Stability and instability of solutions of a nonlocal reaction-diffusion equation when the essential spectrum crosses the imaginary axis

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Abstract. The paper is devoted to an integro-differential equation arising in population dynamics where the integral term describes nonlocal consumption of resources. This equation can have several stationary points and, as it is already well known, a travelling wave solution which provides a transition between them. It is also possible that one of these stationary points loses its stability resulting in appearance of a stationary periodic in space structure. In this case, we can expect a possible transition between a stationary point and a periodic structure. The main goal of this work is to study such transitions. The loss of stability of the stationary point signifies that the essential spectrum of the operator linearized about the wave intersects the imaginary axis. Contrary to the usual Hopf bifurcation where a pair of isolated complex conjugate eigenvalues crosses the imaginary axis, here a periodic solution may not necessarily emerge. To describe dynamics of solutions, we need to consider two transitions: a steady wave with a constant speed between two stationary points, and a periodic wave between the stationary point which loses its stability and the periodic structure which appears around it. Both of these waves propagate in space, each one with its own speed. If the speed of the steady wave is greater, then it runs away from the periodic wave, and they propagate independently one after another.

1 Nonlocal reaction-diffusion equations in population dynamics

The classical logistic equation in population dynamics

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + ku(1-u), \tag{1.1}$$

describes the evolution of population density due to random displacement of individuals (diffusion term) and their reproduction (nonlinear reaction term). This equation is intensively studied beginning from the works by Fischer and Kolmogorov, Petrovskii, Piskunov (see [12] and the references therein). The reproduction terms is proportional to the population density u and to available resources $(1 - u)$. Here 1 is the normalized carrying capacity decreased by consumed resources, which are proportional to the population density. In some biological applications, consumption of resources is proportional not to the point-wise value of the density but to its average value (e.g., [8]):

$$\overline{u}(x) = \frac{1}{2h} \int_{x-h}^{x+h} u(y)dy = \frac{1}{2h} \int_{-\infty}^{\infty} \chi_h(x-y)u(y)dy,$$

where χ_h is the characteristic function of the interval $[-h, h]$. In this case we arrive to the nonlocal reaction-diffusion equation

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + ku(1 - \overline{u}). \tag{1.2}$$

In a more general setting, we consider the integro-differential equation

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + F(u, J(u)), \tag{1.3}$$

where

$$J(u) = \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy,$$

$F(u, v)$ is a sufficiently smooth function of its two arguments. In what follows it is convenient to consider it in the form

$$F(u, J(u)) = f(u)(1 - J(u)),$$

where $f(u)$ is a sufficiently smooth function, $f(u) > 0$ for $u > 0$, $f(0) = 0, f(1) = 1$. We will assume everywhere below that ϕ is a bounded even function with a compact support, and $\int_{-\infty}^{\infty} \phi(y)dy = 1$. Hence $u = 0$ and $u = 1$ are stationary solutions of equation (1.3).

If we look for travelling wave solutions of equation (1.3), it is convenient to substitute $u(x,t) = v(x - ct, t)$. Then

$$\frac{\partial v}{\partial t} = d\,\frac{\partial^2 v}{\partial x^2} + c\,\frac{\partial v}{\partial x} + F(v, J(v)). \tag{1.4}$$

Travelling wave is a stationary nonhomogeneous in space solution of this equation. Generalized travelling wave can be time dependent. The existence of travelling waves was studied in [1]-[5], [7].

An important property of equation (1.3) is that its homogeneous in space stationary solution $u = 1$ can lose its stability resulting in appearance of periodic in space stationary solutions (see, e.g., [6], [8]), [10]). If we consider this problem in a bounded interval, then this solution bifurcates due to a real eigenvalue which crosses the origin. The situation is more complex if we consider this equation on the whole axis. In this case, not an isolated eigenvalue crosses the origin but the essential spectrum. Conventional bifurcation analysis does not allow us to study bifurcations of nonhomogeneous in space solutions.

We will study behavior of solutions in this case by the combination of numerical simulations and stability analysis. Linear stability analysis allows us to determine the stability boundary. Numerical simulations show that solution with a localized initial condition propagates in space. Stability analysis in a weighted space, where the whole spectrum lies in the left half-plane, gives an estimate of the speed of propagation of this solution. Though the spectrum is in the left half-plane, conventional results on nonlinear stability of the solution appear to be unapplicable because the operator does not satisfy the required conditions. We prove a weaker nonlinear stability result of the homogeneous in space solution in a properly chosen weighted space. This is the stability on a half-axis in the coordinate frame moving faster than the propagation of the periodic in space solution.

Next, we study travelling waves connecting the points $u = 0$ and $u = 1$. If the essential spectrum of the operator linearized about a wave crosses the imaginary axis, we cannot use the existing results on the wave stability. We prove the wave stability on a half-axis in some weighted spaces. This result can be interpreted as follows. The wave connecting the points $u = 0$ and $u = 1$ propagates with some speed c_0. The solution which provides the transition from $u = 1$ to a periodic in space solution propagates with some speed c_1. If $c_0 > c_1$, then the wave moves faster and the distance between them increases. This implies the wave stability on the half-axis. Considered on the whole axis, the wave is not stable because of the periodic perturbation, which grows since the essential spectrum is partially in the right-half plane.

2 Spectrum of the operator linearized about a stationary solution

2.1 Equation (1.3)

We analyze the stability of the solution $u = 1$ of equation (1.3). Linearizing this equation about this stationary solution, we obtain the eigenvalue problem:

$$du'' - \int_{-\infty}^{\infty} \phi(x-y)u(y)dy = \lambda u. \tag{2.1}$$

Applying the Fourier transform, we have

$$\lambda_d(\xi) = -d\xi^2 - \tilde{\phi}(\xi),$$

where $\tilde{\phi}(\xi)$ is the Fourier transform of the function $\phi(x)$. We will assume that it is a real-valued, even, bounded and continuous function, which is not everywhere positive. An example where these conditions are satisfied is given by the following function ϕ:

$$\phi(x) = \begin{cases} 1/(2N) & , \quad -N \le x \le N \\ 0 & , \quad |x| > N \end{cases}$$

Thus, $\tilde{\phi}(0) = 1/\sqrt{2\pi}$, and there exist one or more intervals where $\tilde{\phi}(\xi)$ is negative. Hence we can make some conclusions about the structure of the function $\lambda_d(\xi)$. There exists $d = d_c$ such that

$$\lambda_d(\xi) < 0, \quad \xi \in \mathbb{R}, \quad d > d_c$$

and

$$\lambda_{d_c}(\xi) \le 0, \quad \xi \in \mathbb{R}, \quad \lambda_{d_c}(\pm\xi_0) = 0$$

for some $\xi_0 > 0$. Finally, $\lambda_d(\xi) > 0$ in some intervals of ξ for $0 < d < d_c$.

These assumptions signify that the essential spectrum of the operator

$$L_0 u = du'' - \int_{-\infty}^{\infty} \phi(x-y)u(y)dy$$

is in the left-half plane for $d > d_c$ and it is partially in the right-half plane for $d < d_c$.

2.2 Equation (1.4)

Consider next equation (1.4). As above, we linearize it about the solution $u = 1$ and obtain the eigenvalue problem

$$du'' + cu' - \int_{-\infty}^{\infty} \phi(x-y)u(y)dy = \Lambda u. \tag{2.2}$$

Applying the Fourier transform, we get

$$\Lambda_d(\xi) = -d\xi^2 + ci\xi - \tilde{\phi}(\xi).$$

From the properties of the function $\lambda_d(\xi)$ it follows that $\Lambda_d(\xi)$ is in the left-half plane of the complex plane for $d > d_c$. For $d = d_c$, it has two values, $\pm ci\xi_0$ at the imaginary axis (Figure 1), and it is partially in the right-half plane for $d < d_c$. Thus, the essential spectrum passes to the right-half plane when the parameter d decreases and crosses the critical value $d = d_c$.

Let us introduce the function $v(x) = u(x)\exp(-\sigma x)$, where $\sigma > 0$ is a constant. We substitute the function $u(x) = v(x)\exp(\sigma x)$ into (2.2):

$$dv'' + (c + 2d\sigma)v' + (d\sigma^2 + c\sigma)v - e^{-\sigma x}\int_{-\infty}^{\infty} \phi(x-y)v(y)e^{\sigma y}dy = \Lambda v.$$

Denote

$$\psi(x) = \phi(x)e^{-\sigma x}.$$

Then

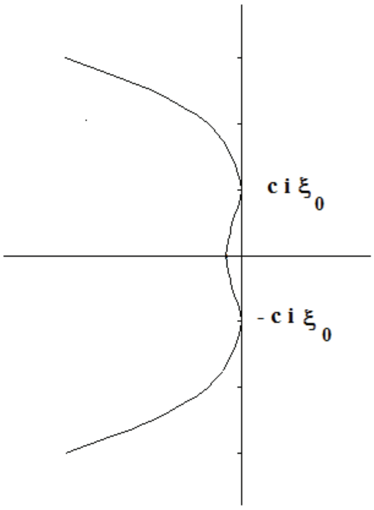


Figure 1: Schematic representation of the part $\lambda_+(\xi)$ of the essential spectrum for $d = d_c$.

$$\Lambda_{d,\sigma}(\xi) = -d\xi^2 + (c + 2d\sigma)i\xi + d\sigma^2 + c\sigma - \tilde{\psi}(\xi), \quad \xi \in \mathbb{R},$$

where $\tilde{\psi}(\xi)$ is the Fourier transform of the function $\psi(x)$.

Put $\sigma = -c/(2d)$. Then

$$\Lambda_{d,\sigma}(\xi) = -d\xi^2 - \tilde{\psi}(\xi) - \frac{c^2}{4d}, \quad \xi \in \mathbb{R}.$$

If

$$\frac{c^2}{4d} > \sup_{\xi} \mathrm{Re}\,(-d\xi^2 - \tilde{\psi}(\xi)), \tag{2.3}$$

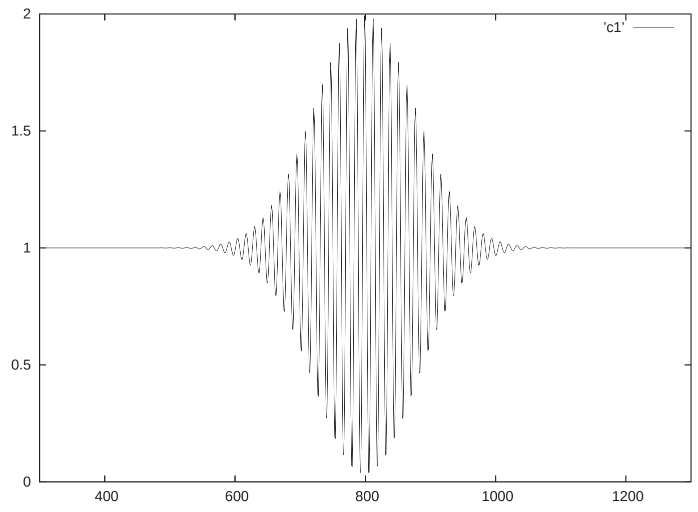
then the spectrum is in the left-half plane and the solution of the linear equation

$$\frac{\partial v}{\partial t} = d\,\frac{\partial^2 v}{\partial x^2} + (c + 2d\sigma)\,\frac{\partial v}{\partial x} + (d\sigma^2 + c\sigma)v - e^{-\sigma x}\int_{-\infty}^{\infty} \phi(x-y)v(y,t)e^{\sigma y}dy \tag{2.4}$$

will converge to zero as $t \to \infty$ uniformly in x. Hence condition (2.3) gives an estimate of the speed of propagation of the perturbation for the equation

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} - \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy. \tag{2.5}$$

Figure 2 shows an example of numerical simulations of a propagating perturbation.



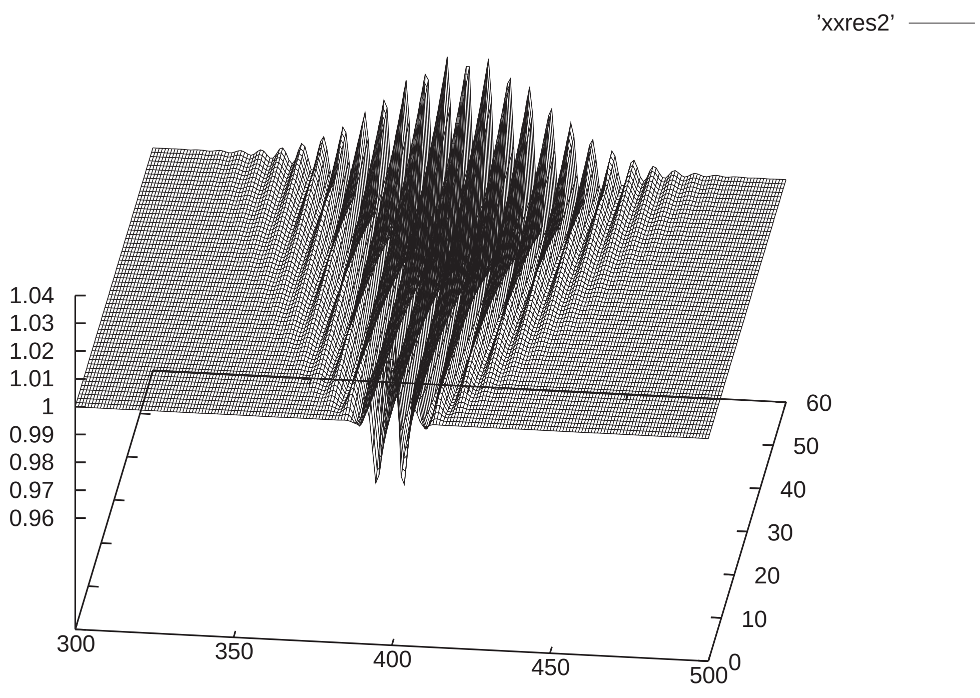


Figure 2: Snapshot of solution $u(x,t)$ of equation (2.5) (top). The same solution as a function of two variables. The perturbation spreads to the left and to the right with a constant speed (bottom).

2.3 Essential spectrum of the operator linearized about a wave

Suppose that equation (1.4) has a stationary solution $w(x)$ with the limits $w(-\infty) = 0, w(\infty) = 1$. Suppose that the wave propagates from the right to the left, that is $c < 0$ and the solution $u(x,t) = w(x - ct)$ converges to 1 uniformly on every bounded set.

We linearize equation (1.4) about $w(x)$ and obtain the eigenvalue problem

$$du'' + cu' + a(x)u - b(x)J(u) = \lambda u, \tag{2.6}$$

where

$$a(x) = f'(w(x))(1 - J(w)), \quad b(x) = f(w(x)).$$

The essential spectrum of the operator

$$Lu = du'' + cu' + a(x)u - b(x)J(u)$$

is given by two curves on the complex plane:

$$\lambda_-(\xi) = -d\xi^2 + ci\xi + f'(0), \qquad \lambda_+(\xi) = -d\xi^2 + ci\xi - \tilde{\phi}(\xi), \quad \xi \in \mathbb{R}.$$

The curve $\lambda_-(\xi)$ is a parabola. It lies in the left-half plane if $f'(0) < 0$ and it is partially in the right-half plane if $f'(0) > 0$. The second curve $\lambda_+(\xi)$ coincides with $\Lambda_d(\xi)$ considered in Section 2.2. It is located completely in the left-half plane for $d > d_c$ and it is partially in the right-half plane for $d < d_c$. It is shown schematically in Figure 1 for the critical value $d = d_c$.

Let us now introduce a weight function $g(x)$. We assume that it is positive, sufficiently smooth and such that

$$g(x) = \begin{cases} e^{\sigma_+ x} & , \quad x \geq 1 \\ e^{\sigma_- x} & , \quad x \leq -1 \end{cases}, \tag{2.7}$$

where the exponents $\sigma_\pm$ will be specified below. We substitute $u(x) = v(x)g(x)$ into (2.6):

$$d(gv'' + 2v'g' + vg'') + c(v'g + vg') + avg - b\int_{-\infty}^{\infty} \phi(x-y)v(y)g(y)du = \lambda vg$$

or

$$dv'' + (c + 2dg_1)v' + (dg_2 + cg_1)v - b\int_{-\infty}^{\infty} \phi(x-y)\gamma(x,y)v(y)dy = \lambda v,$$

where

$$g_1(x) = \frac{g'(x)}{g(x)}, \quad g_2(x) = \frac{g''(x)}{g(x)}, \quad \gamma(x,y) = \frac{g(y)}{g(x)}. \tag{2.8}$$

The essential spectrum of the operator

$$Mv = dv'' + (c + 2dg_1)v' + (dg_2 + cg_1 + a)v - b\int_{-\infty}^{\infty} \phi(x-y)\gamma(x,y)v(y)dy \tag{2.9}$$

is given by the following expressions:

$$\begin{aligned} \lambda_\sigma^-(\xi) &= -d\xi^2 + (c + 2d\sigma_-)i\xi + (d\sigma_-^2 + c\sigma_- + f'(0)) \quad , \\ \lambda_\sigma^+(\xi) &= -d\xi^2 + (c + 2d\sigma_+)i\xi + d\sigma_+^2 + c\sigma_+ - \tilde{\psi}(\xi), \quad \xi \in \mathbb{R} \quad . \end{aligned} \tag{2.10}$$

The second expression coincides with $\Lambda_{d,\sigma}(\xi)$ (Section 2.2). When we study stability of waves, we will choose, when it is possible, $\sigma_\pm$ in such a way that the essential spectrum lies in the left-half plane.

3 Nonlinear stability

Consider the equation (cf. (1.4))

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + c\,\frac{\partial u}{\partial x} + f(u)\left(1 - \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy\right). \tag{3.1}$$

Let us look for its solution in the form $u(x,t) = w(x) + v(x,t)g(x)$, where $w(x)$ is a stationary solution and $g(x)$ is a weight function. Then

$$\frac{\partial v}{\partial t} = d\,\frac{\partial^2 v}{\partial x^2} + (c + 2dg_1)\,\frac{\partial u}{\partial x} + (dg_2 + cg_1)v + B(v), \tag{3.2}$$

where the nonlinear operator $B(v)$ writes

$$B(v) = g^{-1}f(w + vg)\left(1 - \int_{-\infty}^{\infty} \phi(x-y)g(y)v(y,t)dy - \int_{-\infty}^{\infty} \phi(x-y)w(y)dy\right) -$$

$$g^{-1}f(w)\left(1 - \int_{-\infty}^{\infty} \phi(x-y)w(y)dy\right) =$$

$$-f(w+vg)\int_{-\infty}^{\infty} \phi(x-y)\gamma(x,y)v(y,t)dy + \frac{f(w+vg) - f(w)}{vg}\,v\,\left(1 - \int_{-\infty}^{\infty} \phi(x-y)w(y)dy\right) =$$

$$-f(w+vg)\int_{-\infty}^{\infty} \phi(x-y)\gamma(x,y)v(y,t)dy + f'(w + \theta vg)\,v\,\left(1 - \int_{-\infty}^{\infty} \phi(x-y)w(y)dy\right),$$

where $\theta(x) \in (0,1)$ is some function. Then

$$B(v) = B_0(v) + B_1(v),$$

where

$$B_0(v) = -f(w) \int_{-\infty}^{\infty} \phi(x-y)\gamma(x,y)v(y,t)dy + f'(w)\; v\; \left(1 - \int_{-\infty}^{\infty} \phi(x-y)w(y)dy\right)$$

is a linear and

$$B_1(v) = -(f(w+vg) - f(w)) \int_{-\infty}^{\infty} \phi(x-y)\gamma(x,y)v(y,t)dy +$$

$$(f'(w+\theta vg) - f'(w))\; v\; \left(1 - \int_{-\infty}^{\infty} \phi(x-y)w(y)dy\right) \tag{3.3}$$

a nonlinear operators.

Hence we can write (3.2) as

$$\frac{\partial v}{\partial t} = M_0 v + B_0 v + B_1(v), \tag{3.4}$$

where

$$M_0 v = d\,\frac{\partial^2 v}{\partial x^2} + (c + 2dg_1)\,\frac{\partial u}{\partial x} + (dg_2 + cg_1)v.$$

Note that $M = M_0 + B_0$, where M is the operator introduced in Section 2.3.

We note that the function $\gamma(x, y)$ under the integral is bounded since $(x - y)$ is bounded in the support of the function ϕ. However, if the weight function $g(x)$ is unbounded, then the nonlinear operator $B_1(v)$ does not satisfy a Lipschitz condition because $f(w + vg)$ and $f'(w + \theta vg)$ do not satisfy it (considered as operators acting on v). Therefore we cannot apply conventional results on stability of solutions [11], [12]. We will use a weaker result presented in the next section.

3.1 An abstract theorem on stability of stationary solutions

Consider the evolution equation

$$\frac{\partial v}{\partial t} = Av + T(v), \tag{3.5}$$

where A is a linear operator acting in a Banach space E, $T(v)$ is a nonlinear operator acting in the same space. Suppose that A is a sectorial operator and its spectrum lies in the half-plane Re $\lambda < -\beta$, where β is a positive number, and the operator T satisfies the estimate

$$\|T(v)\| \le K\|v\|, \tag{3.6}$$

where K is a constant independent of v. Suppose next that there exists a mild solution of equation (3.5), that is a function $v(t) \in E$ which satisfies the equation

$$v(t) = e^{A(t-t_0)}v(t_0) + \int_{t_0}^{t} e^{A(t-s)}T(v(s))ds. \tag{3.7}$$

Since the operator A is sectorial, then we have the estimate

$$\|e^{At}v\| \le Re^{-\beta t}\|v\| \tag{3.8}$$

with some constant $R \ge 1$. Let $0 < \alpha < \beta$. Suppose that

$$2RK < \beta - \alpha. \tag{3.9}$$

We will show that $\|v\|$ converges to zero. Similar to the estimates in the proof of Theorem 5.1.1 in [11] [1], we obtain from (3.7):

$$\|v(t)\| \le R\|v(t_0)\| + KR \sup_{s\in(t_0,t)} \|v(s)\| \int_{t_0}^{t} e^{-\beta(t-s)}ds.$$

Hence

$$\sup_{s\in(t_0,t)} \|v(s)\| \le R\|v(t_0)\| + \frac{1}{2} \sup_{s\in(t_0,t)} \|v(s)\|.$$

Therefore $\|v(t)\| \le 2R\|v(t_0)\|$ for all $t \ge t_0$, that is the norm of the solution remains uniformly bounded.

On the other hand, from the same equation we obtain

$$\|v(t)\| \le Re^{-\alpha(t-t_0)}\|v(t_0)\| + KR \int_{t_0}^{t} e^{-\beta(t-s)}\|v(s)\|ds.$$

Let

$$\omega(t) = \sup_{s\in(t_0,t)} \|v(s)\|e^{\alpha(s-t_0)}.$$

Then from the last inequality

$$\|v(t)\|e^{\alpha(t-t_0)} \le R\|v(t_0)\| + KR \int_{t_0}^{t} e^{-\beta(t-s)}e^{\alpha(t-t_0)}\|v(s)\|ds \le$$

$$R\|v(t_0)\| + KR\,\omega(t) \int_{t_0}^{t} e^{-(\beta-\alpha)(t-s)}ds \le R\|v(t_0)\| + \frac{1}{2}\,\omega(t).$$

Hence $\omega(t) \le 2R\|v(t_0)\|$. Thus the norm of the solution exponentially converges to zero.

[1] We cannot directly use the theorem because the nonlinear operator does not satisfy a Lipschitz condition. This is the reason why we suppose in addition the existence of a mild solution.

Existence of a mild solution Suppose that $f(0) = 0$ and the initial condition of the Cauchy problem for equation (3.1) is non-negative. Then the solution is also non-negative, and the solution of this equation can be estimated from above by the solution of the equation

$$\frac{\partial z}{\partial t} = d\,\frac{\partial^2 z}{\partial x^2} + c\,\frac{\partial z}{\partial x} + f(z). \tag{3.10}$$

If $f(z) \leq m$ for all $z \geq 0$, then the supremum of the solution of the last equation can be estimated by $\exp(mt)\sup_x |z(x,0)|$. Therefore, for every $T > 0$, there exists the classical solution $u(x,t)$ of equation (3.1). Then the operator $B_1(v)$ can be written as

$$B_1(v,t) = -(f(u) - f(w)) \int_{-\infty}^{\infty} \phi(x-y)\gamma(x,y)v(y,t)dy +$$

$$(f'((1-\theta)w + \theta u) - f'(w))\, v\, \left(1 - \int_{-\infty}^{\infty} \phi(x-y)w(y)dy\right),$$

where $u(x,t)$ is considered as a given function. Taking into account that $\gamma(x,y)$ is uniformly bounded, this operator satisfies a Lipschitz condition with respect to v and a Hölder condition with respect to t. Then we can affirm the existence of a mild solution of equation (3.4).

We note that if $f(z_0) = 0$ for some $z_0 > 0$, and the initial condition $z(x,0)$ is such that $0 < z(x,0) < z_0$ for all $x \in \mathbb{R}$, then the solution also satisfies these inequalities for all t: $0 < z(x,t) < z_0$. This provides the boundedness of the solution $u(x,t)$ of equation (3.1).

3.2 Stability of the homogeneous solution

Consider the homogeneous solution $w(x) = 1$. Let $g(x) = e^{\sigma x}$ and $\sigma = -c/(2d)$. Then

$$Mv = d\,\frac{\partial^2 v}{\partial x^2} - \frac{c^2}{4d}\,v\, - f(1)\int_{-\infty}^{\infty} \phi(x-y)e^{-\sigma(x-y)}v(y,t)dy,$$

$$B_1(v) = -(f(1+vg) - f(1)) \int_{-\infty}^{\infty} \phi(x-y)e^{-\sigma(x-y)}v(y,t)dy.$$

We recall that $f(1) = 1$ but we keep it here for convenience. We multiply the equation

$$\frac{dv}{dt} = Mv$$

by v and integrate over $\mathbb{R}$:

$$\frac{d\|v\|^2}{dt} = 2\int_{-\infty}^{\infty} \Psi(\xi)(\tilde{v}(\xi))^2 d\xi,$$

where

$$\Psi(\xi) = -d\xi^2 - \frac{c^2}{4d} - f(1)\tilde{\psi}(\xi),$$

$\tilde{\psi}(\xi)$ is the Fourier transform of the function $\psi(x) = \phi(x)e^{-\sigma x}$. Suppose that

$$\sup_{\xi} \Psi(\xi) < -\beta, \tag{3.11}$$

where β is a positive constant. Then

$$\frac{d\|v\|^2}{dt} \le -2\beta\|v\|^2,$$

where $\|v\|$ is the norm in $L^2(\mathbb{R})$. Hence $\|v(t)\| \le e^{-\beta(t-t_0)}\|v(t_0)\|$.

We next estimate the operator $B_1(v)$. Denote by $V(f)$ the maximal variation of the function f,

$$V(f) = \sup_{x,y} |f(x) - f(y)|.$$

Then

$$\|B_1(v)\|^2 \le (V(f))^2 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \psi(x-y)v(y,t)dy \right)^2 dx = (V(f))^2 \int_{-\infty}^{\infty} \left| \tilde{\psi}(\xi)\tilde{v}(\xi,t) \right|^2 d\xi \le$$

$$\left(V(f) \sup_{\xi} |\tilde{\psi}(\xi)| \right)^2 \|v\|^2.$$

Thus, we have determined the values of the constants in estimate (3.9):

$$R = 1, \quad K = V(f) \sup_{\xi} |\tilde{\psi}(\xi)|$$

and β is determined by (3.11). We have proved the following theorem.

Theorem 3.1. *Let the maximum of the function*

$$\Psi_0(\xi) = -d\xi^2 - f(1)\mathrm{Re}\,\tilde{\psi}(\xi)$$

be attained at $\xi = \xi_0$ and $\Psi_0(\xi_0) > 0$. Suppose that $c < 0$ and $c^2 > 4d\Psi_0(\xi_0)$. If for some α such that

$$0 < \alpha < \frac{c^2}{4d} - \Psi_0(\xi_0) \tag{3.12}$$

the estimate

$$2V(f) \sup_{\xi} |\tilde{\psi}(\xi)| < \frac{c^2}{4d} - \Psi_0(\xi_0) - \alpha \tag{3.13}$$

holds, then the stationary solution $u = 1$ of the equation (3.1) is asymptotically stable with weight and the following convergence occurs:

$$\|(u(x,t)-1)e^{(c/2d)x}\| \le 2Re^{-\alpha t}\ \|(u(x,0)-1)e^{(c/2d)x}\|, \quad t \ge 0.$$

We note that condition (3.12) provides linear stability of the stationary solution in the sense that the spectrum of the linearized problem lies in the left-half plane. However, it may not be sufficient for nonlinear stability understood as convergence of the solution of nonlinear problem to the stationary solution. This implies the additional condition (3.13).

3.3 Stability of waves

Consider the equation

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + c\,\frac{\partial u}{\partial x} + f(u)(1 - J(u)) \tag{3.14}$$

in $\mathbb{R}$. Assume that it has a stationary solution $w(x)$ with the limits $w(\pm\infty) = w_\pm$ at infinity. This function satisfies the equation

$$dw'' + cw' + f(w)(1 - J(w)) = 0, \quad w(\pm\infty) = w_\pm. \tag{3.15}$$

If $f'(0) \ne 0$, then it decays exponentially at $-\infty$ with the exponent μ which can be found from the equation

$$d\mu^2 + c\mu + f'(0) = 0. \tag{3.16}$$

If $f'(0) < 0$, then there is a unique positive value of μ which provides a bounded solution at $-\infty$:

$$\mu_1 = -\frac{c}{2d} - \sqrt{\frac{c^2}{4d^2} - \frac{f'(0)}{d}}.$$

If $f'(0) > 0$, then there are two positive values:

$$\mu_{1,2} = -\frac{c}{2d} \pm \sqrt{\frac{c^2}{4d^2} - \frac{f'(0)}{d}}, \quad \mu_1 > \mu_2 > 0$$

($c < 0$). We will restrict ourselves to the latter. It corresponds to the monostable case where the waves exist for all positive speeds greater or equal to some minimal speed [4], [12].

We recall the operator linearized about the wave:

$$Lu = du'' + cu' + a(x)u - b(x)J(u),$$

where

$$a(x) = f'(w)(1 - J(w)), \;\; b(x) = f(w).$$

The part of its essential spectrum corresponding to $-\infty$ is given by the curve

$$\lambda_-(\xi) = -d\xi^2 + ci\xi + f'(0), \quad x \in \mathbb{R}.$$

Since we assume that $f'(0) > 0$, then it is partially in the right-half plane. We recall that it is obtained as a set of all complex λ for which the equation

$$L^- u \equiv du'' + cu' + f'(0)u = \lambda u$$

has a bounded solution in $\mathbb{R}$. Let us substitute $u = e^{\sigma_- x} v$ in this equation. Then

$$dv'' + (c + 2d\sigma_-)v' + (d\sigma_-^2 + c\sigma_- + f'(0))v = \lambda v.$$

Therefore the essential spectrum

$$\lambda_\sigma^-(\xi) = -d\xi^2 + (c + 2d\sigma_-)i\xi + d\sigma_-^2 + c\sigma_- + f'(0), \quad \xi \in \mathbb{R}$$

is completely in the left-half plane if

$$d\sigma_-^2 + c\sigma_- + f'(0) < 0.$$

There exists a value of σ_- such that this condition is satisfied if $c^2 > 4df'(0)$. We assume that this condition is satisfied, and

$$-\frac{c}{2d} - \sqrt{\frac{c^2}{4d^2} - \frac{f'(0)}{d}} < \sigma_- < -\frac{c}{2d} + \sqrt{\frac{c^2}{4d^2} - \frac{f'(0)}{d}}. \tag{3.17}$$

Consider the weighted norm

$$\|w\|_{C_{\sigma_-}(\mathbb{R})} = \sup_x |(1 + e^{-\sigma_- x})w(x)|.$$

The wave $w(x)$ is bounded in this norm if (a) $w(x) - w_+ \sim e^{\mu_1 x}$ as $x \to -\infty$ and unbounded if (b) $w(x) - w_+ \sim e^{\mu_2 x}$ as $x \to -\infty$.

In the case of the reaction-diffusion equation [12] and for some nonlocal reaction-diffusion equations [4] the wave with the minimal speed behaves as (a) and all waves with greater speeds behave as (b). In what follows we will be interested in waves with sufficiently large speeds. Therefore we will consider the case (b) where the wave does not belong to the weighted space.

We look for the solution of equation (3.14) (the same as (3.1)) in the form $u = w + vg$, where the weight function g is given by (2.7). Then v satisfies the equation

$$\frac{\partial v}{\partial t} = Mv + B_1(v), \tag{3.18}$$

where the operator M is given by (2.9) and the operator $B_1(v)$ by (3.3). The essential spectrum of the operator M is given by the curves (2.10). If condition (3.17) is satisfied, then the curve $\lambda_\sigma^-(\xi)$ lies in the left-half plane. This is also true for $\lambda_\sigma^+(\xi)$ if

$$-d\xi^2 + d\sigma_+^2 + c\sigma_+ - \text{Re}\,\tilde{\psi}(\xi) < 0, \quad \xi \in \mathbb{R}.$$

We recall that $f(1) = 1$ and $\tilde{\psi}(\xi)$ is the Fourier transform of the function $\phi(x)e^{-\sigma_+ x}$. In order to show its dependence on σ_+, we will denote it by $\tilde{\psi}(\xi;\sigma_+)$. Set

$$\Psi(\xi;\sigma_+) = -d\xi^2 + d\sigma_+^2 + c\sigma_+ - \text{Re}\,\tilde{\psi}(\xi;\sigma_+).$$

Assumption 1. The maximum of the function $\Psi(\xi;0)$ is positive and there exists such σ_+ that the maximum of the function $\Psi(\xi;\sigma_+)$ is negative.

This assumption means that the essential spectrum of the operator L is partially located in the right-half plane and that for some σ_+ the essential spectrum of the operator M lies completely in the left-half plane.

Discrete spectrum. Let us now discuss the structure of the discrete spectrum of the operator M. It can be directly verified that the operator L has a zero eigenvalue with the corresponding eigenfunction w'. By virtue of the assumptions on the wave $w(x)$, its derivative does not belong to the weighted space $C_g(\mathbb{R})$ with the norm

$$\|v\|_g = \sup_x \left| \frac{v}{g} \right|.$$

Hence if zero eigenvalue of the operator L is simple, then the operator M does not have zero eigenvalues. This observation justifies the following assumption.

Assumption 2. All eigenvalues of the operator M lie in the half-plane $\text{Re}\,\lambda < -\beta'$ with some positive β'.

Nonlinear operator. The operator $B_1(v)$ admits the estimate

$$\|B_1(v)\| \le K\|v\| \tag{3.19}$$

in the $C(\mathbb{R})$ norm where

$$K = V(f)\sup_{x,y}|\phi(x-y)\gamma(x,y)| + V(f')\sup_x|1 - J(w)|. \tag{3.20}$$

We can now formulate the main theorem of this section. Its proof follows from the result of Section 3.1.

Theorem 3.2. Suppose that there exists a solution of problem (3.15) such that $w(x) - w_+ \sim e^{\mu_2 x}$ as $x \to -\infty$. Let Assumptions 1 and 2 be satisfied, and the spectrum of the operator M lie in the half-plane $\text{Re}\,\lambda < -\beta$ with some positive β. Suppose that $2RK < \beta - \alpha$, where α is a positive constant, $0 < \alpha < \beta$, K is given by (3.20) and R is the constant in the estimate

$$\|e^{Mt}v(t)\| \le Re^{-\beta t}\|v(0)\|.$$

Then the following estimate holds:

$$\|(u(x,t) - w(x))g(x)\| \leq 2Re^{-\alpha t} \, \|(u(x,0) - w(x))g(x)\|, \quad t \geq 0.$$

3.4 Numerical examples

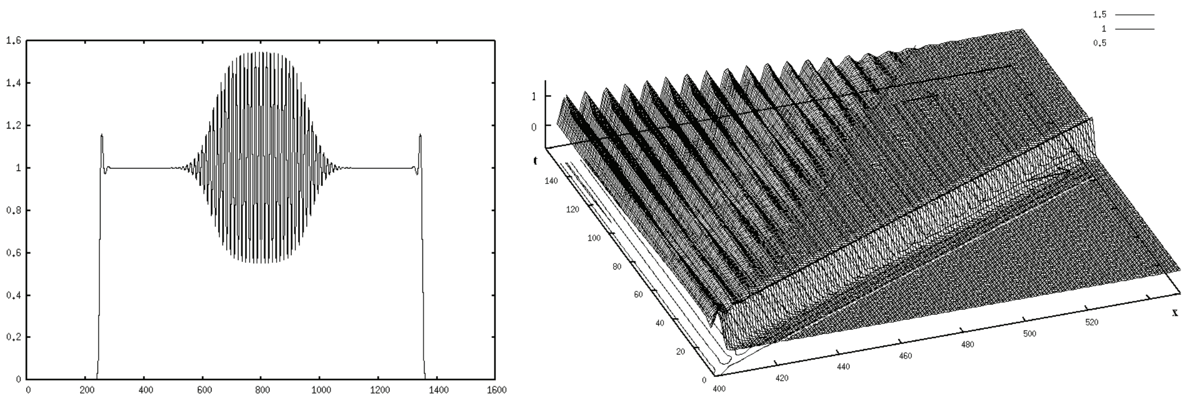


Figure 3: Periodic wave moves slower than the wave between the stationary points. Solution $u(x,t)$ as a function of x for a fixed t. The same solution as a function of two variables.

Figure 3 shows an example of numerical simulations of wave propagation. The wave between 0 and 1 propagates faster than the wave between 1 and the periodic in space stationary solution. The distance between them grows. The first wave is stable on a half-axis. This is in agreement with the result of the previous section. At the same time, this travelling wave is unstable in the uniform norm on the whole axis.

We present here some example of two-dimensional numerical simulations. Qualitatively, they are similar to the 1D case. However, in comparison with the 1D case, there is an additional parameter, the form of the support of the function $\phi(x, y)$. Consider an initial condition with a bounded support in the center of the computational domain. Then we observe a circular travelling wave propagating from the center outside. A spatial structure emerges behind the wave. The pics of the density form a regular square grid in the case of the square support of ϕ (Figure 4, left). In the case of the circular support of the function ϕ, emerging structures are also circular (Figure 4, right). Depending on the parameters, these can be just circles or pics forming circles. Waves and structures in the case of an elliptic support are shown in Figure 5. They can form parallel strips or strips followed by emerging pics.

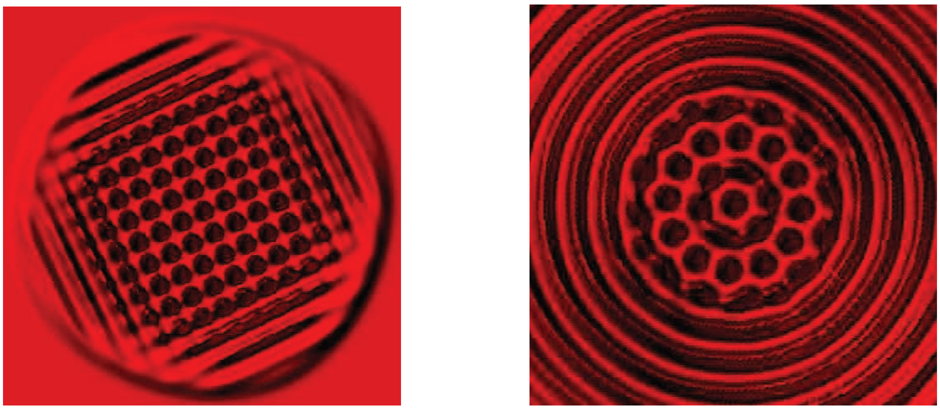


Figure 4: Propagation of circular wave and formation of a structure behind the wave. The maxima of the density form a square grid in the case of a square support of ϕ (left) and a circular structure in the case of a circular support (right).

4 Conclusions

Nonlocal reaction-diffusion equations provide a mechanism of pattern formation in biology, different in comparison with well known and widely discussed Turing structures, mechanochemistry and chemotaxis. In the 1D case and a symmetric kernel ϕ, these are space periodic stationary solutions. In the 2D case, emerging structures depend on the form of the support of the kernel. We observed rectangular and circular structures, strips, and some others.

Transition from the unstable solution $u = 1$ to a periodic in space structure occurs as a periodic travelling wave. Its speed of propagation can be estimated by means of a stability theorem. Indeed, we prove that in a moving frame with a sufficiently large speed, the solution $u = 1$ is stable on a half-axis. This means that if the reference frame moves faster than the periodic structure then, in this reference frame, the solution $u = 1$ invades the whole axis. Propagation of a periodic structure is a result of a bifurcation where the essential spectrum crosses the imaginary axis.

The model considered in this work accounts for two wave types. The first one corresponds to the transition between two stationary solutions, $u = 0$ and $u = 1$. The second one effectuates the transition between $u = 1$ and a periodic structure. Their relative speeds determine the mode of propagation. These two transitions can propagate one after another or they can merge resulting in appearance of a periodic wave.

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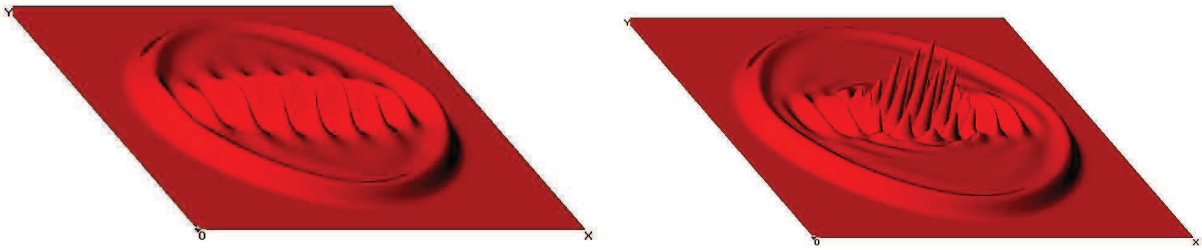

Figure 5: Travelling waves and structures forming behind the waves in the case of elliptic support with two different ratios of the main axis: 2 (left) and 1.5 (right).

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