An inverse problem with single measurement generated by a plane wave

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Abstract

Global uniqueness of inverse problem for a hyperbolic PDE with the unknown potential and a single incident plane wave is a well known long standing open question. This question is addressed for the case when derivatives with respect to variables orthogonal to this plane wave are expressed via finite differences.

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1 Introduction

For the sake of brevity we consider here only the 3-D case. All theorems below have almost identical formulations and proofs for the $n-$D case with $n \geq 2$. Below $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Let the function $a \in C^2(\mathbb{R}^3)$ and is bounded in $\mathbb{R}^3$ together with its derivatives. Consider the Cauchy problem

$$u_{tt} = \Delta u + a(\mathbf{x})\, u, (\mathbf{x}, t) \in \mathbb{R}^3 \times (0, T), \tag{1.1}$$

$$u(\mathbf{x}, 0) = 0, u_t(\mathbf{x}, 0) = \delta(z). \tag{1.2}$$

Conditions (1.1), (1.2) mean that the wave field u is initialized by the plane wave at the plane $\{z = 0\}$. This plane wave propagates along the $z-$axis. Let $A_x, A_y, T = const. > 0$. Define the strip G as

$$\begin{aligned} G &= \{\mathbf{x} : x \in (0, A_x), y \in (0, A_y)\}, \ G_T = G \times (0, T), \\ S_T &= \{z = 0, x \in (0, A_x), y \in (0, A_y)\} \times (0, T). \end{aligned}$$

Inverse Problem 1 (IP1). *Assume that the function $a(\mathbf{x})$ is unknown in G. Determine the coefficient $a(\mathbf{x})$ for $x \in G$, assuming that the following two functions $r(x,t), s(x,t)$ are given*

$$u\,|_{S_T} = r(\mathbf{x}, t), \ u_z\,|_{S_T} = s(\mathbf{x}, t). \tag{1.3}$$

Uniqueness for IP1 is a well known long standing open question. IP1 is a Coefficient Inverse Problem (CIP) with single measurement data. We use the term "single measurement" since only a single direction of the incident plane wave is used. Also, (1.3) is the backscattering data. The main challenge is the single measurement, not the backscattering. Assume for a moment that the Fourier transform with respect to t can be applied to the function u, and the resulting function $p(\mathbf{x}, k)$ satisfies the equation

$$\Delta p + k^2 p + a(\mathbf{x}) p = -\delta(z) \tag{1.4}$$

as well as radiation conditions at the infinity. Then one can derive from (1.3) and (1.4) the statement of a well known CIP in the frequency domain. In accordance with the paper [7], the question about the uniqueness of this CIP was posed by I.M. Gelfand in 1954. In [7] this question was addressed for the case of infinitely many measurements. We also mention works [3,4], where a CIP for a more general hyperbolic equation was considered and solution was constructed for the case of infinitely many measurements.

As to the case of CIPs with single measurement data in $n-D$ $(n \geq 2)$, currently only one class of uniqueness theorems for them is known. All these theorems are proven under the assumption that at least one initial condition does not vanish in the entire domain of interest Ω. They are proven via Carleman estimates. The idea of application of Carleman estimates to proofs of uniqueness results for CIPs with single measurement data was originated in [9] with many follow up publications of many authors, see, e.g. [2,5,10-15] and references cited there for some of these works; the most recent survey of these results can be found in [20].

In this paper the uniqueness question is addressed for a closely related inverse problem. Specifically we assume that derivatives with respect to (x, y) are written via finite differences with the grid step sizes (h_1, h_2). Numbers h_1, h_2 do not tend to zero. However, derivatives with respect to z, t are written in the usual form. Under these conditions we prove uniqueness.

First, we prove a Carleman estimate, which contains a new feature of the positivity of a certain integral over the characteristic curve. Next, a combination of this new feature with a modification of the method of [9] leads to the desired uniqueness result. It is worthy to mention here that discrete Carleman estimates are attracting an interest nowadays, see, e.g. [8]. However, they were not yet used for proofs of uniqueness of discrete CIPs. A discrete Carleman estimate is not used here.

Both classical forward problems for PDEs as well as CIPs are often solved numerically by the Finite Difference Method (FDM). However, there is a substantial difference between forward and inverse problems. Indeed, classical forward problems are well-posed. Because of this, an important topic of study in this case is the convergence of the solution obtained by the FDM to the actual solution when the grid step size tends to zero [19]. However, the computational experience of the author [16,17] shows that, because of the ill-posedness, there is of a little help to investigate the convergence of FDM-based numerical methods for the case when the spatial step size h_{sp} tends to zero. Unlike well-posed problems, in the ill-posed case h_{sp} should usually be limited from the below. The reason of this limitation is that h_{sp} serves as an implicit regularization parameter in most cases. Furthermore, it is an important observation of numerical studies that h_{sp} usually cannot be significantly decreased. For example, while accurate computational results were obtained in [16] for the case $h_{sp} = 0.01$, an attempt to decrease h_{sp} to 0.005 led to a significant degradation of images, see Remark 7.1 in [16]. A similar observation took place in subsection 5.2 of [17]. Therefore, h_{sp} should be bounded from the below by a positive constant, which is the case of the current work.

As to the application of IP1, consider the following analog of the Laplace transform [18]

$$U(x,t) = \frac{1}{2\sqrt{\pi}t^{3/2}} \int_0^\infty u(x,\tau)\,\tau \exp\left(-\frac{\tau^2}{4t}\right) d\tau := \mathbb{L}u. \tag{1.4}$$

It is well known that [18]

$$\begin{aligned} c(\mathbf{x})\, U_t &= \Delta U + a(\mathbf{x})\, U, \\ U(\mathbf{x},0) &= \delta(z). \end{aligned} \tag{1.5}$$

The transform (1.4) is one-to-one. Thus, if the uniqueness of IP1 would be proven, then the uniqueness of a similar inverse problem for equation (1.5) would be proven as well, and vice versa. On the other hand, equation (1.5) governs light propagation in a turbid medium, such as, e.g. biological medium. In this case $U(\mathbf{x},t)$ is the light intensity and $(-a(\mathbf{x})) > 0$ is the absorption coefficient of light [1]. The absorption coefficient contains an information about the blood content, which is important for medical imaging.

2 Formulations Of Results

Consider partitions of intervals $x \in (0, A_x)$, $y \in (0, A_y)$ in small subintervals with step sizes h_1 and h_2 respectively,

$$0 = x_0 < x_1 < ... < x_{N_1} = A_x,\ 0 = y_0 < y_1 < ... < y_{N_2} = A_y, \tag{2.1}$$

$$x_i - x_{i-1} = h_1,\ y_j - y_{j-1} = h_2,\ h := (h_1, h_2),\ h_0 = \min(h_1, h_2); N_1, N_2 > 2. \tag{2.2}$$

Hence, we have obtained the grid $G_h = \{(x,y) : x = ih_1, y = jh_2\}_{(i,j)=(0,0)}^{(N_1,N_2)}$. Consider a vector function $f^h(z,t)$ defined on this grid, $f^h(z,t) = \{f_{i,j}(z,t)\}_{(i,j)=(0,0)}^{(N_1,N_2)}$. For two vector functions $f^h(z,t), g^h(z,t)$ define

$$g^h(z,t)\, f^h(z,t) := \{k_{i,j}(z,t)\}_{(i,j)=(0,0)}^{(N_1,N_2)},\ k_{i,j}(z,t) = g_{i,j}(z,t) \cdot f_{i,j}(z,t). \tag{2.3}$$

Denote

$$\left(f^h(z,t)\right)^2 := \sum_{(i,j)=(0,0)}^{(N_1,N_2)} f_{i,j}^2(z,t).$$

Let B be a Banach space of functions depending on (z,t). Denote B_h the space of above vector functions $f^h(z,t)$ such that $f_{i,j}(z,t) \in B, \forall (i,j) \in [0,N_1] \times [0,N_2]$. We define the norm $\left\| f^h(z,t) \right\|_{B_h}$ in this space as

$$\left\| f^h \right\|_{B_h} := \left(\sum_{(i,j)=(0,0)}^{(N_1,N_2)} \left\| f_{i,j} \right\|_B^2 \right)^{1/2}.$$

We define finite difference second derivatives $\partial_{x,h}^2 f^h(z,t)$ and $\partial_{y,h}^2 f^h(z,t)$ with respect to x and y respectively in the usual way as

$$\partial_{x,h}^2 f^h(z,t) = \left\{ \partial_{x,h}^2 f_{i,j}(z,t) \right\}_{(i,j)=(0,0)}^{(N_1,N_2)}, \ \partial_{y,h}^2 f^h(z,t) = \left\{ \partial_{y,h}^2 f_{i,j}(z,t) \right\}_{(i,j)=(0,0)}^{(N_1,N_2)},$$

$$\partial_{x,h}^2 f_{i,j}(z,t) := \frac{1}{h_1^2} \begin{cases} f_{i-1,j}(z,t) - 2f_{i,j}(z,t) + f_{i+1,j}(z,t) \text{ if } i \neq 0, i \neq N_1, \\ f_{i,j}(z,t) - 2f_{i+1,j}(z,t) + f_{i+2,j}(z,t) \text{ if } i = 0, \\ f_{i,j}(z,t) - 2f_{i-1,j}(z,t) + f_{i-2,j}(z,t) \text{ if } i = N_1 \end{cases}$$

and similarly for $\partial_{y,h}^2 f^h(z,t)$. Hence, if a function $g(x,y,z,t)$ has continuous derivatives up to the fourth order with respect to x, then $\partial_{x,h}^2 g_{i,j}(z,t)$ approximates $g_{xx}(x,y,z,t)$ at the point $(x,y) = (ih_1, jh_2)$ with the accuracy $O(h_1^2), h_1 \to 0$ in the case when $ih_1 \neq 0, A_x$. And it approximates with the accuracy $O(h_1), h_1 \to 0$ in the case when $ih_1 = 0, A_x$. Similarly for the y-derivative. Next, we define the finite difference Laplace operator as

$$\begin{aligned} \Delta_h f_{i,j}(z,t) &: = \partial_z^2 f_{i,j}(z,t) + \Delta_{h,x,y} f_{i,j}(z,t), \\ \Delta_{h,x,y} f_{i,j}(z,t) &: = \partial_{x,h}^2 f_{i,j}(z,t) + \partial_{y,h}^2 f_{i,j}(z,t), \end{aligned}$$

$$\begin{aligned} \Delta_h f^h(z,t) &: = \left\{ \Delta_h f_{i,j}(z,t) \right\}_{(i,j)=(0,0)}^{(N_1,N_2)} \\ &: = \left(\partial_z^2 f_{i,j}(z,t) \right)_{(i,j)=(0,0)}^{(N_1,N_2)} + \left\{ \Delta_{h,x,y} f_{i,j}(z,t) \right\}_{(i,j)=(0,0)}^{(N_1,N_2)} \\ &: = \partial_z^2 f^h(z,t) + \Delta_{h,x,y} f^h(z,t). \end{aligned}$$

Define $a^h(z) := \left\{ a_{i,j}(z) \right\}_{(i,j)=(0,0)}^{(N_1,N_2)}$. Rewrite the problem (1.1), (1.2) in the finite difference form as

$$u_{tt}^h = \Delta^h u^h + a^h(z) u^h, (z,t) \in \mathbb{R} \times (0,T), \tag{2.4}$$

$$u^h(z,0) = 0, u_t^h(z,0) = \delta(z), \tag{2.5}$$

where the product $a^h(z) u^h$ is understood as in (2.3).

Inverse Problem 2 (IP2). *Let the vector function $u^h(z,t)$ be the solution of the problem (2.4), (2.5). Determine the vector function $a^h(z)$ assuming that the following two vector functions $r^h(t), s^h(t)$ are given*

$$u^h(0,t) = r^h(t), \ u_z^h(0,t) = s^h(t), \ t \in (0,T). \tag{2.6}$$

Theorems 1 and 2 are main results of this paper.

Theorem 1. *Let the vector function $a^h(z) \in C^1(\mathbb{R})$ and is bounded in $\mathbb{R}$. Then there exists unique solution of the forward problem (2.4), (2.5) of the form*

$$u_{i,j}(z,t) = \frac{1}{2}H(t-|z|) + \overline{u}_{i,j}(z,t), (i,j) \in [0,N_1] \times [0,N_2], \tag{2.7}$$

where $H(z)$ is the Heaviside function and the function $\overline{u}_{i,j}$ is such that

$$\overline{u}_{i,j} \in C^3(t \geq |z|), \overline{u}_{i,j}(z,t) = 0 \text{ for } t \in (0,|z|]. \tag{2.8}$$

Theorem 2. *Suppose that there exists two pairs of vector functions $\left(u_1^h(z,t), a_1^h(z)\right)$, $\left(u_2^h(z,t), a_2^h(z)\right)$ such that $a_1^h, a_2^h \in C^1(\mathbb{R})$ and vector functions u_1^h,u_2^h are solutions of the problem (2.4), (2.5) with a_1^h and a_2^h respectively of the form (2.7), (2.8). In addition, assume that both vector functions u_1^h, u_2^h satisfy the same conditions (2.6). Let $R>0$ be an arbitrary number and $T>2R$. Then $a_1^h(z) = a_2^h(z)$ for $|z|<R$ and $u_1^h(z,t) = u_2^h(z,t)$ for*

$$(z,t) \in \left\{ \left(1-\frac{R}{T}\right)|z| + \frac{R}{T}t < R, t>0 \right\}.$$

3 Proofs of Theorems 1 and 2

3.1 Proof of Theorem 1

Denoting temporarily $f_{i,j}(z,t) = \Delta_{h,x,y}u_{i,j} + a_{i,j}u_{i,j}$, rewrite (2.4), (2.5) as

$$\partial_t^2 u_{i,j} = \partial_z^2 u_{i,j} + f_{i,j}(z,t), \tag{3.1}$$

$$u_{i,j}(z,0) = 0, \partial_t u_{i,j}(z,0) = \delta(z). \tag{3.2}$$

Hence, D'Alembert formula implies that for $(i,j) \in [0,N_1] \times [0,N_2]$

$$u_{i,j}(z,t) = \frac{1}{2}H(t-|z|) + \frac{1}{2}\int_0^t d\tau \int_{\tau-t+z}^{t-\tau+z} \left(\Delta_{h,x,y}u_{i,j} + a_{i,j}u_{i,j}\right)(\xi,\tau)\,d\xi. \tag{3.3}$$

In (3.3) the integration is carried out over the triangle $\Delta(z,t)$ in the (ξ,τ)−plane, where $\Delta(z,t)$ has vertices at $(\xi_1,\tau_1) = (z-t,0), (\xi_2,\tau_2) = (z,t)$ and $(\xi_3,\tau_3) = (z+t,0)$. The set of equations (3.3) considered for $(i,j) \in [0,N_1] \times [0,N_2]$ is a linear Volterra system of coupled integral equations. It can be solved iteratively as

$$u_{i,j}^{(n)}(z,t) = \frac{1}{2}H(t-|z|) + \frac{1}{2}\int_0^t d\tau \int_{\tau-t+z}^{t-\tau+z} \left(\Delta_{h,x,y}u_{i,j}^{(n-1)} + a_{i,j}u_{i,j}^{(n-1)}\right)(\xi,\tau)\,d\xi.$$

Let $\max_{i,j} \sup_R |a_{i,j}(z)| \leq M, M = const. > 0$. The standard technique for Volterra equations leads to the following estimate

$$\left| u_{i,j}^{(n)}(z,t) \right| \leq \sum_{n=0}^{\infty} \frac{(Ct)^n}{n!}, z \in \mathbb{R}, t > 0 \tag{3.4}$$

where the constant $C = C(h, M)$. Hence, there exists a solution of the integral equation (3.3) such that this solution is bounded for any t. Let $\overline{u}_{i,j}(z,t) = u_{i,j}(z,t) - H(t - |z|)/2$. Then (3.3) implies that the function $\overline{u}_{i,j}(z,t) \in C^1(t \geq |z|)$. Differentiating equation for $\overline{u}_{i,j}(z,t)$, one obtains that $\overline{u}_{i,j}(z,t) \in C^3(t \geq |z|)$, which is the same as (2.8). Thus, the set of functions $u_{i,j}(z,t)$, which solve (3.3), satisfy conditions (3.1), (3.2), (2.7) and (2.8).

Next, let in (3.3) $t < |z|$. Then the rectangle $\Delta(z,t)$ is located below $\{\tau = |\xi|\}$ and above $\{\tau = 0\}$. Hence, we obtain from (3.3)

$$u_{i,j}(z,t) = \frac{1}{2} \int_0^t d\tau \int_{\tau-t+z}^{t-\tau+z} (\Delta_{h,x,y} u_{i,j} + a_{i,j} u_{i,j})(\xi, \tau) \, d\xi \text{ for } t < |z|.$$

Iterating, we obtain from here, similarly with (3.4) that

$$|u_{i,j}(z,t)| \leq \frac{(Ct)^n}{n!}, n = 1, 2, ...$$

Hence, $u_{i,j}(z,t) = 0$ for $t < |z|$. The same way uniqueness of the problem (3.3) can be proven. $\square$

Because of this theorem, we can consider functions $u_{i,j}(z,t)$ only above the characteristic line $\{t = |z|\}$ in the (z,t) plane. Hence, consider new functions $w_{i,j}(z,t) = u_{i,j}(z, t+z), z > 0$. The domain $\{t > z, z > 0\}$ becomes now $\{t > 0, z > 0\}$. Using (2.4) and (2.6)-(2.8), we obtain

$$\partial_z^2 w_{i,j} - 2\partial_z \partial_t w_{i,j} = -\Delta_{h,x,y} w_{i,j} + a_{i,j}(z) w_{i,j}, (z,t) \in \{t > 0, z > 0\}, \tag{3.5}$$

$$w_{i,j}(z,0) = \frac{1}{2}, \tag{3.6}$$

$$w_{i,j}(0,t) = r_{i,j}(t), \ \partial_z w_{i,j}(0,t) = s_{i,j}(t), \ t \in (0,T), \tag{3.7}$$

$$w_{i,j} \in C^3(z, t \geq 0). \tag{3.8}$$

4 The Carleman Estimate

Consider parameters $\alpha \in (0, 1/2), \beta, \nu > 0$ as well as a sufficiently large parameter $\lambda > 1$. We will choose λ later. Consider the functions $\psi(z,t), \varphi(z,t)$,

$$\psi(z,t) = z + \alpha t + 1, \ \varphi(z,t) = \exp\left(\lambda \psi^{-\nu}\right). \tag{4.1}$$

Define the domain D_β as

$$D_\beta = \{(z,t) : z,t > 0, \psi(z,t) < 1+\beta\}. \tag{4.2}$$

Then

$$\partial D_\beta = \cup_{i=1}^{3} \partial_i D_\beta, \tag{4.3}$$

$$\partial_1 D_\beta = \{t = 0, z \in (0,\beta)\}, \tag{4.4}$$

$$\partial_2 D_\beta = \left\{ z = 0, 0 < t < \frac{\beta}{\alpha} \right\}, \tag{4.5}$$

$$\partial_3 D_\beta = \{z,t > 0, \psi(z,t) = 1+\beta\}, \tag{4.6}$$

$$\varphi(z,t) \mid_{\partial_3 D_\beta} = \exp\left(\lambda (1+\beta)^{-\nu}\right) = \min_{\overline{D}_\beta} \varphi(z,t). \tag{4.7}$$

When applying the Carleman estimate of Lemma 1, we will have Dirichlet and Neumann data at $\partial_2 D_\beta$, as it follows from (3.7). At $\partial_3 D_\beta$ the function $\varphi(z,t)$ attains its minimal value, which is one of key points of any Carleman estimate. As to $\partial_1 D_\beta$, we will not have any data at $\partial_1 D_\beta$ when applying Lemma 1. Also, $\partial_1 D_\beta$ is not a level curve of the function $\varphi(z,t)$. Therefore, to make our Carleman estimate valuable, we should prove that the integral over $\partial_1 D_\beta$, which occurs due to the Gauss formula, is non-negative, see the second line of (4.8). The latter is the main new feature of Lemma 1.

Lemma 1 (Carleman estimate). *Let* $\alpha \in (0,1/2)$ and $\beta, \nu > 0$. *Then there exist constants* $\lambda_0 = \lambda_0(\alpha,\beta,\nu) > 1, C = C(\alpha,\beta,\nu) > 0$ *such that the following Carleman estimate holds*

$$\begin{aligned}\int\limits_{D_\beta} (u_{zz} - 2u_{zt})^2 \varphi^2 dzdt \geq C\lambda \int\limits_{D_\beta} \left(u_z^2 + u_t^2 + \lambda^2 u^2\right) \varphi^2 dzdt \\ +C\lambda \int\limits_{\partial_1 D_\beta} \left(u_z^2 + \lambda^2 u^2\right)(z,0)\, \varphi^2(z,0)\, dz \\ -C\lambda^3 \exp\left[2\lambda (\beta+1)^{-\nu}\right] \int\limits_{\partial_3 D_\beta} \left(u_z^2 + u_t^2 + u^2\right) dS,\end{aligned} \tag{4.8}$$

$$\forall \lambda \geq \lambda_0, \forall u \in C^2\left(\overline{D}_\beta\right) \cap \left\{u : u \mid_{\partial_2 D_\beta} = \partial_z u \mid_{\partial_2 D_\beta} = 0\right\}. \tag{4.9}$$

Proof. In this proof $C = C(\alpha,\beta,\nu) > 0$ denotes different positive constants. Consider a new function $v = u\varphi$ and express $u_{zz} - 2u_{zt}$ via v. We have

$$\begin{aligned}
u &= v \exp\left(-\lambda \psi^{-\nu}\right), \\
u_z &= \left(v_z + \lambda\nu\psi^{-\nu-1} v\right) \exp\left(-\lambda\psi^{-\nu}\right), \\
u_{zz} &= \left[v_{zz} + 2\lambda\nu\psi^{-\nu-1} v_z + \lambda^2\nu^2\psi^{-2\nu-2}\left(1 - \frac{(\nu+1)}{\lambda\nu}\psi^\nu\right) v\right] \exp\left(-\lambda\psi^{-\nu}\right), \\
u_{zt} &= \left[v_{zt} + \alpha\lambda\nu\psi^{-\nu-1} v_z + \lambda\nu\psi^{-\nu-1} v_t + \alpha\lambda^2\nu^2\psi^{-2\nu-2}\left(\left(1 - \frac{(\nu+1)}{\lambda\nu}\psi^\nu\right)\right) v\right] \exp\left(-\lambda\psi^{-\nu}\right),
\end{aligned}$$

$$u_{zz}-2u_{zt} = \left[v_{zz}-2v_{zt}+(1-2\alpha)\lambda^2\nu^2\psi^{-2\nu-2}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v\right]\exp\left(-\lambda\psi^{-\nu}\right)$$
$$+\left(2(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z-2\lambda\nu\psi^{-\nu-1}v_t\right)\exp\left(-\lambda\psi^{-\nu}\right).$$

Hence,

$$(u_{zz}-2u_{zt})^2\varphi^2 \geq 2y_2y_1-2y_3y_1, \tag{4.10}$$
$$y_1=\left[v_{zz}-2v_{zt}+(1-2\alpha)\lambda^2\nu^2\psi^{-2\nu-2}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v\right],$$
$$y_2=2(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z,\ \ y_3=2\lambda\nu\psi^{-\nu-1}v_t.$$

We have

$$2y_2y_1=4(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z\left[v_{zz}-2v_{zt}+(1-2\alpha)\lambda^2\nu^2\psi^{-2\nu-2}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v\right]$$
$$=\partial_z\left(2(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z^2\right)+2(1-\alpha)\lambda\nu(\nu+1)\psi^{-\nu-2}v_z^2$$
$$+\partial_t\left(-4(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z^2\right)-4\alpha(1-\alpha)\lambda\nu(\nu+1)\psi^{-\nu-2}v_z^2$$
$$+\partial_z\left[2(1-\alpha)(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v^2\right]$$
$$+6(1-\alpha)(1-2\alpha)\lambda^3\nu^3(\nu+1)\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^{\nu}\right)v^2.$$

Thus,

$$\begin{gathered}2y_2y_1=2(1-\alpha)(1-2\alpha)\lambda\nu(\nu+1)\psi^{-\nu-2}v_z^2\\ +6(1-\alpha)(1-2\alpha)\lambda^3\nu^3(\nu+1)\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^{\nu}\right)v^2\\ +\partial_t\left(-4(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z^2\right)\\ +\partial_z\left[2(1-\alpha)\lambda\nu\psi^{-\nu-1}v_z^2+2(1-\alpha)(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v^2\right].\end{gathered} \tag{4.11}$$

Next we estimate $-2y_3y_1$,

$$-2y_3y_1=-4\lambda\nu\psi^{-\nu-1}v_t\left[v_{zz}-2v_{zt}+(1-2\alpha)\lambda^2\nu^2\psi^{-2\nu-2}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v\right]$$
$$=\partial_z\left(-4\lambda\nu\psi^{-\nu-1}v_tv_z\right)+4\lambda\nu\psi^{-\nu-1}v_{zt}v_z-4\lambda\nu(\nu+1)\psi^{-\nu-2}v_tv_z$$
$$+\partial_z\left(4\lambda\nu\psi^{-\nu-1}v_t^2\right)+4\lambda\nu(\nu+1)\psi^{-\nu-2}v_t^2$$
$$+\partial_t\left[-2(1-2\alpha)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^{\nu}\right)v^2\right]$$
$$-6\alpha(1-2\alpha)\lambda^3\nu^3(\nu+1)\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^{\nu}\right)v^2.$$

Next,

$$4\lambda\nu\psi^{-\nu-1}v_{zt}v_z = \partial_t\left(2\lambda\nu\psi^{-\nu-1}v_z^2\right) + 2\alpha\lambda\nu\left(\nu+1\right)\psi^{-\nu-2}v_z^2.$$

Hence,

$$\begin{aligned} -2y_3y_1 &= 2\lambda\nu\left(\nu+1\right)\psi^{-\nu-2}\left(\alpha v_z^2 - 2v_tv_z + 2v_t^2\right) \\ &-6\alpha\left(1-2\alpha\right)\lambda^3\nu^3\left(\nu+1\right)\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^\nu\right)v^2 \\ &+\partial_t\left[2\lambda\nu\psi^{-\nu-1}v_z^2 - 2\left(1-2\alpha\right)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^\nu\right)v^2\right] \\ &+\partial_z\left[-4\lambda\nu\psi^{-\nu-1}v_tv_z + 4\lambda\nu\psi^{-\nu-1}v_t^2\right]. \end{aligned} \tag{4.12}$$

Summing up (4.11) and (4.12) and taking into account (4.11), we obtain

$$\begin{aligned} \left(u_{zz}-2u_{zt}\right)^2\varphi^2 &\geq 2y_2y_1 - 2y_3y_1 \\ &= 2\lambda\nu\left(\nu+1\right)\psi^{-\nu-2}\left[\left(1-2\alpha+3\alpha^2\right)v_z^2 - 2v_tv_z + 2v_t^2\right] \\ &+6\left(1-2\alpha\right)^2\lambda^3\nu^3\left(\nu+1\right)\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^\nu\right)v^2 \\ &+\partial_t\left[-2\left(1-2\alpha\right)\lambda\nu\psi^{-\nu-1}v_z^2 - 2\left(1-2\alpha\right)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^\nu\right)v^2\right] \\ &+\partial_z\left[2\left(1-\alpha\right)\lambda\nu\psi^{-\nu-1}v_z^2 + 2\left(1-\alpha\right)\left(1-2\alpha\right)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^\nu\right)v^2\right] \\ &+\partial_z\left[-4\lambda\nu\psi^{-\nu-1}v_tv_z + 4\lambda\nu\psi^{-\nu-1}v_t^2\right]. \end{aligned} \tag{4.13}$$

Obviously there exists a constant $C_1 = C_1(\alpha) > 0$ such that

$$\left(1-2\alpha+3\alpha^2\right)a^2 - 2ab + 2b^2 \geq C_1\left(a^2+b^2\right), \forall\alpha \in \left(0, \frac{1}{2}\right),\ \forall a, b \in \mathbb{R}.$$

Hence, integrating (4.13) over D_β, we obtain

$$\begin{aligned} \int\limits_{D_\beta}\left(u_{zz}-2u_{zt}\right)^2\varphi^2 dzdt &\geq 2\lambda\nu\left(\nu+1\right)C_1\int\limits_{D_\beta}\left(v_z^2+v_t^2\right)\psi^{-\nu-2}dzdt \\ &+6\left(1-2\alpha\right)^2\lambda^3\nu^3\left(\nu+1\right)\int\limits_{D_\beta}\psi^{-3\nu-4}\left(1-\frac{(2\nu+3)}{3\lambda\nu}\psi^\nu\right)v^2dzdt \\ &+\int\limits_{\partial_1 D_\beta}\left[2\left(1-2\alpha\right)\lambda\nu\psi^{-\nu-1}v_z^2 + 2\left(1-2\alpha\right)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^\nu\right)v^2\right]dz \end{aligned} \tag{4.14}$$

$$+\int\limits_{\partial_3 D_\beta}\left[-2\left(1-2\alpha\right)\lambda\nu\psi^{-\nu-1}v_z^2-2\left(1-2\alpha\right)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^\nu\right)v^2\right]\cos\left(n,t\right)dS$$

$$+\int\limits_{\partial_3 D_\beta}\left[2\left(1-\alpha\right)\lambda\nu\psi^{-\nu-1}v_z^2+2\left(1-\alpha\right)\left(1-2\alpha\right)\lambda^3\nu^3\psi^{-3\nu-3}\left(1-\frac{(\nu+1)}{\lambda\nu}\psi^\nu\right)v^2\right]\cos\left(n,z\right)dS$$

$$+\int\limits_{\partial_3 D_\beta}\left[-4\lambda\nu\psi^{-\nu-1}v_tv_z+4\lambda\nu\psi^{-\nu-1}v_t^2\right]\cos\left(n,z\right)dS.$$

Here $\cos\left(n,t\right)$ and $\cos\left(n,z\right)$ are cosines of angles between the unit outward normal vector n at $\partial_3 D_\beta$ and positive directions of t and z axis respectively. Since the number $\nu>0$ is fixed, we can incorporate it in the constant C. Change variables back in (4.14) replacing v with $u=v\varphi$. Then for sufficiently large $\lambda\geq\lambda_0\left(\nu,\beta\right)$ (4.14) implies (4.8). $\square$

5 Proof of Theorem 2

It is sufficient to consider the case $z>0$, since the case $z<0$ is similar. Assume that there exist two pairs of vector functions $\left(u^{1,h}\left(z,t\right),a^{1,h}\left(z\right)\right),\left(u^{2,h}\left(z,t\right),a^{2,h}\left(z\right)\right)$ satisfying conditions of this theorem. Then for $z,t>0$ there exist two pairs of functions $\left(w^{1,h}\left(z,t\right),a^{1,h}\left(z\right)\right),\left(w^{2,h}\left(z,t\right),a^{2,h}\left(z\right)\right)$, where $w^{1,h}\left(z,t\right)=u^{1,h}\left(z,t+z\right),w^{2,h}\left(z,t\right)=u^{2,h}\left(z,t+z\right).$ Denote

$$\begin{aligned}\widetilde{w}^h\left(z,t\right)&=w^{1,h}\left(z,t\right)-w^{2,h}\left(z,t\right)=\left\{\widetilde{w}_{i,j}\left(z,t\right)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)},\\ \widetilde{a}^h\left(z\right)&=a^{1,h}\left(z\right)-a^{2,h}\left(z\right)=\left\{\widetilde{a}_{i,j}\left(z\right)\right\}_{(i,j)=(0,0)}^{(N_1,N_2)}\end{aligned}$$

Then (3.5)-(3.7) imply that

$$\widetilde{w}_{zz}^h-2\widetilde{w}_{zt}^h=-\Delta_{h,x,y}\widetilde{w}^h+a^{1,h}\left(z\right)\widetilde{w}^h+\widetilde{a}^h\left(z\right)w^{2,h}\left(z,t\right),\ \ \left(z,t\right)\in\left\{t>0,z>0\right\}, \tag{5.1}$$

$$\widetilde{w}^h\left(z,0\right)=0, \tag{5.2}$$

$$\widetilde{w}^h\left(0,t\right)=0,\ \partial_z\widetilde{w}^h\left(0,t\right)=0,\ t\in\left(0,T\right). \tag{5.3}$$

By (3.8) $\widetilde{w}^h\in C^3\left(\mathbb{R}\times[0,T]\right).$ Hence, setting in (5.1) $t=0$ and using (5.2), we obtain

$$\widetilde{a}^h\left(z\right)=-4\partial_z\partial_t\widetilde{w}^h\left(z,0\right). \tag{5.4}$$

Let $\widetilde{v}^h\left(z,t\right)=\partial_t\widetilde{w}^h\left(z,t\right),v^{2,h}\left(z,t\right)=\partial_tw^{2,h}\left(z,t\right).$ Differentiating (5.1) with respect to t and using (5.3) and (5.4), we obtain for $\left(z,t\right)\in\left\{t>0,z>0\right\}$

$$\widetilde{v}_{zz}^h-2\widetilde{v}_{zt}^h=-\Delta_{h,x,y}\widetilde{v}^h+a^{1,h}\left(z\right)\widetilde{v}^h-4\partial_z\widetilde{v}^h\left(z,0\right)v^{2,h}\left(z,t\right), \tag{5.6}$$

$$\widetilde{v}^h\left(0,t\right)=0,\ \partial_z\widetilde{v}^h\left(0,t\right)=0,\ t\in\left(0,T\right). \tag{5.7}$$

Since $T > 2R$, then $(0,R/T) \subset (0,1/2)$. In (4.1) choose an arbitrary $\alpha \in (0,R/T)$. Next, set in (4.2) $\beta := R$ and choose an arbitrary $\nu > 0$ in (4.1). Consider equation (5.6) for $\widetilde{v}_{i,j}^{h}$ for an arbitrary pair $(i,j) \in [0,N_1] \times [0,N_2]$. Next, square both sides of the latter equation, multiply by the function $\varphi^2(z,t)$ and integrate over D_R. We obtain with a constant $M = M\left(h_0, \left\|a^{1,h}\right\|_{C[0,R]}, \left\|v^{2,h}\right\|_{C\left(\overline{D}_R\right)}\right) > 0$ depending on listed parameters

$$\int\limits_{D_R} \left(\partial_z^2 \widetilde{v}_{i,j} - 2\partial_z\partial_t \widetilde{v}_{i,j}\right)^2 \varphi^2 dzdt \tag{5.8}$$

$$\leq M \int\limits_{D_R} \left[\widetilde{v}^h(z,t)\right]^2 \varphi^2 dzdt + M \int\limits_{D_R} \left[\widetilde{v}_{i,j}(z,0)\right]^2 \varphi^2 dzdt.$$

Since the function $\varphi^2(z,t)$ is decreasing with respect to t, we obtain from (5.8) and (4.5)

$$\int\limits_{D_R} \left(\partial_z^2 \widetilde{v}_{i,j} - 2\partial_z\partial_t \widetilde{v}_{i,j}\right)^2 \varphi^2 dzdt \tag{5.9}$$

$$\leq M \int\limits_{D_R} \left[\widetilde{v}^h(z,t)\right]^2 \varphi^2 dzdt + M_1 \int\limits_{\partial_1 D_R} \left[\widetilde{v}_{i,j}(z,0)\right]^2 \varphi^2(z,0)\,dz,$$

where the constant $M_1 = M_1(M,R)$.

Applying Lemma 1 to the left hand side of (5.9) and using (5.7), we obtain

$$C\lambda \int\limits_{D_R} \left[(\partial_z \widetilde{v}_{i,j})^2 + (\partial_t \widetilde{v}_{i,j})^2 + \lambda^2 (\widetilde{v}_{i,j})^2\right] \varphi^2 dzdt$$

$$+C\lambda \int\limits_{\partial_1 D_R} \left[(\partial_z \widetilde{v}_{i,j})^2 + \lambda^2 (\widetilde{v}_{i,j})^2\right](z,0)\,\varphi^2(z,0)\,dz$$

$$-C\lambda^3 \exp\left(2\lambda(R+1)^{-\nu}\right) \int\limits_{\partial_3 D_\beta} \left[(\partial_z \widetilde{v}_{i,j})^2 + (\partial_t \widetilde{v}_{i,j})^2 + (\widetilde{v}_{i,j})^2\right] dS$$

$$\leq M \int\limits_{D_R} \left[\widetilde{v}^h(z,t)\right]^2 \varphi^2 dzdt + M_1 \int\limits_{\partial_1 D_R} \left[\widetilde{v}_{i,j}(z,0)\right]^2 \varphi^2(z,0)\,dz.$$

Choose a sufficiently large number $\lambda_0 > 1$ such that

$$\max(M, M_1) < \frac{C\lambda_0^3}{2}. \tag{5.10}$$

Then we obtain from the latter estimate

$$C\lambda \int\limits_{D_R} \left[(\partial_z \widetilde{v}_{i,j})^2 + (\partial_t \widetilde{v}_{i,j})^2 + \lambda^2 (\widetilde{v}_{i,j})^2\right] \varphi^2 dzdt$$

$$+C\lambda^3 \int\limits_{\partial_1 D_R} (\widetilde{v}_{i,j})^2 (z,0) \varphi^2 (z,0)\, dz$$

$$-C\lambda^3 \exp\left[2\lambda (R+1)^{-\nu}\right] \int\limits_{\partial_3 D_\beta} \left[(\partial_z \widetilde{v}_{i,j})^2 + (\partial_t \widetilde{v}_{i,j})^2 + (\widetilde{v}_{i,j})^2\right] dS$$

$$\leq M \int\limits_{D_R} \left[\widetilde{v}^h (z,t)\right]^2 \varphi^2 dz dt.$$

Summing up these estimates with respect to $(i,j) \in [0, N_1] \times [0, N_2]$ and using (5.10), we obtain a stronger estimate

$$\int\limits_{D_R} \left(\widetilde{v}^h\right)^2 \varphi^2 dz dt \leq C \exp\left[2\lambda (R+1)^{-\nu}\right] \int\limits_{\partial_3 D_\beta} \left[\left(\widetilde{v}_z^h\right)^2 + \left(\widetilde{v}_t^h\right)^2 + \left(\widetilde{v}^h\right)^2\right] dS. \quad (5.11)$$

Let $\varepsilon \in (0, R)$ be an arbitrary number. By (4.1) and (4.2) $\varphi^2 (z,t) > \exp\left[2\lambda (R+1-\varepsilon)^{-\nu}\right]$ in $D_{R-\varepsilon}$ and $D_{R-\varepsilon} \subset D_R$. Hence, making the estimate (5.11) stronger, we obtain

$$\exp\left[2\lambda (R+1-\varepsilon)^{-\nu}\right] \int\limits_{D_{R-\varepsilon}} \left(\widetilde{v}^h\right)^2 dz dt \leq C \exp\left[2\lambda (R+1)^{-\nu}\right] \int\limits_{\partial_3 D_\beta} \left[\left(\widetilde{v}_z^h\right)^2 + \left(\widetilde{v}_t^h\right)^2 + \left(\widetilde{v}^h\right)^2\right] dS.$$

Or

$$\int\limits_{D_{R-\varepsilon}} \left(\widetilde{v}^h\right)^2 dz dt \leq C \exp\left\{-2\lambda \left[(R+1-\varepsilon)^{-\nu} - (R+1)^{-\nu}\right]\right\} \int\limits_{\partial_3 D_\beta} \left[\left(\widetilde{v}_z^h\right)^2 + \left(\widetilde{v}_t^h\right)^2 + \left(\widetilde{v}^h\right)^2\right] dS.$$

Setting here $\lambda \to \infty$, we obtain

$$\int\limits_{D_{R-\varepsilon}} \left(\widetilde{v}^h\right)^2 dz dt = 0. \quad (5.12)$$

Since $\varepsilon \in (0, R)$ is an arbitrary number, then (5.12) implies that $\widetilde{v}^h (z,t) = 0$ in D_R. Since by (5.4) $\widetilde{a}^h (z) = -4\partial_z \partial_t \widetilde{w}^h (z,0) = -4\partial_z \widetilde{v}^h (z,0)$, then $\widetilde{a}^h (z) = 0$ for $z \in (0,R)$. Finally since by (5.2)

$$\widetilde{w}^h (z,t) = \int\limits_0^t \widetilde{v}^h (z,\tau)\, d\tau,$$

then $\widetilde{w}^h (z,t) = 0$ in D_R. $\square$

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