Stochastic perturbations to dynamical systems: a response theory approach

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Abstract

We study the impact of stochastic perturbations to deterministic dynamical systems using the formalism of the Ruelle response theory. We find the expression for the change in the expectation value of a general observable when a white noise forcing is introduced in the system, both in the case of additive and multiplicative noise. We also show that the difference between the expectation value of the power spectrum of an observable in the stochastically perturbed case and of the same observable in the unperturbed case is equal to twice the square of the intensity of the noise times the square of the modulus of the susceptibility describing the frequency-dependent response of the system to perturbations with the same spatial patterns as the considered stochastic forcing. Using Kramers-Kronig theory, it is then possible to derive the real and imaginary part of the susceptibility and thus deduce the Green function of the system for any desired observable. We provide a example of application of our results by considering the spatially extended chaotic Lorenz 96 model. These results clarify the property of stochastic stability of SRB measures in Axiom A flows, provide tools for analysing stochastic parameterisations and related closure ansatz to be implemented in modelling studies, and introduce new ways to study the response of a system to external perturbations.

1. Introduction

In many scientific fields, numerical modelling is taking more and more advantage of supplementing traditional deterministic numerical models with additional stochastic forcings. This has served the overall goal of achieving an approximate but convincing representation of the spatial and temporal scales which cannot be directly resolved. Moreover, stochastic noise is usually thought as a reliable tool for quickening the exploration of the attractor of the deterministic system, due to the additional mixing due to noise, and, in some heuristic sense, to increase the robustness of the model, by getting rid of potentially pathological solutions. This may be especially desirable when computational limitations hinder our ability to perform long simulations. For both of these reasons, climate science is probably the field where the application of the so-called stochastic parameterisations is presently gaining more rapidly momentum for models of various degrees of complexity, including full-blown GCMs; see, e.g. [1], after having enjoyed an early popularity for enriching and increasing the realism of very simple models with few degrees of freedom [2].

In this paper we wish to provide some new analytical results with a rather large degree of generality on the impact of adding stochastic forcings to deterministic system, with the goal of possibly providing useful guidance for closure problems related to large scale application of stochastic parameterisation. We will study this problem by taking advantage of the response theory developed by Ruelle [3-5] for studying the impact of small perturbations to rather general flows. Whereas the theory has been developed for deterministic perturbations, we will adapt its results to perturbations which have a stochastic nature and derive analytical expressions describing how noise changes the expectation value of a general observable. We will also derive new results which show how the changes in the spectra properties of the system due to introduction of the stochastic perturbations can be used to derive the general properties on the frequency-dependent response of the system – more precisely its susceptibility [5] -   thus providing information on the fine structure of the attractor. This is accomplished by applying Kramers-Kronig theory [6,7] to suitably defined spectral functions

2. Response Theory and Stochastic Perturbations

Let's frame our problem in a mathematically convenient framework. Axiom A dynamical systems of the form $dx_i/dt = F_i(x)$ constitute good models for describing natural or artificial systems for the basic reason that they possess a very special kind of invariant measure $\rho_0(dx)$, usually referred to as SRB measure [8]. Such a measure is first of all a physical measure, i.e. for a set of initial of initial conditions of positive Lebesgue measure the time average $\lim_{T\to\infty}\int_0^T dtA(f^t x)$ of any smooth observables *A,* with *f* being the evolution operator of the flow, converges to the expectation value $\rho_0(A) = \int \rho_0(dx)A(x)$. Another remarkable property of $\rho_0(dx)$ is that it is stochastically stable, i.e. it corresponds to the zero noise limit of the invariant measure of the random dynamical system whose zero-noise is the deterministic system $dx_i/dt = F_i(x)$. See [9,10] for a much broader and more refined description of Axiom A systems and SRB measures.

Ruelle [3-5] has shown that one can construct a response theory able to express the change in the expectation value $\delta_{\varepsilon,t}\rho(A) = \rho_{\varepsilon,t}(A) - \rho_0(A)$ of an arbitrary measurable observable *A* when the flow undergoes a small perturbation of the form $dx_i/dt = F_i(x) \to dx_i/dt = F_i(x) + \varepsilon X_i(x)g(t)$, where $X_i(x)$ is a smooth vector field, $g(t)$ is its time modulation and ε is the order parameter of the perturbation (we introduce such a factor in order to clarify the perturbative expansion). The main result is that $\delta_{\varepsilon,t}\rho(A)$ can be written as a power series:

$$
\begin{aligned}
\delta_{\varepsilon,t}\rho(A) &= \varepsilon\int d\tau G_1(\tau)g(t-\tau)+ \\
&+\varepsilon^2\int d\tau_1 d\tau_2 G_2(\tau_1,\tau_2)g(t-\tau_1)g(t-\tau_2)+\sum_{k=3}^{\infty}\varepsilon^k\int d\tau_1...d\tau_k G_k(\tau_1,...,\tau_k)g(t-\tau_1)...g(t-\tau_k)= \\
&=\varepsilon\int\rho_0(dx)\int d\tau\Theta(\tau)X_i\partial_i A(f^{\tau}x)g(t-\tau)+ \\
&+\varepsilon^2\int\rho_0(dx)\int d\tau\Theta(\tau)_1\Theta(\tau_2-\tau_1)X_i\partial_i\left(f^{\tau_2-\tau_1}\left(X_j\partial_j A(f^{\tau_1}x)\right)\right)g(t-\tau_1)g(t-\tau_2)+o(\varepsilon^3)
\end{aligned}
\tag{1}
$$

where $G_k(\tau_1,...,\tau_k)$ is the k^{th} order Green function of the system, which, convoluted k times with the delayed modulation function $g(t)$, describes the contribution to the response resulting from interaction at the k^{th} order of nonlinearity of the perturbation term. In the previous formula, we have presented the first two terms of the infinite series. Two fundamental properties of formula given in Eq. (1) are that at all orders 1) the Green function is causal, and 2) the contribution to the response is written as expectation value over the unperturbed state of a an observable for which we have n explicit expression. Since $\rho_0(dx)$ is a physical measure, in practical terms the Green function can be evaluated by producing long numerical integration of the dynamical system $dx_i/dt = F_i(x)$ and evaluating suitably the time averages. In this direction, efficient algorithms have been proposed, e.g. in [11]. While the theory strictly applies only to Axiom A systems, the chaotic hypothesis [12] suggests that, when we consider systems with many degrees of freedom and analyse smooth observables, the behaviour is typically close to that of Axiom A systems. Therefore, a much wider applicability beyond the mathematical limits reached so far is reasonable. As an example, numerical evidences suggest that Ruelle's theory provides consistent results for the linear and nonlinear response of the Lorenz 63 system [13] and the Lorenz 96 system [15].

We now assume that the perturbation is modulated by white noise, so that $g(t)=\eta(t)=dW(t)/dt$, where $W(t)$ is a Wiener process. Therefore, $\langle\eta(t)\rangle=0$ and $\langle\eta(t)\eta(t')\rangle=\delta(t-t')$, where the symbol $\langle\bullet\rangle$ describes the operation of averaging over the realizations of the stochastic processes. Note that, since the invariant measure $\rho_0(dx)$ is stochastically stable, as discussed above, it makes sense to compute the impact of weak stochastic perturbations. Since we are actually dealing with a stochastic dynamical system, we redefine our response as $\langle\delta_{\varepsilon,t}\rho(A)\rangle$. When the statistical properties of $\eta(t)$ are taken into consideration, we obtain the following formula for the response of the system:

$$
\langle\delta_{\varepsilon}\rho(A)\rangle=\varepsilon^2\int d\tau_1 G_2(\tau_1,\tau_1)+o(\varepsilon^4)=1/2\,\varepsilon^2\int\rho_0(dx)\int d\tau_1\Theta(\tau_1)X_i\partial_i X_j\partial_j A(f^{\tau_1}x)+o(\varepsilon^4)\,. \tag{2}
$$

All odd order terms are vanishing because of the symmetry properties of the noise, and the leading term results to be proportional to the square of the order parameter ε times the time-independent expectation value of an observable on the unperturbed measure. The factor ½ emerges from evaluating $\Theta(\tau)$ taking a symmetric limit for $\tau \to 0$. As expected, the response has only a static component, as no time-dependence is present. Moreover, the fact that $\langle \delta_\varepsilon \rho(A) \rangle$ depends smoothly on the intensity of noise and vanishes for noise of vanishing intensity is in agreement with the stochastic stability of the SRB measure $\rho_0(dx)$.

A brief diversion to the special case of non-singular invariant measures. If the invariant measure $\rho_0(dx)$ is absolutely continuous with respect to Lebesgue $\rho_0(dx) = \rho_0(x)dx$, as in the case of equilibrium systems, by performing an integration by parts described in [2-4], we obtain for the leading term:

$$\langle \delta_\varepsilon \rho(A) \rangle \approx -\varepsilon^2 \int d\tau \Theta(\tau_1) \int \rho_0(x) dx (\partial_i X_i) X_j \partial_j A(f^{\tau_1} x) \tag{4}$$

where $\partial_i X_i = 1/\rho_0 \, \partial_i(\rho_0 X_i)$ [5], so that the leading term can be written as proportional to the 0th moment of the correlation of two suitably defined observables *B* and *C:*

$$\langle \delta_\varepsilon \rho(A) \rangle \approx -\varepsilon^2 \int d\tau \int \rho_0(dx) B(x) C(f^{\tau_1} x) \tag{5}$$

where $B = \partial_i X_i = 1/\rho_0 \, \partial_i(\rho_0 X_i)$, while $C = X_j \partial_j A(f^{\tau_1} x)$ is such that its expectation value is exactly the linear the Green function of the system - see Eq. (1). This can be loosely interpreted as a second-order version of the fluctuation-dissipation theorem.

Let's now consider the ensemble average over the probability space of the considered stochastic processes of the expectation value of the time correlation of the response of the system. We then consider the following expression:

$$\begin{aligned} \left\langle \int d\sigma \delta_{\varepsilon,\sigma} \rho(A) \delta_{\varepsilon,\sigma-t} \rho(A) \right\rangle &= \varepsilon^2 \int d\sigma d\tau_1 d\tau_2 G_1(\sigma - \tau_1) G_1(\sigma - t - \tau_2) \langle \eta(\tau_1) \eta(\tau_2) \rangle + o(\varepsilon^4) \\ &= \varepsilon^2 \int d\sigma G_1(\sigma) G_1(\sigma - t) + o(\varepsilon^4) \end{aligned} \tag{6}$$

Where, as shown in Eq. (1), $G_1(\tau)=\int \rho_0(dx)\int d\tau\Theta(\tau)X_i\partial_i A(f^\tau x)$. By applying the Fourier transform to both members and using the Wigner-Khinchin theorem, we obtain:

$$\left\langle\left|\delta_{\varepsilon,\omega}\rho(A)\right|^2\right\rangle \approx 2\varepsilon^2\left|\chi_1(\omega)\right|^2, \tag{7}$$

where the susceptibility $\chi_1(\omega)$ is the Fourier Transform of the $G_1(t)$. Therefore, the ensemble average of the power spectrum of the response to the random perturbation is, just like in the case of equilibrium systems (see e.g. the explicit expression for the forced and damped linear oscillator) approximately proportional to the square of the modulus of the linear susceptibility via the square of the order parameter ε. Considering the chaotic nature of the unperturbed flow and the fact that the stochastic perturbation is a white noise, after some algebraic manipulations we derive that:

$$\left\langle P_{\varepsilon,\omega}(A)\right\rangle - P_\omega(A) \approx \left\langle\left|\delta_{\varepsilon,\omega}\rho(A)\right|^2\right\rangle \approx 2\varepsilon^2\left|\chi_1(\omega)\right|^2 \tag{8}$$

i.e., the ensemble average of the power spectrum of the response is up to the second order in ε equal to the difference between the expectation value of the power spectrum of the observable *A* of the perturbed flow and the power spectrum of the observable *A* in the unperturbed flow. Equation (8) is much more useful than Eq. (7) because the left hand side member can be observed much more easily. By measuring experimentally the difference between the power spectrum of an observable in the presence of noise (for several realizations of the noise) and in absence of noise, it is possible to reconstruct the square of modulus of the susceptibility of the system.

As widely discussed in [3-5,13-15], the susceptibility is an analytic function in the upper complex ω-plane, so that it obeys Kramers-Kronig relations [6,7]. Using analyticity, we can derive $\chi_1(\omega)$ from $\left|\chi_1(\omega)\right|$. In fact, we can write $\chi_1(\omega)=\left|\chi_1(\omega)\right|e^{i\varphi(\omega)}$ and, by taking the logarithm to both members, we obtain $\log[\chi_1(\omega)]=\log\left[\left|\chi_1(\omega)\right|\right]+i\varphi(\omega)$. The function $\log[G_1(\omega)]$ is also analytic in the upper complex ω-plane, so that it also obeys Kramers-Kronig relations. Therefore, from the knowledge of the real part for all values of ω we can obtain the value of its imaginary par via a Hilbert transform (and vice versa). In this case, the real part is given by $\log\left[\left|\chi_1(\omega)\right|\right]=1/2\log\left[\left|\chi_1(\omega)\right|^2\right]$, which can be derived by the analysis of the power spectra in the perturbed and unperturbed case using Eq. (8)., whereas the imaginary part, obtained using Kramers-Kronig relations, will give the phase function $\varphi(\omega)$. Note that such standard reconstruction technique is widely used in optics to derive the index of refraction of a material from its reflectivity [16]. The strategy of taking the logarithm of the susceptibility falls into some troubles if the function $G_1(\omega)$ possesses some zeros in the upper complex ω-plane. At any rate, both exact and approximate numerical techniques have been devised to take care of this rather special case [16]. Once we have obtained $\varphi(\omega)$, we can reconstruct $\chi_1(\omega)$, and thus derive full information on the linear response of the system to perturbations with the same spatial pattern and with general temporal modulation. The Green function of the system $G_1(t)$ can be obtained using an inverse Fourier transform and, as discussed in [15], it can be used to predict finite and infinite time horizon response of the system to perturbations with the same spatial structure but general time modulation.

We now analyse the case of a more complex pattern of stochastic forcing, so that the perturbation can be written as $dx_i/dt = F_i(x) \rightarrow dx_i/dt = F_i(x) + \sum_{j=1}^{p} \varepsilon_j X_i^j(x)\eta_j(t)$, where we consider p independent perturbative vector flows $X^j(x)$ and p independent white noise $\eta_j(t)$ such that $\langle \eta_j(t) \rangle = 0 \ \ \forall j$ and $\langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij}\delta(t-t')$, Under these conditions, it is straightforward to prove that:

$$\left\langle \delta_{\{\varepsilon\}}\rho(A) \right\rangle = \sum_{k=1}^{p} \varepsilon_k^2 \int d\tau_1 G_2^k(\tau_1,\tau_1) + o(\varepsilon^4) = 1/2 \sum_{k=1}^{p} \varepsilon_k^2 \int \rho_0(dx) \int d\tau_1 \Theta(\tau_1) X_i^k \partial_i X_j^k \partial_j A\left(f^{\tau_1}x\right) + o(\varepsilon^4) \quad (9)$$

where the various Green functions corresponding to the p different perturbation vector fields are indexed with k. Therefore, even if second order terms are considered, thanks to the independence of the p white noise processes, the impact of each of them sums up linearly. When considering the power spectrum of the response, we obtain an analogous result:

$$\left\langle P_{\varepsilon,\omega}(A) \right\rangle - P_\omega(A) \approx \left\langle \left| \delta_{\{\varepsilon\},\omega}\rho(A) \right|^2 \right\rangle \approx \sum_{k=1}^{p} 2\varepsilon_k^2 \left| \chi_1^k(\omega) \right|^2 . \quad (10)$$

Also in this case the impact of the various stochastic forcings sum up linearly, so that the square moduli of the various Green functions are weighted with the square of the intensity of the noise. It is obvious that by repeating the experiments under varying conditions for the intensity of the p white noise processes, we can disentangle the square modulus of each Green functions and, as described above, eventually of the full Green function. Note that, instead, the application of the Kramers-Kronig relations to the logarithm of the right hand side member of Eq. (10) cannot in genera be used to derive the real and imaginary part of the "effective" susceptibility $\sum_{k=1}^{p}\varepsilon_k\chi_1^k(\omega)$, unless, e.g. the various susceptibilities are identical to each other as in the case of presence of special symmetry properties in the forcings and in the system .

When considering the results on the spectral properties of the response shown in Eqs. (7), (8), (10), the findings presented in this paper are strikingly similar to what would be derived in the much elementary case of systems possessing simple attractors such as fixed points or in the even more basic case of linear dynamical systems. This is the case because we are only exploiting the formal properties of the linear response formula, which, as discussed in [3, 14], are the same in linear, in general equilibrium cases, and in general non-equilibrium cases.

3. A Numerical Experiment

In order to provide further support for our findings, we provide a simple but nontrivial example of numerical investigation along the lines detailed above. We consider the Lorenz 96 model [17,18], which, describes the evolution of a generic atmospheric variable defined in N equally spaced grid points along a latitudinal circle and provides a simple, unrealistic but conceptually satisfying representation of some basic atmospheric processes, such as advection, dissipation, and forcing. This model has a well recognised prototypical value in data assimilation [19, 20], predictability studies [21,22] and has been investigated in detail in terms of linear response theory [11, 15]. The model is defined by the differential equations $dx_i/dt = x_{i-1}(x_{i+1}-x_{i-2})-x_i+F$ with i=1,...,N and the index i is cyclic so that $x_{i-N}=x_i=x_{i+N}$. In order to provide results directly comparable with obtained in [15], we perturb the system with an additive white noise, so that $F\rightarrow F+\varepsilon\eta(t)$ for all grid points i. We take as observables the "intensive energy" $e=\frac{1}{2N}\sum_{j=1}^{N}x_i^2$ and "intensive momentum" $m=\frac{1}{N}\sum_{j=1}^{N}x_i$ of the system and consider the corresponding linear susceptibilities $\chi_{e,N}^{(1)}(\omega)$ and $\chi_{m,N}^{(1)}(\omega)$. We choose standard conditions ($F=8.0$, $N=40$, see discussion in [15] explaining how results can be generalized for all values of F and N as long as the system is chaotic), select $\varepsilon=0.5$ and integrate each of the 1000 members of the ensemble of realisations of the stochastic process for 1000 time units, which correspond to about 5000 days [17,18]. We choose a computationally very inexpensive experiment (all runs have been completed in less than one day on a commercial laptop using MATLAB) and use a rather sub-optimal way to estimate the power spectra, such as taking the square of the fast Fourier transform of the signal, in order to prove the robustness of our approach. The obtained results do not depend on the intensity of the noise for, e.g., $\varepsilon \leq 1$, except that performing simulations using a larger value of ε improves the signal-to-noise ratio. The squared modulus of the susceptibility $\left|\chi_{e,N}^{(1)}(\omega)\right|^2$ obtained using Eq. (8) (blue line) and its direct estimate drawn from the data for published in [15] (black line) are in excellent agreement, so that in Figure 1a we shift vertically one of the two curves to improve readability. We also plot (red line) the high-frequency asymptotic behaviour $\left|\chi_{e,N}^{(1)}(\omega)\right|^2 \approx \rho_0(m)^2/\omega^2$ derived analytically using the short-time expansion of the Green function in [15], in order to show how accurately the methodology presented in this paper is able to capture the response of the system to high frequency perturbations. Similarly, in Figure 1b) we present the corresponding results for $\left|\chi_{m,N}^{(1)}(\omega)\right|^2$, which feature a comparable degree of accuracy throughout the spectral range. In this case, the asymptotic behaviour is $\left|\chi_{m,N}^{(1)}(\omega)\right|^2 \approx 1/\omega^2$ Note that the signals obtained in this work and shown in Figs. 1a,b) are much cleaner and cover a much wider spectral range than what obtained in [15] after carefully running a number of runs larger by about three orders of magnitudes, each with a periodic forcing of different frequency.

4. Conclusions

In this paper we have shown on one side that the impact of stochastic perturbations in the form of additive or multiplicative white noise to deterministic dynamical systems can be effectively studied using the Ruelle response theory. We have shown that, in agreement with the fact that SRB measure feature stochastic stability, the impact of stochastic forcings on the expectation value of a general observable vanishes with vanishing intensity of the noise, and is proportional to the variance of the noise. Moreover, using the second order Ruelle response theory, we have been able to obtain an explicit expression for the proportionality coefficient.

On the other side, we have shown that performing experiments on a given system with and without adding stochastic perturbation provides a new way to access information on its response to more general forcings. The different between the expectation value of the power spectra of an observable in the stochastically perturbed and unperturbed case is proportional to the square modulus of the corresponding susceptibility function of the system via the squared intensity of the system. Then, by using the Kramers-Kronig formalism, we can obtain the susceptibility of the system and, via inverse Fourier Transform, its Green function, which allows us to project perturbations into the future with a finite and infinite time horizon. At practical level, with only one ensemble of runs for the perturbed and unperturbed model we can derive the susceptibility and the

Green function for any desired observable. Therefore, the results exposed in this paper allow for bridging the response of the system to very fast perturbations (in the form of white noise) to its behaviour and predictability at all time scales.

We have clarified some of these results by resorting to an example, namely by considering the “intensive energy” and “intensive momentum” observables for the Lorenz 96 model in standard and stochastically perturbed conditions, where a simple additive white noise is taken into account. We have found that the quality in terms of signal to noise of the obtained squared modulus of the susceptibility and its spectral range are much wider than what derived using directly the response theory with periodic perturbations, even if the computational cost is in the present case much lower. This suggest a very efficient general way to derive the susceptibility of a system bypassing the correct but somewhat cumbersome procedure shown in [13,15].

The direct application of the fluctuation-dissipation theorem to the unperturbed deterministic system is not possible for general deterministic non equilibrium steady state systems described by Axiom A flows. Therefore, the analysis of the internal fluctuations of the unperturbed system does not allow for obtaining the properties of the response to external perturbations – see discussion in [5, 13, 15]. In physical terms, this marks the difference between quasi-equilibrium and non-equilibrium systems [15]. Instead, adding stochastic perturbations smoothens the invariant measure and thus allows for the applicability of the fluctuation-dissipation theorem [23]. This is the fundamental reasons why we are able to obtain the results summarized in Eqs (7) and (8). We hope that this paper may provide stimulation, on one side, for providing a more rigorous analysis of stochastic perturbations on complex systems like the climate’s, and, on the other side, to further investigate the relevance of Ruelle response theory and of its spectral counterpart based upon Kramers-Kronig relations for studying the response of general systems to perturbations. Future development include the investigations of forcings whose stochastic components have memory, and the even more general case where we consider a non-separable in space and time random forcing of the form $X(x,t)$.

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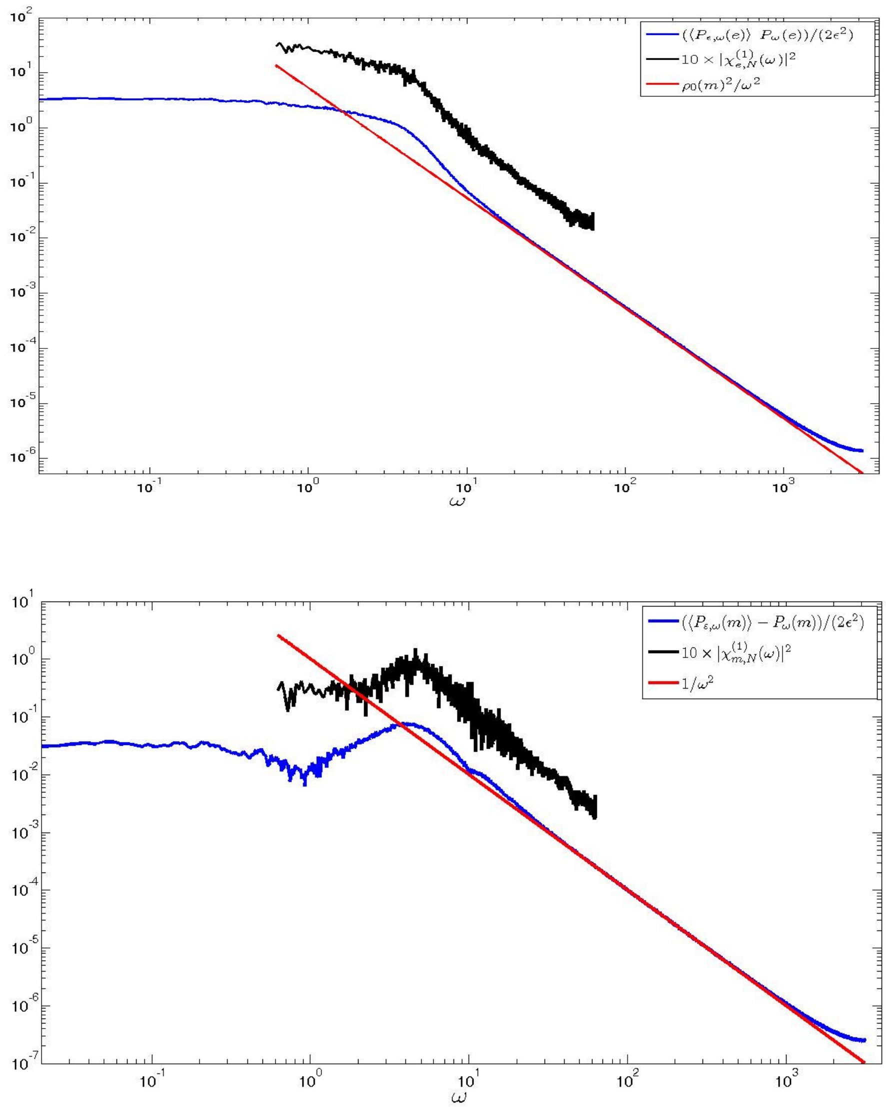

a)

b)

Figure 1: How to reconstruct the square modulus of the susceptibility using stochastic perturbations. a) Blue line: evaluation of $\left|\chi^{(1)}_{e,N}(\omega)\right|^2$ via analysis of the impact of stochastic forcing using the formula given in Eq. (8). Black line: direct evaluation of $\left|\chi^{(1)}_{e,N}(\omega)\right|^2$ (data taken from [15]). The curve has been shifted (see legend) to improve readability. Red line: asymptotic behaviour derived analytically in [15]. b) Same as a), but for the function $\left|\chi^{(1)}_{m,N}(\omega)\right|^2$.