

KAM THEORY FOR QUASI-PERIODIC EQUILIBRIA IN 1-D QUASIPERIODIC MEDIA

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ABSTRACT. We consider Frenkel-Kontorova models corresponding to 1 dimensional quasicrystals.

We present a KAM theory for quasi-periodic equilibria. The theorem presented has an *a-posteriori* format. We show that, given an approximate solution of the equilibrium equation, which satisfies some appropriate non-degeneracy conditions, then, there is a true solution nearby. This solution is locally unique.

Such a-posteriori theorems can be used to validate numerical computations and also lead immediately to several consequences a) Existence to all orders of perturbative expansion and their convergence b) Bootstrap for regularity c) An efficient method to compute the breakdown of analyticity.

Since the system does not admit an easy dynamical formulation, the method of proof is based on developing several identities. These identities also lead to very efficient algorithms.

Quasi-periodic solutions, quasicrystals, hull functions, KAM theory [2000] 70K43, 52C23, 37A60, 37J40, 82B20

1. INTRODUCTION

We consider Frenkel-Kontorova models with a quasi-periodic potential. In these models, the state of a system is given by a sequence $\{q_i\}_{i \in \mathbb{Z}}$ of numbers and the physical states are selected to be the critical points of a formal energy

$$(1) \quad \mathcal{S}(\{u\}_{i \in \mathbb{Z}}) = \sum_{n \in \mathbb{Z}} \frac{1}{2} (u_n - u_{n+1})^2 - V(u_n)$$

The critical points of \mathcal{S} are obtained by taking formal derivatives of \mathcal{S} and setting them to zero $\frac{\partial}{\partial u_n} \mathcal{S}(\{u\}_{i \in \mathbb{Z}}) = 0$. That is

$$(2) \quad u_{n+1} + u_{n-1} - 2u_n + V'(u_n) = 0.$$

In the classical Frenkel-Kontorova model [FK39, ALD83], V is a periodic function $V(u_n + 1) = V(u_n)$ —we choose units so that the period is normalized to 1.

In this paper, however, we choose V to be a quasi-periodic function

$$V(\theta) = \widehat{V}(\theta\alpha)$$

where $\widehat{V} : \mathbb{T}^d \rightarrow \mathbb{R}$ and $\theta \in \mathbb{R}$, $\alpha \in \mathbb{R}^d$ will be an irrational vector, that is $k \cdot \alpha \neq 0$ when $k \in \mathbb{Z}^d - \{0\}$ where $d \geq 2$. Later, we will also assume further non-resonance conditions. One example to keep in mind could be

$$V(\theta) = a \cdot \sin(2\pi\theta) + b \cdot \sin(2\pi\sqrt{2}\theta)$$

where $a, b \in \mathbb{R}$.

Our goal in this paper is to prove an analogue of KAM theorem. See Theorem 11, 12. Under some appropriate conditions, we show that there are smooth families of quasi-periodic solutions. See Section 1.1 for precise definitions of these families. The proof of the theorems is rather constructive and leads to efficient algorithms that are being implemented.

There are several physical interpretations of FK models. The original motivation [FK39] was dislocations in solids. In the interpretation of [ALD83] u_i are the positions of a deposited material over a substratum. The interaction of the atoms with the substratum is modeled by the term V . The periodicity of V considered in [ALD83] corresponds to a periodic substratum (e.g. a crystal) and the quasi-periodic models considered here could appear in cleaved faces of crystals or in quasi-crystals.

In the interpretation of deposition, the existence of quasiperiodic solutions implies the existence of a continuum of equilibria, so that the system can *slide*. In contrast, if the KAM tori are not present, the system is pinned. There have been numerical explorations of these issues in [vE99, vEFRJ99, vEFJ01, vEF02, RJ97, RJ99]. In particular, the above references pay special attention to the boundary of the set of parameters for which there is an analytic solution (breakdown of analyticity, Aubry transition), which corresponds physically to the boundary between sliding and pinning.

Note that the physical argument to obtain sliding only requires a continuous family of solutions, whereas it has been found that often the boundary is described by the breakdown of an analytic family. In Section 6.2 we will show that all the families which are smooth enough are indeed analytic. From the mathematical point of view this leaves open the possibility that there are regimes where the solutions have only a rather low regularity. In the case of twist mappings, using renormalization group, this regime has found to exist but being a codimension one surface [Koc08].

In the mathematical literature, quasi-periodic Frenkel-Kontorova models have been considered in [GGP06, AP10], which use mainly topological methods to study the existence of orbits with rotation number. In the periodic case, the critical points of the energy, i.e. the configurations solving (2) can be identified with orbits of a dynamical system on the annulus $\mathbb{T} \times \mathbb{R}$.

In the quasi-periodic case, no such identification is easy. [GGP06] shows that the quasi-periodic case can be considered as a dynamical system on a Cantor set (Delone set).

To the best of our knowledge, there is no systematic Aubry-Mather theory analogue to that of the periodic 1-D Frenkel-Kontorova systems (existence of minimizing Aubry-Mather well ordered minimizers). It seems possible that one could get analogues of the theory of minimizing invariant measures. See [Bur87, Bur88, Bur90].

1.1. Hull function. We will be interested in solutions of (2) given by a hull function

$$u_n = h(n \cdot \omega)$$

where $\omega \in \mathbb{R}$ and

$$h(\theta) = \theta + \tilde{h}(\theta)$$

with

$$\tilde{h}(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{h}_k e^{2\pi i k \cdot \theta \alpha}$$

equivalently,

$$\tilde{h}(\theta) = \hat{h}(\theta \alpha)$$

where $\hat{h} : \mathbb{T}^d \rightarrow \mathbb{R}$ is a function

$$\hat{h}(\sigma) = \sum_{k \in \mathbb{Z}^d} \hat{h}_k e^{2\pi i k \cdot \sigma}.$$

We denote the set of \tilde{h} of this type by $QP(\alpha)$. Later on, we always use the notation $\sigma = \theta \alpha$ for variables in \mathbb{T}^d .

Then (2) is equivalent to

$$(3) \quad h(\theta + \omega) + h(\theta - \omega) - 2h(\theta) + \partial_\alpha V(\alpha \cdot h(\theta)) = 0$$

where $\partial_\alpha V \equiv (\alpha \cdot \nabla)V$. We write (3) in terms of \hat{h} which is

$$(4) \quad \hat{h}(\sigma + \omega \alpha) + \hat{h}(\sigma - \omega \alpha) - 2\hat{h}(\sigma) + \partial_\alpha V(\sigma + \alpha \cdot \hat{h}(\sigma)) = 0$$

1.2. External forces. We will find it convenient to add an external parameter to the equilibrium equation (4) and to generalize to situations where the forces do not derive from a potential. So we will be looking for solutions of

$$(5) \quad \mathcal{E}[\hat{h}, \lambda](\theta) \equiv \hat{h}(\sigma + \omega \alpha) + \hat{h}(\sigma - \omega \alpha) - 2\hat{h}(\sigma) + \widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma)) + \lambda = 0$$

where $\widehat{U} : \mathbb{T}^d \rightarrow \mathbb{R}$ and \hat{h} is as before, $\lambda \in \mathbb{R}$. Note that, both \hat{h} and λ are unknown.

Adding the extra term λ will allow us to study the equation (5) for arbitrary periodic \widehat{U} even if \widehat{U} is not the gradient of some periodic function. Later, a very simple argument developed in Section 5 Lemma 22, will show

that, when \widehat{U} has a variational structure, then $\lambda = 0$. Hence, in the case that $\widehat{U} = \partial_\alpha \widehat{V}$, the equation (5) is equivalent to (2).

Remark 1. This procedure of adding an extra parameter and showing that it vanishes when there are geometric properties similar to the extra parameter method introduced in [Mos67] to prove “translated curve theorems” and developed later in [Yoc92, Sev99]. The main advantage is that the extra term allows a more efficient iterative procedure. The proofs of existence of perturbative expansions are also considerably easier.

Our main results will be “a-posteriori” theorems (see Theorem 11 and 12 for more details) which show that if we are given a pair $[\hat{h}, \lambda]$ that solves (5) very approximately (and provided ω, α satisfy some appropriate Diophantine condition and $[\hat{h}, \lambda]$ satisfy some non-degeneracy condition), then there is a true solution close by. We note that we do not need that the system is “close to integrable”, we only need that there is an approximate solution of the equilibrium equations. Of course, in the case that the system is close to integrable, the solutions in the integrable case are approximate solutions. One can, however obtain approximate solutions in other cases, for example, the result can validate numerically produced approximate solutions.

The method of proof is based on an iterative procedure of Nash-Moser type using the quadratic convergence to overcome the small divisors. The quadratic convergence will be based on some geometric cancellations that come from the variational structure of the problem.

We note that the method of proof leads to very efficient algorithms, which we present in Section 4.2 and which are implemented in [BHdLL11]. We note that the method is based on the Lagrangian proof in [Ran87, LM01] and in [dLL01, Section 5.3]. In [dLL08], it is shown that, in the classical case, when V is periodic, the method of proof extends to interactions which are not nearest neighbors. We hope that such an extension is possible in the quasi-periodic case considered here. This extension to interactions which are not nearest neighbors is interesting because the most physical models involve such interactions.

The precise formulation of Theorem 11 and Theorem 12 requires us to make precise the sizes of functions, Diophantine condition, etc. which we take up in Section 2.1.

Note that the solution of (5) are not unique. If $[\hat{h}(\sigma), \lambda]$ is a solution, for any $\beta \in \mathbb{R}$, $[\hat{h}(\sigma + \beta\alpha) + \beta, \lambda]$ is a solution. Hence, by choosing β , we can always choose our solution normalized in such a way that

$$(6) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T \tilde{h}(\theta) d\theta \equiv \int_{\mathbb{T}^d} \hat{h}(\sigma) d\sigma = 0.$$

Note that the choice of β that accomplishes (6) is unique.

We will indeed establish that the solution of (5) and (6) is unique.

2. FUNCTION SPACES AND PRELIMINARY ESTIMATES

In Section 2.1, we collect several standard definitions of spaces and present some preliminary results on these spaces. In Section 2.2 we present definitions of the Diophantine properties we will use in this paper. In Section 2.3 we present well known estimates for cohomology equations, which are the basis of the KAM procedure.

2.1. Spaces. Given a function

$$\tilde{h}(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{h}_k e^{2\pi i k \cdot \alpha \theta}$$

it is natural to consider the function

$$\hat{h}(\sigma) = \sum_{k \in \mathbb{Z}^d} \hat{h}_k e^{2\pi i k \cdot \sigma}$$

where $\sigma \in \mathbb{C}^d / \mathbb{Z}^d$.

Clearly, if the above sums converge pointwise (in our applications they will converge in much stronger senses), we have

$$(7) \quad \tilde{h}(\theta) = \hat{h}(\alpha \theta)$$

We will find it more convenient to define spaces and norms using the function \hat{h} .

Following [CdIL10] we will find it convenient to use at the same time spaces of analytic functions and Sobolev spaces. Using the abstract results in [CdIL10] this leads automatically to several interesting corollaries such as bootstrap of regularity of solutions and a numerically verifiable criterion for the breakdown of analytic solutions. See Section 6.

Definition 2. *If the series $\hat{h}(\eta) = \sum_{k \in \mathbb{Z}^d} \hat{h}_k e^{2\pi i k \cdot \eta}$ defines an analytic function on $D_\rho \equiv \{ \eta \mid |Im(\eta_i)| < \rho \}$ which extends continuously to \overline{D}_ρ , we denote*

$$(8) \quad \|\hat{h}\|_\rho = \sup_{\eta \in \overline{D}_\rho} |\hat{h}(\eta)|$$

We denote by \mathcal{A}_ρ the space of such functions. As it is well known, \mathcal{A}_ρ is a Banach space when endowed with (8). Clearly for $\rho' < \rho$, $\mathcal{A}_\rho \subset \mathcal{A}_{\rho'}$. By the maximum principle, $\|f\|_\rho$ is monotone increasing in ρ for $f \in \mathcal{A}_\rho$.

For the sake of convenience we will also introduce \mathcal{A}_ρ^r , the Banach space of functions whose r derivative is in \mathcal{A}_ρ , endowed with the norm $\|f\|_{\mathcal{A}_\rho^r} = \|D^r f\|_{\mathcal{A}_\rho} + \langle f \rangle$ where $\langle f \rangle$ denotes the average of f . By Cauchy bounds, if $\rho' < \rho$, then $\mathcal{A}_{\rho'}^r \subset \mathcal{A}_\rho^r$.

We also define

$$(9) \quad \|\hat{h}\|_{H^r}^2 = \sum_{k \in \mathbb{Z}^d} |\hat{h}_k|^2 (1 + |k|^2)^r$$

and denote by H^r the spaces of functions for which (9) is finite. It is well known that H^r is a Banach space and indeed a Hilbert space.

2.1.1. *Some elementary properties.* There are several well-known properties of those spaces. We just note

- Cauchy estimates for analytic functions:

$$\begin{aligned} \|D^l \hat{h}\|_{\rho-\delta} &\leq C(l, d) \cdot \delta^{-l} \cdot \|\hat{h}\|_{\rho} \\ |\hat{h}_k| &\leq e^{-2\pi \cdot |k| \cdot \rho} \cdot \|\hat{h}\|_{\rho}. \end{aligned}$$

- Interpolation inequalities:

– Analytic case:

Lemma 3 (Hadamard 3-circle theorem, see [Ste70]).

$$\|\hat{h}\|_{\rho} \leq \|\hat{h}\|_{\rho-\delta}^{\frac{1}{2}} \cdot \|\hat{h}\|_{\rho+\delta}^{\frac{1}{2}}.$$

– Sobolev case:

Lemma 4 (See [Zeh75, Ste70]). For any $0 \leq n \leq j$, $0 \leq \theta \leq 1$, denote $l = (1 - \theta)n + \theta j$, we have for any $\hat{h} \in H^j$:

$$\|\hat{h}\|_l \leq C_{n,j} \cdot \|\hat{h}\|_n^{1-\theta} \cdot \|\hat{h}\|_j^{\theta}.$$

- Banach algebra properties:

– Analytic case:

$$\forall \hat{g}, \hat{h} \in \mathcal{A}_{\rho} : \|\hat{g} \cdot \hat{h}\|_{\rho} \leq \|\hat{g}\|_{\rho} \cdot \|\hat{h}\|_{\rho}.$$

– Sobolev case (see [Ada75]): Let $m > \frac{d}{2}$, there exists a constant K depending only on m , d such that for any $u, v \in H^m$, $u \cdot v \in H^m$ and we have

$$\|u \cdot v\|_{H^m} \leq K \cdot \|u\|_{H^m} \cdot \|v\|_{H^m}.$$

- Composition properties:

– Analytic case (see [dlLO99, dlL08]): Let $\hat{h} \in \mathcal{A}_{\rho}$ and $\Omega \subseteq \mathbb{C}^d$ be a compact set. Take $\iota = \text{dist}(\mathbb{C}^d - \Omega, (Id + \alpha \cdot \hat{h})(\overline{D}_{\rho}))$.

Lemma 5. Let $\widehat{U} : \Omega \rightarrow \mathbb{C}$ be an analytic function $\|\widehat{U}(z)\|_{L^{\infty}(\Omega)} \leq M$. Define the operator Φ acting on analytic functions by $(\Phi[\hat{h}])(\sigma) = \widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma))$.

We present sufficient conditions that ensure that the operator is well defined and differentiable.

* If $\|\hat{h}^* - \hat{h}\|_{\rho} \leq \iota$, then $\Phi_{\hat{h}^*} \in \mathcal{A}_{\rho}$.

* If $\|\hat{h}^* - \hat{h}\|_\rho < \frac{\iota}{2}$, then

$$(D\Phi[\hat{h}]\widehat{\Delta})(\sigma) = \partial_\alpha \widehat{U}(\sigma + \alpha \hat{h}(\sigma))\widehat{\Delta}(\sigma),$$

$$\|\Phi[\hat{h}^*] - \Phi[\hat{h}] - D\Phi[\hat{h}](\hat{h}^* - \hat{h})\|_\rho \leq C \cdot \|\hat{h}^* - \hat{h}\|_\rho^2.$$

– Sobolev case (see [Tay97, CdLL10]):

The following result is a consequence of the Gagliardo-Nirenberg inequalities.

Lemma 6. *Let $f \in C^m$ and assume $f(0) = 0$. Then, for $u \in H^m \cap L^\infty$*

$$\|f(u)\|_{H^m} \leq K_2(\|u\|_{L^\infty})(1 + \|u\|_{H^m}),$$

where $K_2(\lambda) = \sup_{|x| \leq \lambda, \mu \leq m} |D^\mu f(x)|$.

In the case that $m > \frac{d}{2}$, if $f \in C^{m+2}$, we have that

$$(10) \quad \|f \circ (u+v) - f \circ u - Df \circ u \cdot v\|_{H^m} \leq C_{d,m}(\|u\|_{L^\infty})(1 + \|u\|_{H^m})\|f\|_{C^{m+2}}\|v\|_{H^m}^2.$$

The reason for (10) is that we have pointwise

$$\begin{aligned} f \circ (u+v)(x) - f \circ u(x) - Df \circ u(x) \cdot v(x) &= \\ &= \int_0^1 dt \int_0^t ds D^2 f \circ (u + tsv)(x) \cdot u(x) \cdot v(x) \end{aligned}$$

We obtain the desired result using that, by Gagliardo-Nirenberg $D^2 f \circ (u + tsv) \in H^m$ and that the H^m norm is a Banach algebra under multiplication. Therefore, we can estimate the H^m norm of the integrand independently of s, t .

Note that Lemma 6 also gives a formula for the derivative of Φ as the composition with the derivative of \widehat{U} . It follows that if $\widehat{U} \in C^{m+2+\ell}$, then, $\Phi \in C^{\ell+1}$.

Remark 7. In [Zeh75] it is shown that the interpolation inequalities are a consequence of the existence of smoothing operators. which play an important role in the abstract formulation of KAM theorem.

We also note that Lemma 5 is rather elementary. It suffices to show that there are Taylor estimates with uniform constants. (As shown in [dlLO99], it is important that in the domain of \widehat{U} , among any two points x_1, x_2 , it is possible to choose a path γ such that $\ell(\gamma)$, its length satisfies $\ell(\gamma) \leq C|x_1 - x_2|$.)

Actually, one can prove something stronger, namely that the operator Φ is an analytic operator when \widehat{U} is analytic, but we will not need as much.

The following lemma is standard in the theory of quasi-periodic functions.

Lemma 8. *Let α be irrational. Assume that $\sum_{k \in \mathbb{Z}^d} |\hat{h}_k| < \infty$. (So that, by Weierstrass M -test, the series $\tilde{h}(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{h}_k e^{2\pi i k \cdot \alpha \theta}$ converges uniformly over the real line.) Then,*

$$\hat{h}_0 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \tilde{h}(\theta) d\theta.$$

Proof. Given $\epsilon > 0$, we can find N such that

$$\sum_{|k| > N} |\hat{h}_k| \leq \frac{\epsilon}{3}.$$

Hence, for all $T > 0$

$$\left| \frac{1}{2T} \int_{-T}^T \sum_{|k| > N} \hat{h}_k e^{2\pi i k \cdot \alpha \theta} d\theta \right| \leq \frac{\epsilon}{3}.$$

Furthermore,

$$(11) \quad \frac{1}{2T} \int_{-T}^T \sum_{|k| \leq N} \hat{h}_k e^{2\pi i k \cdot \alpha \theta} d\theta = \hat{h}_0 + \sum_{0 < |k| \leq N} \hat{h}_k \frac{e^{2\pi i k \cdot \alpha T} - e^{-2\pi i k \cdot \alpha T}}{2\pi i k \cdot \alpha \cdot 2T}$$

We see that, for all T sufficiently large, we can assume that the term in the sum in (11) (it is a finite sum) is smaller than $\frac{\epsilon}{3}$. This ends the proof of Lemma 8. \square

2.2. Diophantine properties. Given $\alpha \in \mathbb{R}^d$ such that

$$(12) \quad |\alpha \cdot k| \geq \mu |k|^{-\nu}, \quad \forall k \in \mathbb{Z}^d - \{0\}$$

where $|k| = |k_1| + |k_2| + \dots + |k_d|$, we are interested in the numbers $\omega \in \mathbb{R}$ such that

$$(13) \quad |\omega \alpha \cdot k - n| \geq \nu |k|^{-\tau}, \quad \forall k \in \mathbb{Z}^d - \{0\}, n \in \mathbb{Z}.$$

That is, we are interested in the $\omega \in \mathbb{R}$ such that $\omega \alpha$ is a Diophantine vector in the standard sense. Here μ, ν, ν, τ are positive numbers.

We will denote the set of ω satisfying (13) as $\mathcal{D}(\nu, \tau; \alpha)$. We also denote $\mathcal{D}(\tau; \alpha) = \cup_{\nu > 0} \mathcal{D}(\nu, \tau; \alpha)$.

The abundance of Diophantine numbers in subspaces is a subject of great current interest in number theory [Kle01, Kle08]. Nevertheless in our case, it suffices the following elementary result Lemma 9 (which is weaker with respect to the exponents obtained than the results of [Kle01, Kle08]).

Lemma 9. *If $\alpha \in \mathbb{R}^d$ satisfies (12) and $\tau > d + \nu$, then $\mathcal{D}(\tau; \alpha)$ is of full Lebesgue measure.*

Proof. Let $A > 0$. Consider the sets

$$\mathcal{B}_{k,n} = \{\omega \mid |\omega\alpha \cdot k - n| \leq \nu|k|^{-\tau}\} = \{\omega \mid \left| \omega - \frac{n}{\alpha \cdot k} \right| \leq \frac{\nu|k|^{-\tau}}{|\alpha \cdot k|}\}$$

and

$$\mathcal{B}_{k,n,A} = \mathcal{B}_{k,n} \cap [A, 1.01A].$$

Clearly,

$$[A, 1.01A] \setminus \mathcal{D}(\nu, \tau; \alpha) = \cup_{k,n} \mathcal{B}_{k,n,A}$$

So Lemma 9 will be proved when we show

$$|\cup_{k,n} \mathcal{B}_{k,n,A}| \leq \nu \cdot C(A, \tau, \alpha, \nu, \mu).$$

We clearly have that $\mathcal{B}_{k,n}$ is an interval of length

$$|\mathcal{B}_{k,n}| = \frac{2\nu|k|^{-\tau}}{|k \cdot \alpha|}.$$

We also observe that $\mathcal{B}_{k,n,A} = \emptyset$ unless $A + \frac{\nu|k|^{-\tau}}{|k \cdot \alpha|} \leq \frac{n}{k \cdot \alpha} \leq 1.01A + \frac{\nu|k|^{-\tau}}{|k \cdot \alpha|}$ or $A - \frac{\nu|k|^{-\tau}}{|k \cdot \alpha|} \leq \frac{n}{k \cdot \alpha} \leq 1.01A - \frac{\nu|k|^{-\tau}}{|k \cdot \alpha|}$. Also clearly,

$$\#\{n \mid \mathcal{B}_{k,n,A} \neq \emptyset\} \leq 0.02A \cdot |k \cdot \alpha| + 2$$

Hence

$$\begin{aligned} |\cup_{k,n} \mathcal{B}_{k,n,A}| &\leq \sum_{\substack{k,n \\ \mathcal{B}_{k,n,A} \neq \emptyset}} \frac{2\nu|k|^{-\tau}}{|k \cdot \alpha|} \leq \sum_k \frac{2\nu|k|^{-\tau}}{|k \cdot \alpha|} \cdot (0.02A \cdot |k \cdot \alpha| + 2) \\ &\leq 0.02A \cdot 2\nu \cdot \sum_k |k|^{-\tau} + \frac{4\nu}{\mu} \sum_k |k|^{-(\tau-\nu)} \leq \nu \cdot C. \end{aligned}$$

□

2.3. Cohomology equations. To prove our results, as it is standard in KAM theory, we have to study equations of the form

$$(14) \quad \tilde{\phi}(\theta + \omega) - \tilde{\phi}(\theta) = \tilde{\eta}(\theta)$$

where $\tilde{\eta} \in QP(\alpha)$, $\omega \in \mathbb{R}$ are given and $\tilde{\phi}$ is the unknown.

We note that if we use the function $\hat{\phi}(\alpha\theta) = \tilde{\phi}(\theta)$ we see that (14) is equivalent to

$$(15) \quad \hat{\phi}(\sigma + \omega\alpha) - \hat{\phi}(\sigma) = \hat{\eta}(\sigma).$$

The operator solving these equations is unbounded, but it satisfies some ‘‘tame’’ estimates from one space to another that can be overcome by a quadratically convergent algorithm.

Clearly, a necessary solution for the existence of solutions of (15) is

$$\int \hat{\eta}(\sigma) d\sigma = 0.$$

The equation (15) has been considered in KAM theory. The optimal results for analytic functions were proved in [Rüs75] and we just reproduce the results adapted to our notation. The results for Sobolev regularity are very easy.

Lemma 10. *Let $\hat{\eta} \in \mathcal{A}_\rho$ (resp. H^r , $r \geq \tau$) be such that*

$$\int_{\mathbb{T}^d} \hat{\eta}(\sigma) d\sigma = 0.$$

Then, there exists a unique solution $\hat{\phi}$ of (15) which satisfies

$$(16) \quad \int_{\mathbb{T}^d} \hat{\phi}(\sigma) d\sigma = 0.$$

This solution satisfies for any $\rho' < \rho$

$$(17) \quad \|\hat{\phi}\|_{\rho'} \leq C(d, \tau) \cdot \nu^{-1} \cdot (\rho - \rho')^{-\tau} \|\hat{\eta}\|_\rho$$

(resp.

$$(18) \quad \|\hat{\phi}\|_{H^{s-\tau}} \leq C \cdot \nu^{-1} \cdot \|\hat{\eta}\|_{H^s}, \quad \tau \leq s \leq r).$$

Furthermore, any distribution solution of (14) differs from the solution claimed before by a constant.

The method of proof is standard. We take the Fourier coefficients and see that (14) is equivalent to:

$$\hat{\phi}_k (e^{2\pi i k \cdot \alpha \omega} - 1) = \hat{\eta}_k.$$

When $k \neq 0$, we can use that $|e^{2\pi i k \cdot \alpha \omega} - 1|^{-1} \leq C \cdot \text{dist}(k \cdot \alpha \omega, \mathbb{Z})^{-1}$. From this, the result for Sobolev space follows rather straightforwardly. The result for analytic functions – with the optimal exponent quoted above – requires somewhat more elaborate arguments that use not only the upper bounds in the denominators, but also that they are not saturated very often. We refer to [Rüs75] for the original proof and to [dLL01] for a more pedagogical exposition.

3. STATEMENT OF THE MAIN RESULTS

3.1. Statement of the analytic result.

Theorem 11. *Let $h = Id + \tilde{h}$ where $\tilde{h}(\theta) = \hat{h}(\alpha \cdot \theta) = \sum_{k \in \mathbb{Z}^d} \hat{h}_k \cdot e^{2\pi i k \cdot \alpha \theta}$ with $\hat{h}_0 = 0$, $\hat{h} \in \mathcal{A}_\rho^1$ and $\alpha \in \mathbb{R}^d$ such that $\alpha \cdot j \neq 0$, $j \in \mathbb{Z}^d - \{0\}$. Denote $\hat{l} = 1 + \partial_\alpha \hat{h}$ and $T_{-\omega\alpha}(\sigma) = \sigma - \omega\alpha$. We assume*

$$(H1) \text{ Diophantine properties (13): } |\omega\alpha \cdot k - n| \geq \nu |k|^{-\tau}, \quad \forall k \in \mathbb{Z}^d - \{0\}, n \in \mathbb{Z}.$$

- (H2) *Non-degeneracy condition:* $\|\hat{l}(\sigma)\|_\rho \leq N^+$, $\|(\hat{l}(\sigma))^{-1}\|_\rho \leq N^-$ and $|\langle \frac{1}{\hat{l} \circ T_{-\omega\alpha}} \rangle| \geq c$ for some positive constant c where $\langle f \rangle$ denotes the average of the periodic function f .
- (H3) $\|\mathcal{E}[\hat{h}, \lambda]\|_\rho \leq \epsilon$ where \mathcal{E} is defined in (5).
- (H4) *Composition condition:* Take $\iota = \text{dist}(\mathbb{C}^d - \Omega, (Id + \alpha \cdot \hat{h})(\overline{D_\rho}))$ where Ω is the domain of \widehat{U} . We assume that $\|\hat{h}\|_\rho + \rho \leq \frac{1}{2}\iota$.

Assume furthermore that $\epsilon \leq \epsilon^*(N^-, N^+, d, \tau, c, \iota, \|\widehat{U}\|_{C^2(\Omega)}) \cdot \nu^4 \cdot \rho^{4\tau+A}$ where $\epsilon^* > 0$ is a function and $A \in \mathbb{R}^+$ (we will make explicit ϵ^* and A along the proof).

Then, there exists a periodic function \hat{h}^* and $\lambda^* \in \mathbb{R}$ such that

$$\mathcal{E}[\hat{h}^*, \lambda^*] = 0.$$

Moreover,

$$\begin{aligned} \|\hat{h} - \hat{h}^*\|_{\frac{\rho}{2}} &\leq C \cdot \nu^{-2} \rho^{-2\tau-A} \cdot \|\mathcal{E}[\hat{h}, \lambda]\|_\rho, \\ |\lambda - \lambda^*| &\leq C \cdot \|\mathcal{E}[\hat{h}, \lambda]\|_\rho. \end{aligned}$$

The solution $[\hat{h}^*, \lambda^*]$ is the only solution of $\mathcal{E}[\hat{h}^*, \lambda^*] = 0$ with zero average for \hat{h}^* in a ball centered at \hat{h} in $\mathcal{A}_{\frac{3\rho}{8}}$, i.e. $[\hat{h}^*, \lambda^*]$ is the unique solution in the set

$$\left\{ \hat{g} \in \mathcal{A}_{\frac{3\rho}{8}} \mid \langle \hat{g} \rangle = 0, \|\hat{g} - \hat{h}\|_{\frac{3\rho}{8}} \leq \frac{\nu^2 \cdot \rho^{2\tau}}{2\tilde{C}(N^-, N^+, d, \tau, c, C)} \right\}$$

where \tilde{C} will be made explicit along the proof.

3.2. Statement of the Sobolev result.

Theorem 12. Let $m > \frac{1}{2} + 2\tau$ and $\widehat{U} \in C^{m+34\tau+1}$. Let $h = Id + \tilde{h}$ where $\tilde{h}(\theta) = \hat{h}(\alpha \cdot \theta) = \sum_{k \in \mathbb{Z}^d} \hat{h}_k \cdot e^{2\pi i \cdot k \cdot \alpha \theta}$ with $\hat{h} \in H^{m+32\tau}$, $\hat{h}_0 = 0$ for any $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{R}^d$ such that $\alpha \cdot j \neq 0$, $j \in \mathbb{Z}^d - \{0\}$. Denote $\hat{l} = 1 + \partial_\alpha \hat{h}$ and $T_{-\omega\alpha}(\sigma) = \sigma - \omega\alpha$. We assume

- (H1) *Diophantine properties* (13): $|\omega\alpha \cdot k - n| \geq \nu |k|^{-\tau}$, $\forall k \in \mathbb{Z}^d - \{0\}$, $n \in \mathbb{Z}$.
- (H2) *Non-degeneracy condition:* $\|\hat{l}(\sigma)\|_{H^m} \leq N^+$, $\|(\hat{l}(\sigma))^{-1}\|_{H^m} \leq N^-$ and $|\langle \frac{1}{\hat{l} \circ T_{-\omega\alpha}} \rangle| \geq c$ for some positive constant c .
- (H3) $\|\mathcal{E}[\hat{h}, \lambda]\|_{H^m} \leq \epsilon$.

Assume furthermore that $\epsilon \leq \epsilon^*(N^-, N^+, d, \tau, c, \|\widehat{U}\|_{C^{m+34\tau+1}}) \cdot \nu^{-2}$ where $\epsilon^* > 0$ is a function which we will make explicit along the proof. Then, there exists a periodic function $\hat{h}^* \in H^{m-4\tau}$ and $\lambda^* \in \mathbb{R}$ such that

$$\mathcal{E}[\hat{h}^*, \lambda^*] = 0.$$

Moreover,

$$\begin{aligned}\|\hat{h} - \hat{h}^*\|_{H^{m-4\tau}} &\leq C \cdot \nu^{-2}(N^+)^2 \cdot \epsilon, \\ |\lambda - \lambda^*| &\leq C \cdot \epsilon.\end{aligned}$$

The solution $[\hat{h}^*, \lambda^*]$ is the only solution of $\mathcal{E}[\hat{h}^*, \lambda] = 0$ with zero average for \hat{h}^* in a neighborhood of $[\hat{h}, \lambda]$ in $H^{m+4\tau}$, i.e. $[\hat{h}^*, \lambda^*]$ is the unique solution in the set

$$\left\{ \hat{g} \in H^{m+4\tau} \mid \langle \hat{g} \rangle = 0, \|\hat{g} - \hat{h}\|_{H^{m+4\tau}} \leq \frac{\nu^2}{2\tilde{C}(N^-, N^+, d, \tau, c, \|\widehat{U}\|_{C^{m+34\tau+1}}) \cdot C_{m-4\tau, m+4\tau}} \right\}$$

where \tilde{C} will be made explicit along the proof.

Remark 13. The theorems 11 and 12 have the “*a-posteriori*” format of numerical analysis.

Given a function which solves very approximately the invariance equations, then there is a true solution nearby. The needed approximation is quantified in terms of the non-degeneracy condition N^+, N^-, c , (which in numerical analysis are often called *condition numbers*). We note that the condition numbers can be computed in the approximate solution. We emphasize that this formulation does not require that the system is close to integrable.

Of course, the non-degeneracy conditions depend on the function \hat{h} (and on the parameter of the domain ρ in the analytic case. If this can cause confusion – e.g. when we are performing an iterative step – we will use $N^\pm(\hat{h}; \rho)$.

4. PROOF OF MAIN THEOREMS, THEOREM 11 AND THEOREM 12

As indicated in the introduction, the proofs of Theorems 11 and 12 are based on an iterative step that given an approximate solution of (19) will produce a better approximation.

The crux of the proof is to show that, if started with a sufficiently approximate solution, the procedure converges.

As it is very well known in KAM theory, there are arguments that establish the convergence, provided that we show that the iterative procedure satisfies *tame quadratic estimates*. That is, that the norm of the new error is bounded by the square of another norm of the original error (in a smoother space) times a factor that depends on the “loss of regularity”. There are several abstract theorems in this direction, one which is quite well adapted to the problem at hand and which we will use appears in [CdLL10].

In Section 4.1, we will give some motivations for the iterative procedure. (It is a Newton method with a small modification that does not affect the quadratic convergence.)

In Section 4.2, we will formulate the iterative procedure as a succession of elementary sub-steps. We note that these elementary sub-steps can be implemented by very efficient algorithms. If the functions \hat{h} are discretized using N points and appropriate algorithms are used for the mathematical sub-steps, the iterative procedure requires only $O(N)$ storage and $O(N \log N)$ operations.

In Section 4.3, we present estimates for the iterative step. We first present estimates on how much it changes the function. Then, we use the Taylor estimates to show that the error of the improved function is *tame quadratic* in the sense of Nash-Moser theory.

In Section 4.4, we will review the convergence of the procedure, which, as we have indicated before is rather standard, indeed in [CdLL10], there is an abstract theorem designed to cover exactly the problems considered here. Nevertheless, for the sake of completeness in the analytic case, we will also present a very short direct proof of the convergence argument. Since this direct proof is so direct it also leads to good numerical values.

In Section 4.5, we present several considerations that allow us to discuss uniqueness.

Finally, in Section 6, we show that we can obtain several consequences combining the Sobolev and analytic versions of the a-posteriori theorem. Namely, we show that for analytic mappings all sufficiently smooth solutions are analytic, that Lindstedt series converge, and we present a numerically efficient criterion for the breakdown of analyticity. As it was shown in [ALD83], the breakdown of analyticity is associated to the onset of transport properties, so there is some interest in its computation.

Of course, readers interested only in rigorous proofs can safely skip Section 4.1 and the algorithmic considerations in Section 4.2. Readers only interested in algorithms can skip 4.3.

4.1. Motivation for the iterative step. We start from an approximate solution of (5),

$$(19) \quad \hat{h}(\sigma + \omega\alpha) + \hat{h}(\sigma - \omega\alpha) - 2\hat{h}(\sigma) + \widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma)) + \lambda = e$$

where e is to be thought of as “*small*”.

Our goal is to devise a procedure that gives a much more approximate solution. For the moment, we will not make precise the sense in which quantities are small. This will be taken up in Section 4.3.

Given an approximate solution as in (19), the Newton method would consist in finding a solution of

$$(20) \quad \widehat{\Delta}(\sigma + \omega\alpha) + \widehat{\Delta}(\sigma - \omega\alpha) - 2\widehat{\Delta}(\sigma) + \partial_\alpha \widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma)) \cdot \widehat{\Delta}(\sigma) + \delta = -e.$$

Then $[\hat{h} + \widehat{\Delta}, \lambda + \delta]$ will be a better approximate solution.

The equation (20) is hard to study because of the term $\partial_\alpha \widehat{U}(\sigma + \alpha \cdot \widehat{h}(\sigma)) \cdot \widehat{\Delta}(\sigma)$ which is not constant coefficients.

The key observation is that, if we are given (19) we are also given the following equation which is just obtained by taking the derivative of (19) with respect to θ (we recall that $\sigma = \theta\alpha$):

$$(21) \quad \partial_\alpha \widehat{h}(\sigma + \omega\alpha) + \partial_\alpha \widehat{h}(\sigma - \omega\alpha) - 2\partial_\alpha \widehat{h}(\sigma) + \partial_\alpha \widehat{U}(\sigma + \alpha \cdot \widehat{h}(\sigma)) \cdot (1 + \partial_\alpha \widehat{h}(\sigma)) = e'(\theta).$$

Denoting $\widehat{l}(\sigma) = 1 + \partial_\alpha \widehat{h}(\sigma)$, we rewrite (21):

$$(22) \quad \widehat{l}(\sigma + \omega\alpha) + \widehat{l}(\sigma - \omega\alpha) - 2\widehat{l}(\sigma) + \partial_\alpha \widehat{U}(\sigma + \alpha \cdot \widehat{h}(\sigma)) \cdot \widehat{l}(\sigma) = e'(\theta).$$

If we substitute (22) into (20), we obtain that the Newton procedure (20) is equivalent to

$$(23) \quad \widehat{\Delta}(\sigma + \omega\alpha) + \widehat{\Delta}(\sigma - \omega\alpha) + \frac{e'(\theta) - \widehat{l}(\sigma + \omega\alpha) - \widehat{l}(\sigma - \omega\alpha)}{\widehat{l}(\sigma)} \cdot \widehat{\Delta}(\sigma) = -e - \delta.$$

The equation (23) is still hard to solve, but, as we will now see, we can study the equation obtained by omitting the term $e'(\theta) \cdot \widehat{\Delta}(\sigma)$ in (23).

Because both $e'(\theta)$ and $\widehat{\Delta}(\sigma)$ are small, we can hope and we will show that omitting the term $e' \cdot \widehat{\Delta}(\sigma)$ does not affect the quadratic character of the procedure.

The key observation is that the quasi-Newton equation for Δ , δ :

$$(24) \quad \widehat{l}(\sigma) \cdot \widehat{\Delta}(\sigma + \omega\alpha) + \widehat{l}(\sigma) \cdot \widehat{\Delta}(\sigma - \omega\alpha) - \widehat{\Delta}(\sigma) [\widehat{l}(\sigma + \omega\alpha) + \widehat{l}(\sigma - \omega\alpha)] = (-e - \delta) \cdot \widehat{l}(\sigma)$$

is equivalent to the system

$$(25) \quad \begin{pmatrix} \widehat{\Delta} \\ \widehat{l} \end{pmatrix} \circ T_{-\omega\alpha} - \begin{pmatrix} \widehat{\Delta} \\ \widehat{l} \end{pmatrix} = \frac{\widehat{W}}{\widehat{l} \cdot \widehat{l} \circ T_{-\omega\alpha}}$$

$$(26) \quad \widehat{W} \circ T_{\omega\alpha} - \widehat{W} = \widehat{l} \cdot (e + \delta).$$

In fact, from (25), we get

$$\widehat{W} = \widehat{\Delta} \circ T_{-\omega\alpha} \cdot \widehat{l} - \widehat{\Delta} \cdot \widehat{l} \circ T_{-\omega\alpha}$$

and

$$\widehat{W} \circ T_{\omega\alpha} = \widehat{\Delta} \cdot \widehat{l} \circ T_{\omega\alpha} - \widehat{\Delta} \circ T_{\omega\alpha} \cdot \widehat{l}$$

By (26), we can easily get the equivalence.

The key point is that the equations (25) and (26) are of the form (14), and can be studied using the theory in Section 2.3.

We write $\widehat{W} = \widehat{W}^0 + \overline{\widehat{W}}$ where \widehat{W}^0 is a function with zero average and $\overline{\widehat{W}}$ is a number. That is, we decompose \widehat{W} into its average and the zero

average part. Both are unknowns. Now we describe the procedure to solve the system (25), (26).

We first choose δ to be the unique value that makes the average of the right-hand-side of (26) zero. Then, we can apply Lemma 10 to find \widehat{W}^0 solving (26). We note that there is only one choice of δ and then, \widehat{W}^0 is determined uniquely, by the condition that it solves (26) and that it has zero average. The only solutions of (26) differ from it by a constant.

Then we observe that $\overline{\widehat{W}} = -\frac{\langle \frac{\widehat{W}^0}{\widehat{l} \circ T - \omega \alpha} \rangle}{\langle \frac{1}{\widehat{l} \circ T - \omega \alpha} \rangle}$ is the only possible value of the average of solutions of (26) that makes the right-hand-side of (25) with zero average. Then, we can apply again Lemma 10 to find $\widehat{\frac{\Delta}{\widehat{l}}}$ solving (25). This solution is unique up to the addition of constant. Once we have $\widehat{\frac{\Delta}{\widehat{l}}}$, we obtain $\widehat{\Delta}$ is obtained just multiplying by \widehat{l} . Note that the Δ is thus determined uniquely up to the addition of a constant multiple of \widehat{l} . In particular, $\widehat{\Delta}$ is unique when we impose the normalization that it has zero average.

4.2. Formulation of the iterative step.

Algorithm 14. Given $\hat{h} : \mathbb{T}^d \rightarrow \mathbb{R}$, $\lambda \in \mathbb{R}$ with $\hat{h}(\sigma) = \sum_{k \in \mathbb{Z}^d} \hat{h}_k e^{2\pi i k \sigma}$ and $\tilde{h}(\theta) = \hat{h}(\alpha \theta)$ for $\theta \in \mathbb{R}$ and any irrational vector $\alpha \in \mathbb{R}^d$, $d \geq 2$, we will calculate:

- 1) Let $\mathcal{L} = \hat{h}(\sigma + \alpha \omega) + \hat{h}(\sigma - \alpha \omega) - 2\hat{h}(\sigma)$. In Fourier components $\widehat{\mathcal{L}}_k = 2(\cos \omega \alpha \cdot k - 1)\hat{h}_k$.
- 2) We calculate $\widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma))$.
- 3) So we can calculate $e = \mathcal{L} + \widehat{U} + \lambda$.
- 4) Calculate $\widehat{l} = 1 + \partial_\alpha \hat{h}$. In Fourier components $\widehat{l}_k = \delta_{k,0} + 2\pi i k \cdot \alpha \cdot \hat{h}_k$ where $\delta_{0,k}$ is the Kronecker delta.
- 5) Let $f = \widehat{l} \cdot e$.
- 6) Choose $\delta = -\langle f \rangle$.
- 7) Denote $b = \widehat{l} \cdot (e + \delta)$.
- 8) Solve the cohomology equation (26) for \widehat{W}^0 with zero average. That is, $\widehat{W}_k^0 = \frac{b_k}{2(\cos \omega \alpha \cdot k - 1)}$.
- 9) Take $\overline{\widehat{W}} = -\frac{\langle \frac{\widehat{W}^0}{\widehat{l} \circ T - \omega \alpha} \rangle}{\langle \frac{1}{\widehat{l} \circ T - \omega \alpha} \rangle}$.
- 10) Calculate $\widehat{W} = \widehat{W}^0 + \overline{\widehat{W}}$.
- 11) Solve the cohomology equation (25). Find $\tilde{\beta}$ satisfying $\tilde{\beta} - \tilde{\beta} \circ T_{-\omega \alpha} = \frac{\widehat{W}}{\widehat{l} \circ T_{-\omega \alpha}}$. That is, $\tilde{\beta}_k = \frac{a_k}{2(\cos \omega \alpha \cdot k - 1)}$ where $a = \frac{\widehat{W}}{\widehat{l} \circ T_{-\omega \alpha}}$.
- 12) We obtain $\Delta = (\tilde{\beta} + \bar{\beta}) \cdot \widehat{l}$ where $\bar{\beta}$ is chosen to be $-\langle \tilde{\beta} \cdot \widehat{l} \rangle / \langle \widehat{l} \rangle$ so that $\langle \Delta \rangle = 0$.

Remark 15. It is important to note that the procedure also shows that the solution of (25) and (26) is unique up to the addition of a constant multiple of \hat{l} to $\hat{\Delta}$.

Hence, if we choose the solutions of (25), (26) which satisfy the normalization $\langle \hat{\Delta} \rangle = 0$, the solutions are unique.

Of course, since the system (25), (26) is equivalent to (24), the same considerations apply to (24).

Remark 16. Note that if we consider functions discretized by their values at N points and by N Fourier coefficients, steps 1), 4), 8), 11) are fast in Fourier coefficients (they require $O(N)$ operations) while the other steps are fast (they require $O(N)$ operations) in the representation of the function by its values at points. Of course, once we know the representation in space or in Fourier coefficients we can use the Fast Fourier transform which requires $O(N \log N)$ operations to compute the other.

Note also that if we discretize the function as above, the iterative step only requires to store several functions, and therefore we only need to store $O(N)$ numbers.

We, thus obtain a quadratical convergent algorithm, with $O(N)$ storage requirements and $O(N \log(N))$ operations. In contrast, a Newton method would require $O(N^2)$ storage to store a matrix and $O(N^3)$ operations to solve the linear equations (there are faster algorithms [Knu81] to solve linear equations but they do not seem to be practical). In practice the present algorithm with $N = 10^7$ can run comfortably on a modest desktop machine.

Remark 17. It is important to note that $[\hat{\Delta}, \delta]$, the outcome of the algorithm depends linearly on $e \equiv \mathcal{E}[\hat{h}, \lambda]$.

Hence, we will write

$$(27) \quad [\hat{\Delta}, \delta] = \eta[\hat{h}, \lambda]e$$

The operator η is called an “*approximate right inverse*” in Nash-Moser theory. See, for example [Zeh75].

Notice that the estimates for the improved solution can be written in a symbolic way as estimating $\mathcal{E}[[\hat{h}, \lambda] + \eta[\hat{h}, \lambda]\mathcal{E}[\hat{h}, \lambda]]$, which, using Taylor expansion (up to quadratic errors) becomes

$$(28) \quad \mathcal{E}[\hat{h}, \lambda] + D\mathcal{E}[\hat{h}, \lambda]\eta[\hat{h}, \lambda]\mathcal{E}[\hat{h}, \lambda]$$

In the Newton method, we would choose η in such a way that (28) vanishes. As pointed out in [Mos66, Zeh75], it suffices that the norm of (28) can be bounded by the square of another norm of $\mathcal{E}[\hat{h}, \lambda]$.

4.3. Estimates on the quasi-Newton step. In this section we show that the Quasi-Newton method specified in Algorithm 14 produces more approximate solutions. We will present two versions of the estimates, one in

analytic spaces and another one in Sobolev spaces. The goal is to obtain that the new error is quadratic in the original error even if in a weaker norm. We note that the analytic estimates presented are a bit more delicate and involve a condition, (34).

4.3.1. *Some useful identities.* We start by remarking an elementary identity that will be used for both the analytic and the Sobolev estimates:

$$(29) \quad \hat{l} \cdot (D_1 \mathcal{E}[\hat{h}, \lambda] \widehat{\Delta}) - \widehat{\Delta} \cdot (D_1 \mathcal{E}[\hat{h}, \lambda] \hat{l}) = -\hat{l} \cdot (\mathcal{E}[\hat{h}, \lambda] + \delta).$$

where D_1 denote the derivative with respect to the first variable.

We also have the following identity obtained just adding and subtracting terms in the definition of $\mathcal{E}[\hat{h} + \widehat{\Delta}, \lambda + \delta]$ and grouping them.

$$(30) \quad \begin{aligned} & \mathcal{E}[\hat{h} + \widehat{\Delta}, \lambda + \delta] \\ &= \mathcal{E}[\hat{h}, \lambda] + \widehat{\Delta}(\sigma + \omega\alpha) + \widehat{\Delta}(\sigma - \omega\alpha) - 2\widehat{\Delta}(\sigma) + \delta \\ & \quad + \widehat{U}(\sigma + \alpha \cdot (\hat{h} + \widehat{\Delta})(\sigma)) - \widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma)) \\ &= \mathcal{E}[\hat{h}, \lambda] + (-\mathcal{E}[\hat{h}, \lambda]) + \frac{\hat{l}(\sigma + \omega\alpha) + \hat{l}(\sigma - \omega\alpha) - 2\hat{l}(\sigma)}{\hat{l}(\sigma)} \cdot \widehat{\Delta}(\sigma) \\ & \quad + \widehat{U}(\sigma + \alpha \cdot (\hat{h} + \widehat{\Delta})(\sigma)) - \widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma)) \\ &= \frac{e'(\theta) - \partial_\alpha \widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma)) \cdot \hat{l}(\sigma)}{\hat{l}(\sigma)} \cdot \widehat{\Delta}(\sigma) + \widehat{U}(\sigma + \alpha \cdot (\hat{h} + \widehat{\Delta})(\sigma)) - \widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma)) \\ &= e' \cdot \frac{\widehat{\Delta}(\sigma)}{\hat{l}(\sigma)} + \widehat{U}(\sigma + \alpha \cdot (\hat{h} + \widehat{\Delta})(\sigma)) - \widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma)) - \partial_\alpha \widehat{U}(\sigma + \alpha \cdot \hat{h}) \cdot \widehat{\Delta}(\sigma) \\ &\equiv e' \cdot \frac{\widehat{\Delta}(\sigma)}{\hat{l}(\sigma)} + R \end{aligned}$$

where $R = \widehat{U}(\sigma + \alpha \cdot (\hat{h} + \widehat{\Delta})(\sigma)) - \widehat{U}(\sigma + \alpha \cdot \hat{h}(\sigma)) - \partial_\alpha \widehat{U}(\sigma + \alpha \cdot \hat{h}) \cdot \widehat{\Delta}(\sigma)$. Clearly, R is the remainder of the Taylor estimate in the composition studied in Lemma 5 and Lemma 6.

4.3.2. *Estimates for the iterative step in analytic spaces.* We now observe that for any $\rho' < \rho$, by (17), we obtain using (26)

$$\|\widehat{W}^0\|_{\rho'} \leq C(d, \tau) \cdot \nu^{-1} \cdot (\rho - \rho')^{-\tau} \cdot N^+ \cdot \|e\|_\rho.$$

Since the average of \widehat{W} is obtained in step 9) of Algorithm (14), we have the estimate for $\overline{\widehat{W}}$:

$$|\overline{\widehat{W}}| \leq c \cdot \|\widehat{W}^0\|_{\rho'} \cdot (N^-)^2 \leq c \cdot (N^-)^2 \cdot N^+ \cdot C(d, \tau) \cdot \nu^{-1} \cdot |\rho - \rho'|^{-\tau} \cdot \|e\|_\rho.$$

Therefore, we obtain the estimates for \widehat{W} :

$$\|\widehat{W}\|_{\rho'} \leq M \cdot \nu^{-1} \cdot (\rho - \rho')^{-\tau} \cdot \|e\|_{\rho}$$

where $M = (c \cdot (N^-)^2 + 1) \cdot N^+ C(d, \tau)$. The important point is that the constant is uniform provided \widehat{h} stays in a neighborhood in $\|\cdot\|_{\rho}$ norm.

Again for $\rho'' < \rho'$, by (17), we have

$$\|\widetilde{\Delta}\|_{\rho''} \leq C(d, \tau) \cdot \nu^{-1} \cdot (\rho' - \rho'')^{-\tau} \cdot (N^-)^2 \cdot \|\widehat{W}\|_{\rho'} \leq M' \cdot (\rho - \rho')^{-\tau} \cdot (\rho' - \rho'')^{-\tau} \cdot \|e\|_{\rho}.$$

So, we have

$$\|\widetilde{\beta}\|_{\rho''} \leq N^+ \cdot M \cdot \nu^{-2} (\rho - \rho')^{-\tau} \cdot (\rho' - \rho'')^{-\tau} \cdot \|e\|_{\rho}$$

and

$$(31) \quad \|\Delta\|_{\rho''} \leq N^+ \cdot N^- M \cdot \nu^{-2} (\rho - \rho')^{-\tau} \cdot (\rho' - \rho'')^{-\tau} \cdot \|e\|_{\rho}.$$

Similarly, using Cauchy estimates, we obtain for $\rho''' < \rho''$

$$(32) \quad \|\Delta\|_{\rho''} \leq N^+ \cdot N^- M \cdot \nu^{-2} (\rho'' - \rho''')^{-1} (\rho - \rho')^{-\tau} \cdot (\rho' - \rho'')^{-\tau} \cdot \|e\|_{\rho}.$$

If we take $\rho - \rho' = \rho' - \rho''$ in (31) and $\rho - \rho'' = \rho' - \rho'' = \rho - \rho''$ (and redefine ρ''') in (32) we obtain:

$$(33) \quad \begin{aligned} \|\widehat{\Delta}\|_{\rho''} &\leq M' \cdot \nu^{-2} (\rho - \rho'')^{-2\tau} \cdot \|e\|_{\rho} \\ \|\widehat{\Delta}\|_{\rho''} &\leq M' \cdot \nu^{-2} (\rho - \rho'')^{-2\tau-1} \cdot \|e\|_{\rho}. \end{aligned}$$

If we have that $\|\widehat{\Delta}\|_{\rho''} \leq \iota/2$, which, by (33) is implied by

$$(34) \quad M' \cdot \nu^{-2} (\rho - \rho'')^{-2\tau} \cdot \|e\|_{\rho} \leq \iota/2,$$

we can define $\widehat{U}(\sigma + \alpha\widehat{h} + \widehat{\Delta}(\sigma))$ and indeed apply Taylor's estimate we obtain

$$(35) \quad \|R\|_{\rho''} \leq \sup \widehat{U} \cdot \|\widehat{\Delta}\|_{\rho''}^2 \leq M \nu^{-4} (\rho'' - \rho)^{-2\tau} \|e\|_{\rho}^2.$$

The first term in the right-hand-side of (30) is estimated using the Cauchy estimates and the previous estimates on $\|\widehat{\Delta}\|$.

$$\begin{aligned} \left\| e' \cdot \frac{\widehat{\Delta}}{\widehat{I}} \right\|_{\rho''} &\leq (\rho - \rho'')^{-1} \|e\|_{\rho} N^- N^+ \cdot M' \cdot \nu^{-2} (\rho - \rho'')^{-2\tau} \cdot \|e\|_{\rho} \\ &= M (\rho - \rho'')^{-2\tau-1} \nu^{-2} \|e\|_{\rho}^2 \\ &\leq M (\rho - \rho'')^{-4\tau-1} \nu^{-4} \|e\|_{\rho}^2. \end{aligned}$$

The last estimate is done with the purpose of simplifying the expressions, but it is obviously wasteful. Note that $\tau \geq 1$ and that the estimates above are delicate only when $\rho - \rho''$, ν are small.

Finally, putting together the estimates for the two terms in the right-hand-side of (30), we have:

$$(36) \quad \|\mathcal{E}[\hat{h} + \widehat{\Delta}, \lambda + \delta]\|_{\rho''} \leq C \cdot \nu^{-4} (\rho - \rho'')^{-4\tau} \|\mathcal{E}[\hat{h}, \lambda]\|_{\rho}^2.$$

Therefore, we have proved the following

Lemma 18. *In the hypothesis of Theorem 11.*

Assume that (34) holds. Then, the improved function obtained applying Algorithm (14), satisfies (36).

As it is well-known in KAM theory, the above estimates imply that the iterative procedure can be repeated indefinitely and the resulting sequence converges to a function satisfying the claims of Theorem 11. Indeed, the paper [CdL10] contains an abstract theorem that immediately applies to this situation. We will discuss this in more detail in Section 4.4.

4.3.3. *Sobolev estimates for the iterative step.* Let $s > \frac{d}{2}$. According to the algorithm 8), we have

$$\begin{aligned} \|b\|_{H^s} &= \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\hat{l}_k \cdot (e_k + \delta_{k,0})|^2 \\ &= \sum_{k \in \mathbb{Z}^d - \{0\}} (1 + |k|^2)^s \cdot |\hat{l}_k \cdot e_k|^2 \\ &\leq \|f\|_{H^s} \leq K \cdot \|\hat{l}\|_{H^s} \cdot \|e\|_{H^s}. \end{aligned}$$

By (18), we obtain

$$\|\widehat{W}^0\|_{H^{s-\tau}} \leq C \cdot \nu^{-1} \cdot \|b\|_{H^s} \leq C \cdot \nu^{-1} \cdot K \cdot N^+ \cdot \|e\|_{H^s}.$$

We get

$$\|\overline{\widehat{W}}\| \leq c \cdot (N^-)^2 \|\widehat{W}^0\|_{H^{s-\tau}}$$

and

$$\|\widehat{W}\|_{H^{s-\tau}} \leq C \cdot \nu^{-1} \cdot K \cdot N^+ \cdot \|e\|_{H^s}.$$

So we have

$$\|\tilde{\Delta}\|_{H^{s-2\tau}} \leq C \cdot \nu^{-1} K^2 \cdot (N^-)^2 \cdot \|\widehat{W}\|_{H^{s-\tau}} \leq C \cdot \nu^{-2} \cdot N^+ \cdot (N^-)^2 \cdot \|e\|_{H^s}.$$

Hence,

$$\|\widehat{\Delta}\|_{H^{s-2\tau}} \leq C \cdot \nu^{-2} \cdot (N^+)^2 \cdot (N^-)^2 \cdot \|e\|_{H^s}.$$

We recall that the approximate inverse of the derivative $\eta[\hat{h}, \lambda]$ is just the result of applying the algorithm in Section 4.2, i.e. $[\widehat{\Delta}, \delta] = \eta[\hat{h}, \lambda] \mathcal{E}[\hat{h}, \lambda]$. We have proved the following lemma:

Lemma 19. *Let $s > \frac{d}{2} + 2\tau$. Then we have*

$$\|\eta[\hat{h}, \lambda] \mathcal{E}[\hat{h}, \lambda]\|_{H^{s-2\tau}} \leq C \cdot \nu^{-2} \cdot (N^+)^2 \cdot (N^-)^2 \cdot \|e\|_{H^s}.$$

We will also need estimates on $(D_1\mathcal{E}[\hat{h}, \lambda]\eta[\hat{h}, \lambda] - Id)(\mathcal{E}[\hat{h}, \lambda] + \delta)$.

Lemma 20.

$$\begin{aligned} & \| (D_1\mathcal{E}[\hat{h}, \lambda]\eta[\hat{h}, \lambda] - Id)(\mathcal{E}[\hat{h}, \lambda] + \delta) \|_{H^{s-2\tau}} \\ & \leq C \cdot \nu^{-2} \cdot (N^+)^2 \cdot (N^-)^3 \cdot \|\mathcal{E}[\hat{h}, \lambda]\|_{H^{s-2\tau-1}} \|\mathcal{E}[\hat{h}, \lambda]\|_{H^s}. \end{aligned}$$

Proof. By the definition of $\eta[\hat{h}, \lambda]$, we know $\widehat{\Delta} = -\eta[\hat{h}, \lambda](\mathcal{E}[\hat{h}, \lambda] + \delta)$. Hence,

$$\begin{aligned} & (D_1\mathcal{E}[\hat{h}, \lambda]\eta[\hat{h}, \lambda] - Id)(\mathcal{E}[\hat{h}, \lambda] + \delta) \\ & = -D_1\mathcal{E}[\hat{h}, \lambda]\widehat{\Delta} - \mathcal{E}[\hat{h}, \lambda] - \delta = \frac{\widehat{\Delta} \cdot D_1\mathcal{E}[\hat{h}, \lambda]}{\hat{I}}. \end{aligned}$$

So we have that

$$\begin{aligned} & \| (D_1\mathcal{E}[\hat{h}, \lambda]\eta[\hat{h}, \lambda] - Id)(\mathcal{E}[\hat{h}, \lambda] + \delta) \|_{H^{s-2\tau}} \\ & \leq C \cdot \nu^{-2} \cdot (N^+)^2 \cdot (N^-)^3 \cdot \|\mathcal{E}[\hat{h}, \lambda]\|_{H^{s-2\tau-1}} \|\mathcal{E}[\hat{h}, \lambda]\|_{H^s}. \end{aligned}$$

□

4.4. Convergence of the procedure. The existence of solutions both in the analytic case and in the Sobolev case is deduced from the estimates in Section 4.3 followed by Nash-Moser estimates.

Indeed in [CdLL10], one can find an abstract Nash-Moser implicit function theorem which is tailored to the theorems 11 and 12.

In this section, we reproduce the theorem from [CdLL10] and explain why it is applicable. We note that the the theorem has several corollaries which are of physical interest and we present them in Section 6.

For the sake of completeness, in Section 4.4.2, we present a direct proof of the convergence in the analytic case.

4.4.1. An abstract implicit function theorem. In [CdLL10, Appendix A] one can find a proof of the following result, Theorem 21. This is an abstract theorem that applies to operators in scales of Banach spaces, which have smoothing operators.

In [CdLL10] one can also find a verification that the Sobolev spaces and analytic spaces considered indeed have smoothing operators (one can take $S_t \sum_k \hat{h}_k e^{2\pi i k \sigma} = e^{-t|k|} \hat{h}_k e^{2\pi i k \sigma}$). The regularity properties of the operator entering in the assumptions of Theorem 21 follow immediately for the composition properties presented in Section 2.1.1 and, specially Lemma 6.

Theorem 21. *Let $m > 2\tau$ and \mathcal{X}^r for $m \leq r \leq m + 34\tau$ be a scale of Banach spaces with smoothing operators as shown in [CdLL10]. Let \mathcal{B}_r be the unit ball in \mathcal{X}^r , $\tilde{\mathcal{B}}_r = \hat{h} + \mathcal{B}_r$ the unit ball translated by $\hat{h} \in \mathcal{X}^r$ and $\mathcal{B}(\mathcal{X}^r, \mathcal{X}^{r-2\tau})$*

is the space of bounded linear operators from \mathcal{X}^r to $\mathcal{X}^{r-2\tau}$. Consider a map $\mathcal{F} : \tilde{\mathcal{B}}_r \rightarrow \mathcal{X}^{r-2\tau}$ and $\eta : \tilde{\mathcal{B}}_r \rightarrow \mathcal{B}(\mathcal{X}^r, \mathcal{X}^{r-2\tau})$ satisfying the following:

- (i) $\mathcal{F}(\tilde{\mathcal{B}}_r \cap \mathcal{X}^r) \subset \mathcal{X}^{r-2\tau}$ for $m \leq r \leq m + 34\tau$.
- (ii) $\mathcal{F}|_{\tilde{\mathcal{B}}_r \cap \mathcal{X}^r} : \tilde{\mathcal{B}}_r \cap \mathcal{X}^r \rightarrow \mathcal{X}^{r-2\tau}$ has two continuous Fréchet derivatives, both bounded by some constant M , for $m \leq r \leq m + 34\tau$.
- (iii) $\|\eta[\widehat{\Delta}]\mathcal{F}[\widehat{\Delta}]\|_{\mathcal{X}^{r-2\tau}} \leq C \cdot \|\mathcal{F}[\widehat{\Delta}]\|_{\mathcal{X}^r}$, $\widehat{\Delta} \in \tilde{\mathcal{B}}_r$, for $r = m - 2\tau$, $m + 32\tau$.
- (iv) $\|(D\mathcal{F}[\widehat{\Delta}]\eta[\widehat{\Delta}] - Id)\mathcal{F}[\widehat{\Delta}]\|_{\mathcal{X}^{r-2\tau}} \leq C \cdot \|\mathcal{F}[\widehat{\Delta}]\|_{\mathcal{X}^r}^2$, $\widehat{\Delta} \in \tilde{\mathcal{B}}_r$, for $r = m$.
- (v) $\|\mathcal{F}[\widehat{\Delta}]\|_{\mathcal{X}^{m+32\tau}} \leq C \cdot (1 + \|\widehat{\Delta}\|_{\mathcal{X}^{m+34\tau}})$, $\widehat{\Delta} \in \tilde{\mathcal{B}}_m$.

Then if $\|\mathcal{F}[\widehat{h}]\|_{\mathcal{X}^{m-2\tau}}$ is sufficiently small, there exists $\widehat{h}^* \in \mathcal{X}^m$ such that $\mathcal{F}[\widehat{h}^*] = 0$. Moreover, $\|\widehat{h} - \widehat{h}^*\|_{\mathcal{X}^m} < C \cdot \|\mathcal{F}[\widehat{h}]\|_{\mathcal{X}^{m-2\tau}}$.

We recall that the method of proof of Theorem 21, following [Sch60] is to modify the quasi-Newton step adding a smoothing step. That is, one constructs a sequence $[\widehat{h}_{n+1}, \lambda_{n+1}] = [\widehat{h}_n, \lambda_n] + S_{t_n}\eta[\widehat{h}_n, \lambda_n]\mathcal{E}[\widehat{h}_n, \lambda_n]$. The choices of t_n have to be carefully chosen so that the quadratic convergence (in some norm) is maintained. The main difference between Theorem 21 and the result in [Sch60] is that Theorem 21 includes the fact that η is an approximate inverse and not an inverse.

The estimates showing that η is indeed an approximate inverse are the estimates obtained in Section 4.3.

4.4.2. A direct proof of the convergence in the analytic case. Since the estimates in the analytic case are so easy, we present a direct proof. As we will see, the estimates are rather easy to verify. The main difficulty is the order of the choices.

We start with an approximate solution $[\widehat{h}_0, \lambda_0]$ with $\widehat{h} \in \mathcal{A}_{\rho_0}^1$.

Since we will have to change the function through an iterative procedure, we note that the condition numbers N^+ , N^- , c depend on the functions we are considering, nevertheless, they are uniform in a \mathcal{A}_ρ^1 neighborhood.

We start by choosing a number $\gamma > 0$ such that that in neighborhood of size γ in $\mathcal{A}_{\rho_0}^1$, we have that $N^\pm \leq 2N^\pm(\widehat{h}_0)$, $c \leq 2c(\widehat{h}_0)$.

The following algebraic identities will be useful in estimating the change of non-degeneracy conditions in the iterative step.

$$\begin{aligned}
N^+(h; \rho_0) &\equiv \|1 + \partial_\alpha \hat{h}\|_{\rho_0} \\
&\leq N^+(\hat{h}_0; \rho_0) \|\partial_\alpha(\hat{h} - \hat{h}_0)\|_{\rho_0} \\
N^-(\hat{h}; \rho_0) &\equiv \|(1 + \partial_\alpha \hat{h})^{-1}\|_{\rho_0} \\
&\leq N^-(\hat{h}_0; \rho_0) + \|\partial_\alpha(\hat{h} - \hat{h}_0)\|_{\rho_0} N^-(h_0; \rho_0) N^-(\hat{h}; \rho) \\
(37) \quad |c(\hat{h}) - c(\hat{h}_0)| &= \left| \left\langle \frac{1}{\hat{l} \cdot \hat{l} \circ T_{-\omega\alpha}} - \frac{1}{\hat{l}_0 \cdot \hat{l}_0 \circ T_{-\omega\alpha}} \right\rangle \right| \\
&= \left| \left\langle \frac{\hat{l}_0(\hat{l}_0 - \hat{l}) \circ T_{-\omega\alpha} + (\hat{l} - \hat{l}_0)}{\hat{l}_0 \hat{l}_0 \circ T_{-\omega\alpha} \hat{l} \circ T_{-\omega\alpha}} \right\rangle \right| \\
&\leq [N^-(\hat{h}; \rho_0) N^-(\hat{h}_0; \rho_0)]^2 (\|\hat{l}\|_{\rho_0} + \|\hat{l}_0\|_{\rho_0}) \|\hat{l} - \hat{l}_0\|_{\rho_0}.
\end{aligned}$$

Hence we can find a number $\gamma > 0$ depending only on the non-degeneracy conditions N^\pm, c so that all the functions in a ball of radius γ in $\mathcal{A}_{\rho_0}^1$ centered at \hat{h}_0 , have non-degeneracy constants not larger than twice the non-degeneracy assumptions of \hat{h}_0 .

More generally, we have, by the same argument that if $\|\hat{h} - \hat{h}_0\|_{\mathcal{A}_\rho^1} \leq \gamma$, then, $N^\pm(\hat{h}; \rho) \leq 2N^\pm(\hat{h}_0, \rho)$.

The key estimates are, as follows to show that, with some convenient choices of radii, which we do at the outset, the iterative process can be applied indefinitely and indeed it converges. We will use (37) to show that the non-degeneracy constants do not deteriorate much.

We denote by

$$(38) \quad \rho_n = \rho_{n-1} - \frac{\rho_0}{4} 2^{-n} = \rho_0 \left(1 - \frac{1}{4} \sum_{i=0}^n 2^{-i}\right)$$

and provided that we can apply the iterative step (that is, provided that (34) applies with the choices of ρ_n in (38), we define for $n \geq 1$, $[\hat{h}_n, \lambda_n] = [\hat{h}_{n-1}, \lambda_{n-1}] + \eta[\hat{h}_{n-1}, \lambda_{n-1}] \mathcal{E}[\hat{h}_{n-1}, \lambda_{n-1}]$.

We denote by M the constant in Lemma 18 corresponding to twice the degeneracy assumptions corresponding to the original function.

If (34) applies n times, for typographical simplicity, we denote $\epsilon_i = \|\mathcal{E}[\hat{h}_i, \lambda_i]\|_{\rho_i}$ we see that

$$\begin{aligned}
\epsilon_n &\leq Mv^{-2}\rho_0^{-4\tau}2^{(n-1)4\tau}\epsilon_{n-1}^2 \leq (M\rho_0^{-4\tau}v^{-2})^{1+2}2^{(n-1)4\tau+2(n-2)4\tau}\epsilon_{n-2}^{2^2} \\
&\dots \\
(39) \quad &\leq (Mv^{-2}\rho_0^{-4\tau})^{1+2+\dots+2^n}2^{(n-1)4\tau+2(n-2)4\tau+\dots+2^{n-1}4\tau}\epsilon_0^{2^n} \\
&\leq (Mv^{-2}\rho_0^{-4\tau})^{2^{n+1}}2^{8\tau^{2^n}}\epsilon_0^{2^n}.
\end{aligned}$$

We see that if $(Mv^{-2}\rho_0^{-4\tau})^22^{8\tau}\epsilon_0 < 1$, the right-hand-side of (39) decreases faster than any exponential. Indeed the factor can be made as small as desired by assuming that ϵ_0 is small enough.

If we apply n -times the inductive step, we see that the distance from the range of h_n to the complement of the domain of definition of \widehat{U} is at least

$$\begin{aligned}
\iota - \sum_{i=0}^n \|\Delta_i\|_{\rho_n} &\geq \iota - \sum_{i=0}^n \|\Delta_i\|_{\rho_i} \geq \iota - \sum_{i=0}^n M'v^{-2}\rho_0^{-4\tau}2^{i4\tau}\epsilon_i \\
&\geq \iota - \sum_{i=0}^n M'v^{-2}\rho_0^{-4\tau}(A\epsilon_0)^{2^i}.
\end{aligned}$$

Note that if ϵ_0 is small enough, this is bounded from below by $\iota\frac{3}{4}$ independent of n .

According to Lemma 18, the only thing we have to verify is (34), which with the choices of radii that we have made amounts to:

$$M' \cdot v^{-2}\rho_0^{-2\tau}2^{n4\tau}\epsilon_n \leq \iota/4.$$

We note that this condition is satisfied independently of n if n is large enough.

Using (37), we have:

$$\begin{aligned}
N^+(\hat{h}_n, \rho_n) &\leq N^+(h_{n-1}, \rho_n) + \|D\Delta_n\|_{\rho_n} \leq N^+(h_{n-1}, \rho_{n-1}) + Mv^{-2}\rho_0^{4\tau+1}2^{4\tau(n-1)}\epsilon_{n-1} \\
&\leq N^+(h_0, \rho_0) + Mv^{-2}\rho_0^{4\tau+1} \sum_{i=0}^{n-1} 2^{4\tau(n-1)}(A\epsilon_0)^{2^i}
\end{aligned}$$

and similarly for N^-, c . Therefore, under smallness conditions on ϵ_0 , we get that the non-degeneracy conditions do not change by a factor 2 from the original one, so that the induction hypothesis are satisfied.

In summary, under just three smallness conditions in ϵ_0 , which can be assessed just looking at the non-degeneracy conditions, we conclude that the iterative step can be carried out infinitely often and that the assumptions on the non-degeneracy constants make in the estimates for the step remain valid.

We also note that since $\rho_n \geq \rho_0/2$, we have

$$\begin{aligned} \|h_N - \hat{h}_0\|_{\rho_0/2} + |\lambda_n - \lambda_0| &\leq \sum_{n=1}^N \|\hat{h}_n - \hat{h}_{n-1}\|_{\rho_0/2} + |\lambda_n - \lambda_{n-1}| \\ &\leq \|\hat{h}_N - \hat{h}_0\|_{\rho_n} \leq \sum_{n=1}^N (A\epsilon_0)^{2^{2n}} 2^{(4\tau+1)n} M\nu^{-4} \rho_0^{-4\tau} \end{aligned}$$

which establishes the quantitative claims made for the result.

4.5. Uniqueness of the solution. In this section, we establish the uniqueness claims for the Theorems 11, 12. We note that the proof is very elementary and only uses the theory of linearized solutions as well as the interpolation inequalities in Section 2.1.1.

4.5.1. *Uniqueness for the analytic case.* If $\|\hat{h}^* - \hat{h}^{**}\|_{\frac{\rho}{4}}, |\lambda^* - \lambda^{**}|$ is sufficiently small and $\mathcal{E}[\hat{h}^*, \lambda^*] = \mathcal{E}[\hat{h}^{**}, \lambda^{**}] = 0$, by Taylor's theorem and Lemma 5, we have

$$(40) \quad 0 = \mathcal{E}[\hat{h}^{**}, \lambda^{**}] - \mathcal{E}[\hat{h}^*, \lambda^*] = D_1 \mathcal{E}[\hat{h}^*, \lambda^*](\hat{h}^{**} - \hat{h}^*) + (\lambda^{**} - \lambda^*) + R$$

where $\|R\|_{\frac{\rho}{4}} \leq C \cdot \|\hat{h}^{**} - \hat{h}^*\|_{\frac{\rho}{4}}^2$.

Now, denoting as before $\hat{l} = 1 + \partial_\alpha \hat{h}^*$ and recalling that

$$D_1 \mathcal{E}[\hat{h}^*, \lambda^*] \cdot \hat{l} = \frac{d}{d\theta} \mathcal{E}[\hat{h}^*, \lambda^*] = 0$$

we can write the equation (40) as:

$$(41) \quad \hat{l} \cdot (D_1 \mathcal{E}[\hat{h}^*, \lambda^*](\hat{h}^{**} - \hat{h}^*)) - (\hat{h}^{**} - \hat{h}^*) \cdot (D_1 \mathcal{E}[\hat{h}^*, \lambda^*] \hat{l}) = -\hat{l}R.$$

The proof of uniqueness is based on uniqueness of the solution of the system (25) and (26). By the estimates in Section 4.3, we conclude that for any $0 < \rho'' < \rho' < \frac{\rho}{4}$,

$$\|\hat{h}^{**} - \hat{h}^*\|_{\rho''} \leq C(N^-, N^+, d, \tau, c) \cdot \nu^{-2} \cdot (\rho - \rho')^{-\tau} \cdot (\rho' - \rho'')^{-\tau} \|R\|_{\rho}$$

Take $\rho'' = \frac{\rho}{8}$ and $\rho' = \frac{3}{16}\rho$.

In the analytic case, we obtain

$$\begin{aligned} \|\hat{h}^{**} - \hat{h}^*\|_{\frac{\rho}{8}} &\leq \tilde{C} \cdot \nu^{-2} \cdot \rho^{-2\tau} \cdot \|\hat{h}^{**} - \hat{h}^*\|_{\frac{\rho}{4}}^2 \\ &\leq \tilde{C} \cdot \nu^{-2} \cdot \rho^{-2\tau} \cdot \|\hat{h}^{**} - \hat{h}^*\|_{\frac{\rho}{8}} \cdot \|\hat{h}^{**} - \hat{h}^*\|_{\frac{3}{8}\rho}, \end{aligned}$$

where $\tilde{C} > 0$ is a constant depending on N^-, N^+, d, τ, c, C . The last inequality holds by Lemma 3. So when $\|\hat{h}^* - \hat{h}\|_{\frac{3}{8}\rho}$ small enough, we obtain $\hat{h}^{**} = \hat{h}^*$, $\lambda^{**} = \lambda^*$. This completes the proof of uniqueness of the solution in Theorem 11 for the analytic case.

4.5.2. *Uniqueness for the Sobolev case.* Instead of applying Hadamard 3-circle theorem for the analytic case, we use the interpolation inequality for Sobolev case (Lemma 4).

Following the proof in Section 4.5.1, we will have

$$\begin{aligned} \|\hat{h}^{**} - \hat{h}^*\|_{H^{m-4\tau}} &\leq \tilde{C} \cdot \nu^{-2} \cdot \|\hat{h}^{**} - \hat{h}^*\|_{H^m}^2 \\ &\leq \tilde{C} \cdot \nu^{-2} \cdot C_{m-4\tau, m+4\tau} \cdot \|\hat{h}^{**} - \hat{h}^*\|_{H^{m-4\tau}} \cdot \|\hat{h}^{**} - \hat{h}^*\|_{H^{m+4\tau}}. \end{aligned}$$

This completes the proof of uniqueness of the solution in Theorem 12 for the Sobolev case.

5. VANISHING LEMMA

In this section we prove

Lemma 22. *Consider a solution of (5) with the stated periodicity condition. If*

$$\widehat{U} = \partial_\alpha V$$

then $\lambda = 0$.

Proof. The proof is very simple. We multiply (5) by $h'(\theta)$ and compute $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T$ of all the terms. We note that this produces the formula

$$(42) \quad \lambda = - \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \widehat{U}(h(\theta)) \cdot h'(\theta) d\theta.$$

In fact, we observe that

$$h(\theta + \omega) + h(\theta - \omega) - 2h(\theta) = \tilde{h}(\theta + \omega) + \tilde{h}(\theta - \omega) - 2\tilde{h}(\theta) \in QP(\alpha).$$

Similarly,

$$h'(\theta) = 1 + \tilde{h}'(\theta) \in QP(\alpha).$$

Hence,

$$[h(\theta + \omega) + h(\theta - \omega) - 2h(\theta)] \cdot h'(\theta) \in QP(\alpha)$$

and we have that

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [h(\theta + \omega) + h(\theta - \omega) - 2h(\theta)] \cdot h'(\theta) \\ &= \sum_{k \in \mathbb{Z}^d - \{0\}} -\hat{h}_k \cdot 2(\cos(2\pi\omega k \cdot \alpha) - 1) \cdot \hat{h}_{-k} 2\pi i(k \cdot \alpha) + \hat{h}_0(2 \cos(2\pi\omega 0 \cdot \alpha) - 1) \\ &= 0. \end{aligned}$$

The first equality is true because of Lemma 8 and the fact that the sum is Cauchy formula for the $k = 0$ coefficient of the integrand (as we will see below). The fact that the sum is 0 is clear because it is antisymmetric in k .

In fact, note that

$$[\tilde{h} \circ T_\omega + \widehat{\tilde{h} \circ T_{-\omega}} - 2\tilde{h}]_k = 2(\cos(2\pi\omega k \cdot \alpha) - 1)\hat{h}_k,$$

in particular, the coefficient vanishes for $k = 0$, and

$$[\widehat{h'}]_k = \delta_{0,k} + 2\pi i k \cdot \alpha \cdot \hat{h}_k.$$

We have using Cauchy formula for the Fourier series of the product

$$\begin{aligned} & [([\tilde{h} \circ T_\omega + \widehat{\tilde{h} \circ T_{-\omega}} - 2\tilde{h}] \cdot h')]_0 \\ &= \sum_{k \in \mathbb{Z}^d} 2(\cos(2\pi\omega k \cdot \alpha) - 1)\hat{h}_k \cdot [\delta_{0,k} - 2\pi i k \cdot \alpha \cdot \hat{h}_{-k}] \\ &= \sum_{k \in \mathbb{Z}^d - \{0\}} -\hat{h}_k \cdot 2(\cos(2\pi\omega k \cdot \alpha) - 1) \cdot \hat{h}_{-k} 2\pi i(k \cdot \alpha) + \hat{h}_0(2 \cos(2\pi\omega 0 \cdot \alpha) - 1). \end{aligned}$$

We also observe that

$$\int_{-T}^T \partial_\alpha V(\alpha h(\theta)) \cdot h'(\theta) d\theta = V(\alpha h(T)) - V(\alpha h(-T)).$$

So it is bounded independent of T . When we divide the integral by $2T$ and take the limit $T \rightarrow \infty$. We obtain 0. This ends the proof of Lemma 22. \square

6. SEVERAL FURTHER CONSEQUENCES OF THE FORMALISM

As pointed out in [CdIL10], once one has an a-posteriori theorem with local uniqueness in analytic and Sobolev spaces, there are more or less automatically several consequences which could be of interest for applications and which we now make explicit for our case.

6.1. Existence of perturbative expansions to all orders and their convergence. If we consider models in which the interaction has a small parameter, i.e. the interaction term is given by $\epsilon \widehat{U}$, it is interesting to know whether one can write formal power series $\hat{h}_\epsilon = \sum_n \epsilon^n \hat{h}^n$, $\lambda_\epsilon = \sum_n \epsilon^n \lambda^n$ which solve (5) in the sense of power series as well as the normalization condition (6). Furthermore it is interesting to show that that the series converges. These power series for hull functions are very similar to the Lindstedt series in mechanics.

We will show that, when the frequencies are Diophantine, the solution to both questions is affirmative. Series exist to all orders and converge.

6.1.1. *Existence of Lindstedt series to all orders.* We first argue that one can find the solution to (5) in the sense of power series.

If we substitute the power series and match like powers of ϵ , we obtain a hierarchy of equations for the coefficients of the perturbation. At order ϵ^0 we obtain:

$$(43) \quad \hat{h}^0(\sigma + \omega\alpha) + \hat{h}^0(\sigma - \omega\alpha) - 2\hat{h}^0(\sigma) + \lambda^0 = 0$$

which implies that $\lambda^0 = 0$, \hat{h}^0 is a constant. Because of the normalization (6), we have $\hat{h}^0 = 0$.

At order ϵ^1 , we obtain

$$(44) \quad \hat{h}^1(\sigma + \omega\alpha) + \hat{h}^1(\sigma - \omega\alpha) - 2\hat{h}^1(\sigma) + \widehat{U}(\sigma) + \lambda^1 = 0.$$

This equation is very similar to the equations studied in Section 2.3. Indeed, in Fourier series, it is equivalent to

$$\hat{h}_k^1 2(\cos(2\pi\omega k \cdot \alpha) - 1) = \widehat{U}_k + \delta_{0,k}\lambda^1.$$

We see that we can determine $\lambda^1 = -\widehat{U}_0$. \hat{h}_0^1 is not determined by (44) but the normalization (6) sets it to $h_0^1 = 0$. All the other Fourier coefficients can be determined and indeed if \widehat{U}_k is analytic in some domain then \hat{h}_k^1 is analytic in a slightly smaller domain.

In general, at order n , we obtain:

$$(45) \quad \hat{h}^n(\sigma + \omega\alpha) + \hat{h}^n(\sigma - \omega\alpha) - 2\hat{h}^n(\sigma) + R_n(\sigma) + \lambda^n = 0$$

where R_n is a polynomial expression in $\hat{h}^1, \hat{h}^2, \dots, \hat{h}^{n-1}$ with coefficients which are derivatives of \widehat{U} . So, we can assume by induction that R_n is known. The equation (45) is of the same form as (44) and the same analysis shows that we can get a unique solution for \hat{h}^n and, hence, recover the hypothesis.

6.1.2. *Convergence of the formal power series.* The fact that the formal power series converges is a very easy consequence of the fact that there are analytic families $\hat{h}_\epsilon, \lambda_\epsilon$, which solve the equations. This is a general fact, which is a consequence of the formalism and we go over the proof rather quickly. See [CdIL10, GEdIL08].

We just note that the method we have used works just as well for complex functions. We note that for all ϵ small enough, there is a solution. (Note that we can take $\hat{h} = 0, \lambda = 0$ as an approximate solution if ϵ is small). So, it suffices to show that this solution depends differentiably on the complex parameter ϵ . We follow the standard practice in analysis of first obtaining a guess of the derivative and then proving it indeed satisfies the definition of derivative.

For a fixed ϵ we can guess $\frac{d}{d\epsilon}\hat{h}_\epsilon$, $\frac{d}{d\epsilon}\lambda_\epsilon$ because if they existed, they should satisfy

$$(46) \quad \begin{aligned} & \frac{d}{d\epsilon}\hat{h}_\epsilon(\sigma + \omega\alpha) + \frac{d}{d\epsilon}\hat{h}_\epsilon(\sigma - \omega\alpha) - 2\frac{d}{d\epsilon}\hat{h}_\epsilon(\sigma) \\ & + \epsilon\partial_\alpha\widehat{U}(\sigma + \alpha\hat{h}_\epsilon)\frac{d}{d\epsilon}\hat{h}_\epsilon(\sigma) + \widehat{U}(\sigma + \alpha\hat{h}_\epsilon) + \frac{d}{d\epsilon}\lambda_\epsilon = 0. \end{aligned}$$

The method used in Section 4.1 shows that the equation (46) for $\frac{d}{d\epsilon}\hat{h}_\epsilon(\sigma)$ can be transformed into a constant coefficient equation (note that, by assumption h_ϵ is an exact solution of (5)).

Now, to prove that this guess indeed is the derivative, we just note that $\|\mathcal{E}_{\epsilon+\mu}(\hat{h}_\epsilon + \mu\frac{d}{d\epsilon}\hat{h}_\epsilon)\|_{\rho-\eta} \leq C|\mu|^2$. Then, applying the a-posteriori format and the local uniqueness, we conclude $\|\hat{h}_{\epsilon+\mu} - \hat{h}_\epsilon - \mu\frac{d}{d\epsilon}\hat{h}_\epsilon\|_{(\rho-\eta)/2} \leq C\mu^2$.

6.2. Bootstrap of regularity. In this section we state the theorem of bootstrap of regularity and omit the proof. See [CdIL10] for more details.

Theorem 23. *Let $\hat{h} \in H^m$, λ be a solution of (5) with \widehat{U} analytic.*

Assume that m is large enough (depending only on the Diophantine exponent). Then, \hat{h} is analytic.

The idea of the proof is very simple. We can take a truncation of the Fourier series as an approximate solution. Of course, these are analytic functions. If the decrease of the Fourier series is fast enough, it is possible to use the analytic theorem and conclude that there is an analytic solution. By the uniqueness in Sobolev spaces, this must be the original solution.

In Sobolev regularity this is restated as Theorem 6.8 in [CdIL10]. In [SZ88, GEDIL08], one can find a similar argument for C^r classes. The argument for C^r classes in the later papers is somewhat more involved since it obtains sharper results by using better approximations than truncating.

6.3. A practical numerical criterion for the analyticity breakdown. The above considerations lead to a very practical and reliable way to compute thresholds of breakdown of analytic solutions.

Observe that by now, we have efficient algorithms to compute the invariant tori, given approximate solutions. This, of course, immediately leads to a continuation algorithm. Since we have an a-posteriori theorem, we can be sure that, the approximate solutions produced numerically (which satisfy the invariance equation up to a few units of round-off error) correspond to true solutions if they satisfy the non-degeneracy conditions.

Therefore, a practical algorithm to compute the threshold of breakdown is to implement the continuation method and monitor the non-degeneracy conditions.

In many occasions it happens that the only condition that fails is that $\|\hat{h}\|_{H^m}$ becomes very large. In that case, one can argue that the solutions experience a breakdown because if there were analytic tori in a neighborhood of parameters, the Sobolev norms would remain bounded. Similar methods for the periodic Frenkel-Kontorova models and models with long range interactions have been implemented in [CdLL10].

Some implementations of the method are already in progress [BHdLL11] and the results will be reported elsewhere.

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