.

LEMMA 6.6. *Assume that $(\mathbf{t},K)\in\Xi\times\mathrm{Emb}(\mathbb{T}^n_\rho,\mathcal{B}_0,C^1)$, for some $\rho_0/2<\rho\le\rho_0$. Let $(\Delta_t,\Delta_K)\in\mathbb{R}^m\times(\mathrm{A}(\mathbb{T}^n_{\rho_1},C^1))^{2n}$, for some $\rho_0/2<\rho_1<\rho$, with $\|\Delta_K\|_{\rho_1,C^1}\le 1$. Let $Q_{(\mathbf{t},K)}$ be defined in* (6.6) *and assume that it is invertible. Then, there exists a constant $\kappa$, depending on $n$, $m$, $\tau$, $c_{\mathrm{symp}}$, $\beta_2$ and polynomially on the components of $\mathbf{n}_\rho(\mathbf{t},K)$ (see* (6.7)*), such that if*

$$\kappa\,\gamma^{-2}\rho_1^{-2\tau}\,\|(\Delta_t,\Delta_K)\|_{\rho_1,C^1}<\min(\mathrm{dist}(\mathbf{t},\partial\Xi),\mathrm{dist}(K(\mathbb{T}^n_\rho),\partial\mathcal{B}_0),\,1)\,. \tag{6.21}$$

*Then $\mathbf{t}_1=\mathbf{t}+\Delta_t\in\Xi$, $K_1=K+\Delta_K\in\mathrm{Emb}(\mathbb{T}^n_{\rho_1},\mathcal{B}_0,C^1)$ and $Q_{(\mathbf{t}_1,K_1)}$ is also invertible. Moreover, the following estimates hold:*

$$\left\|(G_K)^{-1}-(G_{K_1})^{-1}\right\|_{\rho_1}\le\kappa\,\|\Delta_K\|_{\rho_1,C^1}\,, \tag{6.22}$$

$$\left|(Q_{(\mathbf{t},K)})^{-1}-(Q_{(\mathbf{t}_1,K_1)})^{-1}\right|\le\kappa\,\gamma^{-2}\rho_1^{-2\tau}\,\|(\Delta_t,\Delta_K)\|_{\rho_1,C^1}\,. \tag{6.23}$$

PROOF. To prove that $K_1=K+\Delta_K\in\mathrm{Emb}(\mathbb{T}^n_{\rho_1},\mathcal{B}_0,C^1)$ it is sufficient to verify that $G_{K_1}=\mathrm{D}_\theta K_1(\theta)^\top G(K_1(\theta))\mathrm{D}_\theta K_1(\theta)$ is invertible. It is easy to prove that $\Delta_K G=G_K-G_{K_1}$ satisfies

$$\|\Delta_K G\|_{\rho_1}\le\kappa\,\|\Delta_K\|_{\rho_1,C^1}\,, \tag{6.24}$$

where $\kappa$ is a constant depending on $n$, $\tau$, $c_{\mathrm{symp}}$ and polynomially on $\|\mathrm{D}_\theta K\|_\rho$ and $\|\mathrm{D}_\theta K_1\|_{\rho_1}$. Notice that, since the dependence of $\kappa$ in (6.24) on $\|\mathrm{D}_\theta K_1\|_{\rho_1}$ is polynomial and $\|\Delta_K\|_{\rho_1,C^1}$ is bounded by 1, $\kappa$ can be considered as a constant depending on $n$, $\tau$, $c_{\mathrm{symp}}$ and polynomially on $\|\mathrm{D}_\theta K\|_\rho$.

Hence, if $G_K$ is invertible and $\|\Delta_K\|_{\rho_1,C^1}$ is sufficiently small, then (using Neumann series) $G_{K_1}$ is invertible and the estimate (6.22) holds.

Notice that the operator $\mathcal{R}_\omega$ in Section 2.5 is linear and bounded. Moreover, from (6.8) we obtain the following inequalities:

$$\|\mathrm{D}_K Z(\mathbf{t},K)-\mathrm{D}_K Z(\mathbf{t}_1,K_1)\|\le\beta_2\,\|(\Delta_t,\Delta_K)\|_{\rho_1}\,,$$
$$\|\mathrm{D}_{\mathbf{t}} Z(\mathbf{t},K)-\mathrm{D}_{\mathbf{t}} Z(\mathbf{t}_1,K_1)\|\le\beta_2\,\|(\Delta_t,\Delta_K)\|_{\rho_1}\,.$$

Next, using the definition of $Q_{(\mathbf{t},K)}$ in (6.6) and performing some computations one shows that there is a constant $\kappa$, depending on $n$, $\tau$, $c_{\mathrm{symp}}$, $\beta_2$ and polynomially on the components of $\mathbf{n}_\rho(\mathbf{t},K)$ (see (6.7)), and $\|\Delta_K\|_{\rho_1,C^1}$ such that

$$|\Delta_K Q|\le\kappa\,\gamma^{-2}\rho_1^{-2\tau}\,\|(\Delta_t,\Delta_K)\|_{\rho_1,C^1} \tag{6.25}$$

where

$$\Delta_K Q=Q_{(\mathbf{t},K)}-Q_{(\mathbf{t}_1,K_1)}\,.$$

Using the fact that $\|\Delta_K\|_{\rho_1,C^1}$ is bounded by 1, we can make the constant $\kappa$ in (6.25) independent of $\|\Delta_K\|_{\rho_1,C^1}$.

Using Neumann series we have that if $Q_{(\mathbf{t},K)}$ is invertible and the norm $\|(\Delta_t, \Delta_K)\|_{\rho_1,C^1}$ is sufficiently small, then $Q_{(\mathbf{t}_1,K_1)} = Q_{(\mathbf{t},K)} - \Delta_K Q$ is invertible and moreover

$$(Q_{(\mathbf{t}_1,K_1)})^{-1} = (Q_{(\mathbf{t},K)})^{-1} + \\ + \left(I_{2n+m} - (Q_{(\mathbf{t},K)})^{-1}\Delta_K Q\right)^{-1} (Q_{(\mathbf{t},K)})^{-1}\Delta_K Q (Q_{(\mathbf{t},K)})^{-1},$$

this and estimate (6.25) imply (6.23). □

**6.3.3. Proof of the existence of a solution of** (6.2)**.** In this section we prove of Part a) of Theorem 6.2. Using Lemma 6.3, Lemma 6.6 we show that a Newton-like step produces an error which is quadratic with respect to the previous one. The proof of the convergence of the Newton-like method is standard in the literature (see e.g. [**18, 74**]). Throughout this section $\zeta_0 \in \mathbb{R}^m$ is assumed to be fixed.

The initial approximate solution $(\zeta_0; \mathbf{t}_0, K_0)$ provides the zero-step of the iterative procedure. Moreover, we recall that the matrix $Q_{(\mathbf{t}_0,K_0)}$ is assumed to be invertible.

Assume that for $i \geq 0$, $(\zeta_0; \mathbf{t}_i, K_i)$ is an approximate solution of (6.2), with $(\mathbf{t}_i, K_i) \in \Xi \times \mathrm{Emb}(\mathbb{T}^n_{\rho_i}, \mathcal{B}_0, C^1)$ satisfying the hypothesis of Lemma 6.3. Then, Lemma 6.3 implies the existence of an approximate solution $(\Delta_{t\,i}, \Delta_{K\,i})$ of the linearized equations:

$$\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta_0; \mathbf{t}_i, K_i) \begin{pmatrix} \Delta_{t\,i} \\ \Delta_{K\,i} \end{pmatrix} = -\mathcal{F}(\zeta_0; \mathbf{t}_i, K_i)\,, \tag{6.26}$$

satisfying the following estimates:

$$|\Delta_{t\,i}| \leq \gamma^{-2}\kappa_i\, \rho_i^{-2\tau}\ \|\mathcal{F}(\zeta_0; \mathbf{t}_i, K_i)\|_{\rho_i}\,, \tag{6.27}$$

$$\|\Delta_{K\,i}\|_{\rho_i - 2\delta_i} \leq \gamma^{-2}\kappa_i\, \delta_i^{-2\tau}\left(1 + \gamma^{-2}\rho_i^{-2\tau}\right)\ \|\mathcal{F}(\zeta_0; \mathbf{t}_i, K_i)\|_{\rho_i}\,, \tag{6.28}$$

$$\|\mathrm{D}\Delta_{K\,i}\|_{\rho_i - 3\delta_i} \leq \gamma^{-2}\kappa_i\, \delta_i^{-2(\tau+1)}\left(1 + \gamma^{-2}\rho_i^{-2\tau}\right)\|\mathcal{F}(\zeta_0; \mathbf{t}_i, K_i)\|_{\rho_i}\,, \tag{6.29}$$

where $0 < \delta_i < \rho_i/3$ and $\kappa_i$ is a positive constant depending on $n$, $\tau$, $m$, $c_{\mathrm{symp}}$ and polynomially on the components of $\mathbf{n}_{\rho_i}(K_i, \mathbf{t}_i)$ (see (6.7)).

If $\|\mathcal{F}(\zeta_0; \mathbf{t}_i, K_i)\|_{\rho_i}$ is sufficiently small, such that $(\Delta_{t\,i}, \Delta_{K\,i})$ satisfies the quantitative estimates of Lemma 6.6 ($\|\mathcal{F}(\zeta_0; \mathbf{t}_0, K_0)\|_{\rho_0}$ does), then for any $0 < \delta_i < \rho_i/3$, $\mathbf{t}_{i+1} = \mathbf{t}_i + \Delta_{t\,i} \in \Xi$, $K_{i+1} = K_i + \Delta_{K\,i} \in \mathrm{Emb}(\mathbb{T}^n_{\rho_i - 3\delta_i}, \mathcal{B}_0, C^1)$ and $Q_{(\mathbf{t}_{i+1},K_{i+1})}$ is invertible.

Remark 6.7. The dependence on the iterative step of the constant $\kappa_i$ in equations (6.27), (6.28), (6.29) relies on the polynomial dependence on the components of $\mathbf{n}_{\rho_i}(\mathbf{t}_i, K_i)$ (see (6.7)). We control the increments $(\mathbf{t}_i, \Delta_{K\,i})$ in such a way that

$$\|(\mathbf{t}_i - \mathbf{t}_0, K_i - K_0)\|_{\rho_i, C^1} < \min(\mathrm{dist}(\mathbf{t}_0, \partial\Xi), \mathrm{dist}(K_0(\mathbb{T}^n_{\rho_0}), \partial\mathcal{B}_0), 1)\,.$$

Hence, $\kappa_i$ is bounded by a uniform constant $\kappa$ that depends on $n$, $\tau$, $m$, $c_{\mathrm{symp}}$ and polynomially on the components of $\mathbf{n}_{\rho_0}(\mathbf{t}_0, K_0)$.

To iterate this procedure it is sufficient to show that the error decreases and that the quantitative assumptions in Lemma 6.3 are satisfied by the new approximate solution $(\mathbf{t}_{i+1}, K_{i+1})$. Moreover, we need to specify a choice of the loss of domain at each step (i.e. $\delta_i$). We follow the choices of [**50, 74**]. Fix $0 < \delta < \rho_0/12$ and for $i \geq 0$ define

$$\delta_i = \delta\, 2^{-i}, \qquad \rho_{i+1} = \rho_i - 3\delta_i\,.$$

In the following lemma we estimate the error after a step of the modified Newton method described above.

LEMMA 6.8. *Let $\omega \in \mathcal{D}_n(\gamma,\tau)$, for some $\gamma > 0$ and $\tau \geq n$. There exists a constant $\kappa$, depending on $n$, $\tau$, $m$, $c_{\rm symp}$, $\beta_2$ and polynomially on components of $\mathbf{n}_{\rho_0}(\mathbf{t}_0, K_0)$ (see (6.7)) such that*

$$\|\mathcal{F}(\zeta_0;\mathbf{t}_{i+1},K_{i+1})\|_{\rho_i-2\delta_i} \leq \kappa\,\gamma^{-4}\,\delta_i^{-4\tau}\left(1+\gamma^{-2}\rho_i^{-2\tau}\right)^2\|\mathcal{F}(\zeta_0;\mathbf{t}_i,K_i)\|^2_{\rho_i}\,.$$

PROOF. This is a consequence of Taylor's Theorem, Lemma 6.3 and Remark 6.7. □

The proof of Part a) of Theorem 6.2 is finished following standard steps in KAM theory (c.f. [**18, 74**]).

**6.3.4. Proof of local uniqueness.** The local uniqueness of solutions of (6.2), with $\zeta_0$ fixed, is a consequence of Remark 6.4.

Let $(\zeta_0;\mathbf{t}_1,K_1)$ and $(\zeta_0;\mathbf{t}_0,K_0)$ be as in Part b) of Theorem 6.2. Define $\Delta_t = \mathbf{t}_1 - \mathbf{t}_0$ and $\Delta_K = K_1 - K_0$ then using Taylor's Theorem we have:

$$\begin{aligned}\mathcal{F}(\zeta_0;\mathbf{t}_1,K_1) =& \mathcal{F}(\zeta_0;\mathbf{t}_0,K_0)+\\ &+\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta_0;\mathbf{t}_0,K_0)(\Delta_t,\Delta_K)+\mathcal{R}(\mathbf{t}_0,K_0)(\Delta_t,\Delta_K)^{\otimes 2}\,,\end{aligned}$$

where

$$\left\|\mathcal{R}(\mathbf{t}_0,K_0)(\Delta_t,\Delta_K)^{\otimes 2}\right\|_{\rho_0} \leq \beta_2\,\|(\Delta_t,\Delta_K)\|^2_{\rho_0}\,. \tag{6.30}$$

Using that $\mathcal{F}(\zeta_0;\mathbf{t}_i,K_i) = 0$ for $i = 1,2$, one obtains

$$\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta_0;\mathbf{t}_0,K_0)(\Delta_t,\Delta_K) = -\mathcal{R}(\mathbf{t}_0,K_0)(\Delta_t,\Delta_K)^{\otimes 2}\,.$$

Lemma 6.3, Remark 6.4 and estimate (6.30) imply that, for any $0 < \delta < \rho_0/2$:

$$\|(\Delta_t,\Delta_K)\|_{\rho_0-2\delta} \leq \kappa\,\gamma^{-2}\delta^{-2\tau}\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)\|(\Delta_t,\Delta_K)\|^2_{\rho_0}\,, \tag{6.31}$$

where $\kappa$ is a constant depending on $n$, $\tau$, $m$, $c_{\rm symp}$, $\beta_2$ and polynomially on components of $\mathbf{n}_{\rho_0}(\mathbf{t}_0,K_0)$.

Fix $0 < \delta < \rho_0/8$. For $i \in \mathbb{N}$, define $\delta_i = \delta\,2^{-i}$ and $\rho_{i+1} = \rho_i - 2\delta_i$. Applying (6.31) repeatedly for $j = 0,\ldots,i$, we have

$$\|(\Delta_t,\Delta_K)\|_{\rho_i} \leq \left(2^{2\tau}\right)^{2^i-(i+1)}\left(\kappa\,\gamma^{-2}\delta^{-2\tau}\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)\right)^{2^i-1}\|(\Delta_t,\Delta_K)\|^{2^i}_{\rho_0}\,,$$

for some constant $\kappa$. From which we have that if

$$2^{2\tau}\kappa\,\gamma^{-2}\delta^{-2\tau}\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)\|(\Delta_t,\Delta_K)\|_{\rho_0} < 1\,,$$

then $(\mathbf{t}_1,K_1) = (\mathbf{t}_0,K_0)$ on $\mathbb{T}^n_{\rho_0-4\delta}$. The analyticity assumption implies $(\mathbf{t}_1,K_1) = (\mathbf{t}_0,K_0)$ on $\mathbb{T}^n_{\rho_0}$. This finishes the proof of the local uniqueness of solutions of (6.2).

**6.3.5. Proof of the smooth dependence on $\zeta$ of the solutions.** Throughout this section we assume that the hypotheses of Theorem 6.2 hold and that $(\zeta_0; \mathbf{t}_0, K_0)$ is as in Part c) of Theorem 6.2. In particular, we assume that $\mathcal{F}(\zeta_0; \mathbf{t}_0, K_0) = 0$. Here we show the existence of implicit solutions $(\zeta; \mathbf{t}(\zeta), K_\zeta)$ of (6.2) for $\zeta$ near $\zeta_0$. We also show that these solutions depend smoothly on $\zeta$. The existence of $(\mathbf{t}(\zeta), K_\zeta)$ for $\zeta$ sufficiently near to $\zeta_0$, proved in Lemma 6.9, is a consequence of Part a) of Theorem 6.2. The smooth dependence on $\zeta$ is obtained by finding explicitly local Taylor expansions (Lemma 6.11) and applying the Converse Taylor Theorem (see e.g. page 88 in [**1**]).

LEMMA 6.9. *For any $0 < \delta < \rho_0/12$ there exists a neighborhood of $\zeta_0$, $D_0$, and a function*

$$\begin{array}{rccl} \Phi: & D_0 & \longrightarrow & \Xi \times \mathrm{Emb}(\mathbb{T}^n_{\rho_0 - 6\delta}, \mathcal{B}_0, C^1) \\ & \zeta & \longrightarrow & \Phi(\zeta) = (\mathbf{t}(\zeta), K_\zeta)\,, \end{array}$$

*such that $\Phi(\zeta_0) = (\mathbf{t}_0, K_0)$, $\mathcal{F}(\zeta; \Phi(\zeta)) = 0$ and $Q_\zeta = Q_{(\mathbf{t}(\zeta), K_\zeta)}$ is invertible for any $\zeta \in D_0$. Moreover, the following estimates hold:*

$$|\mathbf{t}(\zeta) - \mathbf{t}_0| \;\leq\; \kappa\,\gamma^{-2}\,\rho_0^{-2\tau}\;|\,\zeta - \zeta_0|\,, \tag{6.32}$$

$$\|K_\zeta - K_0\|_{\rho_0 - 6\delta} \;\leq\; \kappa\,\gamma^{-2}\,\delta^{-2\tau}\,\left(1 + \gamma^{-2}\rho_0^{-2\tau}\right)|\,\zeta - \zeta_0|\,, \tag{6.33}$$

*where $\kappa$ is a constant, depending on $n$, $\tau$, $m$, $c_{\mathrm{symp}}$, $\beta_2$ and polynomially on the components of $\mathbf{n}_{\rho_0}(\mathbf{t}_0, K_0)$ (see (6.7)).*

PROOF. This is a consequence of parts a) and b) of Theorem 6.2. Indeed, for $\zeta$ sufficiently close to $\zeta_0$ we have

$$\mathcal{F}(\zeta; \mathbf{t}_0, K_0) = \begin{pmatrix} 0 \\ \zeta_0 - \zeta \end{pmatrix}\,,$$

Then, Part a) of Theorem 6.2 yields a positive constant $\kappa$, depending on $n$, $\tau$, $m$, $c_{\mathrm{symp}}$, $\beta_2$ and polynomially on the components of $\mathbf{n}_{\rho_0}(\mathbf{t}_0, K_0)$ (see (6.7)), such that if

$$\kappa\,\gamma^{-4}\,\delta^{-4\tau}\left(1 + \gamma^{-2}{\rho_0}^{-2\tau}\right)^2\,|\zeta - \zeta_0| < c_0,$$

where

$$c_0 = \min\left[\mathrm{dist}\,(\mathbf{t}_0, \partial\,\Xi)\,,\, \mathrm{dist}\left(K_0\left(\mathbb{T}^n_{\rho_0}\right), \partial\mathcal{B}_0\right),\; 1\right],$$

then there exists $(\mathbf{t}(\zeta), K_\zeta, ) \in \Xi \times \mathrm{Emb}(\mathbb{T}^n_{\rho_0 - 6\delta}, \mathcal{B}_0, C^1)$ such that:

$$\mathcal{F}(\zeta; \mathbf{t}(\zeta), K_\zeta) = 0\,,$$

and such that estimates (6.32) and (6.33) hold. Next, let $\zeta \in D_0$ and assume that $\left(\tilde{\mathbf{t}}(\zeta), \tilde{K}_\zeta\right) \in \Xi \times \mathrm{Emb}(\mathbb{T}^n_{\rho_0 - 6\delta}, \mathcal{B}_0, C^1)$ is such that

$$\mathcal{F}(\zeta; \tilde{\mathbf{t}}(\zeta), \tilde{K}_\zeta) = 0.$$

and such that estimates (6.32) and (6.33) hold. From the local uniqueness of the solutions of (6.2) (see Part b) in Theorem 6.2 and estimates (6.32) and (6.33)) it is clear that $(\tilde{\mathbf{t}}(\zeta), \tilde{K}_\zeta) = (\mathbf{t}(\zeta), K_\zeta)$. □

REMARK 6.10. Let $D_0$ and $(\mathbf{t}(\zeta), K_\zeta)$ be as Lemma 6.9. It is clear that if $0 < \delta < \rho_0/12$ is fixed, then $D_0$ can be chosen in such a way that, for any $\zeta \in D_0$,

$$\|\Phi(\zeta) - \Phi(\zeta_0)\|_{\rho_0 - 6\delta, C^1} < \min\left[\mathrm{dist}\,(\mathbf{t}_0, \partial\,\Xi)\,,\, \mathrm{dist}\left(K_0\left(\mathbb{T}^n_{\rho_0}\right), \partial\mathcal{B}_0\right),\, 1\right]\,.$$

Hence if a constant $\kappa$ depends in a polynomial way on the components of $\mathbf{n}_{\rho_0-6\delta}(K_\zeta,\mathbf{t})$ (see (6.7)), it can be assumed that $\kappa$ depends in a polynomial way on the components of $\mathbf{n}_{\rho_0}(K_0,\mathbf{t}_0)$.

We now prove that the function $\Phi$ in Lemma 6.9 is smooth. For $j\geq 1$, we denote by $L^j_{\rm sym}(\mathbb{C}^m,\mathbb{C}^d)$ the set of symmetric continuous $j$-linear transformations from $(\mathbb{C}^m)^j$ to $\mathbb{C}^d$. For $\rho>0$, fixed, denote by $L^j_{\rm sym}\left(\mathbb{C}^m,(\mathrm{A}(\mathbb{T}^n_\rho,C^1))^{2n}\right)$ the set of continuous $j$-linear transformations from $(\mathbb{C}^m)^j$ to $(\mathrm{A}(\mathbb{T}^n_\rho,C^1))^{2n}$, which can be identified with $C^1_\rho(\mathbb{T}^n,L^j_{\rm sym}\left(\mathbb{C}^m,\mathbb{C}^{2n}\right))$. Let $K^j\in C^1(\mathbb{T}^n_\rho,L^j_{\rm sym}\left(\mathbb{C}^m,\mathbb{C}^{2n}\right))$ and $\zeta^1,\ldots,\zeta^j\in\mathbb{C}^m$, with $\zeta^i=(\zeta^i_1,\ldots,\zeta^i_m)$ for $i=1,\ldots,j$. We use the following notation

$$K^j(\theta)\,[\zeta^1,\zeta^2,\ldots,\zeta^j]=\sum_{\substack{k=(k_1,\ldots,k_j)\\1\leq k_i\leq m}}K^j_k(\theta)\,\zeta^1_{k_1}\,\cdots\zeta^j_{k_j}\,,$$

where $K^j_k\in(\mathrm{A}(\mathbb{T}^n_\rho,C^1))^{2n}$. By $\mathrm{D}_\theta K^j$ we mean

$$\mathrm{D}_\theta K^j(\theta)\,[\zeta^1,\zeta^2,\ldots,\zeta^j]=\sum_{\substack{k=(k_1,\ldots,k_j)\\1\leq k_i\leq m}}\mathrm{D}_\theta K^j_k(\theta)\,\zeta^1_{k_1}\,\cdots\zeta^j_{k_j}\,.$$

In the following lemma we use Lindstedt expansions and parts a) and b) of Theorem 6.2 to prove that the function $\Phi$ in Lemma 6.9 have a local Taylor expansion. The standard notation $\zeta^{\otimes j}=[\zeta,\ldots,\zeta]\in(\mathbb{R}^m)^j$ is used.

LEMMA 6.11. *Fix $0<\delta<\rho_0/44$ and define $\tilde\rho_0=\rho_0-8\delta$ and $\delta_0=\delta$. For $j=1,\ldots,r$, define $\delta_j=\delta\,2^{-j}$ and $\tilde\rho_j=\tilde\rho_{j-1}-4\,\delta_{j-1}$ and $\rho_r=\tilde\rho_r-6\delta$.*

*Let $D_0$ and $\Phi$ be as in Lemma 6.9. Then, there exists an open neighborhood $D\subseteq D_0$ of $\zeta_0$, such that $\Phi:D\to\Xi\times\mathrm{Emb}(\mathbb{T}^n_{\tilde\rho_0},\mathcal{B}_0,C^1)$ is continuous and for each $j=1,\ldots,r$ there is a continuous function*

$$\Phi^j:D\to L^j_{\rm sym}\left(\mathbb{C}^m,\mathbb{C}^m\right)\times L^j_{\rm sym}\left(\mathbb{C}^m,(\mathrm{A}(\mathbb{T}^n_{\tilde\rho_j},C^1))^{2n}\right),$$

*satisfying*

$$\left\|\Phi^j(\tilde\zeta)\right\|_{\tilde\rho_j,C^1}\leq\;2^{(2\tau+1)\,b_j}\left(\kappa\,\gamma^{-2}\delta^{-(2\tau+1)}(1+\gamma^{-2}\rho_0^{-2\tau})\right)^{a_j}\;,\tag{6.34}$$

*where $a_j=j!\,\sum\limits_{i=1}^{j}\frac{1}{i!}$ and $b_j=j!-1$.*

*Moreover, if $\tilde\zeta\in D$ and $\left|\zeta-\tilde\zeta\right|$ is sufficiently small, then $\Phi^{\leq r}(\tilde\zeta;\zeta)\in\Xi\times\mathrm{Emb}(\mathbb{T}^n_{\tilde\rho_r},\mathcal{B}_0,C^1)$, with*

$$\Phi^{\leq r}(\tilde\zeta;\zeta)=\Phi(\tilde\zeta)+\sum_{j=1}^{r}\frac{1}{j!}\,\Phi^j(\tilde\zeta)(\zeta-\tilde\zeta)^{\otimes j}\,,\tag{6.35}$$

*and*

$$\left\|\Phi(\zeta)-\Phi^{\leq r}(\tilde\zeta;\zeta)\right\|_{\rho_r}\leq\kappa_r\,\gamma^{-2}\,\delta^{-2\tau}\,\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)\,\left\|R^r_{\mathcal{F}}(\tilde\zeta;\zeta)\right\|_{\tilde\rho_r}\,\left|\zeta-\tilde\zeta\right|^r\,,\tag{6.36}$$

*where*

$$\lim_{\zeta\to\tilde\zeta}\left\|R^r_{\mathcal{F}}(\tilde\zeta;\zeta)\right\|_{\tilde\rho_r}=0\,.\tag{6.37}$$

*The constants $\kappa$ and $\kappa_r$ in* (6.34) *and* (6.36) *depend on $n$, $\tau$, $m$, $c_{\rm symp}$, $\beta_r$, $r$, and polynomially on the components of $\mathbf{n}_{\rho_0}(\mathbf{t}_0, K_0)$ (see* (6.7)*).*

Proof. First we prove the continuity of $\Phi$. Let $\zeta_1, \zeta_2 \in D_0$, then applying Taylor Theorem to $\mathcal{F}(\zeta_2; \Phi(\zeta_1))$ around $(\zeta_1; \Phi(\zeta_1))$ we obtain the following equality:

$$\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta_1; \Phi(\zeta_1))\,(\Phi(\zeta_2) - \Phi(\zeta_1)) = \Psi^0(\zeta_1, \zeta_2),$$

where

$$\left\|\Psi^0(\zeta_1, \zeta_2)\right\|_{\rho_0 - 6\delta_0} \leq \kappa\, |\zeta_1 - \zeta_2| \ ,$$

where $\kappa$ is a constant depending on $\beta_2$ (we assume that the diameter of $D_0$ is finite). Then Lemma 6.3 and Remark 6.4 imply

$$\|\Phi(\zeta_1) - \Phi(\zeta_2)\|_{\rho_0 - 8\delta_0} \leq \ \kappa\, \gamma^{-2}\, \delta_0^{-2\tau} \left(1 + \gamma^{-2} \rho_0^{-2\tau}\right) |\zeta_1 - \zeta_2| \ , \tag{6.38}$$

where $\kappa$ is a constant depending on $n$, $\tau$, $m$, $c_{\rm symp}$ and polynomially on the components of $\mathbf{n}_{\rho_0}(\mathbf{t}_0, K_0)$ (see (6.7) and Remark 6.10). This proves the continuity of $\Phi$.

The transformations $\Phi^j(\tilde\zeta)$ for $j = 1, \dots, r$ and $\tilde\zeta \in D_0$ are computed recursively in such a way that

$$\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\tilde\zeta; \Phi(\tilde\zeta))\ \Phi^j(\tilde\zeta) = -\mathcal{F}^j(\tilde\zeta)\,, \tag{6.39}$$

where

$$\mathcal{F}^j(\tilde\zeta) = \mathrm{D}^j_\zeta \left(\mathcal{F}(\zeta; \Phi^{\leq (j-1)}(\tilde\zeta; \zeta))\right)_{|\zeta = \tilde\zeta} . \tag{6.40}$$

with

$$\Phi^{\leq (j-1)}(\tilde\zeta; \zeta) = \Phi(\tilde\zeta) + \sum_{i=1}^{j-1} \frac{1}{i!} \Phi^i(\tilde\zeta)(\zeta - \tilde\zeta)^{\otimes\, i} .$$

This formally implies

$$\mathrm{D}^j_\zeta \left(\mathcal{F}(\zeta; \Phi^{\leq r}(\tilde\zeta; \zeta))\right)_{|\zeta = \tilde\zeta} = 0\ , \quad j = 0, \dots, r\,. \tag{6.41}$$

Equation (6.39) for $j = 1$ is

$$\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\tilde\zeta; \Phi(\tilde\zeta))\, \Phi^1(\tilde\zeta)\ =\ \begin{pmatrix} O_{2n\times m} \\ I_m \end{pmatrix} . \tag{6.42}$$

Existence and uniqueness of $\Phi^1(\tilde\zeta)$ is guaranteed by Lemma 6.3 and Remark 6.4. Moreover, the following estimates hold:

$$\left\|\Phi^1(\tilde\zeta)\right\|_{\tilde\rho_0 - 2\delta_0} \leq \kappa\, \gamma^{-2} \delta_0^{-2\tau} \left(1 + \gamma^{-2} \rho_0^{-2\tau}\right), \tag{6.43}$$

$$\left\|\Phi^1(\tilde\zeta)\right\|_{\tilde\rho_0 - 3\delta_0, C^1} \leq c_1\,,$$

with

$$c_1 = \kappa\, \gamma^{-2} \delta_0^{-(2\tau+1)} \left(1 + \gamma^{-2} \rho_0^{-2\tau}\right),$$

where $\kappa$ is a constant, depending on $n$, $\tau$, $m$, $c_{\rm symp}$, and polynomially on the components of $\mathbf{n}_{\rho_0}(K_0, \mathbf{t}_0)$ (see Remark 6.10).

We now show that $\Phi^1 : D_0 \to L^1_{\rm sym}\left(\mathbb{C}^m, (\mathrm{A}(\mathbb{T}^n_{\tilde\rho_1}, C^1))^{2n}\right) \times L^1_{\rm sym}\left(\mathbb{C}^m, \mathbb{C}^m\right)$ is continuous. Let $\zeta_1, \zeta_2 \in D_0$ then from (6.42) we have

$$\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta_1; \Phi(\zeta_1)) \left(\Phi^1(\zeta_1) - \Phi^1(\zeta_2)\right) = \Psi^1(\zeta_1, \zeta_2)\,, \tag{6.44}$$

where

$$\Psi^1(\zeta_1,\zeta_2) = \left(\mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta_2;\Phi(\zeta_2)) - \mathrm{D}_{(\mathbf{t},K)}\mathcal{F}(\zeta_1;\Phi(\zeta_1))\right)\Phi^1(\zeta_2)\,.$$

Notice that from (6.43) and the smoothness of $\mathcal{F}$ we have

$$\left\|\Psi^1(\zeta_1,\zeta_2)\right\|_{\tilde\rho_0-2\delta_0} \le \kappa\,\gamma^{-2}\delta_0^{-2\tau}\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)\,|\zeta_1-\zeta_2|\;, \tag{6.45}$$

where $\kappa$ is a constant depending on $n$, $\tau$, $m$, $c_{\mathrm{symp}}$, $\beta_r$ and polynomially on the components of $\mathbf{n}_{\rho_0}(K_0,\mathbf{t}_0)$ (see Remark 6.10).

Hence, from Lemma 6.3 (see also Remark 6.4), equality (6.44) and estimate (6.45) we have that the following estimate holds for any $\zeta_1,\zeta_2\in D_0$:

$$\left\|\Phi^1(\zeta_1)-\Phi^1(\zeta_2)\right\|_{\tilde\rho_0-4\delta_0} \le d_1\;|\zeta_1-\zeta_2|$$

where

$$d_1 = \kappa\,\gamma^{-4}\,\delta_0^{-4\tau}\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)^2\;.$$

This proves that $\Phi^1$ is continuous on $D_0$.

From Lemma 6.6 we have that if $\left|\zeta-\tilde\zeta\right|$ is sufficiently small, then $\Phi^{\le 1}(\tilde\zeta;\zeta)\in \mathrm{Emb}(\mathbb{T}^n_{\tilde\rho_1},\mathcal{B}_0,C^1)\times\Xi$ (hence, the function $\mathcal{F}(\zeta;\Phi^{\le 1}(\tilde\zeta;\zeta))$ is well-defined) and moreover

$$\mathrm{D}^i_\zeta\left(\mathcal{F}(\zeta;\Phi^{\le 1}(\tilde\zeta;\zeta))\right)_{|\zeta=\tilde\zeta} = 0\,,\quad i=0,1\;.$$

Let $2\le j\le r$, and assume that we have computed $\Phi^1,\dots,\Phi^{j-1}$ with the following properties :

(i) there are constants $1\le c_1\le\cdots\le c_{j-1}$ and $1\le d_1\le\cdots\le d_{j-1}$ such that for any $\zeta_1,\zeta_2\in D_0$

$$\left\|\Phi^i(\zeta_1)\right\|_{\tilde\rho_i,C^1} \le\; c_i,\qquad i=1,\dots,j-1\,, \tag{6.46}$$

$$\left\|\Phi^i(\zeta_1)-\Phi^i(\zeta_2)\right\|_{\tilde\rho_i} \le d_i\;|\zeta_1-\zeta_2|\;,\qquad i=1,\dots,j-1\,, \tag{6.47}$$

(ii) $\Phi^{\le(j-1)}(\tilde\zeta;\zeta)\in\mathrm{Emb}(\mathbb{T}^n_{\tilde\rho_{j-1}},\mathcal{B}_0,C^1)\times\Xi$,

(iii) $\mathrm{D}^i_\zeta\left(\mathcal{F}(\zeta;\Phi^{\le(j-1)}(\tilde\zeta;\zeta))\right)_{|\zeta=\tilde\zeta} = 0\;,\quad i=0,\dots,j-1\;.$

We perform the step $j$, i.e. find $\Phi^j$ and show that it satisfies (i), (ii) and (iii) for $j$. Let $\tilde\zeta\in D_0$ be fixed. Using the chain rule (see e.g. page 87 in [**1**]), $\mathcal{F}^j(\tilde\zeta)$ is a polynomial expression, of degree $j$, of $\Phi^1(\tilde\zeta),\dots,\Phi^{j-1}(\tilde\zeta)$, whose coefficients are linear combinations of the components of the derivatives $\mathrm{D}^i_{(\mathbf{t},K;\zeta)}\mathcal{F}(\tilde\zeta;\Phi(\tilde\zeta))$ for $1\le i\le j$. Then, using (6.46) we have:

$$\left\|\mathcal{F}^j(\tilde\zeta)\right\|_{\tilde\rho_{j-1}} \le \kappa\;(c_{j-1})^j\;, \tag{6.48}$$

where $\kappa$ depends on $r$ and $\beta_r$.

Now, from Lemma 6.9 we know that $Q_{\tilde\zeta}$ is invertible. Then, using Lemma 6.3 and Remark 6.4 we have that there exists a unique $\Phi^j(\tilde\zeta)=(K^j(\tilde\zeta),\mathbf{t}^j(\tilde\zeta))$ in $L^j_{\mathrm{sym}}\left(\mathbb{C}^m,(\mathrm{A}(\mathbb{T}^n_{\tilde\rho_{j-1}-3\delta_{j-1}},C^1))^{2n}\right)\times L^j_{\mathrm{sym}}\left(\mathbb{C}^m,\mathbb{C}^m\right)$ such that equality (6.39) holds. Moreover, using Lemma 6.3, Cauchy estimates and estimate (6.48) we have

$$\left\|\Phi^j(\tilde\zeta)\right\|_{\tilde\rho_{j-1}-2\delta_{j-1}} \le \kappa\,\gamma^{-2}\delta_{j-1}^{-2\tau}\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)(c_{j-1})^j\;,$$

$$\left\|\mathrm{D}_\theta\,K^j(\tilde\zeta)\right\|_{\tilde\rho_{j-1}-3\delta_{j-1}} \le \kappa\,\gamma^{-2}\delta_{j-1}^{-(2\tau+1)}\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)(c_{j-1})^j\;,$$

where we have used that $\tilde{\rho}_{j-1} > \rho_0/2$ and $\kappa$ is a constant, depending on $n$, $\tau$, $c_{\text{symp}}$, $\beta_2$, $\beta_r$ and polynomially on the components of $\mathbf{n}_{\rho_0}(K_0, \mathbf{t}_0)$ (see Remark 6.10). To obtain estimate (6.46) for $j$, we define

$$c_j = \kappa\, \gamma^{-2} \delta_{j-1}^{-(2\tau+1)} \left(1 + \gamma^{-2} \rho_0^{-2\tau}\right) (c_{j-1})^j \,.$$

This proves estimate (6.34). We now prove the continuity of $\Phi^j$. From (6.39) we have

$$\mathrm{D}_{(\mathbf{t},K)} \mathcal{F}(\zeta_1; \Phi(\zeta_1)) \left(\Phi^j(\zeta_1) - \Phi^j(\zeta_2)\right) = \Psi^j(\zeta_1, \zeta_2)\,, \tag{6.49}$$

where

$$\begin{aligned}\Psi^j(\zeta_1, \zeta_2) = \mathcal{F}^j(\zeta_2) - \mathcal{F}^j(\zeta_1) + &\\ \left(\mathrm{D}_{(\mathbf{t},K)} \mathcal{F}(\zeta_2; \Phi(\zeta_2)) - \mathrm{D}_{(\mathbf{t},K)} \mathcal{F}(\zeta_1; \Phi(\zeta_1))\right) \Phi^j(\zeta_2)\,.&\end{aligned}$$

Notice that $\mathcal{F}^j : D_0 \to L^j_{\text{sym}} \left(\mathbb{C}^m, (\mathrm{A}(\mathbb{T}^n_{\tilde{\rho}_{j-1}}, C^1))^{2n}\right) \times L^j_{\text{sym}} (\mathbb{C}^m, \mathbb{C}^m)$ is continuous. Indeed, $\mathcal{F}^j(\zeta_i)$ is a polynomial expression of $\Phi^1(\tilde{\zeta})$, $\Phi^2(\tilde{\zeta}) \ldots, \Phi^{j-1}(\tilde{\zeta})$, of degree $j$, whose coefficients are linear combinations of the components of the derivatives $\mathrm{D}^i_{(\mathbf{t},K;\zeta)} \mathcal{F}(\tilde{\zeta}; \Phi(\tilde{\zeta}))$ for $1 \leq i \leq j$. Moreover, $\Phi^1, \ldots, \Phi^{j-1}$ are continuous on $D_0$. Hence, the following estimate holds:

$$\left\|\mathcal{F}^j(\zeta_1) - \mathcal{F}^j(\zeta_2)\right\|_{\rho_{j-1}} \leq \hat{d}_j \, |\zeta_1 - \zeta_2| \,,$$

where $\hat{d}_j$ depends on $n$, $m$, $\beta_r$ and polynomially on $d_1$, $\ldots$, $d_{j-1}$, $c_1$, $\ldots$, $c_{j-1}$. Then,

$$\left\|\Psi^j(\zeta_1, \zeta_2)\right\|_{\tilde{\rho}_{j-1} - 2\delta_{j-1}} \leq \hat{\hat{d}}_j \, |\zeta_1 - \zeta_2| \,,$$

where $\hat{\hat{d}}_j$ depends on $n$, $m$, $\beta_r$ and polynomially on $d_1$, $\ldots$, $d_{j-1}$, $c_1$, $\ldots$ ,$c_j$.

Hence, using equality (6.49), Lemma 6.3 and Remark 6.4 we have, for any $\zeta_1, \zeta_2 \in D_0$,

$$\left\|\Phi^j(\zeta_1) - \Phi^j(\zeta_2)\right\|_{\tilde{\rho}_{j-1} - 4\delta_{j-1}} \leq d_j \, |\zeta_1 - \zeta_2| \,,$$

with

$$d_j = \hat{\hat{d}}_j \, \kappa \, \gamma^{-2} \, \delta_{j-1}^{-2\tau} (1 + \gamma^{-2} \rho_0^{-2\tau}),$$

where $\kappa$ is a constant, depending on $n$, $m$, $\tau$, $c_{\text{symp}}$, $\beta_2$, $\beta_r$ and polynomially on the components of $\mathbf{n}_{\rho_0}(K_0, \mathbf{t}_0)$ (see Remark 6.10). This proves that $\Phi^j$ is continuous on $D_0$.

As a consequence of Lemma 6.6 we have that if $\left|\zeta - \tilde{\zeta}\right|$ is sufficiently small, then $\Phi^{\leq j}(\tilde{\zeta}; \zeta) \in \mathrm{Emb}(\mathbb{T}^n_{\tilde{\rho}_j}, \mathcal{B}_0, C^1) \times \Xi$ and moreover the following holds:

$$\mathrm{D}^i_\zeta \left(\mathcal{F}(\zeta; \Phi^{\leq j}(\tilde{\zeta}; \zeta))\right)_{|\zeta = \tilde{\zeta}} = 0 \,, \quad i = 0, \ldots, j\,.$$

This finishes the step $j$.

Iterating the above procedure, we obtain $\Phi^1(\tilde{\zeta})$,$\ldots$, $\Phi^r(\tilde{\zeta})$ satisfying (6.39) and such that if $\Phi^{\leq r}(\tilde{\zeta}; \zeta)$ is defined by (6.35) then for $\left|\zeta - \tilde{\zeta}\right|$ sufficiently small, $\Phi^{\leq r}(\tilde{\zeta}; \zeta) \in \mathrm{Emb}(\mathbb{T}^n_{\tilde{\rho}_r}, \mathcal{B}_0, C^1) \times \Xi$ and (6.41) holds.

We now prove (6.36). Let $\tilde{\zeta} \in D_0$ be fixed. Assume that $\left|\zeta - \tilde{\zeta}\right|$ is sufficiently small such that $\Phi^{\leq r}(\tilde{\zeta}; \zeta) \in \mathrm{Emb}(\mathbb{T}^n_{\tilde{\rho}_r}, \mathcal{B}_0, C^1) \times \Xi$. Then, by construction, $\Phi^{\leq r}(\tilde{\zeta}; \zeta)$

is an approximate solution of (6.2) with error $\mathcal{F}(\zeta;\Phi^{\leq r}(\tilde{\zeta};\zeta))$. From Taylor Theorem (see e.g. page 88 in [**1**]) and the construction of $\Phi^{\leq r}(\tilde{\zeta},\zeta)$ we have

$$\mathcal{F}(\zeta;\Phi^{\leq r}(\tilde{\zeta};\zeta)) = R^r_{\mathcal{F}}(\tilde{\zeta};\zeta)\,(\zeta-\tilde{\zeta})^{\otimes r}\,, \tag{6.50}$$

where $R^r_{\mathcal{F}}(\tilde{\zeta};\zeta)$ satisfies (6.37).

Moreover, since $\Phi(\tilde{\zeta}) = (\mathbf{t}(\tilde{\zeta}), K_{\tilde{\zeta}})$ is non-degenerate with respect to (6.2) (see Definition 6.1), Lemma 6.6 implies that, if $\left|\zeta-\tilde{\zeta}\right|$ is sufficiently small, $\Phi^{\leq r}(\tilde{\zeta};\zeta)$ is also nondegenerate. Hence, if $\left|\zeta-\tilde{\zeta}\right|$ is sufficiently small, Part a) of Theorem 6.2 yields a solution of (6.2),

$$\hat{\Phi}(\zeta) = \left(\hat{\mathbf{t}}(\zeta), \hat{K}_\zeta\right) \in \Xi\times\mathrm{Emb}(\mathbb{T}^n_{\tilde{\rho}_r-6\delta},\mathcal{B}_0,C^1),$$

which is nondegenerate. Moreover from (6.50) and (6.37) we have

$$\begin{aligned}\left\|\Phi^{\leq r}(\tilde{\zeta};\zeta)\ -\hat{\Phi}(\zeta)\right\|_{\tilde{\rho}_r-6\delta} &\leq \\ &\leq \kappa_r\,\gamma^{-2}\,\delta^{-2\tau}\,\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)\left\|R^r_{\mathcal{F}}(\tilde{\zeta};\zeta)\right\|_{\tilde{\rho}_r}\left|\zeta-\tilde{\zeta}\right|^r,\end{aligned} \tag{6.51}$$

where $\kappa_r$ depends on $n$, $\tau$, $m$, $r$, $c_{\mathrm{symp}}$, $\beta_2$, $\beta_r$ and polynomially on the components of $\mathbf{n}_{\rho_0}(\mathbf{t}_0, K_0)$ (see Remark 6.10).

Definition (6.35) and estimates (6.38), (6.46) and (6.51) imply

$$\begin{aligned}\left\|\Phi(\zeta)-\hat{\Phi}(\zeta)\right\|_{\tilde{\rho}_r-6\delta} &\leq \left\|\Phi(\zeta)-\Phi(\tilde{\zeta})\right\|_{\tilde{\rho}_r-6\delta} \\ &\quad+\left\|\Phi(\tilde{\zeta})-\Phi^{\leq r}(\tilde{\zeta};\zeta)\right\|_{\tilde{\rho}_r-6\delta} \\ &\quad+\left\|\Phi^{\leq r}(\tilde{\zeta};\zeta)-\hat{\Phi}(\zeta)\right\|_{\tilde{\rho}_r-6\delta} \\ &\leq \kappa\,\gamma^{-2}\delta^{-2\tau}\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)\left|\zeta-\tilde{\zeta}\right| \\ &\quad+\sum_{j=1}^{r} c_j\left|\zeta-\tilde{\zeta}\right|^j \\ &\quad+\kappa\,\gamma^{-2}\,\delta^{-2\tau}\,\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)\left|\zeta-\tilde{\zeta}\right|^r,\end{aligned}$$

where we have also used (6.37).

Then, from Part b) of Theorem 6.2 we have: $\hat{\Phi}(\zeta) = (\hat{K}(\zeta),\hat{\mathbf{t}}(\zeta)) = \Phi(\zeta)$, for $|\zeta-\zeta_0|$ sufficiently small. Estimate (6.36) follows from (6.51). This finishes the proof of Lemma 6.11. □

The differentiability of the function given in Lemma 6.9 is a consequence of Lemma 6.11 and the Converse Taylor Theorem (see e.g. [**38**] or page 88 in [**1**]). This finishes the proof of Part c) of Theorem 6.2.

CHAPTER 7

# A Transformed Tori Theorem

The main result of this chapter is Theorem 7.4. This is a KAM result on the existence of parametric FLD. More concretely, given a Hamiltonian deformation $f:\mathcal{U}\times\mathcal{A}_0\to\mathcal{A}$ and $s\in\mathbb{N}$ such that $0\le s\le n$, we introduce an $s$-dimensional (modifying) parameter by embedding $f$ into a family of Hamiltonian deformations $g:\mathcal{U}\times\Lambda\times\mathcal{A}_0\to\mathcal{A}$ in such a way that $g_{(\mu,\lambda)}=f_\mu$ if and only if $\lambda=0$. Theorem 7.4 gives sufficient conditions for the existence of a parametric FLD $\mathbf{K}:\mathcal{U}_0\times\mathrm{D}\times\mathbb{T}^n\to\Lambda\times\mathcal{A}_0$, with base sets D and $\Lambda$ and parameter $\mu\in\mathcal{U}_0$, in such a way that the hypotheses of Theorem 4.11 hold.

The parameter $\lambda$ is introduced to weaken the twist condition and it can be viewed as a generalization of the Moser's modifying term [**51**]. To deal with the non-uniqueness of the parameterization of invariant tori and for technical convenience, we also introduce an $n$-dimensional parameter $\sigma$, that we call dummy. As we show below, the dummy parameter is not relevant when working with invariant tori. For a motivation and a detailed explanation of the role of the modifying and dummy parameters, see Section 5.2.

Theorem 7.4 is formulated and proved in the analytic category. That is, the symplectic deformations and the parameterizations of tori are assumed to be real-analytic with respect to, respectively, the phase space variable and the angle variable. The dependence on the parameters is assumed to be smooth. A smooth version of Theorem 7.4 can be obtained from Theorem 7.4 and the Moser's *smoothing technique* [**26, 49, 50, 74**] but this is not done here because it involves several technicalities.

Throughout this chapeter $\mathcal{B}_0$ and $\mathcal{B}$ are complex strips of $\mathcal{A}_0$ and $\mathcal{A}$, respectively, with $\mathcal{B}_0\subset\mathcal{B}$ and the components of $\Omega$, $\Omega^{-1}$, $a$, $J$, $J^{-1}$ and $G$ are in $\mathrm{A}(\mathcal{B},C^2)$, with $C^1$-norms on $\mathcal{B}$ bounded by a positive constant denoted by $c_{\mathrm{symp}}$.

## 7.1. Nondegeneracy condition

Since Theorem 7.4 involves a KAM procedure, a nondegeneracy condition is need. The nondegeneracy condition imposed here is motivated in Section 5.2. Let $\mathcal{U}\subset\mathbb{R}^k$, $\Lambda\subset\mathbb{R}^s$ and $\Sigma\subset\mathbb{R}^n$ be open, with $0\le s\le n$. Assume that $f:\mathcal{U}\times\mathcal{A}_0\to\mathcal{A}$ is a Hamiltonian deformation, $h:\Lambda\times\mathcal{A}_1\to\mathcal{A}$ is a modifying family and $d:\Sigma\times\mathcal{A}_2\to\mathcal{A}$ is a dummy family, such that $f_\mu(\mathcal{A}_0)\subset\mathcal{A}_1$ and $h_\lambda(\mathcal{A}_1)\subset\mathcal{A}_2$ for all $\mu\in\mathcal{U}$ and all $\lambda\in\Lambda$. Define $m=n+s+k$, $\Xi=\mathcal{U}\times\Sigma\times\Lambda\subset\mathbb{R}^m$, $\mathbf{t}=(\mu,\sigma,\lambda)$ and $\mathbf{f}:\Xi\times\mathcal{A}_0\to\mathcal{A}$ be defined by

$$\mathbf{f}(\mathbf{t},z)=(d_\sigma\circ h_\lambda\circ f_\mu)(z)\,.$$

Let $K \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A}_0)$ and let $L_K$ and $N_K$ be given by (3.2) and (3.3), respectively. Let $\mathcal{M}^h$ be the moment map of $h$, we use the following notation:

$$\begin{aligned} T_{(\mathbf{t},K)}(\theta) &= N_K(\theta+\omega)^\top \, \Omega(K(\theta+\omega)) \, \mathrm{D}_z \mathbf{f}(\mathbf{t}, K(\theta)) \, N_K(\theta), \\ B^h_{(\lambda,K)}(\theta) &= (\mathrm{D}_z \mathcal{M}^h(\lambda, K(\theta+\omega)) M_K(\theta+\omega) J_0)^\top . \end{aligned}$$

Let $\mathcal{R}_\omega$ be the one-bite solver operator (see Section 2.5). assume that $\mathcal{R}_\omega T_{(\mathbf{t},K)}$ and $\mathcal{R}_\omega B^h_{(\lambda,K)}$ are smooth. Define

$$\begin{aligned} \bar{\mathrm{T}}_{(\mathbf{t},K)} &= \left\langle T_{(\mathbf{t},K)}(\theta) \right\rangle, \\ P^{12}_{(\mathbf{t},K)} &= \left\langle \mathrm{D}_z \mathcal{M}^h(\lambda, K(\theta)) \left( N_K(\theta) - L_K(\theta) \, \mathcal{R}_\omega T_{(\mathbf{t},K)}(\theta)^\top \right) \right\rangle^\top, \\ P^{21}_{(\mathbf{t},K)} &= \left\langle \mathrm{D}_z \mathcal{M}^h(\lambda, K(\theta)) \left( N_K(\theta) - L_K(\theta) \, \mathcal{R}_\omega T_{(\mathbf{t},K)}(\theta) \right) \right\rangle, \\ P^{22}_{(\mathbf{t},K)} &= \langle \mathrm{D}_\lambda \mathcal{M}^h(\lambda, K(\theta)) \rangle + \left\langle (\mathcal{R}_\omega B^{h,x}_{(\lambda,K)}(\theta))^\top B^{h,y}_{(\lambda,K)}(\theta) \right\rangle \\ &\quad - \left\langle (\mathcal{R}_\omega B^{h,y}_{(\lambda,K)}(\theta))^\top B^{h,x}_{(\lambda,K)}(\theta) \right\rangle \\ &\quad + \left\langle (\mathcal{R}_\omega B^{h,y}_{(\lambda,K)}(\theta))^\top T_{(\mathbf{t},K)}(\theta) \, \mathcal{R}_\omega B^{h,y}_{(\lambda,K)}(\theta) \right\rangle, \end{aligned}$$

and

$$P_{(\mathbf{t},K)} = \begin{pmatrix} \bar{\mathrm{T}}_{(\mathbf{t},K)} & P^{12}_{(\mathbf{t},K)} \\ P^{21}_{(\mathbf{t},K)} & P^{22}_{(\mathbf{t},K)} \end{pmatrix}. \tag{7.1}$$

DEFINITION 7.1. The pair $(\lambda_*, K_*)$ is *$h$-nondegenerate* with respect to $f_{\mu_*}$ and $\omega$, if the $(n+s)\times(n+s)$ matrix $P_{(\mathbf{t}_*,K_*)}$ is invertible, with $\mathbf{t}_* = (\mu_*, 0, \lambda_*,)$ and $P_{(\mathbf{t},K)}$, defined in (7.1).

REMARK 7.2. In Definition 7.1 the dummy parameters are not involved because $d_0$ is the identity. Moreover, it is always possible to introduce a modifying parameter in such a way that any integrable system is nondegenerate in the sense of Definition 7.1.

REMARK 7.3. If $s = 0$, the only modifying deformation is the family with the identity as its unique element. In such a case, a torus $K_*$ is nondegenerate with respect to $f_{\mu_*}$ and $\omega$, in the sense of Definition 7.1, if and only if $K_*$ is twist with respect to $f_{\mu_*}$ and $\omega$ (see Definition 3.7). Moreover, if $s > 0$ the nondegeneracy condition in Definition 7.1 is weaker than the standard twist condition (see Appendix 5.2).

## 7.2. Statement of the Transformed Tori Theorem

The following result is the KAM part of our method to study bifurcations of invariant tori.

THEOREM 7.4. *Let $\omega \in \mathcal{D}_n(\gamma, \tau)$, for some $\gamma > 0$ and $\tau \geq n$. Let $k, s \in \mathbb{N}$ and $r \in \mathbb{R}$ be such that $0 \leq s \leq n$ and $r \geq 2$. Let $\mathcal{U} \subset \mathbb{R}^k$, $\Lambda \subset \mathbb{R}^s$ and $\Sigma \subset \mathbb{R}^n$ be open, with $0 \in \Lambda$ and $0 \in \Sigma$. Assume that $f : \mathcal{U}\times\mathcal{A}_0 \to \mathcal{A}$ is a Hamiltonian deformation, $h : \Lambda\times\mathcal{A}_1 \to \mathcal{A}$ is a modifying family and $d : \Sigma\times\mathcal{A}_2 \to \mathcal{A}$ is a dummy family, such that $f_\mu(\mathcal{A}_0) \subset \mathcal{A}_1$ and $h_\lambda(\mathcal{A}_1) \subset \mathcal{A}_0$ for all $\mu \in \mathcal{U}$ and all $\lambda \in \Lambda$. Take $m = n+s+k$, $\Xi = \mathcal{U}\times\Sigma\times\Lambda \subset \mathbb{R}^m$ and $\mathbf{t} = (\mu, \sigma, \lambda)$. Let $\mathbf{f} : \Xi\times\mathcal{A}_0 \to \mathcal{A}$ be defined* *by* $\mathbf{f}(\mathbf{t}, z) = (d_\sigma \circ h_\lambda \circ f_\mu)(z)$. *Assume that, for any* $\mathbf{t} \in \Xi$, $\mathbf{f_t} \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^2)$ *and moreover for* $i = 0, \ldots, r$

$$\beta_i = \max\left( \sup_{(\mathbf{t},z)\,\in\,\Xi\times\mathcal{B}_0} \left|\mathrm{D}^i_{(\mathbf{t},z)}\mathbf{f}(\mathbf{t}, z)\right|, \sup_{(\lambda,z)\,\in\,\Lambda\times\mathcal{B}} \left|\mathrm{D}^i_{(\lambda,z)}\mathcal{M}^h(\lambda, z)\right|\right) < \infty .$$

*Let* $\rho_0 > 0$ *be fixed and such that* $\gamma\rho_0^\tau < 1$. *Assume that* $(\lambda_0, K_0) \in \Lambda\times\mathrm{Emb}(\mathbb{T}^n_{\rho_0}, \mathcal{B}_0, C^1)$ *is h-nondegenerate with respect to* $f_{\mu_0}$ *and* $\omega$, *with* $\mu_0 \in \mathcal{U}$ *fixed (see Definition 7.1). Define*

$$p_0 = \langle\, \mathcal{M}^h\,(\lambda_0, K_0(\theta)))\,\rangle\ .$$

*Then there exists a positive constant* $\kappa$, *depending on* $r$, $\tau$, $c_{\mathrm{symp}}$, $\beta_2$ *and polynomially on* $\|\mathrm{D}_\theta K_0\|_{\rho_0}$, $\left\|\left(G_{K_0}\right)^{-1}\right\|_{\rho_0}$, *and* $\left|(P_{((\mu_0,0,\lambda_0),K_0)})^{-1}\right|$ *such that if*

$$\kappa\,\gamma^{-4}\,\delta_0^{-4\tau}\left(1+\gamma^{-2}\rho_0{}^{-2\tau}\right)^2\,\|h_{\lambda_0}\circ f_{\mu_0}\circ K_0 - K_0\circ \mathrm{R}_\omega\|_{\rho_0} < c_0,$$

*where*

$$c_0 = \min\left(\mathrm{dist}\left(K_0\left(\mathbb{T}^n_{\rho_0}\right), \partial\mathcal{B}_0\right), \mathrm{dist}\left((0,0,\mu_0), \partial\Xi\right), 1\right),$$

*and* $0 < \delta_0 < \rho_0/44$ *is fixed, then there exist* $\mathcal{U}_0\times\mathrm{D} \subset \mathcal{U}\times\mathbb{R}^s$ *an open neighborhood of* $(\mu_0, p_0)$ *and a* $C^r$*-smooth function*

$$\begin{array}{rccl} \mathbf{K}: & \mathcal{U}_0\times\mathrm{D}\times\mathbb{T}^n & \longrightarrow & \Lambda\times\mathcal{A}_0 \\ & (\mu, p, \theta) & \longrightarrow & (\lambda(\mu, p), K(\mu, p, \theta))\,, \end{array}$$

*such that the following hold:*

**a)** *For any* $\mu \in \mathcal{U}_0$, $\mathbf{K}_\mu(p, \theta) = \mathbf{K}(\mu, p, \theta)$ *is a parametric FLD with base sets* $\mathrm{D}$ *and* $\Lambda$ *such that:*

$$\begin{array}{rcl} h_{\lambda(\mu,p)}\circ f_\mu\circ K_{(\mu,p)} - K_{(\mu,p)}\circ\mathrm{R}_\omega & = & 0, \\ \langle K^x(\mu,p,\theta)-\theta\rangle & = & 0, \\ \langle\mathcal{M}^h(\lambda(\mu,p), K(\mu,p,\theta))\rangle & = & p\,. \end{array}$$

**b)** *For any* $(\mu, p) \in \mathcal{U}_0\times\mathrm{D}$, $K_{(\mu,p)} \in \mathrm{Emb}(\mathbb{T}^n_{\rho_0-22\delta_0}, \mathcal{B}_0, C^1)$ *and*

$$|\lambda(\mu,p) - \lambda(\mu_0,p_0)|\,\leq \kappa\,\gamma^{-2}\,\rho_0^{-2\tau}\,\left|\,(\mu,p)-(\mu_0,p_0)\right|\,,$$

$$\left\|K_{(\mu,p)} - K_{(\mu_0,p_0)}\right\|_{\rho_0-22\delta_0} \leq \kappa\,\gamma^{-2}\,\delta_0^{-2\tau}\,\left(1+\gamma^{-2}\rho_0^{-2\tau}\right)|\,(\mu,p)-(\mu_0,p_0)|\,.$$

**c)** *If* $(\lambda_0, K_0)$ *is* $(h_{\lambda_0}\circ f_{\mu_0})$*-invariant with frequency* $\omega$, *then*

$$\mathbf{K}(\mu_0, p_0) = (\lambda_0, K_0).$$

*Moreover,* $\mathbf{K}$ *is locally unique in the sense that if there is*

$$\begin{array}{rccl} \mathbf{K}': & \mathcal{U}'_0\times\mathrm{D}'\times\mathbb{T}^n & \longrightarrow & \Lambda\times\mathcal{A}_0 \\ & (\mu, p, \theta) & \longrightarrow & (\lambda'(\mu, p), K'(\mu, p, \theta))\,, \end{array}$$

*satisfying the same properties as* $\mathbf{K}$, *then* $\mathbf{K}'(\mu,p,\theta) = \mathbf{K}(\mu,p,\theta)$ *for all* $\theta \in \mathbb{T}^n$ *and* $(\mu, p)$ *in a neighborhood of* $(\mu_0, p_0)$, *contained in* $(\mathcal{U}'_0\times\mathrm{D}')\cap(\mathcal{U}_0\times\mathrm{D})$.

Notice that, in Theorem 7.4, $\mathbf{K}$ depends on the modifying deformation $h$. For notational simplicity this dependence is not included in the notation.

Remark 7.5. If $s = 0$, Theorem 7.4 is a KAM result under twist condition (see Remark 7.3). In this case, the use of a dummy deformation gives a generalization of the Translated Curve Theorem of [**9, 19, 57**] as follows.

Let $g \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^2)$, not necessarily exact. Assume that the following hold.

i) $g(\mathcal{B}_0) \subset \mathcal{B}'_0 \subset \mathcal{B}$, with $\mathcal{B}_0$ a complex strip of an annulus $\mathcal{A}'_0 \subset \mathcal{A}$.
ii) There exists $\varphi \in \mathrm{Symp}_{C^2}(\mathcal{B}'_0, \mathcal{B})$ such that $\mathrm{C}^\varphi = -\mathrm{C}^g$ (i.e. $\varphi \circ g$ is exact symplectic);
iii) $K_0 \in \mathrm{Emb}(\mathbb{T}^n_{\rho_0}, \mathcal{B}_0, C^1)$ is an approximately $(\varphi \circ g)$-invariant twist torus with frequency $\omega \in \mathcal{D}_n(\gamma, \tau)$;
iv) $d$ is a dummy deformation with base $\Sigma$ such that $(d_\sigma \circ \varphi \circ g) \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^2)$ for all $\sigma \in \Sigma$;
v) $(d_\sigma \circ \varphi \circ g)$, $K_0$, $\rho_0$, $\delta_0$ and $\omega$ satisfy the hypotheses of Theorem 7.4 (with $s = 0$).

Then, Theorem 7.4 guarantees the existence of $K_* \in \mathrm{Emb}(\mathbb{T}^n_{\rho_0 - 6\delta_0}, \mathcal{B}_0, C^1)$ satisfying the following equations:

$$\varphi(g(K_*(\theta))) = K_*(\theta + \omega),$$
$$\langle K_*^x(\theta) - \theta \rangle = 0\,.$$

That is, $g$ maps $K_*(\mathbb{T}^n)$ into $\varphi^{-1}(K_*(\mathbb{T}^n))$. In particular, if $\mathcal{A} = \mathbb{T}^n \times U$, with $U \subset \mathbb{R}^n$ open and simply connected, is endowed with the standard symplectic form $\boldsymbol{\omega}_0$ and $\varphi$ is a translation in the $y$-direction on the standard annulus, then $K_*(\mathbb{T}^n)$ is a *translated* torus for $g$.

## 7.3. Proof of the Transformed Tori Theorem

Here we show that the Transformed Tori Theorem (Theorem 7.4) is a consequence of general parametric result, Theorem 6.2. We apply Theorem 6.2 to the following equation

$$\mathcal{F}(\zeta; \mathbf{t}, K) = 0\,, \tag{7.2}$$

where $\zeta = (\mu, q, p) \in \mathcal{U} \times \mathbb{R}^n \times \mathbb{R}^s$, $\mathbf{t} = (\mu, \sigma, \lambda)$ and

$$\mathcal{F}(\zeta; \mathbf{t}, K) = \begin{pmatrix} \mathbf{f_t} \circ K - K \circ \mathrm{R}_\omega \\ Z(\mathbf{t}, K) - \zeta \end{pmatrix}\,, \tag{7.3}$$

with

$$Z(\mathbf{t}, K) = \begin{pmatrix} \mu \\ \langle K^x(\theta) - \theta \rangle \\ \langle\, \mathcal{M}^h\,(\lambda, K)\,\rangle \end{pmatrix}. \tag{7.4}$$

Without loss of generality, we assume that $\langle K_0^x(\theta) - \theta \rangle = 0$. Then, if $\mathbf{t}_0 = (\mu_0, 0, \lambda_0)$, and $\zeta_0 = (\mu_0, 0, p_0)$, $(\zeta_0; \mathbf{t}_0, K_0)$ is an approximate solution of (7.2). Let us verify that $(\zeta_0; \mathbf{t}_0, K_0)$ satisfies the nondegeneracy condition in Theorem 6.2.

Lemma 7.6. *Let $\omega \in \mathcal{D}_n(\gamma, \tau)$, for some $\gamma > 0$ and $\sigma \geq n$. The pair $(\lambda_0, K_0) \in \Lambda \times \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^1)$ is $h$-nondegenerate with respect to $f_{\mu_0}$ and $\omega$ (see Definition 7.1) if and only if $((\mu_0, 0, \lambda_0), K_0)$ is nondegenerate with respect to the functional* (7.3) *(see Definition 6.1).*

Proof. This follows substituting our case in Definition 6.1. We just need to observe that if $Z$ is given by (7.4), then

$$\mathrm{D}_\mathbf{t} Z(\mathbf{t}, K)\delta = \begin{pmatrix} I_k & O_{k\times n} & O_{k\times s} \\ O_{n\times k} & O_{n\times n} & O_{n\times s} \\ O_{s\times k} & O_{s\times n} & \langle\, \mathrm{D}_\lambda \mathcal{M}^h(\lambda, K(\theta))\,\rangle \end{pmatrix} \delta,$$

and

$$\mathrm{D}_K Z(\mathbf{t},K)\,[\Delta_{\scriptscriptstyle K}] = \begin{pmatrix} O_{k\times 1} \\ \langle {\Delta_{\scriptscriptstyle K}}^{x} \rangle \\ \langle\, \mathrm{D}_z \mathcal{M}^h(\lambda, K(\theta))\Delta_{\scriptscriptstyle K}\,\rangle \end{pmatrix}.$$

In particular, using that $\mathcal{M}^h$ is 1-periodic in $x$ we have:

$$\mathrm{D}_K Z(\mathbf{t},K)\,[L_{\scriptscriptstyle K}] = \begin{pmatrix} O_{k\times n} \\ I_n \\ \langle\, \mathrm{D}_\theta\,(\mathcal{M}^h_\lambda \circ K)\,\rangle \end{pmatrix} = \begin{pmatrix} O_{k\times n} \\ I_n \\ O_{s\times n} \end{pmatrix}.$$

Next, let $\tilde{\mathcal{M}}^{\mathbf{f}}$ be the local moment map of $\mathbf{f}$ and take

$$B^{\mathbf{f}}_{(\mathbf{t},K)}(\theta) = \left(\mathrm{D}_z \tilde{\mathcal{M}}^{\mathbf{f}}(\mathbf{t}, K(\theta+\omega)) M_{\scriptscriptstyle K}(\theta+\omega) J_0\right)^\top .$$

Write $\tilde{\mathcal{M}}^{\mathbf{f}}$ as follows (see Remark 2.18):

$$\tilde{\mathcal{M}}^{\mathbf{f}}(\mathbf{t},z) = \mathcal{M}^{\mathbf{f}}(\mathbf{t},z) - \mathrm{D}_{\mathbf{t}}\mathrm{C}^{\mathbf{f}}(\mathbf{t})^\top x$$

with $\mathcal{M}^{\mathbf{f}}(\mathbf{t},z)$ 1-periodic on $x$. Then, since $\mathrm{C}^{\mathbf{f}}(\mathbf{t}) = \sigma$, we have

$$\left\langle B^{\mathbf{f},y}_{(\mathbf{t},K)}(\theta)^\top \right\rangle = \mathrm{D}_{\mathbf{t}}\mathrm{C}^{\mathbf{f}}(\mathbf{t}) = \begin{pmatrix} O_{n\times k} & I_n & O_{n\times s} \end{pmatrix}$$

Hence, in our case, the matrix of Definition 6.1 takes the following form

$$\begin{pmatrix} O_n & \langle T_{(\mathbf{t},K)}\rangle & Q_{15} & Q_{13} & Q_{14} \\ O_n & O_n & O_{n\times k} & I_n & O_{n\times s} \\ O_{k\times n} & O_{k\times n} & I_k & O_{k\times n} & O_{k\times s} \\ I_n & Q_{32} & Q_{35} & Q_{33} & Q_{34} \\ O_{s\times n} & Q_{42} & Q_{45} & Q_{43} & Q_{44} \end{pmatrix}. \tag{7.5}$$

Using equality (5.16) is easy to prove that if $\mathbf{t} = (\mu_0, 0, \lambda_0)$, then

$$Q_{14} = P^{12}_{(\mathbf{t},K)}, \quad Q_{42} = P^{21}_{(\mathbf{t},K)}, \quad Q_{44} = P^{22}_{(\mathbf{t},K)},$$

where the $P^{ij}_{(\mathbf{t},K)}$s are as in Definition 7.1. This finishes the proof of Lemma 7.6. □

It is clear that, under the assumptions of Theorem 7.4, the the hypotheses of Theorem 6.2 hold. From Theorem 6.2, there exist a neighborhood of $\zeta_0 = (\mu_0, 0, p_0)$, $\mathcal{U}_0 \times \mathrm{D}_1 \times \mathrm{D} \subset \mathcal{U} \times \mathbb{R}^n \times \mathbb{R}^n$, and a smooth function

$$\zeta \in \mathcal{U}_0 \times \mathrm{D}_1 \times \mathrm{D} \to (\mathbf{t}(\zeta), K_\zeta) \in \Xi \times \mathrm{Emb}(\mathbb{T}^n_{\rho_0 - 22\delta_0}, \mathcal{B}_0, C^1),$$

such that $K_\zeta$ is $\mathbf{f}_{\mathbf{t}(\zeta)}$-invariant with frequency $\omega$. Moreover, from Lemma 3.1, the function $\mathbf{t}(\zeta)$ is of the following form

$$\mathbf{t}(\zeta) = (\mu, 0, \lambda(\mu, q, p)).$$

Finally, the local uniqueness part of Theorem 6.2 yields, for $q$ sufficiently small,

$$K_{(\mu,q,p)} = K_{(\mu,0,p)} \circ \mathrm{R}_q, \qquad \lambda(\mu,q,p) = \lambda(\mu,0,p).$$

Then, by fixing $q$, e.g. $q = 0$, we obtain that the function

$$\mathbf{K}(\mu, 0, p, \theta) = (\lambda(\mu,0,p), K_{(\mu,0,p)}(\theta))$$

satisfies the properties in Theorem 7.4. The local uniqueness of $\mathbf{K}$ follows from Part b) of Theorem 6.2.

# Part 3

# Singularity theory for KAM tori

CHAPTER 8

# Bifurcation theory for KAM tori

Bifurcation theory for quasi-periodic motions is a complicated mathematical problem that naturally arises in mathematical models depending on external parameters. An important step forward in this field has been developed in the –by now classical– *Parametric KAM Theory* [**10, 11, 51**], which has been developed in several contexts: *dissipative, reversible, volume-preserving and Hamiltonian*. This theory provides results on *quasi-periodic stability*: persistence of Cantor sets of quasi-periodic motions with Whitney regularity. Geometrically quasi-periodic motions are tori with internal dynamics an ergodic rotation. The most studied case is the *reducible* one. Roughly, reducibility means that the normal dynamics –to the invariant torus– is constant. Some contributions to the non-reducible case are [**12, 13, 30, 31, 66**].

To unfold quasi-periodic motions, one of the main strategies in classical Parametric KAM Theory is to apply the *unfolding theory* for linear applications [**4**] to the linearized normal dynamics. However, in the Hamiltonian context this technique can bot be used to study bifurcations of maximal dimensional invariant tori with fixed frequency. The reason is that, in this case, the normal frequencies are equal to zero [**48**]. In this chapter we develop a method to study bifurcation of invariant tori, with fixed frequency, using Singularity Theory. Our methodology is based on the theory developed in parts 1 and 2 of this monograph and gives a robust way to construct bifurcation diagrams of KAM tori. It generalizes the BNF procedure used in [**23**] and the available persistence results obtained using the classical Parametric KAM Theory.

## 8.1. Classification of KAM invariant tori

We classify invariant tori in terms of the potential as provided by Singularity Theory. The main results of this section are Theorem 8.5 and Theorem 8.6. Theorem 8.5 is a result of *robustness* of the proposed classification. Theorem 8.6 relates the bifurcation set of the potential with the bifurcation set of invariant tori.

Throughout Section 8.1 we make the following assumptions.

i) $\omega \in \mathbb{R}^n$ is fixed and $\mathrm{R}_\omega$ is ergodic;
ii) $\mathcal{U} \subset \mathbb{R}^k$ is open, $f : \mathcal{U} \times \mathcal{A}_0 \to \mathcal{A}$ is a Hamiltonian deformation, with $f_\mu(\mathcal{A}_0) \subset \mathcal{A}_1$ for all $\mu \in \mathcal{U}$;
iii) $K_* \in \mathrm{Emb}(\mathbb{T}^\mathrm{n}, \mathcal{A}_0)$ is $f_{\mu_*}$-invariant with frequency $\omega$;
iv) $s \in \mathbb{N}$ is fixed and such that $\dim\ker \bar{\mathrm{T}}_{(f_{\mu_*},K_*)} \leq s \leq n$, with $\bar{\mathrm{T}}_{(f_{\mu_*},K_*)} = \left\langle T_{(f_{\mu_*},K_*)} \right\rangle$, the torsion of $K_*$ with respect to $f_{\mu_*}$ and $\omega$;
v) $\Lambda \subset \mathbb{R}^s$ is open, with $0 \in \Lambda$, and $h : \Lambda \times \mathcal{A}_1 \to \mathcal{A}$ is a modifying deformation with moment map $\mathcal{M}^h$ and let $p_* = \langle \mathcal{M}^h(0, K_*) \rangle$.

We classify invariant tori that satisfy the properties in the following definition.

DEFINITION 8.1. We say that $K_*$ is *h-deformable*, with respect to $f$ and $\omega$, if there exist an open neighborhood of $(\mu_*, p_*)$, $\mathcal{U}_0 \times \mathrm{D} \subset \mathcal{U} \times \mathbb{R}^s$, and a locally unique parametric FLD

$$\begin{array}{rccl} \mathbf{K}: & \mathcal{U}_0 \times \mathrm{D} \times \mathbb{T}^n & \longrightarrow & \Lambda \times \mathcal{A}_0 \\ & (\mu, p, \theta) & \longrightarrow & (\lambda(\mu, p), K(\mu, p, \theta))\,, \end{array}$$

with base sets D and $\lambda$ and parameter $\mu \in \mathcal{U}_0$, such that $\mathbf{K}(\mu_*, p_*) = (0, K_*)$ and

$$\begin{aligned} h_{\lambda(\mu,p)} \circ f_\mu \circ K_{(\mu,p)} - K_{(\mu,p)} \circ \mathrm{R}_\omega &= 0\,, \\ \langle K^x(\mu, p, \theta)\rangle &= 0, \\ \langle \mathcal{M}^h(\lambda(\mu, p), K(\mu, p, \theta))\rangle &= p\,. \end{aligned}$$

$\mathbf{K}$ is called *$\mu$-parametric FLD of $K_*$ with respect to $h$, $f$ and $\omega$.*

REMARK 8.2. In Proposition 5.12 it is shown that it is *always* possible to define the modifying deformation $h$ in such a way that $(0, K_*)$ is $h$-nondegenerate with respect to $f_{\mu_*}$ and $\omega$. Hence, under the regularity conditions of Theorem 7.4, it is always possible to define $h$ in such a way that $K_*$ is $h$-deformable with respect to $f$ and $\omega$.

Using Lemma 5.9, it is easy to verify that if $K_*$ is $h$-deformable with respect to $f$ and $\omega$, then the $\mu$-parametric FLD of $K_*$ with respect to $h$, $f$ and $\omega$, satisfies the hypotheses of Theorem 4.11. To emphasize the dependence on $h$ and since $\mathbf{K}$ is locally unique, the parametric potential of $\mathbf{K}$, given in (4.19), will be denoted by $V^{h,f,K_*}$. Let $S^h$ and $S^f$ be the primitive functions of $h$ and $f$, respectively. It is clear that

$$V^{h,f,K_*}(\mu, p) = -p^\top \lambda(\mu, p) - \left\langle S^h_{\lambda(\mu,p)} \circ f_\mu \circ K_{(\mu,p)} \right\rangle - \left\langle S^f_\mu \circ K_{(\mu,p)} \right\rangle. \tag{8.1}$$

The functions $V^{h,f,K_*}$ and $V^{h,f,K_*}(\mu_*, \cdot)$ will be called, respectively, *the $\mu$-parametric potential $K_*$ with respect to $h$, $f$ and $\omega$* and *the potential of $K_*$ with respect to $h$, $f_{\mu_*}$ and $\omega$.*

DEFINITION 8.3. We say that $K_*$ is of *class* $\Upsilon$, with respect to $h$, $f_{\mu_*}$ and $\omega$, if the germ of $V^{h,f,K_*}(\mu_*, \cdot)$ at $p_*$ is of class $\Upsilon$, under $R^+$-equivalence (see Definition B.5).

The classification of $K_*$ given in Definition 8.3 is invariant under canonical change of variables in the following sense.

PROPOSITION 8.4. *Canonical changes of variables on both $f$ and $h$ do not change the parametric potential* (8.1)*. That is, if $\varphi : \mathcal{A}' \to \mathcal{A}$ is an exact symplectomorphism; and $h'$, $f'$ and $K'$ are given by*

$$\begin{aligned} h'_\lambda(z) &= \varphi^{-1} \circ h_\lambda \circ \varphi(z), \\ f'_\mu(z) &= \varphi^{-1} \circ f_\mu \circ \varphi(z), \\ K'_*(\theta) &= \varphi^{-1}(K_*(\theta)). \end{aligned}$$

*Then, the following equality holds:*

$$V^{h',f',K'_*} = V^{h,f,K_*}\,.$$

PROOF. This follows from Proposition 4.6. □

Summarizing, to classify of $K_*$ we perform the following steps:

**Step 1.** Fix $s \in \mathbb{N}$ such that $\dim\ker \bar{\mathrm{T}}_{(f_{\mu_*},K_*)} \leq s \leq n$. Choose a modifying family $h : \Lambda \times \mathcal{A}_1 \to \mathcal{A}$ in such a way that $K_*$ is $h$-deformable with respect to $f$ and $\omega$;
**Step 2** Apply Theorem 7.4 to find the FLD of $K_*$ with respect to $h$, $f$ and $\omega$;
**Step 3** Use Singularity Theory and Theorem 4.11 to classify the critical point $p_*$ of $V^{h,f,K_*}$.

In the following result we give sufficient conditions that guarantee the persistence of an $f_{\mu_*}$-invariant torus, with fixed frequency $\omega$, in such a way that the persistent torus is of the same class of the unperturbed one.

THEOREM 8.5. *Let $r \geq 2$, $\omega \in \mathcal{D}_n(\gamma, \tau)$ and let $E \subset \mathbb{R}^d$ be a neighborhood of zero and $\mathcal{U} \subset \mathbb{R}^k$ be open. Let $f : \mathcal{U} \times \mathcal{A}_0 \to \mathcal{A}$ $g : E \times \mathcal{U} \times \mathcal{A}_0 \to \mathcal{A}$ Hamiltonian deformations, such that $g_{(0,\mu)} = f_\mu$. Let $h : \Lambda \times \mathcal{A}_1 \to \mathcal{A}$ and $d : \Sigma \times \mathcal{A}_2 \to \mathcal{A}$ be, a modifying and a dummy deformation, respectively, with $\Lambda \subset \mathbb{R}^s$ and $\Sigma \subset \mathbb{R}^n$. Assume that for all $(\varepsilon, \mu, \sigma, \lambda) \in E \times \mathcal{U} \times \Sigma \times \Lambda$*

$$d_\sigma \circ h_\lambda \circ g_{(\varepsilon,\mu)} \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^r).$$

*Let $\rho_0 > 0$ be fixed and such that $\gamma\rho_0^\tau < 1$ and $\mu_* \in \mathcal{U}$. Assume that:*

i) *$K_* \in \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^1)$ is an $f_{\mu_*}$-invariant torus with frequency $\omega$;*
ii) *$(0, K_*)$ is $h$-nondegenerate with respect to $f_{\mu_*}$ and $\omega$;*
ii) *$K_*$ is of class $\Upsilon$ with respect to $h$, $f_{\mu_*}$ and $\omega$;*
iii) *the $\mu$-parametric potential of $K_*$ with respect to $h$, $f$ and $\omega$, $V^{h,f,K_*}(\mu, p)$, is a versal unfolding of $V^{h,f,K_*}(\mu_*, \cdot)$ at the singularity $p_* = \langle \mathcal{M}^h(0, K_*(\theta)) \rangle$.*

*Then, for any fixed $\varepsilon \in E$, with $|\varepsilon|$ sufficiently small, there exist $(\mu_\varepsilon, p_\varepsilon)$, close to $(\mu_*, p_*)$, and $K_\varepsilon \in \mathrm{Emb}(\mathbb{T}^n_{\rho/2}, \mathcal{B}_0, C^1)$ such that:*

**a)** *$K_\varepsilon$ is a $g_{(\varepsilon,\mu_\varepsilon)}$-invariant torus, with frequency $\omega$, of class $\Upsilon$ with respect to $h$, $g_{(\varepsilon,\mu_\varepsilon)}$ and $\omega$;*
**b)** *let $g_\varepsilon : \mathcal{U} \times \mathcal{A}_0 \to \mathcal{A}$ be the Hamiltonian deformation given by $g_\varepsilon(\mu, z) = g(\varepsilon, \mu, z)$, then the potential $V^{h,g_\varepsilon,K_\varepsilon}(\mu, p)$ is a versal unfolding of $V^{h,g_\varepsilon,K_\varepsilon}(\mu_\varepsilon, p)$ at the singularity $p_\varepsilon$.*

PROOF. This is a consequence of Theorem 7.4 and Mather's Theorem on the stability of versal unfoldings (see Theorem B.20). □

## 8.2. Local equivalence of Bifurcations diagrams

Assume that $K_* \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A}_0)$ is $h$-deformable, with respect to $f$ and $\omega$, and let $\mathbf{K} = (\lambda, K) : \mathcal{U}_0 \times \mathrm{D} \times \mathbb{T}^n \to \Lambda \times \mathcal{A}_0$ be its $\mu$-parametric FLD, with respect to $h$, $f$ and $\omega$. Because of Theorem 4.11, the bifurcation of the critical points of the $\mu$-parametric potential $V^{h,f,K_*}(\mu, \cdot)$ (given in (8.1)) determine the bifurcations of $f_\mu$-invariant tori, with frequency $\omega$, in the set

$$\{K_{(\mu,p)} : (\mu, p) \in \mathcal{U}_0 \times \mathrm{D}\}.$$

Moreover, under the hypotheses of Part d) of Theorem 4.11, degenerate critical points of the potential correspond to tori with degenerate torsion in the family $\{K_{(\mu,p)} : (\mu, p) \in \mathcal{U}_0 \times \mathrm{D}\}$. Hence, under the hypotheses of Part d) of Theorem 4.11, the bifurcation set obtained by using the potential is 'locally equivalent' to that using the torsion. In particular, the bifurcation set is locally independent of $h$, although potential in (8.1) depends on $h$.

Since tori with nondegenerate torsion (twist tori) persist under small perturbations, bifurcations of invariant tori, with fixed frequency, can occur only when the torsion is degenerate. Let us introduce some definitions. Let $\bar{\mathrm{T}}_{(f,K)}$ be as in Definition 3.7.

i) The *bifurcation diagram* of $f$ and $\omega$ is

$$\mathrm{Diag}_f(\omega) = \{(\mu, K) \in \mathcal{U} \times \mathrm{Lag}(\mathbb{T}^n, \mathcal{A}_0) : f_\mu \circ K = K \circ \mathrm{R}_\omega, \langle K^x(\theta) - \theta \rangle = 0\};$$

ii) The *catastrophe set* of $f$ and $\omega$ is

$$\mathrm{Cat}_f(\omega) = \left\{ (\mu, K) \in \mathrm{Diag}_f(\omega) : \det \bar{\mathrm{T}}_{(f_\mu, K)} = 0 \right\} ,$$

which is stratified into the sets

$$\mathrm{Cat}^i_f(\omega) = \left\{ (\mu, K) \in \mathrm{Diag}_f(\omega) : \dim \ker \bar{\mathrm{T}}_{(f_\mu, K)} = i \right\} ,$$

with $1 \leq i \leq n$;

iii) The *bifurcation set* of $f$ and $\omega$ is

$$\mathrm{Bif}_f(\omega) = \{\mu \in \mathcal{U} : \exists\, K \text{ such that } (\mu, K) \in \mathrm{Cat}_f(\omega)\} .$$

which is stratified into the sets

$$\mathrm{Bif}^i_f(\omega) = \left\{\mu \in \mathcal{U} : \exists\, K \text{ such that } (\mu, K) \in \mathrm{Cat}^{\mathrm{i}}_{\mathrm{f}}(\omega)\right\} ,$$

with $1 \leq i \leq n$.

An immediate consequence of Theorem 4.11 is the following result on th local equivalence of Bifurcation diagrams.

Theorem 8.6. *Let $\mathcal{U} \subset \mathbb{R}^k$ be open and let $f : \mathcal{U} \times \mathcal{A}_0 \to \mathcal{A}$ be a Hamiltonian deformation, such that $f_\mu(\mathcal{A}_0) \subset \mathcal{A}_1$, for all $\mu \in \mathcal{U}$. Let $K_* \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A}_0)$ be $f_{\mu_*}$-invariant with frequency $\omega \in \mathcal{D}_n(\gamma, \tau)$. Fix $s \in \mathbb{N}$ such that $\dim \ker \bar{\mathrm{T}}_{(f_{\mu_*}, K_*)} \leq s \leq n$. Let $\Lambda \subset \mathbb{R}^s$ be open and let $h : \Lambda \times \mathcal{A}_1 \to \mathcal{A}$ be a modifying deformation with moment map $\mathcal{M}^h$. Assume that $K_*$ is $h$-deformable with respect to $f$ and $\omega$. Let $\mathbf{K} : \mathcal{U}_0 \times \mathrm{D} \times \mathbb{T}^n \to \Lambda \times \mathcal{A}_0$ be the $\mu$-parametric FLD of $K_*$, with respect to $h$, $f$ and $\omega$, and let $V^{h,f,K_*}(\mu, p)$ be given by (8.1). Define*

i) *the* bifurcation diagram *of $V^{h,f,K_*}(\mu, p)$ by*

$$\mathrm{Diag}_{V^{h,f,K_*}}(\omega) = \{(\mu, p) \in \mathcal{U}_0 \times \mathrm{D} : \nabla_p V^{h,f,K_*}(\mu, p) = 0\} ;$$

ii) *the* catastrophe set *of $V^{h,f,K_*}(\mu, p)$ by*

$$\mathrm{Cat}_{V^{h,f,K_*}}(\omega) = \{(\mu, p) \in \mathrm{Diag}_{V^{h,f,K_*}}(\omega) : \det \mathrm{Hess}_p V^{h,f,K_*}(\mu, p) = 0\} ,$$

*which is stratified into the sets*

$$\mathrm{Cat}^i_{V^{h,f,K_*}}(\omega) = \{(\mu, p) \in \mathrm{Cat}_{V^{h,f,K_*}}(\omega) : \dim \ker \mathrm{Hess}_p V^{h,f,K_*}(\mu, p) = i\} ;$$

iii) the bifurcation set *of $V^{h,f,K_*}(\mu, p)$ by*

$$\mathrm{Bif}_{V^{h,f,K_*}}(\omega) = \{\mu \in \mathcal{U}_0 : \exists\, p \in \mathrm{D} \textit{ such that } (\mu, p) \in \mathrm{Cat}_{V^{h,f,K_*}}(\omega)\} .$$

*which is stratified into the sets*

$$\mathrm{Bif}^i_{V^{h,f,K_*}}(\omega) = \left\{\mu \in \mathcal{U}_0 : \exists\, p \in \mathrm{D} \textit{ such that } (\mu, p) \in \mathrm{Cat}^i_{V^{h,f,K_*}}(\omega)\right\} .$$

*Then for $(\mu, p)$ sufficiently close to $(\mu_*, p_*)$, we have that $(\mu, p) \in \mathrm{Diag}_{V^{h,f,K_*}}(\omega)$, if and only if $(\mu, K_{(\mu,p)}) \in \mathrm{Diag}_f(\omega)$. Moreover, if $(\mu, p) \in \mathrm{Diag}_{V^{h,f,K_*}}(\omega)$ is such that the matrices $W(\mu, p)$ (given in (4.20)) and $\mathrm{D}_p C(\mu, p)$ (with $C(\mu, p)$ the averaged action of $K_{(\mu,p)}$) are invertible, then the following holds:*

**a)** $(\mu, p) \in \mathrm{Cat}^i_{V^{h,f,K_*}}(\omega)$ *if and only if* $(\mu, K_{(\mu,p)}) \in \mathrm{Cat}^i_f(\omega)$.
**b)** $\mu \in \mathrm{B}^i_{V^{h,f,K_*}}(\omega)$ *if and only if* $\mu \in \mathrm{B}^i_f(\omega)$.

*In particular,* $\mathrm{Diag}_{V^{h,f,K_*}}(\omega)$, $\mathrm{Cat}_{V^{h,f,K_*}}(\omega)$, $\mathrm{Bif}_{V^{h,f,K_*}}(\omega)$, $\mathrm{Cat}^i_{V^{h,f,K_*}}(\omega)$ *and* $\mathrm{Bif}^i_{V^{h,f,K_*}}(\omega)$ *is locally independent of* $h$.

CHAPTER 9

# The close-to-integrable case

Here we apply the results in Chapter 8 to symplectomorphisms that are written as a perturbation of an integrable one and in angle-action coordinates. The construction of the potential in the integrable case is presented in Section 9.1. Although this is straightforward, it provides insight on the role of the parameters. In Section 9.2 we discuss the persistence of invariant tori under classical nondegeneracy conditions, including the Kolmogorov condition and applications to small twist theorems. Section 9.3 contains the study of the non-twist case. Finally, in Section 9.4 we discuss the singularities of the Birkhoff Normal Form (BNF) around an invariant torus. We show that it is possible to define a modifying family in such a way that the singularities of the potential of the torus with respect to such modifying family is close to the BNF.

Throughout this section $\mathbb{A}^n$ is assumed to be endowed with the standard symplectic structure $\boldsymbol{\omega}_0 = \mathrm{d}y \wedge \mathrm{d}x$ and $U \subset \mathbb{R}^n$ is assumed to be open and simply connected.

## 9.1. The integrable case

Let $\omega \in \mathbb{R}^n$ be fixed, and let $f_0 : \mathbb{T}^n \times U \to \mathbb{A}^n$ be the integrable symplectomorphism:

$$f_0(x, y) = \begin{pmatrix} x + \omega + \nabla_y A_0(y) \\ y \end{pmatrix}. \tag{9.1}$$

The function $\hat{\omega} : U \to \mathbb{R}^n$, given by

$$\hat{\omega}(y) = \omega + \nabla_y A_0(y), \tag{9.2}$$

is known as *the frequency map* of $f_0$. Define the Hamiltonian deformation $h : \mathbb{R}^n \times \mathbb{A}^n \to \mathbb{A}^n$

$$h(\lambda, x, y) = \begin{pmatrix} x + \lambda \\ y \end{pmatrix}. \tag{9.3}$$

It is easy to verify that $h$ is a modifying deformation with base $\mathbb{R}^n$ and with moment map $\mathscr{M}^h(x, y) = y$. Define

$$g_\lambda = h_\lambda \circ f_0, \tag{9.4}$$

$$\lambda_0(p) = -\nabla_p A_0(p), \tag{9.5}$$

$$Z(p, \theta) = \begin{pmatrix} \theta \\ p \end{pmatrix}. \tag{9.6}$$

It is clear that the function $\mathbf{K}_0 : U \times \mathbb{T}^n \to \mathbb{R}^n \times (\mathbb{T}^n \times U)$, defined by

$$\mathbf{K}_0(p, \theta) = (\lambda_0(p), Z(p, \theta)),$$

is a $g$-invariant FLD with base sets $U$ and $\mathbb{R}^n$ and frequency $\omega$ (see Definition 4.1). A direct computation shows that the potential of $\mathbf{K}_0$ with respect to $g$ and $\omega$ is given by:

$$V^{g,\mathbf{K}_0}(p) = A_0(p)\,. \tag{9.7}$$

Clearly, $\lambda_0(p) = -\nabla_p V^{g,\mathbf{K}_0}(p)$ . Hence, $Z_{p_*}$ is $f_0$-invariant with frequency $\omega$ if and only if $p_*$ is a critical point of $V^{g,\mathbf{K}_0} = A_0$.

REMARK 9.1. $\bar{\mathrm{T}}(0,p) = \mathrm{D}_p\,\hat{\omega}(p)$ is the torsion of $Z_p$, and satisfies

$$\bar{\mathrm{T}}(0,p) = \mathrm{Hess}_p\, V^{g,\mathbf{K}_0}(p).$$

Hence, $Z_{p_*}$ is a non-twist $f_0$-invariant tori with frequency $\omega$ if and only if $p_*$ is a degenerate critical point of $V^{g,\mathbf{K}_0} = A_0$.

REMARK 9.2. Let $h$ be the modifying deformation given in (9.3) and let $Z_p$ given in (9.6). Then, the moment map of $h$ satisfies:

$$\begin{aligned} \mathrm{D}_z\mathcal{M}^h(Z(p,\theta))\, N_{Z_p}(\theta) &= I_n\,, \\ \mathrm{D}_z\mathcal{M}^h(Z(p,\theta))\, L_{Z_p}(\theta) &= 0_n\,, \end{aligned}$$

where $L_{Z_p}(\theta) = \begin{pmatrix} I_n \\ O_n \end{pmatrix}$ and $N_{Z_p}(\theta) = \begin{pmatrix} O_n \\ I_n \end{pmatrix}$.

## 9.2. Persistence of invariant tori

Let $E \subset \mathbb{R}^{k_1}$ be an open neighborhood of 0. Let $f : E \times \mathbb{T}^n \times U \to \mathbb{A}^n$ be a Hamiltonian deformation with base $E$. For $\varepsilon \in E$, let $f_\varepsilon : \mathbb{T}^n \times U \to \mathbb{A}^n$ be given by $f_\varepsilon(z) = f(\varepsilon, z)$. Assume that for $\varepsilon = 0$ the function $f_\varepsilon$ is given by (9.1) and that $\omega = \hat{\omega}(p_0) \in \mathcal{D}_n(\gamma,\tau)$, for some $p_0 \in U$. Without loss of generality, we assume that $p_0 = 0 \in U$. In what follows, we show that Theorem 8.5 reduces the existence of $f_\varepsilon$-invariant tori, with frequency $\omega$, to the problem of finding critical points of a scalar function.

LEMMA 9.3. *Let $h$ be given by* (9.3)*. Then, for any $\varepsilon \in E$ and any $p \in U$, the pair $(\lambda_0(p), Z_p)$, given in* (9.5)-(9.6)*, is $h$-nondegenerate with respect to $f_\varepsilon$ and $\omega$ (see Definition 7.1).*

PROOF. Let $\varepsilon \in E$ be fixed, from Remark 9.2 one obtains that, in the present case, the matrix in (7.1) takes the following form (see Remark 7.2):

$$\begin{pmatrix} \bar{\mathrm{T}}(0,p) & I_n \\ I_n & O_n \end{pmatrix}, \tag{9.8}$$

where $\bar{\mathrm{T}}(\varepsilon,p) = \langle \mathrm{D}_y\, f^x(\varepsilon, Z(p,\theta))\rangle$ is the torsion of $Z_p$ with respect to $f_\varepsilon$ and $\omega$. This particular form is due to the way the modifying deformation $h$ was defined and to the fact that the symplectic form is the standard one. ☐

REMARK 9.4. Lemma 9.3 holds in the case that $Z_p$ is non-twist and in the case that $\varepsilon$ is not 'sufficiently small'.

The following result reduces the problem of the persistence of $Z_0$ to the problem of finding critical points of the $\varepsilon$-parametric potential of $Z_0$.

THEOREM 9.5. *Let $r \geq 2$ and let $E \subset \mathbb{R}^{k_1}$ be an open neighborhood of 0. Let $f : E\times(\mathbb{T}^n\times U) \to \mathbb{A}^n$ be a Hamiltonian deformation. For $\varepsilon \in E$, let $f_\varepsilon : \mathbb{T}^n\times U \to \mathbb{A}^n$ be given by $f_\varepsilon(z) = f(\varepsilon, z)$. Assume that there are complex strips of $\mathbb{T}^n \times U$ and $\mathbb{A}^n$, $\mathcal{B}_0$ and $\mathcal{B}$ such that $f_\varepsilon \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^r)$, for all $\varepsilon \in E$. Also assume that for $\varepsilon = 0$ the function $f_\varepsilon$ is given by (9.1) and that $\omega = \hat{\omega}(0) \in \mathcal{D}_n(\gamma, \tau)$.*

*Then the torus $Z_0$ is $h$-deformable with respect to $f$ and $\omega$. That is, there exist an open neighborhood of $(0, 0)$, $E_0\times \mathrm{D} \subset E\times U$, and a locally unique parametric FLD with base sets $\mathrm{D}$ and $\lambda$ and parameter $\varepsilon \in E_0$:*

$$\begin{array}{rccl} \mathbf{K}: & E_0\times\mathrm{D}\times\mathbb{T}^n & \longrightarrow & \mathbb{R}^n\times(\mathbb{T}^n\times U) \\ & (\varepsilon, p, \theta) & \longrightarrow & \mathbf{K}(\varepsilon, p, \theta) = (\lambda(\varepsilon, p), K(\varepsilon, p, \theta)), \end{array}$$

*such that:*

$$\begin{aligned} h_{\lambda(\varepsilon,p)} \circ f_\varepsilon \circ K_{(\varepsilon,p)} - K_{(\varepsilon,p)} \circ \mathrm{R}_\omega &= 0, \\ \langle K^x(\varepsilon, p, \theta) - \theta\rangle &= 0, \\ \langle K^y(\varepsilon, p, \theta)\rangle &= p. \end{aligned}$$

*Take*

$$\begin{aligned} L(\varepsilon, p, \theta) &= \mathrm{D}_\theta K(\varepsilon, p, \theta), \\ N(\varepsilon, p, \theta) &= J_0\, \mathrm{D}_\theta K(\varepsilon, p, \theta)(L(\varepsilon, p, \theta)^\top L(\varepsilon, p, \theta))^{-1}, \\ T(\varepsilon, p, \theta) &= N(\varepsilon, p, \theta + \omega)^\top \Omega_0 \mathrm{D}_z(h_{\lambda(\varepsilon,p)} \circ f_\varepsilon)(K(\varepsilon, p, \theta))N(\varepsilon, p, \theta), \\ \bar{\mathrm{T}}(\varepsilon, p) &= \langle T(\varepsilon, p, \theta)\rangle\,, \\ C(\varepsilon, p)^\top &= \int_{\mathbb{T}^n} K^y(\varepsilon, p, \theta)^\top \mathrm{D}_\theta K^x(\varepsilon, p, \theta)\mathrm{d}\theta, \\ W(\varepsilon, p) &= \langle \pi_y\, (N(\varepsilon, p, \theta) - L(\varepsilon, p, \theta)\mathcal{R}_\omega T(\varepsilon, p, \theta))\rangle^\top . \end{aligned}$$

*Then:*

**a)** *$\mathbf{K}(0, p, \theta) = (\lambda_0(p), Z(p, \theta))$, for all $p \in \mathrm{D}$, with $(\lambda_0(p), Z(p, \theta))$ in (9.5)-(9.6), respectively.*

**b)** *Let $S^f$ be the primitive function of $f$, then the $\varepsilon$-parametric potential of $Z_0$, with respect to $h$, $f$ and $\omega$, $V^{h,f,Z_0} : E_0\times\mathrm{D} \to \mathbb{R}$, is given by*

$$V^{h,f,Z_0}(\varepsilon, p) = -p^\top\lambda(\varepsilon, p) - \langle S^f_\varepsilon(K(\varepsilon, p, \theta))\rangle\ . \tag{9.9}$$

**c)** *The following equalities hold:*

$$\begin{aligned} \lambda(\varepsilon, p) &= -\nabla_p V^{h,f,Z_0}(\varepsilon, p), \\ \bar{\mathrm{T}}(\varepsilon, p)\, \mathrm{D}_p C(\varepsilon, p) &= W(\varepsilon, p)\ \mathrm{Hess}_p\, V^{h,f,Z_0}(\varepsilon, p). \end{aligned}$$

*Moreover,*

$$\begin{aligned} V^{h,f,Z_0}(0, p) &= A(p)\,, \\ \mathrm{D}_p C(0, p) &= I_n\,, \\ W(0, p) &= I_n\,. \end{aligned}$$

PROOF. This is a consequence of Theorem 4.11, Theorem 7.4, Lemma 9.3 and the fact that the symplectic form is the standard one.

Indeed, let $h$ be given by (9.3) and let $d : \mathbb{R}^n\times\mathbb{A}^n \to \mathbb{A}^n$ be given by

$$d(\sigma, x, y) = \begin{pmatrix} x \\ y + \sigma \end{pmatrix} . \tag{9.10}$$

It is easy to verify that $d$ is a dummy deformation with base $\mathbb{R}^n$, whose local moment map is $\tilde{\mathcal{M}}^d(x,y) = x$. Let $\Xi \subset E \times \mathbb{R}^n \times \mathbb{R}^n$ be open. Let $\mathbf{f} : \Xi \times (\mathbb{T}^n \times U) \to \mathbb{A}^n$ be defined by

$$\mathbf{f}(\mathbf{t}, z) = (d_\sigma \circ h_\lambda \circ f_\varepsilon)(z). \tag{9.11}$$

Apply Theorem 7.4 to $\mathbf{f}$, with $\mu = \varepsilon$ and initial data $\mu_0 = 0$, $\lambda_0(0) = 0$ and $Z_0$, to obtain $\mathbf{K}$. Then apply Theorem 4.11. □

REMARK 9.6. As a consequence of Theorem 9.5, we have that $f_\varepsilon$-invariant tori with frequency $\omega$ correspond to critical points of the function $V^{h,f,Z_0}(\varepsilon, p)$:

$$\nabla_p V^{h,f,Z_0}(\varepsilon, p) = 0\,. \tag{9.12}$$

REMARK 9.7. Because of Lemma 9.3, for any $p \in U$, $(\lambda(p), Z_p)$ is $h$-nondegenerate with respect to $f$ and $\omega$. Then, under the regularity assumptions on $f$ in Theorem 9.5, it is possible to show that, for each $p_* \in U$, the torus $Z_{p_*}$ is $h$-deformable with respect to $f$ and $\omega$ (see Definition 8.1). More concretely, for each $p_* \in U$, there exist an $\varepsilon$-parametric FLD of $K_{p_*}$ with respect to $h$, $f$ and $\omega$, defined in an an open neighborhood of $(0, p_*)$, $E^{p_*} \times \mathrm{D}^{p_*} \subset E \times U$. Hence, for $\mathrm{D}' \subset U$ open, there is a open set $E' \subset E$ and a locally unique smooth function

$$\begin{array}{rccl} \mathbf{K}' : & E' \times \mathrm{D}' \times \mathbb{T}^n & \longrightarrow & \mathbb{R}^n \times (\mathbb{T}^n \times U) \\ & (\varepsilon, p, \theta) & \longrightarrow & \mathbf{K}'_\varepsilon(p, \theta) = (\lambda'(\varepsilon, p), K'(\varepsilon, p, \theta))\,, \end{array}$$

for any $\varepsilon \in E'$ fixed, $\mathbf{K}'_\varepsilon(p) = \mathbf{K}'(\varepsilon, p)$ is a FLD with base sets $\mathrm{D}'$ and $\mathbb{R}^n$ satisfying:

$$\begin{aligned} h_{\lambda'(\varepsilon,p)} \circ f_\varepsilon \circ K'_{(\varepsilon,p)} - K'_{(\varepsilon,p)} \circ \mathrm{R}_\omega &= 0, \\ \langle \pi_x K'(\varepsilon, p, \theta) - \theta \rangle &= 0, \\ \langle \pi_y K'(\varepsilon, p, \theta) \rangle &= p. \end{aligned}$$

**9.2.1. Persistence under nondegeneracy conditions.** The function $\lambda : E_0 \times \mathrm{D} \to \mathbb{R}^n$ in Theorem 9.5 can be thought of in two ways: *i)* in a parametric version, i.e. the function $\lambda_\varepsilon : \mathrm{D} \to \mathbb{R}^n$, given by $\lambda_\varepsilon(z) = \lambda(\varepsilon, z)$, is considered as a family of functions, indexed by the parameter $\varepsilon \in E$ and *ii)* in a perturbative way, i.e. the function $\lambda_\varepsilon$ is viewed as a perturbation of $\lambda_0(p) = \omega - \hat{\omega}(p)$.

In the perturbative setting, if equation (9.12) has a solution for $\varepsilon = 0$, then, the existence of solutions for $\varepsilon \neq 0$ is guaranteed by sufficient conditions which allow to apply a finite dimensional version of the Implicit Function Theorem. For example, if $\mathrm{D}_p\lambda_0(0)$ has full rank, the standard Implicit Function Theorem implies the existence of a solution $p(\varepsilon)$, which depends differentiability on $\varepsilon$ and $p(0) = 0$. This sufficient condition is equivalent to require $p_0 = 0$ to be a nondegenerate critical point of the potential $V^{h,f,Z_0}(0, p) = A_0(p)$. This is the classical Kolmogorov's condition for the KAM Theorem [**37**].

There are more cases when it is possible to find solutions of (9.12). For example, it is sufficient that the image of a small neighborhood of 0 under the function $\lambda_0(p)$ contains an open set around zero. This is true, in particular if $T_0^\ell \lambda_0(p)$, the Taylor expansion of $\lambda_0(p)$ around $p_0 = 0$, is such that $T_0^\ell(B_\rho(0))$ contains a ball around 0 of radius $C\rho^\ell$, with $B_\rho(0)$ the open ball with center 0 and radius $\rho$. This algebraic condition on the jets of $\lambda_0$ is very close to necessary [**56, 61**]. In terms of the potential $V^{h,f,Z_0}(\varepsilon, p)$, to have solutions of (9.12) it is sufficient, for example, that 0 is an isolated maximum (or minimum) of $V^{h,f,Z_0}(0, p)$. In this case, $V^{h,f,Z_0}(\varepsilon, p)$ will have also critical points (although not necessarily unique). This condition can also be checked from the jet $T_0^\ell V^{h,f,Z_0}(\varepsilon, p)$.

**9.2.2. Perturbation theory and small twist theorems.** To determine solutions of (9.12), in the perturbative setting, we have used only the information on the unperturbed problem, i.e. on the frequency map $\hat{\omega}(y) = \omega + \nabla_y A_0(y)$. However, in several applications it is indispensable to use also the information on the low-order perturbative terms to conclude that (9.12) can be solved. Two important examples of degenerate frequency maps are the Kepler problem in Celestial Mechanics [**39**] and the harmonic oscillators in [**52**].

These are examples of the so called *'small twist theorems'* in which the twist condition vanishes in the unperturbed system but is generated by the perturbation. Motivated by celestial mechanic, small twist theorems were already considered in [**3, 47**].

To obtain solutions of (9.12) we consider $T^{s,\ell}_{(0,0)}\lambda(\varepsilon, p)$, the Taylor expansion of $\lambda(\varepsilon, p)$ in $(\varepsilon, p)$ around $(0, 0)$ to order $s$ in $\varepsilon$ and to order $\ell$ in $p$. A sufficient condition that guarantees the existence of solutions of (9.12), for $\varepsilon$ sufficiently small, is that the image of a small ball around $p_0 = 0$ under $T^{s,\ell}_{(0,0)}\lambda(\varepsilon, p)$ contains a ball of radius $C\eta^s\rho^\ell$ around 0. The jets in $(\varepsilon, p)$ of $\lambda(\varepsilon, p)$ are just the coefficients of the Lindstedt expansions in $(\varepsilon, p)$, of $\lambda$.

As an application, let us consider the perturbation of an *isochronous* integrable system, which has a constant Diophantine frequency vector $\hat{\omega}(p) = \omega$. This is a highly degenerate case, indeed $\hat{\omega}(p) = \omega$ does not satisfy the Rüssmann nondegeneracy condition. The following result provides sufficient conditions for persistence of invariant tori with frequency $\omega$.

THEOREM 9.8. *Let $\omega \in \mathcal{D}_n(\gamma, \tau)$. Let $r \geq 2$ and let $E \subset \mathbb{R}^{k_1}$ be an open neighborhood of 0. Let $f : E \times \mathbb{T}^n \times U \to \mathbb{A}^n$ be a Hamiltonian deformation of the form*

$$f_\varepsilon(x, y) = \begin{pmatrix} x + \omega \\ y \end{pmatrix} + \varepsilon f^1_\varepsilon(x, y) \ . \tag{9.13}$$

*The primitive function of $f$ can be written as $S^f_\varepsilon = \varepsilon S^{f,1}_\varepsilon$. Assume that there are complex strips of $\mathbb{T}^n \times U$ and $\mathbb{A}^n$, $\mathcal{B}_0$ and $\mathcal{B}$, such that $f_\varepsilon \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^r)$, for all $\varepsilon \in E$, with $r \geq 2$. Define*

$$V^{h,f,1}(p) = -p^\top \left\langle \pi_x f^1_0(\theta, p) \right\rangle - \left\langle S^{f,1}_0(\theta, p) \right\rangle \ . \tag{9.14}$$

*Then, if $p_0$ is a nondegenerate critical point of $V^{h,f,1}(p)$ and $\varepsilon$ is sufficiently small, then there is an $f_\varepsilon$-invariant torus with frequency $\omega$.*

PROOF. Let $\lambda(\varepsilon, p)$, $K_{(\varepsilon,p)}$ and $V^{h,f,Z_0}$ be as in Theorem 9.5. Notice that, since $\lambda(0, p) \equiv 0$, we can write $\lambda(\varepsilon, p) = \varepsilon\lambda^1_\varepsilon(p)$. A direct computation yields the first order coefficient of the Lindstedt series in $\varepsilon$ of $\lambda(\varepsilon, p)$:

$$\lambda^1_0(p) = \left\langle \pi_x f^1_0(\theta, p) \right\rangle \ . \tag{9.15}$$

From (9.9), (9.14) and (9.15), it is clear that $V^{h,f,Z_0}(\varepsilon, p)$ can be written as follows

$$V^{h,f,Z_0}(\varepsilon, p) = \varepsilon V^{h,f,1,Z_0}(\varepsilon, p).$$

Moreover $V^{h,f,1,Z_0}(0, p) = V^{h,f,1}(p)$. Hence, for sufficiently small $\varepsilon > 0$, $K_{(\varepsilon,p)}$ is $f_\varepsilon$-invariant if and only if $\lambda^1_\varepsilon(p) = -\nabla_p V^{h,f,1,Z_0}(\varepsilon, p) = 0$. The existence of such solutions is guaranteed by applying a perturbative argument to $V^{h,f,1}(p)$. □

REMARK 9.9. For $f_\varepsilon$ given by (9.13), the tori $K_{(\varepsilon,p)}$ in the proof of Theorem 9.8 can be written as follows:

$$K_{(\varepsilon,p)}(\theta) = Z_p(\theta) + \varepsilon K^1_{(\varepsilon,p)}(\theta)\,.$$

Moreover, the first order coefficient of the Lindstedt series in $\varepsilon$ of $K_{(\varepsilon,p)}$ is

$$K^1_{(0,p)}(\theta) = -\mathcal{R}_\omega f^1_0(\theta,p)\,,$$

where $\mathcal{R}_\omega$ is as in Lemma 2.28.

## 9.3. Unfolding non-twist tori

Let $f_0 : \mathbb{T}^n \times U \to \mathbb{A}^n$ be the integrable system in (9.1). Let $(\lambda_0(p), Z(p,\theta))$ be given by (9.5)-(9.6). A natural classification of $f_0$-invariant tori, with frequency $\omega$, is the classification of the corresponding critical points of $A_0(p)$ as provided by Singularity Theory (see Remark 9.1). In particular, twist (non-twist) $f_0$-invariant tori correspond to non-degenerate (degenerate) critical points of $A_0(p)$. As a consequence of Theorem 8.5, this classification persists under perturbations.

Assume that $\omega = \hat{\omega}(0) \in \mathcal{D}_n(\gamma,\tau)$ and that $Z_0$ is non-twist $f_0$-invariant, with frequency $\omega$. Also assume that $A_0(p)$ has a finitely determined singularity of class $\Upsilon$ at $p_0 = 0$ of co-dimension less than or equal to $k$. Let $A(\mu,p)$ be a versal unfolding of the singularity of $A_0$ at $p_0 = 0$, with unfolding parameter $\mu$ in an open set $\mu \in \mathcal{U} \subset \mathbb{R}^k$. We consider a perturbation of the integrable system with $\mu$-parametric frequency map $\hat{\omega}(\mu, y) = \omega + \nabla_y A(\mu, y)$ as follows. Let $E \subset \mathbb{R}^{k_1}$ be an open neighborhood of 0 and let $g : E\times\mathcal{U}\times(\mathbb{T}^n\times U) \to \mathbb{A}^n$ be a Hamiltonian deformation with base $E\times\mathcal{U}$ such that for all $\mu \in \mathcal{U}$ and $z \in \mathbb{T}^n\times U$ the following holds:

$$g((0,\mu),z) = \begin{pmatrix} x + \omega + \nabla_y A(\mu,y) \\ y \end{pmatrix}\,.$$

From the discussion in Section 9.1, it is clear that if

$$\lambda_0(\mu,p) = -\nabla_p A(\mu,p),$$

then, for any $\mu \in \mathcal{U}$, the function $\mathbf{K}_{(0,\mu)} : U\times\mathbb{T}^n \to \mathbb{R}^n\times(\mathbb{T}^n\times U)$ given by

$$\mathbf{K}_{(0,\mu)}(p,\theta) = (\lambda_0(\mu,p), Z(p,\theta)),$$

is a FLD such that

$$h_{\lambda_0(\mu,p)} \circ g_{(0,\mu)} \circ Z_p = Z_p \circ \mathrm{R}_\omega.$$

Moreover, the corresponding potential is given by $A_\mu(p) = A(\mu,p)$.

Apply Theorem 8.5 to the Hamiltonian deformation $g$ with $h$ and $d$ given in (9.3) and (9.10), respectively. Then, for any $|\varepsilon_0|$ sufficiently small and fixed, there exists $(\mu_{\varepsilon_0}, p_{\varepsilon_0})$, close to $(0,p_0) = (0,0)$, and $K_{p_{\varepsilon_0}}$, a $g_{(\varepsilon_0,\mu_{\varepsilon_0})}$-invariant torus with frequency $\omega$, which is of class $\Upsilon$. Moreover, the potential $V^g_h(\varepsilon,\mu,p)$ satisfies $V^g_h(0,\mu,p) = A(\mu,p)$ and $V^g_h(\varepsilon_0,\mu,p)$ is a versal unfolding of $V^g_h(\varepsilon_0,\mu_{\varepsilon_0},p)$ at $p_{\varepsilon_0}$.

## 9.4. The Birkhoff potential and the potential of an invariant torus

In some papers [**20, 22, 23**], the twist condition has been related to the singularities of the BNF around the invariant torus.

The goal of this section is to show that, if the modifying family $h$ is chosen in a suitable way, then the potential of the BNF around an invariant torus and the potential of the torus, with respect to a suitable modifying deformation, are close.

From the theoretical point of view, given a Hamiltonian system with an invariant torus, the BNF procedure provides local coordinates on which the system is locally close-to-integrable in a neighborhood of the torus, see §9 in [**40**]. Hence the results for integrable systems apply to the integrable part of the BNF. The transformations into BNF require solving repeatedly cohomological equations which leads to the loss of the analyticity domain in the analytic case and to the loss of regularity in the smooth case. We quote without proof an analytic version of the BNF result.

THEOREM 9.10. *Let $\mathcal{A}$ be an annulus, endowed with the symplectic form $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$. Let $\mathcal{A}_0 \subset \mathcal{A}$ be an annulus, and let $\mathcal{B}_0 \subset \mathcal{B}$ be complex strips of $\mathcal{A}_0$ and $\mathcal{A}$, respectively. Let $f \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^0)$ and $\omega \in \mathcal{D}_n(\gamma, \tau)$, for some $\gamma > 0$ and $\tau \geq n$. Let $\rho_0 > 0$ be fixed and such that $\gamma \rho_0^\tau < 1$. Assume that $K_* \in \mathrm{Emb}(\mathbb{T}^n_{\rho_0}, \mathcal{B}_0, C^1)$ is an $f$-invariant torus, with frequency $\omega$. Then, for any $m \in \mathbb{N}$, there exists a tubular neighborhood $\mathcal{A}_m \subset K_*(\mathbb{T}^n)$, and $\varphi^m \in \mathrm{Symp}(\mathcal{B}_m, \mathbb{T}^n_{\mathbb{C}} \times \mathbb{C}^n)$, with $\mathcal{B}_m$ a complex strip of $\mathcal{A}_m$, satisfying the following properties:*

- **a)** $(\varphi^m)^* \boldsymbol{\omega}_0 = \boldsymbol{\omega}$;
- **b)** $\varphi^m(K_*(\theta)) = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$;
- **c)** *There is $r > 0$ and real-analytic functions $\hat{\omega}^m : U_r \to \mathbb{C}^n$ and $R_m : \mathbb{T}^n_{\rho_0/2} \times U_r \to \mathbb{T}^n_{\mathbb{C}} \times \mathbb{C}^n$, with $U_r = \{I \in \mathbb{C}^n : |I| < r\}$, such that*

$$\varphi^m \circ f \circ (\varphi^m)^{-1}(\theta, I) = \begin{pmatrix} \theta + \hat{\omega}^m(I) \\ I \end{pmatrix} + R_m(\theta, I),$$

*with*

$$\sup_{\theta \in \mathbb{T}^n_{\rho_0/2}, I \in U_r} \left| \mathrm{D}^i_I R_m(\theta, I) \right| \leq \kappa_{mi} \left| I \right|^{m+1-i}. \qquad i = 0, \ldots, m$$

*Furthermore, the following hold:*

- **d)** $\hat{\omega}^m(0) = \omega$.
- **e)** *There is a unique real-analytic function $A^m : U_r \to \mathbb{C}$, such that $A^m(0) = 0$ and*

$$\hat{\omega}^m(I) = \omega + \nabla_I A^m(I).$$

- **f)** *$\hat{\omega}^m$ depends only on $\mathrm{D}^j_z f_{|K_*(\mathbb{T}^n)}$, $j \leq m$.*

REMARK 9.11. The function $A^m$ in Part b) of Theorem 9.10 is called *the Birkhoff potential of order* $m$ for $K_*$ with respect to $f$ and $\omega$. Notice that $\hat{\omega}^m(0) = \omega$ implies $|A^m(I)| = O(|I|^2)$.

LEMMA 9.12. *$\mathcal{A}_0 \subset \mathcal{A}$, $\mathcal{B}_0 \subset \mathcal{B}$, $\omega \in \mathcal{D}_n(\gamma, \tau)$ and $\rho_0 > 0$ be as in Theorem 9.10. Assume that $f \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^0)$ and that $K_* \in \mathrm{Emb}(\mathbb{T}^n_{\rho_0}, \mathcal{B}_0, C^1)$ is $f$-invariant with frequency $\omega$. Let $1 \leq m \in \mathbb{N}$ be fixed and let $\mathcal{A}_m \subset \mathcal{A}$ and $\varphi^m \in \mathrm{Symp}(\mathcal{A}_m, \mathbb{T}^n_{\mathbb{C}} \times \mathbb{C}^n)$ be as in Theorem 9.10. Let $L_{K_*}$, $N_{K_*}$ and $T_{(f,K_*)}$ be given by* (3.2), (3.3) *and* (3.7)*, respectively. Then,* $\det \pi_y \left\langle \mathrm{D}_z \varphi^m(K_*(\theta)) N_{K_*} \right\rangle \neq 0$.

PROOF. The first BNF step is just the computation of Weinstein coordinates (see §9 in [**40**]). Then, $\varphi^1$ satisfies

$$\mathrm{D}_z \varphi^1(K_*(\theta)) N_{K_*}(\theta) = \begin{pmatrix} O_n \\ I_n \end{pmatrix},$$

Further steps of BNF will produce $\varphi^m$ close to $\varphi^1$. □

The following result shows that there is a modifying deformation $h$ such that the potential of the torus $K_*$ with respect to $h$, $f$ and $\omega$ is close to the potential of the BNF around $K_*$.

THEOREM 9.13. *Let $\mathcal{A}$ be an annulus, endowed with the compatible triple ($\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}, \boldsymbol{J}, \boldsymbol{g}$). Let $\mathcal{A}_0 \subset \mathcal{A}$ be an annulus, and let $\mathcal{B}_0 \subset \mathcal{B}$ be complex strips of $\mathcal{A}_0$ and $\mathcal{A}$, respectively. Let $f \in \mathrm{Symp}(\mathcal{B}_0, \mathcal{B}, C^r)$, with $r \geq 2$. Let $\omega \in \mathcal{D}_n(\gamma, \tau)$ and $\rho_0 > 0$ be as in Theorem 9.10. Assume that $K_* \in \mathrm{Emb}(\mathbb{T}^n_{\rho_0}, \mathcal{B}_0, C^1)$ is $f$-invariant with frequency $\omega$. Let $1 \leq m \in \mathbb{N}$ be fixed and let $\mathcal{A}_m$, $\varphi^m \in \mathrm{Symp}(\mathcal{A}_m, \mathbb{T}^n_{\mathbb{C}} \times \mathbb{C}^n)$ and $A^m$ be as in Theorem 9.10. Then, there are $\mathrm{D}, \Lambda \subset \mathbb{R}^n$ open neighborhoods of zero, an annulus $\mathcal{A}_1 \subset \mathcal{A}$ and $h : \Lambda \times \mathcal{A}_1 \to \mathcal{A}$, a modifying deformation with base $\Lambda$, such that $K_*$ is $h$-deformable with respect $f$ and $\omega$. Moreover, the potential of $K_*$ with respect to $h$, $f$ and $\omega$, $V^{h,f,K_*} : \mathrm{D} \to \mathbb{R}$, satisfies the following estimate:*

$$|V^{h,f,K_*}(p) - A^m(p))| \leq \kappa \, |p|^{m+1} \,, \tag{9.16}$$

*for some constant $\kappa > 0$.*

PROOF. Let $\hat{h} : \mathbb{R}^n \times \mathbb{A}^n \to \mathbb{A}^n$ be given by

$$\hat{h}(\lambda, \theta, I) = \begin{pmatrix} \theta + \lambda \\ I \end{pmatrix} .$$

The moment map of $\hat{h}$ is $\mathcal{M}^{\hat{h}}(\theta, I) = I$. Let $0 \in \Lambda \subset \mathbb{R}^n$ be open, let $K_*(\mathbb{T}^n) \subset \mathcal{A}_1 \subset \mathcal{A}_m$ be an annulus such that the composition

$$h(\lambda, z) = (\varphi^m)^{-1}(\hat{h}(\lambda, \varphi^m(z))$$

is defined for $(\lambda, z) \in \Lambda \times \mathcal{A}_1$. Let $0 \in \Sigma \subset \mathbb{R}^n$ be open and let $K_*(\mathbb{T}^n) \subset \mathcal{A}_0'$ and $\mathcal{A}_2 \subset \mathcal{A}$ be annuli such that $f(\mathcal{A}_0') \subset \mathcal{A}_1$ and $h_\lambda(\mathcal{A}_1) \subset \mathcal{A}_2$ for all $\lambda \in \Lambda$. Let $d : \Sigma \times \mathcal{A}_2 \to \mathcal{A}$ be a dummy deformation. Define $\mathbf{f} : \Sigma \times \Lambda \times \mathcal{A}_0' \to \mathcal{A}$ by

$$\mathbf{f}(\sigma, \lambda, z) = d_\sigma(h_\lambda(f(z))).$$

Let $\mathcal{M}^h$ be the moment map of $h$. Assume that $\mathcal{B}_0'$ and $\mathcal{B}_1$ are sufficiently small complex strips of $\mathcal{A}_0'$ and $\mathcal{A}_1$, respectively, in such a way that the following holds, for $i = 0, \ldots, r$:

$$\max \left( \sup_{(\mathbf{t}, z) \in \Xi \times \mathcal{B}_0'} \left| \mathrm{D}^i_{(\mathbf{t}, z)} \mathbf{f}(\mathbf{t}, z) \right| , \sup_{z \in \mathcal{B}_1} \left| \mathrm{D}^i_z \mathcal{M}^h(z) \right| \right) < \infty \,.$$

Next, from Lemma 2.21, the moment map of $h$ is given by $\mathcal{M}^h = \mathcal{M}^{\hat{h}} \circ \varphi^m$. This implies

$$\begin{aligned} \langle \mathcal{M}^h(K_*(\theta)) \rangle &= 0, \\ \langle \mathrm{D}_z \mathcal{M}^h(K_*(\theta)) \mathrm{D}_\theta K_*(\theta) \rangle &= O_n, \\ \left\langle \mathrm{D}_z \mathcal{M}^h(K_*(\theta)) N_{K_*}(\theta) \right\rangle &= \pi_y \left\langle \mathrm{D}_z \varphi^m(K_*(\theta)) N_{K_*}(\theta) \right\rangle . \end{aligned}$$

where $N_{K_*}$ is given by (3.3). Then, the matrix in (7.1) takes the following form:

$$\begin{pmatrix} \left\langle T_{(f,K_*)}(\theta) \right\rangle & \pi_y \left\langle \mathrm{D}_z \varphi^m(K_*(\theta)) N_{K_*}(\theta) \right\rangle^\top \\ \pi_y \left\langle \mathrm{D}_z \varphi^m(K_*(\theta)) N_{K_*}(\theta) \right\rangle & O_{n \times n} \end{pmatrix} .$$

From which and Lemma 9.12, we have that $(0, K_*)$ is $h$-nondegenerate with respect to $f$ and $\omega$ (see Definition 7.1). Hence, Theorem 7.4, applied to $\mathbf{f}$, $\omega$, $K_*$ and $h$, implies the existence of an open neighborhood of zero, $\mathrm{D} \subset \mathbb{R}^n$, and a FLD with base sets D and $\Lambda$ (possibly smaller), $\mathbf{K} = (\lambda, K) : \mathrm{D} \times \mathbb{T}^n \to \Lambda \times \mathcal{A}_0$, such that $\mathbf{K}(0,\theta) = (0, K_*(\theta))$ and

$$\begin{aligned} h_{\lambda(p)} \circ f \circ K_p &= K_p \circ \mathrm{R}_\omega \,, \\ \left\langle K_p^x(\theta) - \theta \right\rangle &= 0 \,, \\ \langle \mathcal{M}^h(K(p,\theta)) \rangle &= p \,. \end{aligned}$$

A direct computation shows that, in the present case, the potential of $K_*$ with respect to $h$, $f$ and $\omega$ is

$$\begin{aligned} V^{h,f,K_*}(p) = &- p^\top \lambda(p) - \langle S^f \circ K_p \rangle - \left\langle S^{\varphi^m} \circ (\varphi^m)^{-1} \circ K_p \right\rangle \\ &+ \left\langle S^{\varphi^m} \circ (\varphi^m)^{-1} \circ h_{-\lambda(p)} \circ K_p \right\rangle \,, \end{aligned} \tag{9.17}$$

Define

$$\begin{aligned} g &= \varphi^m \circ f \circ (\varphi^m)^{-1}, \\ \hat{K}_p &= \varphi^m \circ K_p. \end{aligned}$$

Then, the following holds:

$$\hat{h}_{\lambda(p)} \circ g \circ \hat{K}_p = \hat{K}_p \circ \mathrm{R}_\omega \,, \tag{9.18}$$

$$\left\langle \mathcal{M}^{\hat{h}}(\hat{K}_p(\theta)) \right\rangle = p \,. \tag{9.19}$$

From Proposition 8.4 we have that the potential of $\hat{K}_0(\theta) = Z_0(\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$ with respect to $\hat{h}$, $g$ and $\omega$ satisfies

$$V^{\hat{h},g,\hat{K}_0}(p) = V^{h,f,K_*}(p), \tag{9.20}$$

Next, define

$$\hat{f}_m(\theta, I) = \begin{pmatrix} \theta + \omega + \nabla_I A^m(I) \\ I \end{pmatrix} .$$

Then, $Z_p(\theta) = \begin{pmatrix} \theta \\ p \end{pmatrix}$ and $\lambda^m(p) = -\nabla_p A^m(p)$ satisfy:

$$\hat{h}_{\lambda^m(p)} \circ \hat{f}_m \circ Z_p = Z_p \circ \mathrm{R}_\omega \,, \tag{9.21}$$

$$\left\langle Z_p^x(\theta) - \theta \right\rangle = 0 \,, \tag{9.22}$$

$$\left\langle \mathcal{M}^{\hat{h}}(Z_p(\theta)) \right\rangle = p \,. \tag{9.23}$$

Then, from equalities (9.18) and (9.21), Theorem 7.4 and Theorem 9.10 we have, for $|p|$ sufficiently small:

$$\begin{aligned} \left\| \hat{K}_p - Z_p \right\|_{\rho_0/2} &= O(|p|^{m+1}) \,, \\ |\lambda(p) - \lambda^m(p)| &= O(|p|^{m+1}) \,. \end{aligned}$$

This implies

$$\left| V^{\hat{h},g,\hat{K}_0}(p) - A^m(p) \right| \le \kappa \, |p|^{m+1} \,. \tag{9.24}$$

Estimate (9.16) follows from equality (9.20) and estimate (9.24). □

APPENDIX A

# Hamiltonian vector fields

The results in this paper so far deal with the case of symplectomorphisms. A similar theory can be developed for Hamiltonian vector fields. In this appendix, we state and prove the main geometric and analytic properties that are required for this development.

Throughout this appendix, we use the definitions introduced in Section 2.2.4. We also assume that $\mathcal{A}_0 \subset \mathcal{A}$ and $A_0' \subset \mathcal{A}'$ are annuli in $\mathbb{A}^n$, endowed with the compatible triples $(\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}, \boldsymbol{J}, \boldsymbol{g})$ and $(\boldsymbol{\omega'} = \mathrm{d}\boldsymbol{\alpha'}, \boldsymbol{J'}, \boldsymbol{g'})$, respectively. Let $(\Omega = \mathrm{D}a^\top - \mathrm{D}a, J, G)$ and $(\Omega' = \mathrm{D}a'^\top - \mathrm{D}a', J', G')$ be the corresponding representation, corresponding to the triples.

### A.1. One-bite small divisor equations

Let $\omega \in \mathbb{R}^n$, define the linear differential operator

$$\mathcal{L}'_\omega = -\sum_{i=1}^{n} \omega_i \frac{\partial}{\partial \theta_i} \,. \tag{A.1}$$

The analytic core of KAM theory for Hamiltonian vector fields is the following linear equation:

$$\mathcal{L}'_\omega \, u = v \, - \langle v \rangle \,, \tag{A.2}$$

where $v$ is given and the unknown is $u$. A sufficient condition on $\omega$ that guarantees the solvability of (A.2) is the following Diophantine condition:

$$\left| k^\top \omega \right| \geq \gamma \; |k|_1^{-\tau} \,, \qquad \forall \; k \in \mathbb{N} \setminus \{0\}, \tag{A.3}$$

where $\gamma > 0$ and $\tau \geq n-1$, are constants. Denote by $\mathcal{D'}_n(\gamma,\tau)$ the set of $\omega$ that satisfy (A.3) for some $\gamma > 0$ and $\tau \geq n-1$.

Lemma A.1. *Let $\omega \in \mathcal{D'}_n(\gamma,\tau)$, for some $\gamma > 0$ and $\tau \geq n-1$. Let $\ell \in \mathbb{R}$ be not an integer be such that $\ell - \tau > 0$ is not an integer. Then, for any $C^\ell$-function $v : \mathbb{T}^n \to \mathbb{R}$, there exists a unique $C^{\ell-\tau}$-function, $u : \mathbb{T}^n \to \mathbb{R}$ with zero-average satisfying equation* (A.2)*. We denote such a solution by $u = \mathcal{R}'_\omega v$.*

Lemma A.2 (Rüssmann estimates). *There exists a positive constant $c_R$, depending only on $n$ and $\tau$, such that for any $v \in \mathrm{A}(\mathbb{T}^n_\rho, C^0)$, with $\rho > 0$, there exists a unique zero-average solution $u$, denoted by $u = \mathcal{R}'_\omega v$, of equation* (A.2)*. Moreover, $u \in \mathrm{A}(\mathbb{T}^n_{\rho-\delta}, C^0)$ for any $0 < \delta < \rho$, and*

$$\|u\|_{\rho-\delta} \leq c_R \; \gamma^{-1} \delta^{-\tau} \, \|v\|_\rho \,.$$

From Lemma A.1, it is clear that:

$$\mathcal{R}'_\omega \, \mathcal{L}'_\omega \, u \quad = \quad u - \langle u \rangle \,, \tag{A.4}$$

$$\mathcal{L}'_\omega \, \mathcal{R}'_\omega \, v \quad = \quad v - \langle v \rangle \;. \tag{A.5}$$

Moreover, the following equality holds:

$$\langle (\mathcal{R}'_\omega u) v + u\, (\mathcal{R}'_\omega v) \rangle = 0\,. \tag{A.6}$$

The above definitions for $\mathcal{L}'_\omega$ and $\mathcal{R}'_\omega$ extend component-wise to vector and matrix-valued functions. These extensions also satisfy Lemma A.2 and equalities (A.4), (A.5) and (A.6).

## A.2. Automatic reducibility of invariant tori

The following result is the vector field version of Lemma 3.1.

LEMMA A.3. *Let $\omega \in \mathbb{R}^n$ be rationally independent. Let $X_{\tilde f}(z) = \Omega(z)^{-1} \nabla_z \tilde f(z)$ be a local Hamiltonian vector field with local Hamiltonian function $\tilde f : \tilde{\mathcal{A}}_0 \to \mathbb{R}$. Assume that $K \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A}_0)$ is invariant for the Hamiltonian flow with internal dynamics given by the vector field $\dot\theta = \omega$:*

$$X_{\tilde f} \circ K + \mathcal{L}'_\omega K = 0\,, \tag{A.7}$$

*Then:*

- **a)** *$\tilde f$ is a (global) Hamiltonian: $\tilde f = f$ is 1-periodic in $x$.*
- **b)** *$K(\mathbb{T}^n)$ is Lagrangian.*
- **c)** *Let $L_K, N_K : \mathbb{T}^n \to \mathbb{R}^{2n\times n}$ be given by*

$$\begin{aligned} L_K(\theta) &= \mathrm{D}_\theta K(\theta), \\ N_K(\theta) &= J(K(\theta))\, \mathrm{D}_\theta K(\theta)\; G_K(\theta)^{-1}, \end{aligned}$$

*where $G_K$ is given in (2.11), and let $M_K : \mathbb{T}^n \to \mathbb{R}^{2n\times 2n}$ be given by*

$$M_K(\theta) = \begin{pmatrix} L_K(\theta) & N_K(\theta) \end{pmatrix}. \tag{A.8}$$

*Then, the vector bundle morphism induced by $M_K$:*

$$\begin{aligned} \boldsymbol{M}_K : \quad \mathbb{T}^n \times \mathbb{R}^{2n} &\longrightarrow \mathrm{T}_{\mathcal{K}}\mathcal{A}_0 \\ (\theta, \xi) &\longrightarrow (K(\theta), M_K(\theta)\xi) \end{aligned}$$

*is an isomorphism such that $\boldsymbol{M}_K^* \boldsymbol{\omega} = \boldsymbol{\omega}_0$. In particular,*

$$M_K(\theta)^{-1} = -\Omega_0\; M_K(\theta)^\top\; \Omega(K(\theta)). \tag{A.9}$$

- **d)** *Transformation by $\boldsymbol{M}_K$ reduces the linearized dynamics to a block-triangular matrix:*

$$M_K(\theta)^{-1} \Big( \mathrm{D}_z X_{\tilde f}(K(\theta)) M_K(\theta) + \mathcal{L}'_\omega M(\theta) \Big) = \begin{pmatrix} O_n & T_{(\tilde f, K)}(\theta) \\ O_n & O_n \end{pmatrix} \tag{A.10}$$

*where*

$$T_{(\tilde f, K)}(\theta) = N_K(\theta)^\top \Omega(K(\theta)) \Big( \mathrm{D}_z X_{\tilde f}(K(\theta)) N_K(\theta) + \mathcal{L}'_\omega N_K(\theta) \Big)\,, \tag{A.11}$$

*with $N_K$ defined in (3.3).*

PROOF. Denote by $\Omega_{ij}(z)$ the $(ij)$-component of $\Omega(z)$. Given a vector $v \in \mathbb{R}^d$, $v_i$ represent the $i$-component of $v$. First, we establish some equalities used in the proof. Equality (A.7) is equivalent to

$$-\,\Omega(K(\theta))\, \mathcal{L}'_\omega K(\theta) = \nabla_z \tilde f(K(\theta))\,. \tag{A.12}$$

Using the fact that $\boldsymbol{\omega}$ is a closed form, one easily shows

$$\frac{\partial \Omega_{rs}}{\partial z_t} + \frac{\partial \Omega_{st}}{\partial z_r} + \frac{\partial \Omega_{tr}}{\partial z_s} = 0\,. \tag{A.13}$$

Using equation (A.12) one shows:

$$\left(\frac{\partial K}{\partial \theta_i}(\theta)\right)^\top \frac{\partial}{\partial \theta_j}\left(\nabla_z \tilde{f}(K(\theta))\right) =$$

$$(A.14)\qquad = -\sum_{r=1}^{2n}\sum_{t,s=1}^{2n} \frac{\partial \Omega_{st}}{\partial z_r}(K(\theta))\, \frac{\partial K_r}{\partial \theta_j}(\theta)\, \frac{\partial K_s}{\partial \theta_i}(\theta)\, \mathcal{L}'_\omega K_t(\theta) - \sum_{t,s=1}^{2n} \Omega_{st}(K(\theta))\, \frac{\partial K_s}{\partial \theta_i}(\theta)\; \mathcal{L}'_\omega \left(\frac{\partial K_t}{\partial \theta_j}(\theta)\right).$$

Moreover, it is clear that

$$(A.15)\qquad \left(\frac{\partial K}{\partial \theta_i}(\theta)\right)^\top \frac{\partial}{\partial \theta_j}\left(\nabla_z \tilde{f}(K(\theta))\right) = \left(\frac{\partial K}{\partial \theta_j}(\theta)\right)^\top \frac{\partial}{\partial \theta_i}\left(\nabla_z \tilde{f}(K(\theta))\right).$$

Next, from $\Omega(z)\mathcal{X}_{\tilde{f}}(z) = \nabla_z \tilde{f}(z)$ we obtain:

$$(A.16)\qquad \frac{\partial^2 \tilde{f}}{\partial z_i \partial z_j}(z) = \sum_{s=1}^{2n}\left[\frac{\partial \Omega_{js}}{\partial z_i}(z)\mathcal{X}_{\tilde{f}_s}(z) + \Omega_{js}(z)\,\frac{\partial \mathcal{X}_{\tilde{f}_s}}{\partial z_i}(z)\right].$$

First we prove Part rm b). Using equalities (A.13) ,(A.14), (A.15) and anti-symmetry of $\Omega$, one shows that the components of the matrix $\Omega_K(\theta) = \mathrm{D}_\theta K(\theta)^\top \Omega(K(\theta))\mathrm{D}_\theta K(\theta)$ satisfy

$$\begin{aligned}
(-\mathcal{L}'_\omega\, \Omega_K(\theta))_{ij} &= -\mathcal{L}'_\omega \left(\left(\frac{\partial K}{\partial \theta_i}(\theta)\right)^\top \Omega(K(\theta))\left(\frac{\partial K}{\partial \theta_j}(\theta)\right)\right) \\
&= \sum_{m=1}^{n}\sum_{r,s,t=1}^{2n}\left[\frac{\partial \Omega_{st}}{\partial z_r}(K(\theta))\, \frac{\partial K_r}{\partial \theta_m}(\theta)\, \frac{\partial K_s}{\partial \theta_i}(\theta)\, \frac{\partial K_t}{\partial \theta_j}(\theta)\right. \\
&\qquad - \frac{\partial \Omega_{st}}{\partial z_r}(K(\theta))\, \frac{\partial K_r}{\partial \theta_i}(\theta)\, \frac{\partial K_t}{\partial \theta_j}(\theta)\, \frac{\partial K_s}{\partial \theta_m}(\theta) \\
&\qquad \left. - \frac{\partial \Omega_{st}}{\partial z_r}(K(\theta))\frac{\partial K_r}{\partial \theta_j}(\theta)\, \frac{\partial K_s}{\partial \theta_i}(\theta)\, \frac{\partial K_t}{\partial \theta_m}(\theta)\right]\omega_m \\
&= 0.
\end{aligned}$$

This implies $\mathcal{L}'_\omega \Omega_K(\theta) = 0$. Moreover, since the linear flow of $\dot{\theta} = \omega$ is ergodic, we have $\Omega_K(\theta) = \langle \Omega_K(\theta)\rangle$. In the proof of Lemma 3.1 we showed $\langle \Omega_K(\theta)\rangle = 0$. This proves Partb).

We now prove Part a). Let $\mathrm{C}^{\tilde{f}} \in \mathbb{R}^n$ be the infinitesimal Calabi invariant of $\tilde{f}$ and let $f : \mathcal{A}_0 \to \mathbb{R}$ be 1-periodic in $x$ and such that (see Definition 2.14)

$$\tilde{f}(z) = -x^\top \mathrm{C}^{\tilde{f}} + f(z)\,.$$

From Part b) of Lemma A.3 and (A.12) we have:

$$(A.17)\qquad \mathrm{D}_\theta(\tilde{f} \circ K) = 0\,.$$

Notice that from (A.17) we have that the torus $K(\mathbb{T}^n)$ lies in an energy level. Part a) Lemma A.3 follows from (A.17) and the following equality:

$$\mathrm{C}^{\tilde{f}} = -\left\langle \mathrm{D}_\theta(\tilde{f} \circ K)\right\rangle^\top.$$

Part c) is a consequence of Part b) (see Lemma 3.1).

We now prove Part d). From (3.2) and (3.3) $L_K(\theta) = \mathrm{D}_\theta K(\theta)$ and $N_K(\theta) = J(K(\theta))L_K(\theta)G_K(\theta)^{-1}$. Taking derivatives with respect to $\theta$ in (A.7) we have

$$\mathrm{D}_z \mathcal{X}_{\tilde f}(K(\theta))L_K(\theta) = -\mathcal{L}'_\omega L_K(\theta)\,. \tag{A.18}$$

Then, performing some computations and using (A.9) and (A.18) we obtain

$$M(\theta)^{-1}\left(\mathrm{D}_z\mathcal{X}_{\tilde f}(K(\theta))M(\theta) + \mathcal{L}'_\omega M(\theta)\right) = \begin{pmatrix} O_n & T_{(\tilde f,K)}(\theta) \\ O_n & P_{(\tilde f,K)}(\theta)\end{pmatrix},$$

where

$$P_{(\tilde f,K)}(\theta) = -L_K(\theta)^\top \Omega(K(\theta))\left(\mathrm{D}_z\mathcal{X}_{\tilde f}(K(\theta))N_K(\theta) + \mathcal{L}'_\omega N_K(\theta)\right).$$

To prove that $P_K = 0$ we use the following equalities:

$$L_K(\theta)^\top \Omega(K(\theta))\, J(K(\theta))L_K(\theta) = -G_K(\theta), \tag{A.19}$$

$$G_K(\theta)\,\mathcal{L}'_\omega\left(G_K(\theta)^{-1}\right) = -\left(\mathcal{L}'_\omega\, G_K(\theta)\right)\, G_K(\theta)^{-1}\,, \tag{A.20}$$

where $G_K$ is defined in (2.11). Using equalities (A.19) and (A.20) one shows

$$\begin{aligned} -L_K(\theta)^\top\Omega(K(\theta))\mathcal{L}'_\omega N_K(\theta)G_K(\theta) = {} & (\mathcal{L}'_\omega L_K(\theta))^\top \Omega(K(\theta))J(K(\theta))L_K(\theta) \\ & + L_K(\theta)^\top(\mathcal{L}'_\omega\Omega(K(\theta)))J(K(\theta))L_K(\theta)\end{aligned}$$

from which we obtain:

$$\begin{aligned} P_K(\theta) = -\, L_K(\theta)^\top \Big[\, & \Omega(K(\theta))\mathrm{D}_z\mathcal{X}_{\tilde f}(K(\theta)) \\ & + \mathrm{D}_z\mathcal{X}_{\tilde f}(K(\theta))^\top\Omega(K(\theta)) \\ & - \mathcal{L}'_\omega\,(\Omega\circ K)(\theta)\,\Big]\; N_K(\theta)\,. \end{aligned}$$

Performing some computations and using equalities (A.13) and (A.16) one shows that the following equality holds for any $i,j \in \{1,\ldots,2n\}$:

$$\begin{aligned} & \Big(\Omega(K(\theta))\mathrm{D}_z\mathcal{X}_{\tilde f}(K(\theta)) + \mathrm{D}_z\mathcal{X}_{\tilde f}(K(\theta))^\top\Omega(K(\theta)) - \mathcal{L}'_\omega(\Omega\circ K)(\theta)\Big)_{ij} \\ & = \sum_{s=1}^{2n}\left[-\frac{\partial\Omega_{is}}{\partial z_j}(K(\theta)) + \frac{\partial\Omega_{js}}{\partial z_i}(K(\theta)) + \frac{\partial\Omega_{ij}}{\partial z_s}(K(\theta))\right]\, \mathcal{X}_{\tilde f\, s}(K(\theta)) \\ & = 0\,. \end{aligned}$$

This proves Part d) of Lemma A.3. □

Definition A.4. Let $\omega \in \mathbb{R}^n$ and let Let $\mathcal{X}_{\tilde f}(z) = \Omega(z)^{-1}\nabla_z \tilde f(z)$ be a local Hamiltonian vector field with local Hamiltonian function $\tilde f : \tilde{\mathcal{A}}_0 \to \mathbb{R}$. The *torsion* of $K \in \mathrm{Emb}(\mathbb{T}^{\mathrm{n}}, \mathcal{A}_0)$, with respect to $\mathcal{X}_{\tilde f}$ and $\omega$, is defined by:

$$\bar{\mathrm{T}}_{(\tilde f,K)} = \left\langle T_{(\tilde f,K)}\right\rangle,$$

where $T_{(\tilde f,K)}$ is given by (A.11). We say that $K$ is *twist* with respect to $f$ and $\omega$ if $\bar{\mathrm{T}}_{(\tilde f,K)}$ is nondegenerate, otherwise we say that $K$ is *non-twist* with respect to $f$ and $\omega$.

Following the proof of Lemma A.3, it can be proved that the geometrical properties of $\mathcal{X}_{\tilde{f}}$-invariant tori, stated in Lemma A.3, are slightly modified when the torus is only approximately $\mathcal{X}_{\tilde{f}}$-invariant. The errors are controlled by the norm of the error $\mathcal{X}_{\tilde{f}} \circ K + \mathcal{L}'_{\omega} K = 0$.

## A.3. Families of Hamiltonians and moment map

Let $\tilde{\mathrm{H}}\mathrm{am}(\mathcal{A}_0)$ denote the set of smooth local Hamiltonian functions on $\tilde{\mathcal{A}}_0$. The vector filed version of symplectic deformations is the following.

DEFINITION A.5. Let $\Xi \subset \mathbb{R}^m$ be open. A *family of local Hamiltonians* with base $\Xi$ is a smooth function $\tilde{g} : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}$ inducing a family of local Hamiltonian functions:

$$\begin{array}{ccc} \Xi & \longrightarrow & \tilde{\mathrm{H}}\mathrm{am}(\tilde{\mathcal{A}}_0) \\ \mathbf{t} & \longrightarrow & \tilde{g}_{\mathbf{t}}(z) = \tilde{g}(\mathbf{t}, z)\,, \end{array} \tag{A.21}$$

The *infinitesimal Calabi invariant* of $\tilde{g}$ is a smooth function $\mathrm{C}^{\tilde{g}} : \Xi \to \mathbb{R}^n$ satisfying

$$\tilde{g}(\mathbf{t}, z) = -\, x^{\top} \mathrm{C}^{\tilde{g}}(\mathbf{t}) + g(\mathbf{t}, z)\,, \tag{A.22}$$

where $g$ is 1-periodic in $x$. If $\mathrm{C}^{\tilde{g}} \equiv 0$, then $\tilde{g} = g$ is called *family of (global) Hamiltonians*.

We now introduce the local moment map of a family of local Hamiltonians.

DEFINITION A.6. Given a family of local Hamiltonians $\tilde{g} : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}$, the corresponding *family of symplectic vector fields* is the function $\mathcal{X}_{\tilde{g}} : \Xi \times \tilde{\mathcal{A}}_0 \to \tilde{\mathcal{A}}_0 \times \mathbb{R}^{2n}$, such that, for any $\mathbf{t} \in \Xi$, the function $\mathcal{X}_{\tilde{g}_{\mathbf{t}}} : \tilde{\mathcal{A}}_0 \to \tilde{\mathcal{A}}_0 \times \mathbb{R}^{2n}$, defined by $\mathcal{X}_{\tilde{g}_{\mathbf{t}}}(z) = \mathcal{X}_{\tilde{g}}(\mathbf{t}, z)$, is the vector field corresponding to the local Hamiltonian $\tilde{g}_{\mathbf{t}}$:

$$\mathrm{i}_{\mathcal{X}_{\tilde{g}_{\mathbf{t}}}} \boldsymbol{\omega} = -\, \mathrm{d}\tilde{g}_{\mathbf{t}}, \tag{A.23}$$

where i denotes the interior product. Notice that for any $\mathbf{t} \in \Xi$, $\mathcal{X}_{\tilde{g}_{\mathbf{t}}}$ is 1-periodic in $x$.

The *infinitesimal primitive function*, corresponding to $\tilde{g}$ is the function $S^{\tilde{g}}$ : $\Xi \times \tilde{\mathcal{A}} \to \mathbb{R}$ defined by

$$S^{\tilde{g}}(\mathbf{t}, z) = a(z)^{\top} \mathcal{X}_{\tilde{g}}(\mathbf{t}, z) - g(\mathbf{t}, z).$$

If $\tilde{g} = g$ is a *family of Hamiltonians*, $\mathcal{X}_{\tilde{g}} = \mathcal{X}_g$ is called a *family of Hamiltonian vector fields*.

Given a family of local Hamiltonians with base $\Xi$, $\tilde{g} : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}$, the corresponding family of symplectic vector fields induces the following function

$$\begin{array}{ccc} \Xi & \longrightarrow & \mathbf{X}(\mathcal{A}_{\mathbf{0}}) \\ \mathbf{t} & \longrightarrow & \mathcal{X}_{\tilde{g}_{\mathbf{t}}}(z) = \mathcal{X}_{\tilde{g}}(\mathbf{t}, z)\,, \end{array} \tag{A.24}$$

where $\mathbf{X}(\mathcal{A}_{\mathbf{0}})$ denotes the set of symplectic vector fields on $\mathcal{A}_0$. Since there will be not risk of confusion, the function in (A.24) will also be denoted by $\mathcal{X}_{\tilde{g}}$.

DEFINITION A.7. Let $\tilde{g} : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}$ be a smooth family of local Hamiltonians with base $\Xi$ and let $\mathcal{X}_{\tilde{g}}$ be its corresponding family of symplectic vector fields.

i) The *generator* of $\tilde{g}$ is the function $\mathcal{G}^{\tilde{\mathrm{g}}} : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}^{2n \times m}$ given by

$$\mathcal{G}^{\tilde{\mathrm{g}}}(\mathbf{t}, z) = \mathrm{D}_{\mathbf{t}} \mathcal{X}_{\tilde{g}}(\mathbf{t}, z)$$

ii) The *local moment map* of $\tilde{g}$ is the function $\tilde{\mathcal{M}}^{\tilde{g}} : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}^m$ defined by

$$\tilde{\mathcal{M}}^{\tilde{g}}(\mathbf{t}, z) = \mathrm{D}_{\mathbf{t}} \tilde{g}(\mathbf{t}, z)^\top .$$

If $\tilde{g} = g$ is a family Hamiltonians, then $\tilde{\mathcal{M}}^g$ is called *the moment map* and it will be denoted by $\mathcal{M}^g$.

The following is the vector field version of Lemma 2.19.

LEMMA A.8. *Let $\Xi \subset \mathbb{R}^m$ be open. Let $\tilde{g} : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}$ be a smooth family of local Hamiltonians with local moment map $\tilde{\mathcal{M}}^g$ and family of symplectic vector fields $\mathcal{X}_{\tilde{g}}$. Then, the following equality holds:*

$$\mathrm{D}_{\mathbf{t}} \mathcal{X}_{\tilde{g}}(\mathbf{t}, z) = \Omega(z)^{-1} \mathrm{D}_z \tilde{\mathcal{M}}^g(\mathbf{t}, z)^\top . \tag{A.25}$$

REMARK A.9. The functions $\mathcal{G}^{\mathrm{g}}$ and $\tilde{\mathcal{M}}^g$ in Definition A.7 have a natural geometrical meaning. For $i = 1, \ldots, m$, let $(\tilde{\mathcal{M}}^g_{\mathbf{t}})_i$ and $(\mathcal{G}^{\mathrm{g}}_{\mathbf{t}})_i$ be the $i$-the coordinate of $\tilde{\mathcal{M}}^g_{\mathbf{t}}$ and the $i$-the column of $\mathcal{G}^{\mathrm{g}}_{\mathbf{t}}$, respectively. From equality (A.25) one has that $(\tilde{\mathcal{M}}^g_{\mathbf{t}})_i$ is a local Hamiltonian of the vector field $(\mathcal{G}^{\mathrm{g}}_{\mathbf{t}})_i$:

$$\mathrm{i}_{(\mathcal{G}^{\mathrm{g}}_{\mathbf{t}})_i} \boldsymbol{\omega} = -\,\mathrm{d}(\tilde{\mathcal{M}}^g_{\mathbf{t}})_i \,.$$

We adopt a notational convention similar to that used for symplectic deformations. Namely, families of (local) Hamiltonians are denoted with (dashed) small letters, the (local) moment maps with capital (dashed) letters the (local) generator will be denoted by using (dashed) script capital letters.

For canonical changes of variables the following holds.

LEMMA A.10. *Let $\Xi \subset \mathbb{R}^m$ be open. Let $\tilde{g} : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}$ be a smooth family of local Hamiltonians with local moment map $\tilde{\mathcal{M}}^g$ and family of symplectic vector fields $\mathcal{X}_{\tilde{g}}$. Let $f : \Xi \times \mathcal{A}'_0 \to \mathcal{A}$ be a symplectic deformation, with local moment map $\tilde{\mathcal{M}}^f$ and such that $f_{\mathbf{t}}(\mathcal{A}'_0) = \mathcal{A}_0$, for all $\mathbf{t} \in \Xi$. Define $\tilde{g}' : \Xi \times \tilde{\mathcal{A}}'_0 \to \mathbb{R}$ by*

$$\tilde{g}'_{\mathbf{t}} = \tilde{g}_{\mathbf{t}} \circ f_{\mathbf{t}} \,.$$

*Then, the local moment map of $\tilde{g}'$ is:*

$$\tilde{\mathcal{M}}^{g'}(\mathbf{t}, z) = \tilde{\mathcal{M}}^g(\mathbf{t}, f_{\mathbf{t}}(z))) - \mathrm{D}_z \tilde{\mathcal{M}}^f(\mathbf{t}, f_{\mathbf{t}}(z)))\ \mathcal{X}_{\tilde{f}}(\mathbf{t}, f_{\mathbf{t}}(z)) \,.$$

PROOF. This is a consequence of Lemma 2.19 and Definition A.7. □

## A.4. Potential and moment of an invariant FLD

In this section we state the Hamiltonian vector fields version of Theorem 4.7 and Theorem 4.8.

DEFINITION A.11. Let $\mathrm{D}, \Lambda \subset \mathbb{R}^s$ be open and let $g : \Lambda \times \mathcal{A}_0 \to \mathbb{R}$ be a smooth family of Hamiltonians and let $\mathcal{X}_g : \Lambda \times \mathcal{A}_0 \to \tilde{\mathcal{A}}_0 \times \mathbb{R}^{2n}$ be the corresponding family of Hamiltonian vector fields. A FLD base sets D and $\Lambda$, is a smooth bundle map

$$\begin{array}{rccc} \mathbf{K} : & \mathrm{D} \times \mathbb{T}^n & \longrightarrow & \Lambda \times \mathcal{A}_0 \\ & (p, \theta) & \longrightarrow & (\lambda(p), K(p, \theta)) \,, \end{array}$$

such that, for each $p \in \mathrm{D}$, $K_p \in \mathrm{Lag}(\mathbb{T}^n, \mathcal{A}_0)$. $\mathbf{K}$ is $\mathcal{X}_g$*-invariant with frequency* $\omega$ if for any $p \in \mathrm{D}$, $K_p$ is $\mathcal{X}_{g_{\lambda(p)}}$-invariant with frequency $\omega$:

$$\mathcal{X}_{g_{\lambda(p)}} \circ K_p + \mathcal{L}'_\omega K_p = 0 \,.$$

Let $\mathbf{K} : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ be a smooth FLD with base sets D and $\Lambda$. As in the maps case, we have that the *momentum* and *potential* of $\mathbf{K}$ with respect to $\mathcal{X}_g$ are defined, respectively, by:

$$\mathcal{M}^{g,\mathbf{K}}(p) = \langle \mathcal{M}^g\left(\lambda(p), K(p,\theta)\right)\rangle \,, \tag{A.26}$$

$$V^{g,\mathbf{K}}(p) = -\mathcal{M}^{g,\mathbf{K}}(p)^\top \lambda(p) - \langle S^g\left(\lambda(p), K(p,\theta)\right)\rangle \,. \tag{A.27}$$

where $\mathcal{M}^g$ is the moment map of $g$. Moreover, $\mathbf{K}$ is parameterized by the momentum parameter if the following equality holds for all $p \in \mathrm{D}$:

$$\mathcal{M}^{g,\mathbf{K}}(p) = p\,,$$

where $\mathcal{M}^g$ is the moment map of $g$.

An important property of the momentum and of the potential of a $\mathcal{X}_g$-invariant FLD is that they are invariant under canonical changes of the phase space variable. This is the content of the following result.

PROPOSITION A.12. *Let* $\mathrm{D}, \Lambda \subset \mathbb{R}^s$ *be open and let* $g : \Lambda\times\mathcal{A}_0 \to \mathbb{R}$ *be a smooth family of Hamiltonians, with base* $\Lambda$. *Assume that* $\mathbf{K} = (\lambda, K) : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ *is a smooth* $\mathcal{X}_g$*-invariant FLD with frequency* $\omega$ *and base sets* D *and* $\Lambda$. *Let* $\varphi : \Lambda\times\mathcal{A}' \to \times\mathcal{A}_0$ *be a smooth Hamiltonian deformation such that, for any* $\lambda \in \Lambda$, $\varphi_\lambda(\mathcal{A}'_0) = \mathcal{A}_0$. *Define* $g' : \Lambda\times\mathcal{A}'_0 \to \mathbb{R}$ *and* $\mathbf{K}$ *by*

$$g'_\lambda = g_\lambda \circ \varphi_\lambda \qquad \textit{and} \qquad \mathbf{K}'(p,\theta) = (\lambda(p), \varphi^{-1}_{\lambda(p)}(K(p,\theta))\,).$$

*Then*

$$\mathcal{M}^{g',\mathbf{K}'} = \mathcal{M}^{g,\mathbf{K}}, \qquad \textit{and} \qquad V^{g',\mathbf{K}'} = V^{g,\mathbf{K}}.$$

PROOF. Performing straightforward computations one shows that the following equalities hold:

$$\left\langle \mathcal{M}^{f'}(\mathbf{t}_0, K'_0(\theta)))\right\rangle = \langle \mathcal{M}^f(\mathbf{t}_0, K_0(\theta))\rangle \,,$$
$$\left\langle S^{f'}(\mathbf{t}_0, K'_0(\theta)))\right\rangle = \langle S^f(\mathbf{t}_0, K_0(\theta))\rangle \,,$$

from which Proposition A.12 follows. □

The following is the vector field version of Theorem 4.7.

THEOREM A.13. *Let* $\omega \in \mathbb{R}^n$, $\mathrm{D}, \Lambda \subset \mathbb{R}^s$. *Let*

$$\begin{array}{rccc} \mathbf{K}: & \mathrm{D}\times\mathbb{T}^n & \longrightarrow & \Lambda\times\mathcal{A}_0 \\ & (p,\theta) & \longrightarrow & (\lambda(p), K(p,\theta)) \end{array}$$

*be a smooth FLD with base sets* D *and* $\Lambda$. *Let* $g : \Lambda\times\mathcal{A}_0 \to \mathbb{R}$ *be a smooth family of Hamiltonians, with base* $\Lambda$. *Assume that* $\mathbf{K}$ *is* $\mathcal{X}_g$*-invariant with frequency* $\omega$ *and that it is parameterized by the momentum parameter* $p$. *Then the following equality holds:*

$$\lambda(p) = -\nabla_p V^{g,\mathbf{K}}(p)\,.$$

The following is the vector field version of Theorem 4.8

THEOREM A.14. *Let* $\mathrm{D}, \Lambda \subset \mathbb{R}^s$, $\omega \in \mathbb{R}^n$, $\mathbf{K} : \mathrm{D}\times\mathbb{T}^n \to \Lambda\times\mathcal{A}_0$ *and* $g : \Lambda\times\mathcal{A}_0 \to \mathbb{R}$ *be as in Theorem A.13. Let* $C^K$, $L_K$ $N_K$ *and* $M_K$ *be given by* (2.9), (3.2), (3.3) *and* (3.4), *respectively. For* $(p,\theta) \in \mathrm{D}\times\mathbb{T}^n$, *take* $C(p) = C^{K_p}$ $L(p,\theta) = L_{K_p}(\theta)$, $N(p,\theta) = N_{K_p}(\theta)$, $M(p,\theta) = M_{K_p}(\theta)$ *and*

$$T(p,\theta) = N(p,\theta)^\top \Omega(K(p,\theta))\left(\mathrm{D}_z \mathcal{X}_{g_{\lambda(p)}}(K(p,\theta))N(p,\theta) + \mathcal{L}'_\omega N(p,\theta)\right)\,.$$

*Let $\mathcal{M}^g$ be the moment map of $g$. Define*

$$B^g(p,\theta) = \left(\mathrm{D}_z \mathcal{M}^g(\lambda(p), K(p,\theta+\omega))\, M(p,\theta+\omega) J_0\right)^\top .$$

*Assume that $\omega \in \mathcal{D}_n(\gamma,\tau)$, for some $\gamma > 0$ and $\tau \geq n$ and that, for any $p \in \mathrm{D}$, $T_p(\theta) = T(p,\theta)$ and $B^g_p(\theta) = B^g(p,\theta)$ are sufficiently smooth so that $\mathcal{R}'_\omega T_p$ and $\mathcal{R}'_\omega B^g_p$ are smooth, with $\mathcal{R}_\omega$ as in Lemma A.1. Then, $\bar{\mathrm{T}} : \mathrm{D} \to \mathbb{R}^{n\times n}$, given by $\bar{\mathrm{T}}(p) = \langle T(p,\theta)\rangle$, satisfies the following equality:*

$$\bar{\mathrm{T}}(p)\, \mathrm{D}_p C(p) \;=\; W(p)\, \mathrm{Hess}_p\, V^{g,\mathbf{K}}(p)\,,$$

*where*

$$W(p) = \left\langle\, \mathrm{D}_z \mathcal{M}^g(\lambda(p,\theta))\, \left(N(p,\theta) - L(p,\theta)\, \mathcal{R}'_\omega T(p,\theta)^\top\right) \right\rangle^\top .$$

Corollary A.15. *Assume that the hypotheses of Theorem A.13 hold. Then, for any $\lambda_* \in \Lambda$ fixed, the following holds:*

- **a)** *for any $p_* \in \mathrm{D}$, $K_{p*}$ is an $\mathcal{X}_{g_{\lambda_*}}$-invariant tori with frequency $\omega$ if and only if $p_*$ is a critical point of $V^{g,\mathbf{K}}(p) + p^\top \lambda_*$;*
- **b)** *if $K_{p*}$ is $\mathcal{X}_{g_{\lambda_*}}$-invariant, with frequency $\omega$, and the matrices in Theorem A.13, $W_1(p_*)$ and $W_2(p_*)$, are invertible, then the co-rank of $\langle T_{p_*}\rangle$ equals the co-rank of $p_*$ as a critical point of $V^{g,\mathbf{K}}(p) + p^\top\lambda_*$. That is,*

$$\dim\ker\langle T_{p_*}\rangle = \dim\ker \mathrm{Hess}_p V^{g,\mathbf{K}}(p).$$

## A.5. Transformed Tori Theorem

Here we discuss the main ingredients to formulate the vector field version of the results in Chapter 7.

This is the vector field version of the geometrical functions introduced in definition 5.3 and 5.4.

Definition A.16. Let $\Lambda \subset \mathbb{R}^s$ be an open neighborhood of 0, with $0 \leq s \leq n$. A *modifying deformation of vector fields* is a family of Hamiltonian vector fields $\mathcal{X}_h : \Lambda\times\tilde{\mathcal{A}} \to \tilde{\mathcal{A}}\times\mathbb{R}^{2n}$, with Hamiltonians $h : \Lambda\times\tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$, such that:

- i) $\mathcal{X}_{h0} \equiv 0$;
- ii) $\mathrm{C}^h(\lambda) = 0$, for all $\lambda \in \Lambda$;
- iii) $\mathcal{X}_{h\lambda} \circ K = K$ for some $K \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A}_0)$ if and only if $\lambda = 0$.

Definition A.17. Let $\Sigma \subset \mathbb{R}^n$ be an open neighborhood of 0. A *dummy deformation of vector fields* is a family of symplectic vector fields $\mathcal{X}_{\tilde{d}} : \Sigma\times\tilde{\mathcal{A}} \to \tilde{\mathcal{A}}\times\mathbb{R}^{2n}$, with Hamiltonians $\tilde{d} : \Sigma\times\mathcal{A} \to \mathcal{A}$, such that:

- i) $\mathcal{X}_{\tilde{d}0} \equiv 0$;
- ii) $\mathrm{C}^d(\sigma) = \sigma$, for all $\sigma \in \Sigma$.

Example A.18. The family of vector fields corresponding to the family of Hamiltonians $h(\lambda, x, y) = \lambda^\top y$ is an example of modifying deformation of vector fields. The moment map of $h$ is $H(x,y) = y$. If the symplectic structure is the standard one, then $\mathcal{X}_h(\lambda, z) = \begin{pmatrix}\lambda\\ 0\end{pmatrix}$.

Example A.19. The family of vector fields corresponding to the family of Hamiltonians $\tilde{d}(\sigma, x, y) = -\sigma^\top x$ is an example of dummy deformation of vector fields. The local moment map of $\tilde{d}$ is $\tilde{d}(z) = -x$. If the symplectic structure is the standard one, then $\mathcal{X}_{\tilde{d}}(\sigma, z) = \begin{pmatrix}0\\ \sigma\end{pmatrix}$.

Given an open set $\mathcal{U} \subset \mathbb{R}^k$ and a smooth family of Hamiltonians, $f : \mathcal{U} \times \mathcal{A}_0 \to \mathcal{A}$, we embed the corresponding family of vector fields $\mathcal{X}_f$ in a smooth family of symplectic vector fields by introducing modifying and dummy parameters as follows. Let $\Lambda \subset \mathbb{R}^s$, with $0 \le s \le n$, and $\Sigma \subset \mathbb{R}^n$ be open neighborhoods of the origin. Let $\mathcal{X}_h : \Lambda \times \mathcal{A} \to \mathcal{A} \times \mathbb{R}^{2n}$ and $\mathcal{X}_{\tilde{d}} : \Sigma \times \mathcal{A} \to \mathcal{A} \times \mathbb{R}^{2n}$ be a family of, respectively, modifying and dummy vector fields with smooth families (local) Hamiltonians $h : \Lambda \times \mathcal{A} \to \mathcal{A}$ and $\tilde{d} : \Sigma \times \mathcal{A} \to \mathcal{A}$, respectively. Define $\Xi = \mathcal{U} \times \Sigma \times \Lambda$, $\mathbf{t} = (\mu, \sigma, \lambda)$ and the family of local Hamiltonians with base $\Xi$, $\tilde{\mathbf{f}} : \Xi \times \mathcal{A}_0 \to \mathcal{A}$,

$$\tilde{\mathbf{f}}_{\mathbf{t}} = \tilde{d}_\sigma + h_\lambda + f_\mu \,.$$

The family of vector fields $\mathcal{X}_{\tilde{\mathbf{f}}} : \Xi \times \mathcal{A}_0 \to \mathcal{A} \times \mathbb{R}^{2n}$ satisfies

$$\mathcal{X}_{\tilde{\mathbf{f}}(\mu,0,0)} = \mathcal{X}_{f_\mu}, \qquad \forall \mu \in \mathcal{U} \,.$$

Remark A.20. Since $\mathrm{C}^{\tilde{d}}(\sigma) = \sigma$, $\mathrm{C}^h(\lambda) = 0$ and $\mathrm{C}^f(\mu) = 0$ we have

$$\mathrm{C}^{\tilde{\mathbf{f}}}(\mu, \sigma, \lambda) = \sigma.$$

Then $\tilde{\mathbf{f}}_{\mathbf{t}}$ is (global) Hamiltonian function if and only if $\sigma = 0$. Hence, from Lemma A.3 we have that the only elements of the family $\{\mathcal{X}_{\tilde{\mathbf{f}}_{\mathbf{t}}}\}_{\mathbf{t} \in \Xi}$ that might have invariant tori are those for which $\sigma = 0$.

The nondegeneracy condition for the vector field version of Theorem 7.4 goes as follows. Let $K \in \mathrm{Emb}(\mathbb{T}^n, \mathcal{A})$ and let $L_K$ and $N_K$ be given by (3.2) and (3.3), respectively. Let $H : \tilde{\mathcal{A}}_2 \times \Sigma \to \mathbb{R}^s$ be the moment map of $h$. We use the following notation:

$$\begin{aligned} T_{(\mathbf{t},K)}(\theta) &= N_K(\theta)^\top \Omega(K(\theta)) (\mathrm{D}_z \mathcal{X}_{\tilde{\mathbf{f}}_{\mathbf{t}}}(K(\theta)) N_K(\theta) + \mathcal{L}'_\omega N_K(\theta)) \,, \\ B^h_{(\lambda,K)}(\theta) &= (\mathrm{D}_z H(\lambda, K(\theta)) M_K(\theta) J_0)^\top \end{aligned}$$

Let $\mathcal{R}'_\omega$ be as in Lemma A.1, define

$$\begin{aligned} \bar{\mathrm{T}}_{(\mathbf{t},K)} &= \left\langle T_{(\mathbf{t},K)}(\theta) \right\rangle \\ P^{12}_{(\mathbf{t},K)}(\theta) &= \left\langle B^{h,x}_{(\lambda,K)}(\theta) - T_{(\mathbf{t},K)}(\theta) \mathcal{R}'_\omega B^{h,y}_{(\lambda,K)}(\theta) \right\rangle, \\ P^{21}_{(\mathbf{t},K)}(\theta) &= \left\langle B^{h,x}_{(\lambda,K)}(\theta)^\top + B^{h,y}_{(\lambda,K)}(\theta)^\top \mathcal{R}'_\omega T_{(\mathbf{t},K)}(\theta) \right\rangle \\ P^{22}_{(\mathbf{t},K)}(\theta) &= \langle \mathrm{D}_\lambda H(\lambda, K(\theta))) \rangle - \left\langle B^{h,x}_{(\lambda,K)}(\theta)^\top \mathcal{R}'_\omega B^{h,y}_{(\lambda,K)}(\theta) \right\rangle \\ &\quad + \left\langle B^{h,y}_{(\lambda,K)}(\theta)^\top \mathcal{R}'_\omega \left( B^{h,x}_{(\lambda,K)}(\theta) - T_{(\mathbf{t},K)}(\theta) \mathcal{R}'_\omega B^{h,y}_{(\lambda,K)}(\theta) \right) \right\rangle . \end{aligned}$$

Define

$$P_{(\mathbf{t},K)} = \begin{pmatrix} \bar{\mathrm{T}}_{(\mathbf{t},K)} & P^{12}_{(\mathbf{t},K)} \\ P^{21}_{(\mathbf{t},K)} & P^{22}_{(\mathbf{t},K)} \end{pmatrix}. \tag{A.28}$$

Definition A.21. We say that $(\lambda_0, K)$ is $\mathcal{X}_h$-*nondegenerate* with respect to $\mathcal{X}_{f_{\mu_0}}$ and $\omega$, if the $((n+s) \times (n+s))$-dimensional matrix $P_{(\mathbf{t}_0,K)}$ has maximal range, with $\mathbf{t}_0 = (\mu_0, 0, \lambda_0)$ and $P_{(\mathbf{t},K)}$, defined in (A.28).

Remark A.22. If $s = 0$ then the only modifying deformation of vector fields is $\mathcal{X}_h \equiv 0$, and $K$ is nondegenerate with respect to $f_{\mu_0}$ and $\omega$, in the sense of Definition A.21, if and only if $K$ is twist with respect to $\mathcal{X}_{f_{\mu_0}}$ and $\omega$, i.e. the torsion $\left\langle T_{(\mathbf{t}_0,K)} \right\rangle$ is invertible.

A vector field version of Theorem 7.4 can be obtained from the vector field version of Theorem 6.2, discussed in Section A.6.

## A.6. A parametric KAM Theorem

Here we sketch the procedure to obtain the vector field version of the parametric KAM result, Theorem 6.2. It is also possible to obtain a rigorous proof by considering the time-1 map and applying the corresponding result for symplectomorphisms.

Let $m \geq n$ and let $\Xi \subset \mathbb{R}^m$ be open. Let $\tilde{f} : \Xi \times \tilde{\mathcal{A}}_0 \to \mathbb{R}$ be a smooth family of local Hamiltonians and let $\mathcal{X}_{\tilde{f}} : \Xi \times \tilde{\mathcal{A}}_0 \to \mathcal{A} \times \mathbb{R}^{2n}$ be the corresponding family of local Hamiltonian vector fields. Let $\mathcal{B}_0 \subset \mathcal{B}$ be complex strips of $\mathcal{A}_0$ and $\mathcal{A}$, respectively. Assume that for any $\mathbf{t} \in \Xi$, $\mathcal{X}_{\tilde{f}_{\mathbf{t}}}$ is real-analytic on a complex neighborhood of $\mathcal{A}_0$. Let $Z : \mathbb{R}^m \times \Xi \times \mathrm{Emb}(\mathbb{T}^n_\rho, \mathcal{B}_0, C^0) \to \mathbb{R}^m$ be a $C^r$-functional. Define

$$\mathcal{F}(\zeta; \mathbf{t}, K) = \begin{pmatrix} e(\mathbf{t}, K) \\ \nu(\zeta; \mathbf{t}, K) \end{pmatrix} \begin{pmatrix} \mathcal{X}_{\tilde{f}_{\mathbf{t}}} \circ K + \mathcal{L}'_\omega K \\ Z(\mathbf{t}, K) - \zeta \end{pmatrix}, \tag{A.29}$$

with $\mathcal{L}'_\omega$ given in (A.1).

As in the symplectomorphisms case, a KAM procedure to solve the following non-linear equation

$$\mathcal{F}(\zeta; \mathbf{t}, K) = 0 \tag{A.30}$$

depends on the solvability properties of the corresponding linearized equation:

$$(\mathrm{D}_z \mathcal{X}_{\tilde{f}_{\mathbf{t}}} \circ K)\, \Delta_K + \mathcal{L}'_\omega \Delta_K + (\mathrm{D}_{\mathbf{t}} \mathcal{X}_{\tilde{f}_{\mathbf{t}}} \circ K)\, \Delta_t = \varphi, \tag{A.31}$$

$$\mathrm{D}_K Z(\mathbf{t}, K) \Delta_K + \mathrm{D}_{\mathbf{t}} Z(\mathbf{t}, K)\, \Delta_t = \nu. \tag{A.32}$$

Using lemmas A.3 and A.8 one obtains.

LEMMA A.23. *Let $\omega \in \mathbb{R}^n$ be rationally independent. Let $K \in \mathrm{Emb}(\mathbb{T}, \mathcal{A}_0)$ and let $M_K$ be defined by* (A.8). *Let $\mathcal{M}^f$ is the moment map of $f$ and define*

$$\begin{aligned} T_{(\mathbf{t},K)}(\theta) &= N_K(\theta)^\top \Omega(K(\theta)) \left( \mathrm{D}_z \mathcal{X}_{f_{\mathbf{t}}}(K(\theta)) N_K(\theta) + \mathcal{L}'_\omega N_K(\theta) \right), \\ B^f_{(\mathbf{t},K)}(\theta) &= \left( \mathrm{D}_z F_{\mathbf{t}}(K(\theta))\, M_K(\theta) J_0 \right)^\top. \end{aligned}$$

*Assume that $K$ is $\mathcal{X}_{f_{\mathbf{t}}}$-invariant with frequency $\omega$ and let $\Delta_K = M_K\, \xi$. Then, the system* (A.31), (A.32) *is equivalent to*

$$(\hat{T}_{(\mathbf{t},K)} + \mathcal{L}'_\omega)\, \xi + B^f_{(\mathbf{t},K)} \Delta_t = \eta, \tag{A.33}$$

$$\mathrm{D}_K Z(\mathbf{t}, K)\ [M_K \xi] + \mathrm{D}_{\mathbf{t}} Z(\mathbf{t}, K)\ \Delta_t = \nu, \tag{A.34}$$

*where*

$$\hat{T}_{(\mathbf{t},K)}(\theta) = \begin{pmatrix} O_n & T_{(\mathbf{t},K)}(\theta) \\ O_n & O_n \end{pmatrix}, \tag{A.35}$$

*and*

$$\eta(\theta) = -J_0 M_K(\theta)^\top \Omega(K(\theta))\, \varphi(\theta). \tag{A.36}$$

For an approximate solution $(\zeta; \mathbf{t}, K)$ of (A.30), the change of variables $\Delta_K = M_K\, \xi$ reduces the system (A.31), (A.32) to the linear system (A.33), (A.34) except for some terms that can be estimated by the norm of the error $e = \mathcal{X}_{\tilde{f}_{\mathbf{t}}} \circ K + \mathcal{L}'_\omega K$. The proof of this is not included here because it is very similar to the proof of the corresponding result for symplectomorphisms (see Lemma 5.2).

REMARK A.24. Notice the similitude between the linear systems (A.33) and (5.22).

The linear system (A.33), (A.34) has a solution provided a nondegeneracy condition is satisfied. In what follows we describe this. Using (A.4) and (A.33) one shows that any solution of (A.33) has the following form:

$$\xi = (I_{2n} - \mathcal{R}'_\omega \hat{T})\xi_0 - \mathcal{R}'_\omega (B - \hat{T}\mathcal{R}'_\omega B)\Delta_t + \mathcal{R}'_\omega(\eta - \hat{T}\mathcal{R}'_\omega \eta), \tag{A.37}$$

where for typographical simplicity we have not written the dependence on $(\mathbf{t}, K)$. A direct computation shows that $(\xi, \Delta_t)$ in (A.37) is a solution of (A.33) and (A.34) if and only if the constant $(\xi_0, \Delta_t) \in \mathbb{R}^{2n} \times \mathbb{R}^m$ satisfies the following $(2n+m)$-dimensional linear system:

$$Q_{(\mathbf{t},K)} \begin{pmatrix} \xi_0 \\ \Delta_t \end{pmatrix} = \begin{pmatrix} \left\langle \eta - \hat{T}\,\mathcal{R}'_\omega\,\eta \right\rangle \\ \nu - \mathrm{D}_K Z(\mathbf{t}, K)\left[ M_K \mathcal{R}_\omega(\eta - \hat{T}\,\mathcal{R}'_\omega \eta)\right] \end{pmatrix},$$

with

$$Q_{(\mathbf{t},K)} = \begin{pmatrix} \left\langle \hat{T} \right\rangle & \left\langle B - \hat{T}\mathcal{R}'_\omega B \right\rangle \\ \mathrm{D}_K Z\left[ M_K (I_{2n} - \mathcal{R}'_\omega \hat{T})\right] & \mathrm{D}_{\mathbf{t}} Z - \mathrm{D}_K Z\left[ M_K \mathcal{R}'_\omega (B - \hat{T}\mathcal{R}'_\omega B)\right] \end{pmatrix},$$

where for typographical simplicity we have not written the dependence on $(\mathbf{t}, K)$ of $\hat{T}$ and $B$ and $\mathrm{D}_K Z$ and $\mathrm{D}_{\mathbf{t}} Z$ are evaluated at $(\mathbf{t}, K)$.

The statement and the proof of the vector field version of Theorem 6.2 can be obtained from the above discussion.

APPENDIX B

# Elements of singularity theory

Singularity Theory, started by Poincaré and continued by Whitney, Thom, Mather, Arnold, etc, is nowadays a flourishing area of Mathematics. Here we briefly review the definitions and results on Singularity theory used in this paper. We are interested in the Singularity Theory for critical points of parametric families of functions. For more complete expositions of the subject see [**5, 25, 36, 53**] and [**42, 43, 44, 45, 67, 70, 71**].

The study of local properties of critical points of parametric families of functions and the stability of these properties is performed in the space of $C^\infty$-functions. Similar theory can be developed for finitely differentiable functions.

Let us start with some definitions.

Definition B.1. Given two mappings $f_1 : U_1 \to \mathbb{R}^m$ and $f_2 : U_2 \to \mathbb{R}^m$, write $f_1 \sim_{x_0} f_2$ if there exists a neighborhood of $x_0$ such that $U \subset U_1$ and $U \subset U_2$ and $f_1\big|_U = f_2\big|_U$. The equivalence class of $f$ at $x_0$ is denoted by $[f]_{x_0}$ and called *map-germ* at $x_0$ and its members are called *representatives* of the map-germ. The point $x_0$ is called the *source* of the map-germ, and $y_0 = f(x_0)$ is called the *target* of the map-germ.

Notice that the target of a map-germ does not depend on the representative. If it does not lead to confusion, we refer to $f$ as a map-germ at $x_0$.

Denote by $\mathcal{F}_{x_0}(n, m)$ the set of $C^\infty$ map-germs from $\mathbb{R}^n$ to $\mathbb{R}^m$ at $x_0 \in \mathbb{R}^n$. The notations $[f] = [f]_0$ and $\mathcal{F}(n, m) = \mathcal{F}_0(n, m)$ are also used. The composition of two map-germs, $[f]$ and $[g]$, is defined by the composition of two representatives $[f] \circ [g] = [f \circ g]$.

Definition B.2. A map-germ $[G]_{x_0} \in \mathcal{F}(n, n)$ is *invertible* if $\det \mathrm{D}_x G(x_0) \neq 0$, for any representative $G$.

We are interested in the case $m = 1$. Write $\mathfrak{e}_{x_0}(n) = \mathcal{F}_{x_0}(n, 1)$ and $\mathfrak{e}(n) = \mathcal{F}(n, 1)$.

Remark B.3. $\mathfrak{e}_{x_0}(n)$ is a commutative and associative real algebra with unit element (the germ of the constant function 1). Moreover, $\mathfrak{e}_{x_0}(n)$ is a local algebra: it has a unique maximal ideal,

$$\mathfrak{m}_{x_0}(n) = \{[f]_{x_0} \in \mathfrak{e}_{x_0}(n) \mid f(x_0) = 0\}.$$

Lemma B.4. *Let $k$ be a positive integer, then*

$$\mathfrak{m}_{x_0}(n)^k = \{[f]_{x_0} \in \mathfrak{e}_{x_0}(n) : \mathrm{D}^j f(x_0) = 0,\ j = 0, \dots k-1\}.$$

From Hadamard's Lemma, $\mathfrak{m}(n) = \mathfrak{m}_0(n)$ is generated by the germs of the coordinate functions $x_1, x_2, \dots x_n$.

There are several equivalence relations under which one can classify germs of smooth mappings. We consider here the so-called $R^+$-equivalence relation of germs of smooth real-valued functions.

DEFINITION B.5. Two germs $[f_1]_{x_1} \in \mathscr{e}_{x_1}(n)$ and $[f_2]_{x_2} \in \mathscr{e}_{x_2}(n)$ are $R^+$-equivalent (or right equivalent or just equivalent) if and only if there exist an invertible germ $[G]_{x_1} \in \mathcal{F}_{x_1}(n,n)$ with $G(x_1) = x_2$ such that

$$f_1 = f_2 \circ G + c, \tag{B.1}$$

where $c = f_1(x_1) - f_2(x_2)$ (in suitable neighborhoods of the source points).

DEFINITION B.6. A germ $[f]_{x_0} \in \mathscr{e}_{x_0}(n)$ is *stable* at $x_0$ (or *locally stable*) if there is a neighborhood of the representative $f : U \subset \mathbb{R}^n \to \mathbb{R}$ in $C^\infty(U, \mathbb{R})$ such that any germ $[g]_{x_0} \in \mathscr{e}_{x_0}(n)$ with representative $g$ in this neighborhood, is $R^+$-equivalent to $[f]_{x_0}$.

By a change of variables, the classification of germs of smooth real-valued functions, under $R^+$-equivalence, can be reduced to the study the behavior of *jets*. To formalize these ideas we need the following definitions.

DEFINITION B.7. Let $k \in \mathbb{N}$. Two germs $[f]_{x_0}, [g]_{x_0} \in \mathscr{e}_{x_0}(n)$ are $k$-equivalent (at $x_0$) if they have a contact of order $k$ at $x_0$:

$$f(x) - g(x) = o(|x - x_0|^k).$$

The $k$-equivalence class in $\mathscr{e}_{x_0}(n)$ of a germ $[f]_{x_0}$ is called the $k$-jet of $f$ at the point $x_0$, and it is denoted by $j^k_{x_0} f$.

The $k$-jet can be identified with the Taylor polynomial of $f$ at the point $x_0$ of degree $k$.

DEFINITION B.8. A germ $[f]_{x_0} \in \mathscr{e}_{x_0}(n)$ is $k$-determined (or it is determined by its $k$-jet) if and only if any $k$-equivalent germ is also $R^+$-equivalent. We also say that the $k$-jet of $[f]_{x_0}$, $j^k_{x_0} f$ is *sufficient*.

A germ $[f]_{x_0}$ is *finitely determined* if and only if is $k$-determined for some $k \in \mathbb{N}$.

A first classification of critical points is the following.

DEFINITION B.9. The germ $[f]_{x_0} \in \mathscr{e}_{x_0}(n)$ is *regular* if the source point $x_0$ is a regular point, i.e. $\mathrm{D}f(x_0) \neq 0$. Otherwise the germ is called *singular*.

REMARK B.10. The class of regular germs is formed by submersive germs. Moreover, any regular germ is $R^+$-equivalent to the germ at zero of the projection map: $(x_1, \ldots, x_n) \longrightarrow x_1$. In particular, any regular germ at $x_0$ is 1-determined.

In what follows, we consider the problem of classifying singular germs in $\mathscr{m}_{x_0}(n)^2$.

DEFINITION B.11. A singular germ $[f]_{x_0} \in \mathscr{m}_{x_0}(n)^2$ is nondegenerate if and only if the source point is a nondegenerate critical point of $f$: $\det \mathrm{D}^2 f(x_0) \neq 0$. Otherwise $[f]_{x_0}$ is a degenerate singular germ. The dimension of the kernel of $\mathrm{D}^2 f(x_0)$ is the *co-rank* of the critical point.

The following Morse Lemma classify all the nondegenerate singular germs.

THEOREM B.12 (Morse lemma). *A nondegenerate singular germ* $[f]_{x_0} \in \mathscr{m}_{x_0}(n)^2$ *is* $R^+$*-equivalent to a nondegenerate quadratic germ at* 0

$$(x_1, \ldots, x_n) \to x_1^2 + \cdots + x_s^2 - x_{s+1}^2 - \cdots - x_n^2,$$

*where s is the* signature *(i.e. the number of positive eigenvalues) of* $\mathrm{D}^2 f(x_0)$. *Moreover, the singularity is stable.*

Nondegenerate singular germs, also known as *Morse germs*, are classified by the signature. In particular, given a germ at a nondegenerate critical point $x_0$, all the germs with the same 2-jet at zero, are $R^+$-equivalent.

A simple indicator of the degree of degeneracy of degenerate singular germs is the co-rank of the critical point.

THEOREM B.13 (Parametric Morse Lemma). *A singular germ* $[f]_{x_0} \in \mathfrak{m}_{x_0}(n)^2$ *of co-rank* $k$ *(i.e.* $k = \dim \ker \mathrm{D}^2 f(x_0)$*) is equivalent to a germ at* $0$ *of the form*

$$(x_1, \ldots, x_n) \to f_d(x_1, \ldots, x_k) + f_m(x_{k+1}, \ldots, x_n),$$

*where* $f_d$ *is a degenerate singular germ such that all the second derivatives vanish at* $0$ *(i.e.* $f_d \in \mathfrak{m}(k)^3$*), and* $f_m$ *is a Morse germ at* $0$ *(i.e.* $f_m \in \mathfrak{m}(n-k)^2$ *with* $\det \mathrm{D}^2 f_m(0) \neq 0$*).*

The Parametric Morse Lemma is also known as Morse-Bott Lemma, a proof can be found in [**7**].

Now, we introduce another important indicator of the degeneracy of a singular germ.

DEFINITION B.14. Given a singular germ $[f]_{x_0} \in \mathfrak{m}_{x_0}(n)^2$, its *gradient ideal*, at the point $x_0$, is the ideal generated by the (germs of the) partial derivatives of $f$ at $x_0$:

$$I_{\nabla f, x_0} = \left\langle \left[\frac{\partial f}{\partial x_1}\right]_{x_0}, \ldots, \left[\frac{\partial f}{\partial x_n}\right]_{x_0} \right\rangle$$

The *local algebra* of the singular germ is $Q_{f,x_0} = \mathfrak{m}_{x_0}(n)/I_{\nabla f, x_0}$, and its dimension is referred to as the *co-dimension* of the singular germ: $\mathrm{cod}_{x_0}(f) = \dim Q_{f,x_0}$.

In some references, the definition of the local algebra is $Q_{f,x_0} = \mathfrak{e}_{x_0}(n)/I_{\nabla f, x_0}$, and its dimension is the *multiplicity*, $\mathrm{mul}_{x_0}(f)$. It is clear that $\mathrm{cod}_{x_0}(f) = \mathrm{mul}_{x_0}(f) - 1$.

The co-dimension of a Morse germ is 0. In a certain sense, tractable singularities are those that have finite co-dimension. In such a case, by abuse of language, a *basis* of the local algebra $Q_{f,x_0}$ is a finite set $\{\varphi_i\}_{i=1}^c$ of germs in $\mathfrak{m}_{x_0}(n)^2$ whose projections with respect to the gradient ideal $I_{\nabla f, x_0}$ in $Q_{f,x_0}$ form a basis.

For singularities with finite co-dimension, a 'basis' of the local algebra $Q_{f,x_0}$ is a finite set $\{\varphi_i\}_{i=1}^c$ of germs in $\mathfrak{m}_{x_0}(n)^2$ whose projections with respect to the gradient ideal $I_{\nabla f, x_0}$ in $Q_{f,x_0}$ form a basis.

The following theorem summarizes some results on the classification of finitely determined singularities, due to Mather and Tougeron. Roughly speaking, it states that the classification of critical points can be reduced to a finite dimensional problem, in the case that the germ is $k$-determined (see Definition B.8), for a sufficiently large $k$.

THEOREM B.15.
a) *At a singular germ* $f \in \mathfrak{m}_{x_0}(n)^2$, *either* $\mathrm{cod}_{x_0}(f)$ *is finite and* $f$ *is finitely determined or* $\mathrm{cod}_{x_0}(f)$ *is infinite and* $f$ *is not finitely determined.*
b) *A singular germ* $f \in \mathfrak{m}_{x_0}(n)^2$ *is finitely determined if and only if*

$$\mathfrak{m}_{x_0}(n)^k \subset I_{\nabla f, x_0}$$

*for some $k \in \mathbb{N}$.*

c) *A sufficient condition for $f \in \mathfrak{m}_{x_0}(n)^2$ to be $k$-determined is that*

$$\mathfrak{m}_{x_0}(n)^{k+1} \subset \mathfrak{m}_{x_0}(n)^2 I_{\nabla f, x_0}$$

Morse Lemma ensures that generic real-valued functions have only nondegenerate critical points. Degenerate critical points of a germ can be decomposed into a nondegenerate one under a suitable perturbation. However, when considering parametric families of functions, critical points may persist under perturbation. A simple but illuminating example is the family $x^3 + \mu x$, depending on the parameter $\mu$. For $\mu = 0$, the point $x_0 = 0$ is a degenerate critical point of $x^3$. For $\mu < 0$, there are two nondegenerate critical points, and for $\mu > 0$ there are no critical points. It is a result in Singularity Theory that any family of functions sufficiently close to $x^3 + \mu x$, will have a degenerate critical point for some value of the parameter close to zero (with the same class of equivalence as the original one).

Definition B.16. An *unfolding* of *co-dimension* $r$ of a germ $[f]_{x_0} \in \mathfrak{e}_{x_0}(n)$ is a germ $[F]_{(x_0,0)} \in \mathfrak{e}_{(x_0,0)}(n+r)$ such that $F(x,0) = f(x)$ (locally).

An unfolding $F_\mu$ can be thought of as $\mu$-parametric deformation of $f$ with $F_0 = f$. The dimension of $\mu$ (the number of parameters) is the co-dimension of the unfolding.

Definition B.17. Some point sets associated to an unfolding are:

- The *Bifurcation diagram*:

$$\mathcal{M}_F = \{(x,\mu) \mid \mathrm{D}_x F(x,\mu) = 0\}.$$

- The *catastrophe set*:

$$\mathrm{Cat}_F = \{(x,\mu) \in M_F \mid \det \mathrm{D}_x^2 F(x,\mu) = 0\}.$$

- The *bifurcation set*:

$$\mathrm{Bif}_F = \{\mu \mid \exists x : (x,\mu) \in C_F\}.$$

To study the stability of unfoldings we introduce the following topological classification of unfoldings.

Definition B.18. Let $[F] \in \mathfrak{e}_{(x_0,0)}(n+r)$ and $[F'] \in \mathfrak{e}_{(x_0,0)}(n+r')$ be two unfoldings of the germ $[f] \in \mathfrak{e}_{x_0}(n)$, with parameters $\mu$ and $\mu'$, respectively.

A *morphism* from $[F']$ to $[F]$ is a triple $([G]_{(x_0,0)}, [\theta], [c])$ such that:

(i) $[G]_{(x_0,0)} \in \mathcal{F}_{(x_0,0)}(n+r', n)$ is an unfolding of the identity map in $\mathbb{R}^n$,
(ii) $[\theta] \in \mathcal{F}_0(r', r)$ is a map-germ of the parameters with $\theta(0) = 0$ and
(iii) $[c] \in \mathfrak{e}_0(r')$ is a translation with $c(0) = 0$, such that $F' = F \circ (G, \theta \circ \pi_{\mu'}) + c \circ \pi_{\mu'}$, equivalently:

$$\begin{aligned} F'(x,\mu') &= F(G(x,\mu'), \theta(\mu')) + c(\mu'), \\ G(x,0) &= x,\ \theta(0) = 0,\ c(0) = 0. \end{aligned}$$

If $r = r'$ and the fibered germ $[G, \theta]$ is invertible on $\mathcal{F}_{(x_0,0)}(n+r, n+r)$, we say that $F'$ and $F$ are equivalent (or isomorphic). It is also said that $F'$ is induced from $F$.

Informally, Definition B.18 states that two unfoldings are isomorphic if one can be transformed to the other using changes of variables in both the states and in the parameters.

We recall that, given a topological space $S$ with an equivalence relation $\sim$, an element $x \in S$ is *stable* (with respect to $\sim$) if its equivalence class contains a neighborhood of $x$.

We are interested in studying the stability of unfoldings under the topological equivalence relation given in Definition B.18. A more algebraic classification is the given in the following definition.

DEFINITION B.19. An unfolding $F$ of $f$ is *versal* if any other unfolding $F'$ of $f$ is induced from $F$. A *versal* unfolding of minimal co-dimension is said to be *universal*.

The fundamental result, due to Mather, states that versality gives rise to topological stability. Mather also established that finitely determined germs have versal unfoldings.

THEOREM B.20. a) *A singular germ $f \in \mathfrak{m}_{x_0}(n)^2$ has a versal unfolding if and only if it is finitely determined, which is equivalent to have finite co-dimension.*

b) *Versal unfoldings of a germ are stable (even though the germ is not).*

c) *If $F$ is a universal unfolding of $f$, of co-dimension $c$, then $c = \mathrm{cod}_{x_0} f$. All universal unfoldings are isomorphic.*

An algorithm to construct universal unfolding of singular germs with finite co-dimension $c$ is the following:

- Construct a basis $\varphi_1, \ldots, \varphi_c \in \mathfrak{m}_{x_0}(n)$ of the local algebra $Q_{f,x_0}$;
- A universal deformation is

$$F(x, \mu) = f(x) + \sum_{j=1}^{c} \mu_j \varphi_j(x) \ .$$

The proofs of Theorem B.20 use the geometric notion of transversality of unfoldings, which is equivalent to versality. (Thom Theorem on transversality, finite determinacy, etc. ) As an application of this theory, the following results classifies all singular germs of co-dimension $\leq 4$.

THEOREM B.21. *Let $f \in \mathfrak{m}(n)^2$ be a singular germ at 0, with co-dimension $1 \leq c \leq 4$. Then, $f$ is 6-determined.*

*Moreover, up to changes of sign and the addition of a nondegenerate quadratic form, $f$ is right equivalent to one of the seven germs at 0 described in Table 1.*

The seven singularities described in Theorem B.21 are known as *elementary catastrophes.* They appear generically in families of 4 parameters.

TABLE 1. Elementary Catastrophes

| germ | codim | universal unfolding | popular name |
| --- | --- | --- | --- |
| $x^3$ | 1 | $x^3 + ux$ | fold |
| $x^4$ | 2 | $x^4 + ux^2 + vx$ | cusp |
| $x^5$ | 3 | $x^5 + ux^3 + vx^2 + wx$ | swallow-tail |
| $x^3 + y^3$ | 3 | $x^3 + y^3 + wxy - ux - uy$ | hyperbolic umbilic |
| $x^3 - xy^2$ | 3 | $x^3 - xy^2 + w(x^2 + y^2) - ux - vy$ | elliptic umbilic |
| $x^6$ | 4 | $x^6 + tx^4 + ux^3 + vx^2 + wx$ | butterfly |
| $x^2y + y^4$ | 4 | $x^2y + y^4 + wx^2 + ty^2 - ux - vy$ | parabolic umbilic |

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