L^p-BOUNDS FOR QUASI-GEOSTROPHIC EQUATIONS VIA FUNCTIONAL ANALYSIS

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ABSTRACT. We give a proof of L^p-bounds for the quasi-geostrophic equation and other non-local equations.

The proof uses mainly tools from functional analysis, notably the product formulas (also known as "operator splitting methods") and the Bochner-Pollard subordination identities, hence it could be applicable to other equations.

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1. INTRODUCTION

We consider the following PDE:

$$\begin{cases} (\partial_t + v \cdot \nabla)u = -\kappa(-\Delta)^s u \\ \nabla \cdot v = 0, \end{cases} \tag{1}$$

where we assume v is known and $x \in \mathbb{R}^n$ or $x \in \mathbb{T}^n$ (we can also consider the case $x \in \Omega \subset \mathbb{R}^n$ where Ω is a smooth domain provided that v satisfies some extra conditions: in such case, the equation (1) is supplemented with boundary conditions on $\partial\Omega$). As usual, the time variable is denoted by $t \in \mathbb{R}^+$.

We consider the initial value problem, with the initial condition

$$u(x, 0) = u_0(x). \tag{2}$$

The PDE in (1) is a transport-diffusion equation, and several equations in fluid dynamics reduce to it. In many applications, the vector v depends on the vector u. For the quasi-geostrophic equation in dimension 2 (see [CC04] and references therein) $u = u(x, t)$ has the physical meaning of a potential temperature, $v = v(x, t)$ is the velocity vector, and $\kappa \geqslant 0$ is the viscosity.

In the physical framework, u and v are related by the equation $v = -\nabla^\perp(-\Delta)^{-1/2}u$. On the other hand, in this paper, we will prove the results for a more general v, fixed independently of u. The reason for this generalization is that the relation between u and v complicates the existence and regularity results, but once existence and regularity results are accomplished, then the a-priori bounds can be proved for the v thus obtained in a larger generality.

We suppose that

$$s \in \left(\frac{1}{2}, 1\right), \tag{3}$$

hence $(-\Delta)^s$ is a nonlocal operator (see, e.g., [Lan72, Ste70, Val09] and references therein for further motivations on the fractional Laplacian). As customary, Δ, div and ∇ refer to derivatives in x only, while the derivatives with respect to t will be always written explicitly. The proof presented also works in the case $s = 1$ (see Remark 3.1, but then, in this case, the result is classical).

We assume that

$$\text{for any } T \geqslant 0, \ \sup_{|t|\leqslant T} \big\| v(\cdot,t)\big\|_{C^1} < \infty \tag{4}$$

and

$$\operatorname{div} v = 0. \tag{5}$$

That the L^p-norm (computed over the space variables) of the solution u is non-increasing is one of the main results of [CC04]:

Theorem 1 ([CC04], page 516)**.** *Let v be a fixed vector field. satisfying* (5). *Let u be a smooth solution of* (1) *with either $x \in \mathbb{R}^n$ or $x \in \mathbb{T}^n$.*

Then, for $1 \leqslant p \leqslant \infty$,

$$\|u(\cdot,t)\|_{L^p} \leqslant \|u_0\|_{L^p}. \tag{6}$$

The goal of this note is to give a new, proof of the above result, using only functional analysis tools (such as the Trotter-Lie-Kato product formula and the Bochner-Pollard subordination identity). In contrast with the proof in [CC04] which relies on explicit integral representations of the fractional Laplacian for all the arguments, the present proof includes hypothesis that are functional analysis (Δ generates a contraction semigroup in L^p) and geometric ($e^{t\Delta}$ satisfies a comparsison principle). Hence they apply just as well to the cases in which Δ is a second order uniformly elliptic operator in divergence form. which generates a contration semigroup in L^p. This is, of course, implied by the fact that it is uniformly elliptic and has bounded coefficients.

When Δ a self-adjoint, negative definite operator operator on L^2, we present also a very short proof, which, however, applies only for $p \in [2, \infty]$).

Theorem 2. *The conclusions or Theorem 1 also hold when Δ is replaced by a second order, uniformly elliptic operator in divergence form with uniformly bounded smooth coefficients in $\mathbb{T}^n$ or $\mathbb{R}^n$.*

They also hold when $x \in \Omega \subset \mathbb{R}^n$ is a domain with smooth boundary, v preserves $\partial\Omega$ and u is required to satisfy Dirichlet boundary conditions.

We also note that, if we consider a fixed v, the proof presented here also provides a comparison principle (see § 3.2 for details): that is, if $u_0 \geqslant \tilde{u}_0$, we get $u(t) \geqslant \tilde{u}(t)$. Of course, when the v depends on u, the comparison does not hold.

The idea of the proof is as follows. We introduce the notation, for fixed $t \in \mathbb{R}$, we consider the unbounded linear operators A and B on L^p given by

$$\begin{aligned} (A_t f)(x) &:= -v(x,t) \cdot \nabla f(x) \\ \text{and } \ (Bf)(x) &= -\kappa(-\Delta)^s f(x) \end{aligned} \tag{7}$$

we can construct two objects, namely:

- $\mathcal{A}_s^t$, which is the evolution semiflow of the transport term, and solves (in the usual sense of semiflow theory) the equation $\partial_t \mathcal{A}_x^t = (v \cdot \nabla)\mathcal{A}_x^t$, $\mathcal{A}_s^s = \mathrm{Id}$,
- and the evolution semigroup of the diffusion part e^{tB}.

We will show that

$$\text{each of the two evolutions does not increase the } L^p \text{ norm.} \tag{8}$$

Then, we will construct $\mathcal{G}_s^t$, the evolution semiflow generated of $A_t + B$, and show that it satisfies

$$\mathcal{G}_s^t = \lim_{N\to\infty} \mathcal{A}^t_{s+(t-s)((N-1)/N)} e^{(t-s)/N\, B} \cdots \mathcal{A}_s^{s+(t-a)(1/N)} e^{(t-s)/N\, B}. \tag{9}$$

Formulas of the form (9) have been called "operator splitting methods" in the numerical analysis literature [CMHM78] and some versions of them apply to some non-linear operators too. In our case, they are a time-dependent version of Trotter-Kato product formulas. Then, Theorem 1 will follow by combining (8) and (9).

We will present two proofs of Theorem 1. The first proof is contained in § 2 and it extends the methods of semigroup theory to the case of time dependent vector fields. In a second proof we show that one can reduce to the time independent case by considering functions with one variable more. The second proof, which is presented in § 3.4, is perhaps shorter theoretically, but, for time stepping algorithms, it has the disadvantage that it requires functions of one variable more.

The paper is organised as follows: in § 2, we give a first proof of Theorem 1. Then, we collect in § 3 some further observations, such as the particular case $s = 1$, that is the standard Laplace case, a comment on how to obtain a maximum principle from the results presented in this paper, how to improve the decay bounds in the case of bounded domains, and a second proof of Theorem 1.

2. Proof of Theorem 1

As indicated on the introduction, we will proceed to construct the evolution of the transport part, the diffusion part the full evolution. As well as proving some preliminary properties, including that each of transport evolution and the diffusion evolution satisfy (6). We will also establish some other properties such as strong continuity, and exponential bounds

Then, using these preliminary estimates, we will show that (9) holds from which it is clear that the a-priori bounds (6) hold also for the full evolution.

2.0.1. *Results for the transport evolution.* We will now show that one can construct the evolution of the transport part. That is, we find a strongly continuous family $\mathcal{A}_s^t$ with $\mathcal{A}_s^s = \mathrm{Id}$, such that, for any fixed $f \in W^{1,p}$, $\frac{d}{dt}\mathcal{A}_s^t f = f$ (here the derivative is understood in L^p sense). The technique used for this construction is the rather standard method of characteristics, so we go quickly over it.

Then, we will show that $\mathcal{A}_s^t$ satisfies

$$\|\mathcal{A}_s^t f\|_{L^p} = \|f\|_{L^p}. \tag{10}$$

For these goals, we proceed as follows. For any $x \in \mathbb{R}^n$ (or $x \in \mathbb{T}^n$, or $x \in \Omega \subset \mathbb{R}^n$, if we are in the setting of Theorem 2), let $\Phi_s^t(x)$ be the solution of the ODE

$$\begin{aligned} &\frac{d}{dt}\big(\Phi_s^t(x)\big) = -v\big(\Phi_s^t(x), t\big) \\ &\Phi_s^s(x) = x. \end{aligned}$$

Such solution is well-defined for $|t| \leqslant t_0$, for a suitable $t_0 > 0$ depending on the Lipschitz norm of v (recall (4)).

Of course, it is enough to show that (6) holds for $t \in [0, T]$ and then repeat the argument, so we will take $t \in [t, T]$ and $T > 0$ suitably small from now on.

We note that, by the standard existence and uniqueness of ODE's, we have

$$\Phi_{t'}^{t''} \circ \Phi_t^{t'} = \Phi_t^{t''}. \tag{11}$$

In particular, $\Phi_t^s = (\Phi_s^t)^{-1}$, so that $\frac{d}{dt}\Phi_t^s(x)|_{t=s} = v(x,s)$.

We also observe that,

$$|\Phi_s^t(x) - x| \leqslant \|v\|_{L^\infty}\, t. \tag{12}$$

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we define

$$\Psi_f(x; s, t) := f\big(\Phi_t^s(x)\big).$$

Differentiating with respect to t the identity $\Psi_f(\Phi_s^t x; s, t) = \Psi_f(\Phi_s^t \circ \Phi_t^s x) = f(x)$, and setting $t = s$, we obtain:

$$\nabla \Psi_f(x, s, s) \cdot v(x, s) + \frac{\partial \Psi_f}{\partial t}(x; s, t)|_{t=s} = 0. \tag{13}$$

Notice also that

$$\Psi_f(x; s, s) = f\big(\Phi_s^t(x)\big) = f(x). \tag{14}$$

Thus, we have that

$$\mathcal{A}_s^t f(x) = f\big(\Phi_s^t(x)\big),$$

providing a solution of the evolution equation. It is also standard to prove that the solution is unique.

Also, if $f \in L^p$ and $p \neq \infty$,

$$\int |\mathcal{A}_s^t f(x)|^p \, dx = \int \big|f\big(\Phi_s^t(x)\big)\big|^p \, dx = \int \big|f(x)\big|^p \, dx$$

because by (5) and Liouville Theorem Φ^t is volume preserving. This establishes (10) for $p \in [1, \infty)$. The case of $p = \infty$ of (10) is clear because $\sup_x |f(\Phi^t(x))| = \sup_x |f(x)|$. This finishes the proof of (10) for smooth functions. Extending the results to all functions in L^p is a standard approximation argument.

Now, prove that $\mathcal{A}_s^t$ is strongly continuous. For this, we consider the space Y given by the functions $f \in C^1 \cap L^p$ with $|\nabla f| \in L^p$. Notice that Y is dense in L^p.

Thus, given $f \in L^p$, we take $j \in \mathbb{N}$, $j \geqslant 1$, and $f_j \in Y$ such that $\|f - f_j\|_{L^p} \leqslant 1/j$ and we use (12) and (10) to deduce that

$$\begin{aligned}
\lim_{t\to s} \|\mathcal{A}_s^t f - f\|_{L^p} &= \lim_{j\to\infty} \lim_{t\to 0} \|\mathcal{A}_s^t f - f\|_{L^p} \\
&\leqslant \lim_{j\to\infty} \lim_{t\to 0} \|f_j \circ \Phi^t - f_j\|_{L^p} + \|\mathcal{A}_s^t f - \mathcal{A}_s^t f_j\|_{L^p} + \|f - f_j\|_{L^p} \\
&\leqslant \lim_{j\to\infty} \lim_{t\to 0} \|\nabla f_j\|_{L^p} \|\Phi_s^t(x) - x\|_{L^\infty} + 2\|f - f_j\|_{L^p} \\
&\leqslant \lim_{j\to\infty} \lim_{t\to 0} \|\nabla f_j\|_{L^p} \|v\|_{L^\infty}\, t + \frac{2}{j} \;=\; 0.
\end{aligned}$$

This establishes the strong continuity of $\mathcal{A}_s^t$.

2.1. **Results for e^{tB}.** Now, we study the operator B and the associated semigroup e^{tB}. We will show that

$$\|e^{tB} f\|_{L^p} \leqslant \|f\|_{L^p}, \tag{15}$$

that

$$e^{tB} \text{ is strongly continuous in } L^p \tag{16}$$

and that

$$\|e^{tB}f\|_{W^{1,p}} \leqslant \frac{C}{t^{1/(2s)}}\|f\|_{L^p}. \tag{17}$$

The proofs will make use of the Bochner-Pollard subordination identity (see, e.g., Proposition 4.7 and Lemma 4.10 in [dlLV09] and also [Pol46]) according to which

$$e^{tB} = \int_0^\infty e^{\sigma t^{1/s}\kappa^{1/s}\Delta}\phi_s(\sigma)\,d\sigma, \tag{18}$$

for a suitable nonnegative function ϕ_s given rather explicitly in [Pol46].

We remark that, by taking $t = 0$ in (18), we obtain

$$\int_0^\infty \phi_s(\sigma)\,d\sigma = 1. \tag{19}$$

The main idea of the subordination identity is that we can transfer properties from the heat equation to the fractional Laplacian. For example, the maximum principle of for the heat equation, using the positivity of ϕ_s implies a comparison principle for the $e^{t(-\Delta)^s}$.

One can use several methods to establish the desired properties of the heat equation, either using explicit representations or using more functional analysis [Wid75, Fat83] We will present two methods, using the explicit formulas for the Laplace equation (see § 2.1.1) and using functional analysis tools (see § 2.1.2).

2.1.1. *Properties of the heat kernel in* $\mathbb{R}^n$. In this section, we present the proofs of (15), (16) and (17) using the explicit representation of the heat kernel in $\mathbb{R}^n$.

We define the translation of a function f by a vector y as

$$\tau_y f(x) := f(x-y).$$

It is classical (see, e.g., [Eva98, p. 47]) that

$$e^{t\Delta}f(x) = \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-|y|^2/(4t)}\tau_y f(x)\,dy \tag{20}$$

and therefore, by a standard triangle inequality for integrals,

$$\|e^{t\Delta}f\|_{L^p} \leqslant \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-|y|^2/(4t)}\|\tau_y f\|_{L^p}\,dy = \|f\|_{L^p}. \tag{21}$$

Making use of (18), (19) and (21), we obtain

$$\|e^{tB}f\|_{L^p} \leqslant \int_0^\infty \|e^{\sigma t^{1/s}\kappa^{1/s}\Delta}f\|_{L^p}\,\phi_s(\sigma)\,d\sigma \leqslant \|f\|_{L^p},$$

that is (15).

Now, we prove (16). For this, exploiting (20) and the triangle inequality, we see that

$$\begin{aligned}
\|e^{t\Delta}f - f\|_{L^p} &= \left\|\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-|y|^2/(4t)}\Big(\tau_y f(x) - f(x)\Big)\,dy\right\|_{L^p}\\
&= \left\|\frac{1}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{-|\zeta|^2}\Big(\tau_{\sqrt{4t}\zeta}f(x) - f(x)\Big)\,d\zeta\right\|_{L^p}\\
&\leqslant \frac{1}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{-|\zeta|^2}\Big\|\tau_{\sqrt{4t}\zeta}f - f\Big\|_{L^p}\,d\zeta.
\end{aligned}$$

Therefore, recalling (18),

$$\begin{aligned}\|e^{tB}f-f\|_{L^p} &= \left\|\int_0^\infty \left(e^{\sigma t^{1/s}\kappa^{1/s}\Delta}f-f\right)\phi_s(\sigma)\,d\sigma\right\|_{L^p}\\ &\leqslant \int_0^\infty \left\|e^{\sigma t^{1/s}\kappa^{1/s}\Delta}f-f\right\|_{L^p}\phi_s(\sigma)\,d\sigma\\ &\leqslant \frac{1}{\pi^{n/2}}\int_0^\infty\left[\int_{\mathbb{R}^n} e^{-|\zeta|^2}\left\|\tau_{\sqrt{4\sigma t^{1/s}\kappa^{1/s}}\zeta}f-f\right\|_{L^p}\phi_s(\sigma)\,d\zeta\right]d\sigma.\end{aligned}$$

Accordingly, using the Dominated Convergence Theorem and the continuity of the translations in L^p, we conclude that

$$\lim_{t\to 0^+}\|e^{tB}f-f\|_{L^p}=0,$$

that is (16).

Now we prove (17), with an argument close to the one used in (15) (here C is a constant that may be different from line to line). For this, using (20), and changing variables of integration, we see that

$$\begin{aligned}\partial_{x_j}e^{t\Delta}f(x) &= \partial_{x_j}\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}e^{-|x-y|^2/(4t)}f(y)\,dy\\ &= Ct^{-(n+1)/2}\int_{\mathbb{R}^n}e^{-|x-y|^2/(4t)}\,\frac{x_j-y_j}{\sqrt{t}}\,f(y)\,dy\\ &= Ct^{-(n+1)/2}\int_{\mathbb{R}^n}e^{-|y|^2/(4t)}\,\frac{y_j}{\sqrt{t}}\,f(x-y)\,dy\\ &= Ct^{-(n+1)/2}\int_{\mathbb{R}^n}e^{-|y|^2/(4t)}\,\frac{y_j}{\sqrt{t}}\,\tau_y f(x)\,dy.\end{aligned}$$

Consequently,

$$\|\partial_{x_j}e^{t\Delta}f\|_{L^p}\leqslant Ct^{-(n+1)/2}\int_{\mathbb{R}^n}e^{-|y|^2/(4t)}\,\frac{|y|}{\sqrt{t}}\|\tau_y f\|_{L^p}\,dy=Ct^{-1/2}\|f\|_{L^p}.$$

Hence, recalling (18) and (19),

$$\begin{aligned}\|\partial_{x_j}e^{tB}f\|_{L^p} &\leqslant \int_0^\infty \|e^{\sigma t^{1/s}\kappa^{1/s}\Delta}f\|_{L^p}\phi_s(\sigma)\,d\sigma\\ &\leqslant C\int_0^\infty t^{-1/(2s)}\|f\|_{L^p}\phi_s(\sigma)\,d\sigma\\ &= Ct^{-1/(2s)}\|f\|_{L^p}.\end{aligned}$$

This, together with (15), proves (17).

2.1.2. *Results for $e^{-t(-\Delta)^s}$ using functional analysis.* Considerable insight can be obtained also using functional analysis tools, besides the more explicit representations.

Though we focus on the case in which Δ is the standard Laplacian, our argument also works if we take Δ to be a general second order, uniformly elliptic operator (to wit, in the setting of Theorem 2). We note that the domain of Δ is the space $W^{1,p}$, which is defined using the standard Laplacian (supplemented with the appropriate Dirichlet boundary conditions in the case of a domain Ω).

We define $e^{t\Delta}$ via the Hille-Phillips-Yoshida theorem. This gives that $e^{t\Delta}$ is a contraction semigroup on L^p.

We recall that, following standard practice, contraction semigroup means $\|e^{t\Delta}\|\leqslant 1$ and we say that an operator A is a contraction when $\|A\|<1$ We will also use the name evolution semiflow for a family $\mathcal{A}_s^t$, $0 \leqslant s \leqslant t$ of linear operators satisfying $\mathcal{A}_t^{t''} = \mathcal{A}_{t'}^{t''} \mathcal{A}_t^{t'}$, $\mathcal{A}_s^s = \mathrm{Id}$, and we call it a contraction evolution semiflow when $\|\mathcal{A}_s^t\| \leqslant 1$.

The abstract theory of semigroups (see, for example [Sho97, Tay97]) gives that $\|e^{t\Delta}\Delta\|_{\mathcal{L}(L^p,L^p)} \leqslant Ct^{-1}$ or equivalently $\|e^{t\Delta}\|_{\mathcal{L}(W^{1,p},L^p)} \leqslant Ct^{-1}$.

Using the subordination identity, we find that we can define the semigroup $e^{-t(-\Delta)^s}$ and that we have:

$$
\begin{aligned}
&\|e^{-t(-\Delta)^s}\Delta\|_{\mathcal{L}(L^p,L^p)} \leqslant Ct^{-1} \\
&\|e^{-t(-\Delta^s}\|_{\mathcal{L}(W^{1,p},L^p)} \leqslant \|e^{-t(-\Delta^s}\|_{\mathcal{L}(W^{2s,p},L^p)} \leqslant Ct^{-1}
\end{aligned} \tag{22}
$$

The evolution generated by second order elliptic operators satisfy the maximum principle. That is, $\|e^{t\Delta} f\|_{L^\infty} \leqslant \|f\|_{L^\infty}$.

Using the subordination identity (see (18) and (19)) and the fact that $\phi_s \geqslant 0$, we obtain that

$$
\|e^{-t(-\Delta)^s} f\|_{L^\infty} \leqslant \|f\|_{L^\infty}.
$$

It is worth noticing that one important particular case of the set-up is when $p = 2$ and $-\Delta$ is selfadjoint. In this case, one can use the functional calculus of selfadjoint operators to define the exponential.

Finally, we note that

$$
\frac{d}{dt}\|e^{-t(-\Delta)^s} f\|_{L^2}^2 = 2\langle -(-\Delta)^s e^{-t(-\Delta)^s} f, e^{-t(-\Delta)^s} f\rangle \leqslant 0 \tag{23}
$$

Then the result for all L^p, $2 \leqslant p \leqslant \infty$ follows from Marzinkiewicz interpolation theorem (see, e.g., [Ste70]).

The smoothing bound in (16) follows because $e^{t\Delta}\Delta$ is, in the spectral representation, just the multiplication by xe^{tx} for $L^2([-\infty, 0], \mu)$, and $e^{-t(-\Delta)^s}\Delta$ is the multiplication by $e^{-t|x|^s}x$. Accordingly, the desired bounds follow by taking the supremum of the multipliers.

2.2. **Results for the evolution semiflow generated by $A_t + B$.** Our goal is now to show that $A_t + B$ generates an evolution $\mathcal{G}_s^t$ which satisfies the flow equation and it is strongly continuous. As a byproduct – which we will not explicitly use – one has that this evolution is smoothing and therefore, compact. We will also establish several regularity properties that will be needed in the proof of the product formula (9) in § 2.3.

We will formulate a fixed point equation for $\mathcal{G}_s^t$. Then, we will show that the fixed point satisfies the flow equation and the desired properties. Similar strategies appear in [Hen81]. The perturbation results use the fact that A_t is lower order than B (and that is why we require (3)).

Given a measurable norm-bounded family of bounded operators $\mathcal{G}_s^t$ from L^p taking values in $W^{1,p}$, we define

$$
\mathcal{T}[\mathcal{G}_s^t] = e^{(t-s)B} + \int_s^t e^{(t-\sigma)B} A_\sigma \mathcal{G}_s^\sigma \, d\sigma \tag{24}
$$

We will find a fixed point for $\mathcal{T}$. The existence of such fixed point follows from the following claim: for small T,

$$
\mathcal{T} \text{ is a contraction over } L^\infty\big([s, s+T], \mathcal{L}(L^p, W^{1,p})\big). \tag{25}
$$

To establish (25), we observe that $\mathcal{T}$ is an affine operator and so it is natural to estimate the norm of its linear part. Indeed,

$$
\|A_t f\|_{L^p} \leqslant C\|f\|_{W^{1,p}}, \tag{26}
$$

thanks to (4). Thus, given any $\Gamma \in L^\infty\big([s,s+T],\mathcal{L}(L^p,W^{1,p})\big)$, we have

$$\begin{aligned}
&\sup_{t\in[s,s+T]}\left\|\int_s^t e^{(t-\sigma)B}A_\sigma\Gamma_s^\sigma\,d\sigma\right\|_{\mathcal{L}(L^p,W^{1,p})}\\
&\quad\leqslant \sup_{t\in[s,s+t]}\int_0^t \|e^{(t-\sigma)B}\|_{\mathcal{L}(L^p,W^{1,p})}\|A_\sigma\|_{\mathcal{L}(W^{1,p},L^p)}\|\mathcal{G}_s^\sigma\|_{\mathcal{L}(L^p,W^{1,p})}\,d\sigma\\
&\quad\leqslant C\sup_{t\in[s,s+T]}\int_s^t (t-\sigma)^{-1/(2s)}\,d\sigma\cdot\sup_{\vartheta\in[s,s+T]}\|\Gamma_2^\vartheta\|_{\mathcal{L}(L^p,W^{1,p})}\\
&\quad\leqslant CT^{(2s-1)/(2s)}\sup_{t\in[s,s+T]}\|\Gamma_s^t\|_{\mathcal{L}(L^p,W^{1,p})}\\
&\quad\leqslant \frac{1}{2}\sup_{t\in[s,s+T]}\|\Gamma_s^t\|_{\mathcal{L}(L^p,W^{1,p})}.
\end{aligned}\tag{27}$$

thanks to (3), (17) and (26), if T is small enough.

Now, we use (27) with $\Gamma := \mathcal{G}-\mathcal{H}$, for any given G, $H\in L^\infty\big([s,s+T],\mathcal{L}(L^p,W^{1,p})\big)$: we have that

$$\begin{aligned}
&\|\mathcal{T}(\mathcal{G})-\mathcal{T}(\mathcal{H})\|_{L^\infty\big([s,s+T],\mathcal{L}(L^p,W^{1,p})\big)} = \sup_{t\in[0,T]}\left\|\int_s^t e^{(t-\sigma)B}A_\sigma(\mathcal{G}-\mathcal{H})_s^\sigma\,d\sigma\right\|_{\mathcal{L}(L^p,W^{1,p})}\\
&\quad\leqslant \frac{1}{2}\|\mathcal{G}-\mathcal{H}\|_{L^\infty\big([s,s+T],\mathcal{L}(L^p,W^{1,p})\big)}.
\end{aligned}\tag{28}$$

This proves (25) and it allows us to take the fixed point of $\mathcal{T}$ that agrees with Id at $t=s$.

We also have that if $f\in \mathrm{Dom}(B)$, then $\frac{d}{dt}\mathcal{G}_s^t f|_{t=s} = (B+A_s)f$. Indeed, because of the fixed point property, we have

$$\frac{d}{dt}\mathcal{G}_s^t f|_{t=s} = \frac{d}{dt}e^{(t-s)B}f + \frac{d}{dt}\int_s^t e^{(t-\sigma)B}A_\sigma\mathcal{G}_s^\sigma\,d\sigma.$$

Since the integrand is continuous, the existence of the derivative follows from the Fundamental Theorem of Calculus, so that $\mathrm{Dom}(B)=\mathrm{Dom}(B+A_s)$. Of course, in our case, the result is even more concrete because $\mathrm{Dom}(B)=W^{2s,p}\subset W^{1,p}$ and $\mathrm{Dom}(A_t)=W^{1,p}$.

Now we establish that the fixed point of $\mathcal{T}$ satisfies the flow property

$$\mathcal{G}_t^{t''} = \mathcal{G}_{t'}^{t''}G_t^{t'}\tag{29}$$

for $0\leqslant t\leqslant t'\leqslant t''$.

To prove (29), we compute it by dividing the integrand into pieces, using the semigroup property of the exponential and the fixed point equation:

$$\begin{aligned}
\mathcal{G}_t^{t''} &= e^{(t''-t)B} + \int_t^{t''} e^{(t''-\sigma)B}A_\sigma\mathcal{G}_t^\sigma\,d\sigma\\
&= e^{(t''-t')B}e^{(t'-t)B} + e^{(t''-t')B}\int_t^{t'} e^{(t'-\sigma)B}A_\sigma\mathcal{G}_t^\sigma\,d\sigma + \int_{t'}^{t''} e^{(t''-\sigma)B}A_\sigma\mathcal{G}_t^\sigma\,d\sigma\\
&= e^{(t''-t')B}\mathcal{G}_{t'}^{t''} + \int_{t'}^{t''} e^{(t''-\sigma)B}A_\sigma\mathcal{G}_t^\sigma\,d\sigma
\end{aligned}\tag{30}$$

On the other hand, multiplying by $\mathcal{G}_t^{t'}$ on the left the equation satisfied by $\mathcal{G}_{t'}^{t''}$ we obtain

$$\mathcal{G}_{t'}^{t''}\mathcal{G}_t^{t'} = e^{(t''-t')B}\mathcal{G}_t^{t'} + \int_{t'}^{t''} e^{t''-\sigma}A_\sigma\mathcal{G}_{t'}^\sigma\mathcal{G}_t^{t'}\,d\sigma\tag{31}$$

Hence, subtracting (30) and (31), we obtain that the function $\mathcal{H}(t'') \equiv \mathcal{G}_t^{t''} - \mathcal{G}_{t'}^{t''}\mathcal{G}_t^{t'}$ satisfies

$$\mathcal{H}(t'') = \int_{t'}^{t''} e^{(t''-\sigma)B} A_\sigma \mathcal{H}(\sigma)\, d\sigma.$$

We have already argued that the right hand side of the above equation is a strict contraction for $t'' - t'$ small. Hence, we conclude that $\mathcal{H} \equiv 0$, and this proves (29).

Notice that since, by construction, $\mathcal{G}_s^t \in L^\infty\big([0,T], \mathcal{L}(L^p, W^{1,p})\big)$, we have that

$$C_0 \geqslant \sup_{t\in[0,T]} \|\mathcal{G}_s^t\|_{\mathcal{L}(L^p,W^{1,p})}, \tag{32}$$

for some $C_0 > 0$.

Now, for $t \geqslant \sigma \geqslant 0$, we recall that $\|e^{(t-\sigma)B}\|_{\mathcal{L}(L^p,L^p)}$ and $\|A\|_{\mathcal{L}(W^{1,p},L^p)}$ are bounded, due to (15) and (26). Therefore, using (32), we conclude that

$$\begin{aligned} \left\|\int_s^t e^{(t-\sigma)B} A_\sigma \mathcal{G}_s^\sigma\, d\sigma\right\|_{\mathcal{L}(L^p,L^p)} &\leqslant \int_s^t \left\|e^{(t-\sigma)B} A_\sigma \mathcal{G}_s^\sigma\right\|_{\mathcal{L}(L^p,L^p)} d\sigma \\ &\leqslant \int_s^t \|e^{(t-\sigma)B}\|_{\mathcal{L}(L^p,L^p)} \|A_\sigma\|_{\mathcal{L}(W^{1,p},L^p)} \|\mathcal{G}_s^\sigma\|_{\mathcal{L}(L^p,W^{1,p})}\, d\sigma \\ &\leqslant \lambda(t-s) \end{aligned} \tag{33}$$

for some

$$\lambda \geqslant 0 \tag{34}$$

and so, using (15) once again,

$$\begin{aligned} \|\mathcal{G}_s^t\|_{\mathcal{L}(L^p,L^p)} &\leqslant \|e^{tB}\|_{\mathcal{L}(L^p,L^p)} + \left\|\int_0^t e^{(t-\sigma)B} A_\sigma G_s^\sigma\, d\sigma\right\|_{\mathcal{L}(L^p,L^p)} \\ &\leqslant 1 + \lambda(t-s) \\ &\leqslant e^{\lambda(t-s)}. \end{aligned} \tag{35}$$

2.3. **Proof of the product formula** (9)**.** The proof follows very closely the argument of [Nel64, Theorem 9], but we present here the short proof since our result differs from the result in [Nel64] in that

- we consider non-autonomous problems,
- we allow that the group $\mathcal{G}$ grows exponentially,
- and we do not assume[1] that it is a contraction semiflow.

The proof that we present below also takes advantage of the fact that $\mathrm{Dom}(A_t+B) = \mathrm{Dom}(B)$ and that, therefore, $\mathrm{Dom}(B)$ is a Banach space with the norm $\|f\|_B \equiv \|Bf\| + \|f\|$. In our case, $\|f\|_B = \|f\|_{W^{2s,p}}$.

We notice that the proof of [Nel64] is simpler than other proofs available in the literature, because it assumes that $A_t + B$ generate an evolution semiflow and that $A_t + B$ is closed, rather than showing it. This is what we have accomplished in the previous sections and now we will take advantage of it.

To simplify the typography, for fixed $0 < s < t \in \mathbb{R}$, $N \in \mathbb{N}$ we introduce the notation $h = (t-s)/N$, $t_i = s + ih$.

[1] Of course one of the conclusions of (9) is that indeed $\mathcal{G}$ is a contraction evolution semiflow when $\mathcal{A}_s^t$, e^{tB} are, but we have not proved it yet.

We observe that

$$\begin{aligned}
&\mathcal{A}^t_{t_{N-1}} \circ e^{hB} \mathcal{A}^{t_{N-1}}_{t_{N-2}} \cdots \mathcal{A}^{t_1}_{t_0} e^{hB} - \mathcal{G}^t_{t_{N-1}} \mathcal{G}^{t_{N-1}}_{t_{N-2}} \cdots \mathcal{G}^{t_1}_{t_0} = \\
&\quad = \left(\mathcal{A}^t_{t_{N-1}} \circ e^{hB} - \mathcal{G}^t_{t_{N-1}} \right) \mathcal{G}^{t_{N-1}}_{t_0} + \\
&\qquad + \mathcal{A}^t_{t_{N-1}} \circ e^{hB} \left(\mathcal{A}^{t_{N-1}}_{t_{N-2}} e^{hB} - \mathcal{G}^{t_{N-1}}_{t_{N-2}} \right) \mathcal{G}^{t_{N-2}}_{t_0} \\
&\qquad + \cdots + \\
&\qquad + \mathcal{A}^t_{t_{N-1}} \circ e^{hB} \mathcal{A}^{t_{N-1}}_{t_{N-2}} \cdots \mathcal{A}^{t_2}_{t_1} e^{hB} \left(\mathcal{A}^{t_1}_{t_0} e^{hB} - \mathcal{G}^{t_1}_{t_0} \right).
\end{aligned} \tag{36}$$

Note that all the terms in (36) contain a factor in parenthesis (where the interesting cancellation will take place) and the other factors are all either $\mathcal{A}$, e^{hB} – whose norm is bounded by 1 – or $\mathcal{G}^{t_i}_s$ – whose norm is bounded by $e^{\lambda(t-s)}$. Accordingly, the product of the norms of all the factors in a term, except for the term in parenthesis, can be bounded by $e^{\lambda(t-s)}$.

Now we proceed to estimate the factors in parenthesis in (36), by computing their action on a vector f.

The main observation of [Nel64] is that if $f \in \mathrm{Dom}(B)$, then

$$\|(\mathcal{A}^{t+h}_t e^t hB - \mathcal{G}^{t+h}_t) f\|_{L^p} \leqslant Ch \|f\|_{W^{2s,p}}$$

and, moreover,

$$(\mathcal{A}^{t+h}_t e^t hB - \mathcal{G}^{t+h}_t) f \tag{37}$$

tends to zero. This observation can be verified here because

$$\begin{aligned}
\mathcal{A}^{t+h}_t f &= (\mathrm{Id} + hA_t) f + o(h), \\
e^{hB} f &= (\mathrm{Id} + hB) f + o(h) \\
\text{and} \qquad \mathcal{G}^{t+h}_t f &= (\mathrm{Id} + h(A_t + B)) f + o(h).
\end{aligned}$$

Moreover, as $h \to 0$, we have that (37) tends uniformly to zero for f on any compact set of $W^{2s,p}$.

We recall that A_t, considered as an operator from $W^{1,p}$ to L^p, depends continuously on t. A fortiori, it is continuous on t when considered from $W^{2s,p}$ to $W^{1,p}$. Hence, we can obtain that the convergence of (37) as h tends to zero is uniform for a compact set of t.

Now, we note that for any $u > 0$, $\mathcal{G}^{s+u}_s$ is a compact operator in $W^{2s,p}$ as it follows from the fact that it maps a ball into the domain of $((-\Delta)^s)^r$ for any r.

Therefore, we obtain that $\|(\mathcal{A}^{t+h}_t e^{hB} - \mathcal{G}^{t+h}_t) \mathcal{G}^t_s f\|_{W^{2s,p}}/h$ tends to zero uniformly in compact sets of t which exclude $t = s$ and for f in a bounded set of $W^{2s,p}$.

Now, we can estimate (36). Indeed, given $\epsilon > 0$ we can select a $t_\star > s$ such that

$$\sum_{t_i < t_*} \|(\mathcal{A}^{t_i+h}_{t_i} e^{hB} - \mathcal{G}^{t_i+h}_{t_i}) \mathcal{G}^{t_i}_s f\|_{L^p} \leqslant Ch \|f\|_{W^{2s,p}} \leqslant \epsilon/3.$$

Then, we have, for $t_i > t_\star$,

$$\|(\mathcal{A}^{t_i+h}_{t_i} e^t hB - \mathcal{G}^{t_i+h}_{t_i}) \mathcal{G}^{t_i}_s\|_{L^p} \leqslant Ch \|f\|_{W^{2s,p}} \leqslant o(1/N)$$

uniformly in i. The desired result follows by observing that in the sum there are at most N terms, so that the sum is less than $No(1/N)$, so that it can be made smaller than $\epsilon/3$ by taking N large enough.

This completes the proof[2] of (9). Consequently, the claim in Theorem 1 follows by combining (9), (10) and (15).

[2]We remark that the above proof of (9) is not so useful for numerical analysis since it uses compactness arguments. Nevertheless, using explicit regularity modulus – which depend on the details of the operators

3. Some final observations

3.1. **The case s= 1.** The case $s = 1$ (i.e. when the fractional Laplacian reduces to the classical Laplacian) can also be established by these methods. In that case, we do not need to use subordination. The L^p bounds of e^{tB} needed are just the L^p bounds of the heat equation.

3.2. **A maximum principle.** Notice that the present proof also has a comparison principle as a corollary.

Indeed, since $e^{t\Delta}f \geqslant 0$ for $f \geqslant 0$ (which is a maximum principle for parabolic equations) the subordination formula gives $e^{tB}f \geqslant 0$. The maximum principle for e^{tA} is clear from the method of characteristics.

With this, we also obtain the maximum principle for $\left(e^{\frac{t}{n}A}e^{\frac{t}{n}B}\right)^n$ and, by the product formula (9), we obtain the maximum principle for $e^{t(A+B)}$.

3.3. **Improved estimates in bounded domains.** If, in Theorem 2, we consider the problem defined on a domain with Dirichlet boundary conditions, some refinements on the decay are possible (in this case, as required by Theorem 2, we need that v has no normal component at the boundary of Ω).

Indeed, to estimate (23), we can use the Poincaré inequality and we obtain

$$\frac{d}{dt}\|e^{-t(-\Delta)^s}f\|_{L^2}^2 \leqslant -\mu\|e^{-t(-\Delta)^s}f\|_{L^2}^2$$

and, therefore

$$\|e^{-t(-\Delta)^s}f\|_{L^2} \leqslant C\exp(-t\mu/2).$$

Using Marcinkiewicz interpolation, with the L^∞ bounds coming from the maximum principle, we obtain

$$\|e^{-t(-\Delta)^s}f\|_{L^p} \leqslant C\exp(-t\mu/p).$$

Then, using that $\mathcal{A}_s^t$ preserves the L^p norm and the product formula (9), we obtain that

$$\|\mathcal{G}_x^s f\|_{L^p} \leqslant C\exp(-t\mu/p).$$

3.4. **Reduction of the time dependent case to the time independent one, and a second proof of Theorem 1.** It is a general fact that one can reduce the time-dependent results to the time independent ones using one variable more. This simplifies several aspects of the theoretical proofs but the operators do not become smoothing in the extra variable. Moreover, such procedure is very unsuitable for numerical applications (though quite useful in a purely mathematical approach).

If we consider the function $u(x,s)$ and define $\hat{v}(x,t) = (v(x,t),1)$. So, the partial differential equation (1) is written as

$$\partial_s\hat{u} = \hat{v}\cdot\hat{u} - \kappa(-\Delta_x)^s\hat{u}. \tag{38}$$

Clearly, (38) is equivalent to (1), but it is autonomous. In contrast to the operator B, the corresponding operator $\hat{B}$ is not smoothing in the extra variables.

Nevertheless, the argument presented for the existence of the evolution semiflows can be formulated as well for the existence of the exponential of $\hat{A}+\hat{B}$.

We have, denoting the extended evolution operators by the hat-superscript,

$$e^{t(\hat{A}+\hat{B})}\hat{u}(x,s) = ([\mathcal{G}_s^{s+t}u(.,s-t)](x), s-t)$$

considered, one can get the more explicit bounds on the convergence. There are many variations for different concrete operators in the literature on splitting methods.

and analogously for the other operators.

It is clear that operators $e^{\hat{A}}$, $e^{t\hat{B}}$ are contraction semigroups on $L^\infty(L^p(\mathbb{R}^n))$.

The formula (9) is equivalent to the Trotter-Kato product formula

$$e^{t(\hat{A}+\hat{B})}\hat{f} = \lim_{n\to\infty}\left(e^{\frac{t}{n}\hat{A}}e^{\frac{t}{n}\hat{B}}\right)^n f.$$

A small detail that we take into account is that, even if $e^{t\hat{A}}$ and $e^{t\hat{B}}$ are contraction semigroups, we only have $\|e^{t(\hat{A}+\hat{B})}\| \leqslant e^{\lambda t}$. Therefore, we have that $e^{t(\hat{A}+\hat{B}-\lambda)}$ is a contraction semigroup. and (9) is equivalent to

$$e^{t(\hat{A}+\hat{B}-l\lambda)}\hat{f} = \lim_{n\to\infty}\left(e^{\frac{t}{n}\hat{A}}e^{\frac{t}{n}(\hat{B}-\lambda)}\right)^n f.$$

But the latter is the standard Trotter-Kato product formula, so we obtain that we can define the exponentials of $\hat{A}$, $\hat{B}$, $(\hat{A}+\hat{B})$. Hence, we can use the proof in [Nel64] as is (i.e., we can skip § 2.3), and thus complete this alternative proof of Theorem 1.

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